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Navier-Stokes flow down an inclined plane: Downward periodic motion

Dedicated to Professor Takeshi Kotake
at the occasion of his sixtieth birthday

By

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1. Introduction

Let us consider two-dimensional motion of a viscous incompressible fluid flowing down an inclined plane under the influence of gravity. The motion is governed by the Navier-Stoke equations. Following [3], we consider fluctuations on a laminar steady motion described by the velocity field,

$$\bar{u}_1=(g\sin\alpha/2\nu)(2h_0x_2-x_2^2)\,,\quad \bar{u}_2=0\,,$$

and the scalar pressure,

$$\bar{p}=\bar{\omega}-\rho g\cos\alpha(x_2-h_0)\,,$$

which takes place in the slab $\{(x_1, x_2)\in \boldsymbol{R}^2;\ 0<x_2<h_0\}$. Here we choose a coordinate system (x_1, x_2), where x_1 is down and x_2 is normal to the plane. The given constants are as follows: g is the acceleration of gravity, α the angle of inclination, ν the kinematic visosity, ρ the density of the fiuid, $\bar{\omega}$ the atmospheric pressure.

In order to formulate the problem for disturbances from the laminar flow, we introduce dimensionless variables. Put $U_0=gh_0^2\sin\alpha/2\nu$ and $p_0=\rho gh_0\sin\alpha$. We take h_0, U_0 and p_0 as the unit for length, velocity and pressure respectively. Then we come to consider the following form of the free boundary problem,

$$(1.1)\qquad \partial_t\eta+(1-\eta^2+u_1)\partial_1\eta-u_2=0 \qquad \text{on } x_2=1+\eta(t, x_1),\ t>0\,,$$

$$(1.2)\qquad \partial_t\eta+(U+u, \nabla)(u+U)=-\nabla\Big(\frac{2}{\mathfrak{R}}p\Big)+\frac{1}{\mathfrak{R}}\Delta u \qquad \text{in } 0<x_2<1+\eta(t, x_1),\ t>0\,,$$

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(1.3) $\partial_1 u_1 + \partial_2 u_2 = 0 \quad \text{in } 0 < x_2 < 1 + \eta(t, x_1), \quad t > 0\,,$

(1.4) $u = 0 \quad \text{on } x_2 = 0\,,$

(1.5) $$(\partial_1 u_2 + \partial_2 u_1 - 2\eta)(1 - (\partial_1 \eta)^2) + 2(\partial_1 \eta)(\partial_2 u_2 - \partial_1 u_2) = 0$$
$$\text{on } x_2 = 1 + \eta(t, x_1),\ t > 0\,,$$

(1.6) $$p - \eta \cot\alpha - \frac{1}{1 + (\partial_1 \eta)^2}(\partial_2 u_2 + (\partial_1 \eta)^2(\partial_1 u_1) - (\partial_1 \eta)(\partial_1 u_2 + \partial_2 u_1 - 2\eta))$$
$$+ \mathcal{W} \csc\alpha \frac{\partial_1{}^2 \eta}{(1 + (\partial_1 \eta)^2)^{3/2}} = 0 \qquad \text{on } x_2 = 1 + \eta(t, x_1),\ t > 0\,.$$

The problem contains two dimensionless quantities:

$$\mathcal{R} = \frac{U_0 h_0}{\nu} = \frac{g h_0{}^3 \sin\alpha}{2\nu}, \quad \mathcal{W} = \frac{T_e}{\rho g h_0{}^2}\,,$$

$\mathcal{R}$ being a Reynolds number, $\mathcal{W}$ a Weber unmber, where T_e is surface tension. $U = (2x_2 - x_2{}^2, 0)$ is the nondimensionalized form of the velocity of the laminar flow. We refer to [3, pp 150-152] for derivation of (1.1)-(1.6). The upper free surface is supposed to be given by the graph $\{(x_1, x_2);\ x_2 = 1 + \eta(t, x_1)\}$ at time $t \geq 0$. The unknowns u and p are defined in $\{(x_1, x_2);\ 0 < x_2 < 1 + \eta(t, x_1)\}$ and there describing the fluctuation on the steady motion.Throughout this paper we assume that the fluctuation is downward periodic, and that, for simplicity, the period is 2π.

The purpose of this paper is to show that, when $\mathcal{R}$ and α is sufficiently small, we can obtain global in time solutions for sufficiently smal initial data. The main result will be given in the last section.

We proceed as follows. We introduce in Sect. 2 notations, function spaces and auxiliary lemmas. In Sect. 3, as in [2], we transform (1.1)-(1.6) to the problem on the fixed domain $\Omega = (0, 2\pi) \times (0, 1)$ in $\boldsymbol{R}^2$. We recall in Sect. 4 the existence of local in time solutions obtained in [10] with some modification for our purpose. We carry out the energy estimates in Sect. 5. Using these we show the existence of global in time solutions and their decay property under the assumptions stated above. The methods used in Sect. 5 and 6 are similar to those in [7]. For other results see [8].

2. Preliminaries

Let $r \geq 0$. For an open set $\mathcal{O}$ in $\boldsymbol{R}^n$, $H^r(\mathcal{O})$ is the usual Sobolev space. (See [1 or 6].) $H^r_{loc}(\mathcal{O})$ is the space of functions which are defined in $\mathcal{O}$ and are in $H^r(\mathcal{O}')$ for any bounded open set $\mathcal{O}'$ in $\mathcal{O}$. Let $\Omega = (0, 2\pi) \times (0, 1)$ in $\boldsymbol{R}^2$. We denote by $H_p{}^r(\Omega)$ the space of functions which are in $H^r_{loc}(\boldsymbol{R} \times (0, 1))$ and are periodic with respect to the first variable x_1 with period 2π. We set $S_F = \partial\Omega \cap \{x_2 = 1\}$ and $S_B = \partial\Omega \cap \{x_2 = 0\}$. We identify S_F with the open interval

$(0, 2\pi)$. $H_p^r(S_F)$ denotes the space of functions which are in $H_{loc}^r(\boldsymbol{R})$ and the periodic with period 2π. Set

$$H_{p0}^r(S_F)=\left\{\phi\in H_p^r(S_F);\ \int_0^{2\pi}\phi=0\right\}.$$

Let $r\geq 1/2$. For $\phi\in H_{p0}^{r-1/2}(S_F)$, we define its extension $\tilde{\phi}$ to Ω by

$$\tilde{\phi}(x_1, x_2)=\frac{1}{\sqrt{2\pi}}\sum_{k\neq 0}\frac{\phi_k}{1+k^2(x_2-1)^2}e^{ikx_1}. \tag{2.1}$$

where $\{\phi_k\}$ is the Fourier coefficients of ϕ.

Lemma 2.1. *Let* $r\geq 1$. *For* $\phi\in H_{p0}^{r-1/2}(S_F)$, $\tilde{\phi}\in H_p^r(\Omega)$.

This is the usual property of extension operator, so we omit the proof. We denote the norms of $H_p^r(\Omega)$ and $H_p^{r'}(S_F)$ by $\|\cdot\|_{r,\Omega}$ and $|\cdot|_{r',S_F}$, respectively. For later use we introduce an integral identity:
Suppose u, $v\in H_p^2(\Omega)$, $q\in H_p^1(\Omega)$, $u=0$ on S_B, and, further, $div\ u=div\ v=0$ in Ω. Then, integration by parts yields

$$\frac{1}{\mathfrak{R}}\int_\Omega(-\Delta v+\nabla(2q))u=\frac{1}{\mathfrak{R}}\langle v, u\rangle+\int_{S_F}S(v, q)u\ , \tag{2.2}$$

where

$$\langle v, u\rangle=\frac{1}{2}\int_\Omega(\partial_j v_k+\partial_k v_j)(\partial_j u_k+\partial_k u_j)$$

and

$$S(v, q)_1=-\frac{1}{\mathfrak{R}}(\partial_1 v_2+\partial_2 v_1)\ ,\quad S(v, q)_2=\frac{2}{\mathfrak{R}}(q-\partial_2 v_2)$$

(See, e.g., [5], Chapter 3., Section 2.)

Here and hereafter we use the summation convention: Sum over repeated indices. The lemma below is crucial.

Lemma 2.2. *Suppose* $u\in H_p^1(\Omega)$ *satisfies* $u=0$ *on* S_B. *Then, there exist positive numbers* K_1, K_2 *such that*

i) $K_1\|\nabla u\|_0^2\leq\langle u, u\rangle$,

ii) $K_2\|u\|_0\leq\langle u, u\rangle^{1/2}$.

For the proof see [4].
In the following we assume that $\mathfrak{R}$ is so small that

$$K_0=\frac{1}{\mathfrak{R}}-2K_2^{-2}>0\ . \tag{2.3}$$

We frequently use the lemma below to estimate the nonlinear terms.

Lemma 2.3. *i) If nonnegative numbers* γ_1, γ_2 *and* γ_3 *satisfy* $\gamma_1+\gamma_2-\gamma_3>1$, *then there is a positive constant* K_3 *such that*

$$\|\phi\psi\|_{\gamma_3}\leq K_3\|\phi\|_{\gamma_1}\|\psi\|_{\gamma_2}, \quad \phi\in H_p^{\gamma_1}(\Omega), \quad \psi\in H_p^{\gamma_2}(\Omega)\,.$$

ii) If nonnegative numbers γ_1, γ_2 *and* γ_3 *satisfy* $\gamma_1+\gamma_2-\gamma_3>1/2$, *then there is a positive constant* K_4 *such that*

$$|\phi\psi|_{\gamma_3}\leq K_4|\phi|_{\gamma_1}|\psi|_{\gamma_2}, \quad \phi\in H_p^{\gamma_1}(S_F), \quad \psi\in H_p^{\gamma_2}(S_F)\,.$$

Proof. Modifying the proof of [9, Lemma 1] slightly, we can show that there is a K such that, if $\gamma_1+\gamma_2-\gamma_3>n/2$, then

$$\|\phi\psi\|_{\gamma_3}\leq K\|\phi\|_{\gamma_1}\|\psi\|_{\gamma_2} \quad \text{for} \quad \phi\in H^{\gamma_1}(\boldsymbol{R}^n),\ \psi\in H^{\gamma_2}(\boldsymbol{R}^n)\,.$$

Using this we can show our case by extending the functions appropriately.

Let B be a Banach space. By $H^s(0, T; B)$ we denote the space of B-valued H^s- functions defined on the interval $(0, T)$. We set $H^{r,r/2}(\Omega)=H^0(0, T; H_p^r(\Omega))\cap H^{r/2}(0, T; H_p^0(\Omega))$ and $H_0^{r',r'/2}(S_F)=H^0(0, T; H_{p0}^{r'}(S_F))\cap H^{r'/2}(0, T; H_{p0}(S_F))$. The space $C^l(t_1, t_2; B)$ is defined in the usual way.

3. Reduction to fixed domain

Let us assume that, at time $t\geq 0$, the time dependent domain

$$\Omega(t)=\{(x_1, x_2);\ 0<x_2<1+\eta(t, x_1))\}$$

is given by a diffeomorphism $\Omega\rightarrow\Omega(t)$ defined by

$$x_1=x_1', \quad x_2=x_2'(1+\tilde{\eta}(t', x')); \tag{3.1}$$

$$t=t'; \quad x'=(x_1', x_2')\in\Omega\,,$$

where $\tilde{\eta}$ is the extension of η to Ω (see (2.1)). Put $\zeta_{jk}=\partial x_j'/\partial_{x_k}$ and $a_{jk}=\mathcal{J}^{-1}\partial x_j/\partial x_k'$, j, $k=1, 2$. Here $\mathcal{J}=\det(\partial x_j/\partial x_k')=1+\partial_2(x_2\tilde{\eta})$. Assume that the unknowns u and p on $\Omega(t)$ are given by the vector field u' and the scalar p' on Ω as follows

$$u_j=a_{jk}u_k', \quad j=1, 2, \quad p(x, t)=p'(x', t')\,.$$

Substitute these into (1.1)-(1.6), then, after some calculation, we obtain

$$\partial_t\eta=-\partial_1\eta+u_2+\eta^2\partial_1\eta \quad \text{on} \quad x_2=1\,, \tag{3.2}$$

$$\partial_t u_1-\frac{1}{\mathfrak{R}}\Delta u_1+(2x_2-x_2^2)\partial_1 u_1+2(1-x_2)u_2+\frac{2}{\mathfrak{R}}\partial_1 p \tag{3.3}$$

$$=f_1(\eta, u)-\frac{2}{\mathfrak{R}}(\partial_2(x_2\tilde{\eta})\partial_1 p-x_2\partial_1\tilde{\eta}\partial_2 p)\,,$$

(3.4)
$$\partial_t u_2 - \frac{1}{\mathcal{R}} \Delta u_2 + (2x_2 - x_2^2)\partial_1 u_2 + \frac{2}{\mathcal{R}} \partial_2 p$$
$$= f_2(\eta, u) - \frac{2}{\mathcal{R}} \left(-x_2 \partial_1 \tilde{\eta} \partial_1 p + \frac{(x_2 \partial_1 \tilde{\eta})^2 - \partial_2(x_2 \tilde{\eta})}{\mathcal{G}} \partial_2 p \right),$$

(3.5)
$$\partial_1 u_1 + \partial_2 u_2 = 0, \quad \text{in } \Omega,$$

(3.6)
$$u = 0 \quad \text{on} \quad x_2 = 0,$$

(3.7)
$$\partial_1 u_2 + \partial_2 u_1 - 2\eta = h_1(\eta, u),$$

(3.8)
$$p - \partial_2 u_2 - (\eta \cot \alpha - \mathcal{W} \csc \alpha \partial_1^2 \eta) = h_2(u, \eta), \qquad \text{on } x_2 = 1.$$

Here we dropped primes " $'$ ". $f_j (j=1, 2)$ in the right hand sides of (3.3)-(3.4) do not contain p, but u, $\tilde{\eta}$ and the derivatives. The same is true for $h_j (j=1, 2)$. Since the diffeomorphism (3.1) depends on t, we have to note that

$$\partial_t = \partial_{t'} - \mathcal{G}^{-1} x_2' \partial_{t'} \tilde{\eta} \partial_2'.$$

From the definitinon of extension it follows that $\partial_{t'} \tilde{\eta} = (\partial_{t'} \eta)^{\sim}$. Hence, by using (3.2), we can replace $\partial_{t'} \tilde{\eta}$ in the right hand sides of (3.3)-(3.4) by the extension of the right hand side of (3.2). In what follows we denote the matrix of coefficients of $\nabla \frac{2}{\mathcal{R}} p$ in (3.3)-(3.4) by $b(\eta)$. For details of this transformation, see [2].

From now on we investigate the solvability of (3.2)-(3.8) with initial condition

(3.9)
$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \quad \eta(\cdot, 0) = \eta_0 \text{ on } S_F.$$

4. Local existence

We first introduce the coordinates

$$t = t', \quad x_1 = x_1' + t', \quad x_2 = x_2'.$$

This makes no essential change in treating local in time solutions. By this coordinate change, $\partial_t + \partial_1$ is transformed to $\partial_{t'}$, and (3.2)-(3.4) become

$$\partial_{t'} \eta = u_2 + \eta^2 \partial_1' \eta,$$
$$\partial_{t'} u_1 - \frac{1}{\mathcal{R}} \Delta u_1 + (1 - x_2^2) \partial_1' u_1 + \cdots = f(\cdots),$$

which can be viewed as the two dimensional and downward periodic case of the problem treated in [10].

We recall

Proposition 4.1. *Assume* $0<\delta<1/4$. *Let* $u_0 \in H_p^{2+2\delta}(\Omega)$ *and* $\eta_0 \in H_{p0}^{5/2+2\delta}$. *Suppose that* u_0 *and* η_0 *satisfy*

$$div\, u_0=0 \quad in \quad \Omega\,, \tag{4.1}$$

$$u_0=0 \quad on \quad S_B\,, \tag{4.2}$$

$$\partial_1 u_{0,2}+\partial_2 u_{0,1}-2\eta_0=h_1(u_0, \eta_0) \quad on \quad S_F\,. \tag{4.3}$$

Fix $T_0>0$ *arbitrarily. Then there exist positive numbers* C_0, ε_0 *depending on* T_0 *such that, if* $J_0 \equiv \|u_0\|_{2+2\delta}+|\eta_0|_{5/2+2\delta} \leq \varepsilon_0$, *then the problem* (3.2)-(3.9) *has a unique solution* (η, u, p) *satisfying*,

$$\eta \in H_0^{7/2+2\delta, 7/4+\delta}(S_F)\,, \quad u \in H^{3+2\delta, 3/2+\delta}(\Omega)\,, \quad \nabla p \in H^{1+2\delta, 1/2+\delta}(\Omega)$$
$$and \ \ p|_{S_F} \in H^{3/2+2\delta, 3/4+\delta}(S_F)\,,$$

and, further,

$$\|(\eta, u, p)\| \leq C_0 E_0\,.$$

Here $\|(\cdots)\|$ is the sum of the corresponding norms.

5. Energy estimates

Fix $T>0$. Suppose that (η, u, p) is a solution of (3.2)-(3.8) for $0 \leq t \leq T$. The purpose of this section is to show

Proposition 5.1. *There are positive constants* α_0, ϵ_1, M, *and* γ *such that, if the angle of inclination* α *is such that* $0<\alpha \leq \alpha_0$, *and if the solution* (η, u, p) *satisfies*

$$\sup_{0 \leq t \leq T} \{|\eta(t)|_{\frac{5}{2}}+\|u(t)\|_2\} \ \leq \ \epsilon_1\,, \tag{5.1}$$

then it holds that

$$|\eta(t)|_3+\|u(t)\|_2 \leq Me^{-\gamma t}\{|\eta_0|_3+\|u_0\|_2\} \tag{5.2}$$

for $0 \leq t \leq T$.

The proof of Proposition 5.1 is divided into several steps. To derive the a priori estimates we assume that the solution is smooth enough, otherwise we only need to use the usual mollification.

We note some estimates of the elliptic boundary value problem of the stationary Stokes system.

Proposition 5.2. *Let* v *and* q *satisfy*

(5.3) $$-\frac{1}{\mathfrak{R}}\Delta v+\frac{2}{\mathfrak{R}}\nabla q=f_0 \quad in\ \Omega\,,$$

(5.4) $$div\ v=0 \quad in\ \Omega\,,$$

(5.5) $$v=0 \quad on\ S_B\,,$$

(5.6) $$v=\varphi \quad on\ S_F$$

then it holds for all $l\geq 0$

(5.7) $$\|v\|_{l+2}+\|\nabla q\|_l\leq C(\|f_0\|_l+|\varphi|_{l+2-\frac{1}{2}})\,.$$

Proposition 5.3. *Let* v *and* q *satisfy* (5.3)-(5.5)

(5.8) $$v_2=\varphi_1 \quad on\ S_F\,,$$

(5.9) $$\partial_1 v_2+\partial_2 v_1=\varphi_2 \quad on\ S_F\,,$$

then it holds for all $l\geq 0$

(5.10) $$\|v\|_{l+2}+\|\nabla q\|_l\leq C(\|f_0\|_l+|\varphi_1|_{l+2-\frac{1}{2}}+|\varphi_2|_{l+1-\frac{1}{2}})\,.$$

These come from the facts that the system (5.3)-(5.4) is elliptic in the sense of Agmon-Douglis-Nirenberg, and that the sets of boundary conditions ((5.5)-(5.6) or (5.5) and (5.8)-(5.9)) satisfy the complementary condition (see [3, page 317]).

I) We now estimate (u, p) in terms of the norms of $\partial_t u$ and $\partial_1{}^j u(j=0, 1, 2)$. We regard (u, p) as a solution of elliptic boundary value problem

(5.11) $$-\frac{1}{\mathfrak{R}}\Delta u+\nabla\left(\frac{2}{\mathfrak{R}}p\right)$$
$$=-\partial_t u-(U,\nabla)u-(u,\nabla)U+f(\eta, u)+b(\eta)\nabla\frac{2}{\mathfrak{R}}p, \text{ in } \Omega\,,$$

(5.12) $$div\ u=0 \quad \text{in } \Omega\,,$$

(5.13) $$u=0 \quad \text{on } S_B\,,$$

(5.14) $$u=u|_{S_F} \quad \text{on } S_F\,.$$

Then by Proposition 5.2 we obtain

(5.15) $$\left\|\frac{1}{\mathfrak{R}}u\right\|_{l+2}+\left\|\nabla\frac{2}{\mathfrak{R}}p\right\|_l$$
$$\leq C\Big\{\|\partial_t u\|_l+\|(U,\nabla)u+(u,\nabla)U\|_l+\|f(\eta, u)\|_l$$

$$+\left\|b(\eta)\nabla\frac{2}{\mathcal{R}}p\right\|_l+|u|_{S_F}|_{l+2-\frac{1}{2}}\Big\}, \quad l=0, 1 .$$

The boundary term on the right can be estimated as follows

$$|u|_{l+2-1/2}\leq C\{|u|_0+|\partial_1{}^{l+1}u|_{\frac{1}{2}}\}\leq C\{\|\nabla u\|_0+\|\nabla\partial_1{}^{l+1}u\|_0\} .$$

We next give the bounds of the norms of the nonlinear terms. From Lemma 2.3 and Lemma 2.1 it follows that

$$\left\|b(\eta)\nabla\frac{2}{\mathcal{R}}p\right\|_l\leq C\|b(\eta)\|_{l+2}\left\|\nabla\frac{2}{\mathcal{R}}p\right\|_l\leq C|\eta|_{5/2}\left\|\nabla\frac{2}{\mathcal{R}}p\right\|_l$$

In $f(\eta, u)$ the terms containing the third order derivatives of $\tilde{\eta}$ can be estimated by using Lemma 2.3 as follows

$$\|C(\tilde{\eta},\nabla\tilde{\eta})(\partial^3_{i,j,k}\tilde{\eta})u_\lambda\|_l\leq C(\epsilon_1)\|\partial^3_{i,j,k}\tilde{\eta}\|_l\|u\|_2\leq C(\epsilon_1)\|u\|_2|\partial_1{}^{l+\frac{5}{2}}\eta|_0 .$$

The terms containinbg second order derivatives of u also can be estimated as follows

$$\|C(\tilde{\eta},\nabla\tilde{\eta})(\partial_\lambda\tilde{\eta})\partial^2_{i,j}u_k\|_l\leq C(\epsilon_1)|\eta|_{\frac{5}{2}}\|u\|_{2+l} .$$

The terms in $f(\eta, u)$ other than the ones referred above have the form

$$C(\tilde{\eta},\nabla\tilde{\eta})u_j\partial_j u_\lambda$$

or

$$C'(\tilde{\eta},\nabla\tilde{\eta})\partial_1{}^{j_1}\partial_2{}^{j_2}\tilde{\eta}\partial_1{}^{k_1}\partial_2{}^{k_2}u_\lambda \quad 0\leq j_1+j_2\leq 2 , \ 0\leq k_1+k_2\leq 1 .$$

In view of the explicit forms of the coefficients $C(\tilde{\eta},\nabla\tilde{\eta})$ and C', we can regard these as bounded coefficients by Sobolev's lemma. Hence we can estimate $\|f(\eta, u)\|_l$, $l=0$, 1, as follows

$$\|f(\eta, u)\|_l\leq C\{\|u\|_2\|u\|_{l+1}+|\partial_1{}^{\frac{7}{2}}\eta|_0\|u\|_2+|\partial_1{}^{\frac{5}{2}}\eta|_0\|u\|_3+|\partial_1{}^{\frac{5}{2}}\eta|_0\|u\|_{2+l}\}$$

Collecting these we obtain

$$\left\|\frac{1}{\mathcal{R}}u\right\|_{l+2}+\left\|\nabla\frac{2}{\mathcal{R}}p\right\|_l \tag{5.16}$$

$$\leq C\Big\{\|\partial_t u\|_l+\|\partial_1 u\|_l+\|u\|_l+\|u\|_2\|u\|_{l+1}+|\partial_1{}^{\frac{7}{2}}\eta|_0\|u\|_2$$

$$+|\partial_1{}^{\frac{5}{2}}\eta|_0\|u\|_3+|\partial_1{}^{\frac{5}{2}}\eta|_0\|u\|_{2+l}+|\eta|_{\frac{5}{2}}\left\|\nabla\frac{2}{\mathcal{R}}p\right\|_l+\|\nabla u\|+\|\nabla\partial_1{}^{l+1}u\|\Big\} .$$

We now need to give the bounds of $|\partial_1{}^{\frac{7}{2}}\eta|_0$ in terms of u and its derivatives. Applying $-\partial_1^{\frac{3}{2}}$ to (3.8), multiplying $\partial_1^{\frac{7}{2}}\eta$ to both sides and integrating the resulting identity over $(0, 2\pi)$, we obtain

$$(\partial_1^{\frac{3}{2}}\eta\cot\alpha - \mathcal{W}\csc\alpha\partial_1^{\frac{7}{2}}\eta, \partial_1^{\frac{7}{2}}\eta)$$
$$=(\partial_1^{\frac{1}{2}}(\partial_1 p - \partial_1\partial_2 u_2), \partial_1^{\frac{7}{2}}\eta)_{S_F} - (\partial_1^{\frac{3}{2}}h_2(\eta, u), \partial_1^{\frac{7}{2}}\eta)_{S_F}.$$

Note that $-\partial_1\partial_2 u_2 = \partial_1^2 u_1$ from (3.5). Integrating by parts in the left hand side and substituting $\partial_1\partial_2 u_2 = -\partial_1^2 u_1$ into the right hand side, we obtain

$$(5.17)\qquad |\partial_1^{\frac{5}{2}}\eta|_{S_F}^2\cot\alpha + \mathcal{W}\csc\alpha|\partial_1^{\frac{7}{2}}\eta|_{S_F}^2$$
$$\le |\partial_1 p + \partial_1^2 u_1|_{\frac{1}{2}}|\partial_1^{\frac{7}{2}}\eta|_{S_F} + |\partial_1^{\frac{3}{2}}h_2||\partial_1^{\frac{7}{2}}\eta|_{S_F}$$
$$\le (\|\nabla p\|_1 + \|\nabla\partial_1^2 u_1\|_0)|\partial_1^{\frac{7}{2}}\eta|_{S_F} + |\partial_1^{\frac{3}{2}}h_2(\eta, u)|_{S_F}|\partial_1^{\frac{7}{2}}\eta|_{S_F}$$

Since $H^{\frac{3}{2}}(0, 2\pi)$ is a Banach algebra (see [1]), taking account of the explicit form of $h_2(\eta, u)$, we can easily estimate $|\partial_1^{\frac{3}{2}}h_2(\eta, u)|_{S_F}$ as follows

$$(5.18)\qquad |\partial_1^{\frac{3}{2}}h_2(\eta, u)|_{S_F} \le C\{|u_1|_{\frac{3}{2}}|\partial_1\eta|_{\frac{3}{2}} + |\partial_1 u_1|_1|\eta|_{\frac{5}{2}} + |u_1|_1|\partial_1^{\frac{7}{2}}\eta|_0$$
$$+ (\|\nabla\partial_1 u\| + \|\nabla\partial_1^2 u\|)|\eta|_{\frac{5}{2}} + \mathcal{W}\csc\alpha|\eta|_{\frac{5}{2}}|\partial_1^{\frac{7}{2}}\eta|_0 + |\eta|_{\frac{3}{2}}|\partial_1\eta|_{\frac{3}{2}}\}.$$

From this and (5.17), we easily obtain

$$(5.19)\qquad |\partial_1^{\frac{7}{2}}\eta|_{S_F} \le C\mathcal{W}^{-1}\sin\alpha\{\|\nabla p\|_1 + \|\nabla\partial_1^2 u\| + (\text{the right hand side of (5.18)})\}.$$

Here we note that the usual trace theorem tells us that

$$|u|_{S_F}|_1 \le C\|u\|_2,$$
$$|u|_{S_F}|_{\frac{3}{2}} \le C(|u|_0 + |\partial_1 u_1|_{\frac{1}{2}}) \le C(\|\nabla u\| + \|\nabla\partial_1 u\|).$$

Combining (5.16) and (5.19), then taking account that $|\eta|_{\frac{5}{2}}$ and $\|u\|_2$ are small enough, we can get

$$(5.20)\qquad \left\|\frac{1}{\mathcal{R}}u\right\|_{l+2} + \left\|\nabla\frac{2}{\mathcal{R}}p\right\|_l$$
$$\le C(\|\partial_t u\|_l + \|\nabla u\| + \|\nabla\partial_1 u\| + \|\nabla\partial_1^2 u\|).$$

II) We now begin to obtain an energy inequality.
1st. Step.) We take the inner product of (3.3)-(3.4) with u, and use the integral identity (2.2) and the fact that

$$((U, \nabla)u, u)_\Omega = 0,$$

to get

$$(5.21)\qquad \frac{1}{2}\frac{d}{dt}\|u\|^2+\frac{1}{\mathfrak{R}}\langle u, u\rangle+2\int_\Omega(1-x_2)u_2u_1$$

$$+\left(-\frac{1}{\mathfrak{R}}\right)\int_{S_F}(\partial_1u_2+\partial_2u_1)u_1+\frac{2}{\mathfrak{R}}\int_{S_F}(p-\partial_2u_2)u_2$$

$$=(f, u)_\Omega+\left(b(\eta)\frac{2}{\mathfrak{R}}\nabla p, u\right)_\Omega.$$

The boundary terms in the left hand side can be rewritten as

$$\left(-\frac{1}{\mathfrak{R}}\right)\int_{S_F}(\partial_1u_2+\partial_2u_2)u_1+\frac{2}{\mathfrak{R}}\int_{S_F}(p-\partial_2u_2)u_2$$

$$=-\frac{1}{\mathfrak{R}}\int_{S_F}(2\eta+h_1)u_1$$

$$+\frac{2}{\mathfrak{R}}\int_{S_F}(\eta\cot\alpha-\mathcal{W}\csc\alpha\partial_1{}^2\eta)(\partial_t\eta+\partial_1\eta-\eta^2\partial_1\eta)+\frac{2}{\mathfrak{R}}\int_{S_F}h_2u_2$$

$$=-\frac{1}{\mathfrak{R}}\int_{S_F}2\eta u_1+\frac{1}{\mathfrak{R}}\frac{d}{dt}\{\cot\alpha|\eta|_{S_F}^2+\mathcal{W}\csc\alpha|\partial_1\eta|_{S_F}^2\}$$

$$+\frac{2}{\mathfrak{R}}(\eta\cot\alpha-\mathcal{W}\csc\alpha\partial_1{}^2\eta,\ -\eta^2\partial_1\eta)_{S_F}+\left(-\frac{1}{\mathfrak{R}}\right)\int_{S_F}h_1u_1++\frac{2}{\mathfrak{R}}\int_{S_F}h_2u_2$$

in view of the boundary conditions (3.7)-(3.8) and the equation for η, (3.2). Thus, using Lemma 2.2 and the assumption (2.3), we have

$$(5.22)\qquad \frac{1}{2}\frac{d}{dt}\|u\|^2+K_0'\|\nabla u\|^2+\frac{1}{\mathfrak{R}}\frac{d}{dt}\{\cot\alpha|\eta|_{S_F}^2+\mathcal{W}\csc\alpha|\partial_1\eta|_{S_F}^2\}$$

$$\leq\frac{2}{\mathfrak{R}}(\eta, u_1)_{S_F}+\frac{2}{\mathfrak{R}}(\eta\cot\alpha-\mathcal{W}\csc\alpha\partial_1{}^2\eta,\ -\eta^2\partial_1\eta)_{S_F}$$

$$+\frac{1}{\mathfrak{R}}\int_{S_F}h_1u_1++\frac{2}{\mathfrak{R}}\int_{S_F}h_2u_2+(f, u)_\Omega+\left(b(\eta)\nabla\frac{2}{\mathfrak{R}}p, u\right)_\Omega.$$

Here we put $K_0'=K_1K_0$.

2nd. Step.) We differentiate (3.3)-(3.4) with respect to x_1 and take the inner product with ∂_1u. Since $(\partial_1\eta, \partial_1u, \partial_1p)$ satisfies (3.8)-(3.9) with the nonlinear terms replaced by their derivatives with respect to x_1, in just the same way as above, we obtain

$$(5.23)\qquad \frac{1}{2}\frac{d}{dt}\|\partial_1u\|^2+K_0'\|\nabla\partial_1u\|^2+\frac{1}{\mathfrak{R}}\frac{d}{dt}\{\cot\alpha|\partial_1\eta|_{S_F}^2+\mathcal{W}\csc\alpha|\partial_1{}^2\eta|_{S_F}^2\}$$

$$\leq \frac{2}{\mathfrak{R}}(\partial_1\eta,\ \partial_1 u_1)_{S_F}+\frac{2}{\mathfrak{R}}(\partial_1\eta\cot\alpha-\mathcal{W}\csc\alpha\partial_1{}^3\eta,\ -\partial_1(\eta^2\partial_1\eta))_{S_F}$$

$$+\frac{1}{\mathfrak{R}}\int_{S_F}\partial_1 h_1\partial_1 u_1+\frac{2}{\mathfrak{R}}\int_{S_F}\partial_1 h_2\partial_1 u_2$$

$$+(\partial_1 f,\ \partial_1 u)_\Omega+\left(\partial_1\left(b(\eta)\nabla\frac{2}{\mathfrak{R}}p\right),\ \partial_1 u\right)_\Omega.$$

3rd. Step.) We next apply ∂_1^2 to (3.3)-(3.4) and take the inner product with $\partial_1{}^2 u$. In a similar way we obtain the corresponding inequality,

$$\frac{1}{2}\frac{d}{dt}\|\partial_1{}^2 u\|^2+K_0'\|\nabla\partial_1{}^2 u\|^2+\frac{1}{\mathfrak{R}}\frac{d}{dt}\{\cot\alpha|\partial_1{}^2\eta|_{S_F}^2+\mathcal{W}\csc\alpha|\partial_1{}^3\eta|_{S_F}^2\} \tag{5.24}$$

$$\leq \frac{2}{\mathfrak{R}}(\partial_1{}^2\eta,\ \partial_1{}^2 u_1)_{S_F}+\frac{2}{\mathfrak{R}}(\partial_1{}^2\eta\cot\alpha-\mathcal{W}\csc\alpha\partial_1{}^4\eta,\ -\partial_1{}^2(\eta^2\partial_1\eta))_{S_F}$$

$$+\frac{1}{\mathfrak{R}}\int_{S_F}\partial_1{}^2 h_1\partial_1{}^2 u_1+\frac{2}{\mathfrak{R}}\int_{S_F}\partial_1{}^2 h_2\partial_1{}^2 u_2$$

$$-(\partial_1 f,\ \partial_1{}^3 u)_\Omega-\left(\partial_1\left(b(\eta)\nabla\frac{2}{\mathfrak{R}}p\right),\ \partial_1{}^3 u\right)_\Omega.$$

Here we briefly show how to estimate the cubic or higher degree terms in the right hand side of (5.24). The boundary term containing $\partial_1{}^4\eta$ can be estimated as follows

$$|(\partial_1{}^4\eta,\ \partial_1{}^2(\eta^2\partial_1\eta))_{S_F}|\leq C|(\partial_1{}^{\frac{7}{2}}\eta,\ \partial_1{}^{\frac{1}{2}}(\eta^2\partial_1\eta))_{S_F}|$$

$$\leq C|\partial_1{}^{\frac{7}{2}}\eta|_0|\eta^2|_{\frac{5}{2}}|\partial_1\eta|_{\frac{5}{2}}$$

by using Lemma 2.3. We have already seen how $f(\eta,\ u)$ can be estimated in terms of u and η in I). Furthermore, in deriving (5.19) through (5.17)-(5.18), one easily see that $|\partial_1{}^{\frac{7}{2}}\eta|_0$ is estimated by the right hand side of (5.20). We can treat the cubic or higher degree terms in (5.22)-(5.24) in a similar, but easier way.

4th. Step.) Finally we differentiate (3.3)-(3.4) in t and take the inner product with $\partial_t u$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u\|^2+K_0'\|\nabla\partial_t u\|^2+\frac{1}{\mathfrak{R}}\frac{d}{dt}\{\cot\alpha|\partial_t\eta|_{S_F}^2+\mathcal{W}\csc\alpha|\partial_1\partial_t\eta|_{S_F}^2\} \tag{5.25}$$

$$\leq \frac{2}{\mathfrak{R}}(-\partial_1\eta+u_2,\ \partial_t u_1)_{S_F}+\frac{2}{\mathfrak{R}}(\eta^2\partial_1\eta,\ \partial_t u_1)_{S_F}$$

$$+\frac{2}{\mathfrak{R}}(\partial_t\eta\cot\alpha-\mathcal{W}\csc\alpha\partial_1\partial_t\eta,\ -\partial_t(\eta^2\partial_1\eta))_{S_F}$$

$$+\frac{1}{\mathcal{R}}\int_{S_F}\partial_t h_1\partial_t u_1+\frac{2}{\mathcal{R}}\int_{S_F}\partial_t h_2\partial_t u_2+(\partial_t f,\ \partial_t u)_\Omega$$

$$+\left(\partial_t b(\eta)\left(\nabla\frac{2}{\mathcal{R}}p\right),\ \partial_t u\right)_\Omega+\left(b(\eta)\partial_t\left(\nabla\frac{2}{\mathcal{R}}p\right),\ \partial_t u\right)_\Omega .$$

Here we used (3.2). Further we use this equation to replace $\partial_t\eta$ in the right by the right side of (3.2). It is easy to see that the matrix $I-b(\eta)$ is invertible, and that $\beta(\eta)=b(\eta)(I-b(\eta))^{-1}$ is positive definite. To deal with the term, $\left(b(\eta)\partial_t\left(\nabla\frac{2}{\mathcal{R}}p\right),\ \partial_t u\right)_\Omega$, we use these facts. We recover $\partial_t\left(\nabla\frac{2}{\mathcal{R}}p\right)$ from the time derivative of (3.3)-(3.4), then substitute this expression into the above inner product. After some calculations we can see

$$\left(b(\eta)\partial_t\left(\nabla\frac{2}{\mathcal{R}}p\right),\ \partial_t u\right)=(\beta(\eta)(-\partial_t^2u+\cdots,\ \partial_t u)_\Omega$$

$$=-\frac{1}{2}\frac{d}{dt}(\beta\partial_t u,\ \partial_t u)+\cdots .$$

The next term which is difficult to treat in the right hand side is

$$J\equiv(A_0(\eta)\partial_t\partial_2^2u_1,\ \partial_t u_k)_\Omega .$$

Integrating by parts we have

$$J=\int_{S_F}A_0\partial_t\partial_2u_1\partial_t u_k-\int_\Omega A_1\partial_t\partial_2u_1\partial_t\partial_2u_k$$

$$\equiv J_1+J_2$$

Note that on S_F we can resolve ∂_2u_1 from the boundary condition (3.7). Using this expression we have

$$|J_1|\le C_{\epsilon_1}(|\partial_1^{\frac{1}{2}}\partial_t u|_{S_F}^2+|\partial_t u|_{S_F}^2+|\partial_1\eta|_{S_F}^2+|u|_{S_F}^2) .$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}((I-\beta(\eta))\partial_t u,\ \partial_o u)+K_0'\|\nabla\partial_t u\|^2\le(-\partial_1\eta+u_2,\ \partial_t u_1)_{S_F}+\cdots . \tag{5.26}$$

5th. Step.) Note the fact that, as stated in *3rd.* and *4th. Steps*, the cubic or higher degree terms are bounded by the square of the right hand side of (5.20) with the coefficients of order ϵ_1. Hence, if ϵ_1 is sufficiently small, adding the inequalities obtained above, we obtain

$$\frac{d}{dt}(\tilde{E}(t)+\Phi(t))+\gamma' F(t) \tag{5.27}$$

$$\leq \frac{2}{\mathcal{R}}(|(\eta, u_1)_{S_F}|+|(\partial_1\eta, \partial_1 u_1)_{S_F}|+|(\partial_1^2\eta, \partial_1^2 u_1)_{S_F}|$$

$$+\kappa|(-\partial_1\eta+u_2, \partial_t u_1)_{S_F}|),$$

where γ' is some positive constant and

$$\tilde{E}\equiv\sum_{j=0}^{2}\|\partial_1^j u\|^2+\kappa((I-\beta)\partial_t u, \partial_t u),$$

$$F\equiv\sum_{j=0}^{2}\|\nabla\partial_1^j u\|^2+\kappa\|\nabla\partial_t u\|^2,$$

$$\Phi\equiv\cot\alpha(\sum_{j=0}^{2}|\partial_1^j\eta|_0^2+\kappa|\partial_t\eta|_0^2)+\mathcal{W}\csc\alpha(\sum_{j=0}^{2}|\partial_1^{j+1}\eta|_0^2+\kappa|\partial_1\partial_t\eta|_0^2)$$

and $\kappa>0$ is to be specified later. In a similar way to obtain the bound of $|\partial_1^{\frac{7}{2}}\eta|_{S_F}$, we can get

$$|\partial_1^2\eta|_{S_F}\leq C\sin\alpha\mathcal{W}^{-1}(\|\nabla u\|+\|\nabla\partial_1 u\|+\|\nabla\partial_1^2 u\|+\|\nabla\partial_t u\|). \tag{5.28}$$

From this we can see that the right hand side of (5.27) is bounded by $F(t)$ by choosing κ and α to be small. Hence,

$$\frac{d}{dt}(\tilde{E}+\Phi)+\gamma'' F\leq 0, \tag{5.29}$$

with some positive constant γ''. Furthermore, combining (5.28) and (3.2) and the Poincaré inequality, we see that

$$\Phi+\tilde{E}\leq CF.$$

After multiplying this by small positive constant, adding to (5.29), we have

$$\frac{d}{dt}(\tilde{E}(t)+\Phi(t))+\gamma_1(\tilde{E}(t)+\Phi(t))\leq 0. \tag{5.30}$$

We finally apply Proposition 5.3 to (u, p) for $l=0$, regarding this as a solution of the stationary Stokes problem, (5.3), (5.4), (5.5) and (5.8)-(5.9). Then we have

$$\left\|\frac{1}{\mathcal{R}}u\right\|_2+\left\|\frac{2}{\mathcal{R}}\nabla p\right\| \tag{5.31}$$

$$\leq C\Big(\left\|f(\eta, u)\right\|+\left\|b(\eta)\nabla\frac{2}{\mathcal{R}}p\right\|+|u_2|_{\frac{3}{2}}+|2\eta+h_2(\eta, u)|_{\frac{1}{2}}$$

$$+\|\partial_t u\|+\|(U,\nabla)u+(u,\nabla)U\|\Big).$$

As we assume that $|\eta|_{\frac{5}{2}}+\|u\|_2$ is small enough, we can conclude from (5.31) that

$$\|u\|_2 \leq C(\|\partial_t u\| + \|u\| + \|\partial_1 u\| + |u_2|_{\frac{3}{2}} + |\eta|_{\frac{5}{2}}) . \tag{5.32}$$

By the solenoidal condition we can estimate the trace norm of u_2 as follows:

$$\begin{aligned} |u_2|_{S_F}|_{\frac{3}{2}} &\leq C|\partial_1 u_2|_{\frac{1}{2}} \leq C\|\nabla \partial_1 u_2\| \\ &\leq C\{\|\partial_1^2 u_2\| + \|\partial_1\partial_2 u_2\|\} = C\|\partial_1^2 u\| . \end{aligned} \tag{5.33}$$

Thus the quantities in the right hand side of (5.32) can be estimated by $\tilde{E}$ and Φ. From this it is easily to see that Proposition 5.1 holds.

6. Global solutions

We now only need to apply the argument in [7] to show our main result.

Theorem 6.1. *Let* (2.3) *hold. Let* a *be* $0 < a \leq a_0$, *where* a_0 *is chosen in Proposition* 5.1. *Then there is a positive constant* ϵ_0 *such that, if* $\eta_0 \in H^3_{p0}(S_F)$ *and* $u_0 \in H_p{}^2(\Omega)$ *satisfy the compatibility conditions* (4.1), (4.2) *and* (4.3) *and further if* $E_0 \equiv \|u_0\|_2 + |\eta_0|_{\frac{5}{2}} \leq \epsilon_0$, *then the problem* (3.2)-(3.9) *has a unique global in time solution* (η, u, p) *such that*

$$\eta \in C(0, \infty; H^3_{p0}(S_F)) \cap L^2(0, \infty; H^{\frac{7}{2}}_{p0}(S_F)) , \tag{6.1}$$

$$u \in C(0, \infty; H^2_p(\Omega)) \cap L^2(0, \infty; H^3_p(\Omega)) , \tag{6.2}$$

$$\begin{aligned} &\nabla p \in C(0, \infty; H^0_p(\Omega)) \cap L^2(0, \infty; H^1_p(\Omega)) , \\ &p|_{S_F} \in C(0, \infty; H^1_{p0}) \cap L^2(0, \infty; H^{\frac{3}{2}}_{p0}) . \end{aligned} \tag{6.3}$$

It has an exponential decay property:

$$|\eta|_3 + \|u\|_2 = O(e^{-\gamma t}) . \tag{6.4}$$

Outline of the proof. By Proposition 5.1 and density argument, we can relax the initial condition in Proposition 4.1. It also follows from the estimates of the tangential derivatives obtained in the proof of Proposition 5.1 and the stationary estimates in Proposition 5.2 that $u \in L^2(0, \infty; H_p{}^3)$ and $\eta \in L^2(0, \infty; H^{\frac{7}{2}}_{p0})$.

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