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On the Cauchy problem for Schrödinger type equations and Fourier integral operators

By

Wataru ICHINOSE

0. Introduction

In this paper we study the L^2 well-posedness of the Cauchy problem for Schrödinger type equations

$$(0.1)\qquad \begin{cases} Lu(t,x)\equiv\frac{1}{i}\partial_t u+H_2(t,x,D_x)u+H_1(t,x,D_x)u=f(t,x) \\ \qquad\qquad \text{on } [s,T]\times R_x^n\ (0\le s<T)\,, \\ u(s,x)=u_0(x)\,. \end{cases}$$

Here, we suppose that the symbol $h_j(t,x,\xi)(j=1,2)$ of pseudo-differential operators $H_j(t,x,D_x)$ are continuous functions on $[0,T]\times R^{2n}$ and C^∞ functions on R^{2n} for each $t\in[0,T]$. Moreover, we impose on $h_2(t,x,\xi)$ the assumptions that $h_2(t,x,\xi)$ is real valued and that

$$(0.2)\qquad \text{if } |\alpha+\beta|\ge 2\,,\quad |\partial_\xi^\alpha\partial_x^\beta h_2(t,x,\xi)|\le C_{\alpha,\beta}$$

holds, where α and β are multi-indices and $C_{\alpha,\beta}$ are constants independent of $(t,x,\xi)\in[0,T]\times R^{2n}$.

One of our aims in the present paper is to give a sufficient condition for the Cauchy probler (0.1) to be L^2 well posed on $[0,T]$. Another aim is to derive the above type of equations from the Maxwell equations. We have studied the above type of equations as typical equations not kowalewskian and not parabolic. If $H_1(t,x,D_x)-H_1(t,x,D_x)^*$ is a uniformally L^2-bounded operator on $[0,T]$, the L^2 well-posedness of (0.1) can be proven easily, where $H_1(t,x,D_x)^*$ denotes the formal adjoint operator in the usual L^2-inner product. So, we are interested about the L^2 well-posedness in the other cases, for example, $h_1(t,x,\xi)=i\sum_{j=1}^n b^j(x)\xi_j+c(x)$ ($b^j(x)$ are real valued functions). In section 4 this interesting type of equations will be derived.

We have already known some results on the L^2 well-posedness of (0.1). Under the situation that $h_2(t,x,\xi)=\frac{1}{2}|\xi|^2$ and

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$$(0.3)\qquad h_1(t, x, \xi)=\sum_{j=1}^n b^j(x)\xi_j+c(x)$$
$$(b^j(x), c(x)\in\mathcal{B}^\infty(R^n),\ j=1, 2, \cdots, n),$$

S. Mizohata in [14] gave a sufficent condition and a necessary condition respectively. See also Mizohata [15]. $\mathcal{B}^\infty(R^n)$ denotes the space of all C^∞ functions on R^n whose derivatives of any order are all bounded. We want to remark that the result obtained in the present paper is more general and better than his even in his situation (see Remark 1.1). In [5] a necessary condition was given under the situation that $h_2(t, x, \xi)$ is given by $\sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j (g^{ij}(x)\in\mathcal{B}^\infty(R_x^n), i, j=1, 2, \cdots, n)$ and $h_1(t, x, \xi)$ is the same symbol as in [14]. In more general, we gave a necessary condition in [6] under the situation that $H_2(t, x, D_x)$ is replaced by the Laplace-Beltrami operator on the general Riemannian manifold and $H_1(t, x, D_x)$ is also done by a complex valued vector field. In [7] a sufficient condition was given under the above each situation in [5] and [6], though we had to impose the strong assumption on $H_1(t, x, D_x)$. There was an announcement of a sufficient condition by S. Tarama in 1988 under the assumptions that $h_2(t, x, \xi)$ is a homogeneous polynomial of degree 2 in only x and ξ and $h_1(t, x, \xi)$ has the form (0.3). The author does not know the precise statement and its proof, because his paper has been unpublished. But, our result in the present paper seems to include his result.

We note that H. Kitada [10] and H. Kitada -H. Kumano-go [11] constructed the fundamental solution of the Cauchy problem (0.1) for Schrödinger equations where $H_1(t, x, D_x)$ disappears. In the present paper we use the Fourier integral operators in [10] and [11] essentially in the proof.

In the forthcoming paper [8] we will give a necessary condition for the L^2 well-posedness of (0.1). We shall state it briefly in Remark 1.2 in the present paper. See the references of [7] for the studies of the other problems for Schrödinger type equations.

The plan of the present paper is as follows. In section 1 we shall state the main theorem (Theorem 1.1) and some examples. The proof of the main theorem will be given in section 2. In section 3 we shall extend the main theorem by using the Fourier integral operator theory and the Egorov theorem in the modified form (Theorems 3.3 and 3.4). There, the relation between L^2 well-posedness and the classical canonical transformations will be studied. Section 4 will be devoted to the derivation of the Schrödinger type equations.

1. Main theorem and examples

Let $(x_1, \cdots, x_n)$ denote a point of R^n and let $\alpha=(\alpha_1, \cdots, \alpha_n)$ be a multi-index whose components α_j are non-negative integers. Then, we use the usual notations:

$$|\alpha|=\alpha_1+\cdots+\alpha_n,\quad x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n},\quad \alpha!=\alpha_1!\cdots\alpha_n!,$$

$$\partial_x{}^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}, \quad D_x{}^\alpha=D_{x_1}^{\alpha_n}\cdots D_{x_n}^{\alpha_n}, \quad \partial_{x_j}=\frac{\partial}{\partial x_j},$$

$$D_{x_j}=\frac{1}{i}\frac{\partial}{\partial x_j}, \quad \langle x\rangle=(1+|x|^2)^{1/2}.$$

Let $\varphi(x)$ and $f_j(x)$ $(j=1, 2, \cdots, n)$ be C^∞ functions on R^n. Then, we denote for $f(x)=(f_1(x), \cdots, f_n(x))$

$$\frac{\partial\varphi}{\partial x}=\left(\frac{\partial\varphi}{\partial x_1}, \cdots, \frac{\partial\varphi}{\partial x_n}\right), \quad \frac{\partial f}{\partial x}=\left(\frac{\partial f_i}{\partial x_j}; {}^{i}_{j}{\downarrow}^{\downarrow}_{=}1, 2, \cdots, n\right),$$

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right).$$

$\frac{D(f)}{D(x)}$ denotes the Jacobian determinant.

Let $\mathcal{S}=\mathcal{S}(R^n)$ be the Schwartz space of rapidly decreasing functions on R^n with semi-norms $|f|_l=\max_{k+|\alpha|\le l}\max_{x\in R^n}\{\langle x\rangle^k|\partial_x{}^\alpha f(x)|\}(l=0, 1, \cdots,)$. The Fourier transformation $\hat{u}(\xi)$ for $u(x)\in\mathcal{S}$ is defined by

$$\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)\mathrm{d}x, \; x\cdot\xi=x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n.$$

We shall define the pseudo-differential operator. We determine the symbol class $T^m=T^m(R^{2n})$ by the set of all C^∞ functions $p(x, \xi)$ satisfying for all multi-indices α and β

$$|\partial_\xi{}^\alpha D_x{}^\beta p(x, \xi)|\le C_{\alpha,\beta}(1+|x|^2+|\xi|^2)^{m/2},$$

where $C_{\alpha,\beta}$ are constants independent of $(x, \xi)\in R^{2n}$. We shall often write $\partial_\xi{}^\alpha D_x{}^\beta p(x, \xi)$ as $p_{(\beta)}^{(\alpha)}(x, \xi)$. The pseudo-differential operator $P=p(x, D_x)$ with a symbol $p(x, \xi)\in T^m$ is defined by

$$Pu(x)=\int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)đ\xi \quad (đ\xi=(2\pi)^{-n}\mathrm{d}\xi)$$

for $u(x)\in\mathcal{S}$. Also, the double symbol class $T^{m,m'}(R^{4n})$ are defined by the set of all C^∞ functions $p(x, \xi, \tilde{x}, \tilde{\xi})$ on R^{4n} satisfying

$$|p_{(\beta;\beta')}^{(\alpha;\alpha')}(x, \xi, \tilde{x}, \tilde{\xi})|\le C_{\alpha,\alpha',\beta,\beta'}(1+|x|^2+|\xi|^2)^{m/2}(1+|\tilde{x}|^2+|\tilde{\xi}|^2)^{m'/2}$$

for all α, α', β and β', where $p_{(\beta;\beta')}^{(\alpha;\alpha')}(x, \xi, \tilde{x}, \tilde{\xi})=\partial_\xi{}^\alpha D_x{}^\beta\partial_{\tilde{\xi}}{}^{\alpha'}D_{\tilde{x}}{}^{\beta'}p(x, \xi, \tilde{x}, \tilde{\xi})$ and $C_{\alpha,\alpha',\beta,\beta'}$ are constants independent of $(x, \xi, \tilde{x}, \tilde{\xi})\in R^{4n}$. The pseudo-differential operator $P=p(x, D_x, \tilde{x}, D_{\tilde{x}})$ with a double symbol $p(x, \xi, \tilde{x}, \tilde{\xi})\in T^{m,m'}$ is defined by

$$Pu(x)=\iiint e^{i(x-\tilde{x})\cdot\xi+i\tilde{x}\cdot\tilde{\xi}}p(x, \xi, \tilde{x}, \tilde{\xi})\hat{u}(\tilde{\xi})đ\tilde{\xi}\mathrm{d}\tilde{x}đ\xi$$

for $u(x)\in\mathcal{S}$. We remark that as for the double symbol $p(x, \xi, \tilde{x}, \tilde{\xi})$ we use the different notation from the usual one in [12] and that symbol classes T^m and $T^{m,m'}$ are different slightly from ones in [5] and [13].

In the present paper we also use the other symbol class $\mathcal{B}^{k,\infty}(R^{2n})$ for $k=0, 1, \cdots$ which was introduced in [11]. We denote by $\mathcal{B}^{k,\infty}(R^{2n})$ the set of all C^∞ functions $p(x, \xi)$ satisfying that $p_{(\beta)}^{(\alpha)}(x, \xi)$ are bounded on R^{2n} for all multi-indices α and β such that $|\alpha+\beta|\geq k$. If $p(x, \xi)\in\mathcal{B}^{1,\infty}(R^{2n})$, it follows from the mean value theorem that $p(x, \xi)$ belongs to $T^1(R^{2n})$. Inductively, we can see that $\mathcal{B}^{k,\infty}(R^{2n})$ is included in $T^k(R^{2n})(k=0, 1,\cdots)$.

Let $\boldsymbol{B}$ be a Fréchet space. Then, we denote by $\mathcal{E}_t^0([0, T]; \boldsymbol{B})$ the space of all $\boldsymbol{B}$-valued continuous functions on $[0, T]$. In the same way, $\mathcal{E}_t^1([0, T]; \boldsymbol{B})$ is defined as the space of all $\boldsymbol{B}$-valued continuously differentiable functions on $[0, T]$. The space of all $\boldsymbol{B}$-valued L^1 functions on $[0, T]$ is denoted by $L_t^1([0, T]; \boldsymbol{B})$.

Definition 1.1. We say that the Caucy problem (0.1) is L^2 well posed on $[0, T]$, if for any s $(0\leq s< T)$, any $u_0(x)\in L^2$ and any $f(t, x)\in L_t^1([0, T]; L^2(R^n))$ there exists one and only one solution $u(t, x)$ of (0.1) in $\mathcal{E}_t^0([s, T]; L^2(R^n))$ in a distribution sense $\left(\text{i.e. } \int_s^T\int f(t, x)\overline{\varphi(t, x)}\,dx\,dt=\frac{1}{i}\int\{u(T, x)\overline{\varphi(T, x)}-u_0(x)\overline{\varphi(s, x)}\}dx+\int_s^T\int u(t, x)\overline{{}^*L\varphi(t, x)}dxdt \text{ is valid for any } \varphi(t, x)\in C_0^\infty([s, T]\times R_x^n)\right)$ and we get the energy inequality

$$\|u(t, \cdot)\|\leq C(T)\left(\|u_0(\cdot)\|+\int_s^t\|f(\theta, \cdot)\|d\theta\right) \quad (s\leq t\leq T)$$

for a constant $C(T)$. Here, ${}^*L$ is the formally adjoint operator of L, $C_0^\infty([s, T]\times R_x^n)$ denotes the space of C^∞ functions on $[s, T]\times R_x^n$ whose supports are compact and $\|\cdot\|$ the L^2 norm.

Let $(q, p)(t, s; y, \xi)=(q_1, \cdots, q_n, p_1, \cdots, p_n)(t, s; y, \xi)$ be the solution of the Hamilton canonical equations for $h_2(t, x, \xi)$ issuing from (y, ξ) at $t=s$, that is,

$$\frac{dq}{dt}=\frac{\partial h_2}{\partial\xi}(t, q, p), \quad \frac{dp}{dt}=-\frac{\partial h_2}{\partial x}(t, q, p), \quad (q, p)|_{t=s}=(y, \xi). \tag{1.1}$$

Theorem 1.1. *Besides the assumptions on $h_j(t, x, \xi)(j=1, 2)$ in introduction we suppose that there exists an $m\geq 0$ satisfying $h_1(t, x, \xi)\in T^m(R^{2n})$ for each $t\in[0, T]$. We also assume that there exists a subset Z in $\{1, 2, \cdots, n\}$ satisfying for all multi-indices α and β*

$$(1.2)\qquad \begin{cases} \left|\dfrac{\partial}{\partial x_j} h_{1(\beta)}^{(\alpha)}(t, x, \xi)\right| \leq C_{\alpha,\beta} \quad (j\in Z), \\ \left|\dfrac{\partial}{\partial \xi_k} h_{1(\beta)}^{(\alpha)}(t, x, \xi)\right| \leq C_{\alpha,\beta} \quad (k\in Z^c), \end{cases}$$

where $C_{\alpha,\beta}$ are constants independent of $(t, x, \xi)\in[0, T]\times R^{2n}$ and Z^c denotes the complementary set of Z in $\{1, 2, \cdots, n\}$. We set

$$(1.3)\qquad w(t, s; y, \xi)=\exp\left\{-i\int_s^t h_1(\theta, q(\theta, s; y, \xi), p(\theta, s; y, \xi))d\theta\right\}.$$

Then, if

$$(1.4)\qquad \sup_{0\leq s\leq t\leq T}\ \sup_{(y,\xi)\in R^{2n}} |\partial_\xi^\alpha D_y^\beta w(t, s; y, \xi)|<\infty$$

are valid for all α and β, the Cauchy problem (0.1) *is L^2 well posed on* $[0, T]$.

Remark 1.1. We apply the above theorem to the equation which Mizohata studied in [14]. Since $h_2(t, x, \xi)=\frac{1}{2}|\xi|^2$ and $h_1(t, x, \xi)$ is given by the form (0.3), we have

$$\begin{aligned} w(t, s; y, \xi)&=\exp\left\{-i\Sigma_j\int_s^t b^j((\theta-s)\xi+y)\xi_j \mathrm{d}\theta\right\} \\ &=\exp\left\{-i\Sigma_j\int_0^{t-s} b^j(\theta\xi+y)\xi_j \mathrm{d}\theta\right\}. \end{aligned}$$

We note that a sufficient condition in [14] needed the existence of the inverse operator of $W(t, s; x, D_x)$ from L^2 space onto L^2 space for each t and s ($0\leq s\leq t\leq T$) basides (1.4). So, our result in Theorem 1.1 gives a better result than his in [14].

Here, we return to Theorem 1.1. We note if we moreover assume the existence of the inverse operator $W(t, s; x, D_x)$ from L^2 space onto L^2 space ($0\leq s\leq t\leq T$) in Theorem 1.1, we can easily obtain the same result as in Theorem 1.1 by modifying the proof in [14]. See [8] for such a proof.

Corollary 1.2. *We suppose the same assumption on $h_2(t, x, \xi)$ as in Theorem* 1.1. *On the other hand, we impose on $h_1(t, x, \xi)$ a stronger assumption than* (1.2) *that*

$$(1.2)'\qquad \text{if } |\alpha+\beta|\neq 0, \quad |h_{1(\beta)}^{(\alpha)}(t, x, \xi)|\leq C_{\alpha,\beta}$$

are valid, where $C_{\alpha,\beta}$ are constants independent of $(t, x, \xi)\in[0, T]\times R^{2n}$. Then, if $w(t, s; y, \xi)$ defined by (1.3) *is bounded concerning (t, s, y, ξ) such that $0\leq s\leq t\leq T$ and $(y, \xi)\in R^{2n}$, the Cauchy problem* (0.1) *is L^2 well posed on* $[0, T]$.

Proof. Apply Lemma 2.1 which will be given in section 2. Then, we can easily see from the assumption (1.2)′ that if $|\alpha+\beta|\neq 0$, (1.4) in Theorem 1.1 are valid automatically. So, if $w(t, s; y, \xi)$ is bounded, (1.4) are valid for all α and β. Hence, we can complete the proof from Theorem 1.1.

Remark 1.2. In a forthcoming paper [8] we shall prove under the same conditions as in Corollary 1.2 that if (0.1) is L^2 well posed on $[0, T]$, there must be a $T_1(0<T_1\leq T)$ such that $w(t, s; y, \xi)$ defined by (1.3) is bounded concerning (t, s, y, ξ) such that $0\leq s\leq t\leq T_1$ and $(y, \xi)\in R^{2n}$. Hence, we obtain a necessary and sufficient condition for (0.1) to be L^2 well posed on $[0, T]$ from this result in [8] and Theorem 1.1, if $h_j(t, x, \xi)$ $(j=1, 2)$ are independent of $t\in[0, T]$ and satisfy the assumptions in Corollary 1.2.

We shall state the proposition before giving examples. Let $Z=\{z_1, z_2, \cdots, z_l\}$ be a subset in $\{1, 2, \cdots, n\}$ and set

$$x_Z=(x_{z_1}, \cdots, x_{z_l}), \quad |Z|=l.$$

We define the transformation from $L^2(R^n_{x'})$ space onto $L^2(R_x{}^n)$ by

$$\int e^{-ix'_Z\cdot x_Z}v(x'_Z, x_{Z^c})dx'_Z \quad (dx'_Z=dx'_{z_1}dx'_{z_2}\cdots dx'_{z_l}) \tag{1.5}$$

for $v(x')\in L^2(R^n_{x'})$. If Z is empty, we mean the identity by (1.5). Here, we represented $x=(x_1, \cdots, x_n)$ symbolically by

$$x=(x_Z, x_{Z^c}). \tag{1.6}$$

We shall use this symbolic representation (1.6) through the present paper. If $Z=\{1, 2, \cdots, n\}$, the above transformation (1.5) becomes a usual Fourier transformation. We denote the transformation (1.5) by $\mathcal{F}_{x'_Z\to x_Z}v(x)$. The inverse operator $(\mathcal{F}^{-1})_{x_Z\to x'_Z}$ of $\mathcal{F}_{x'_Z\to x_Z}$ can be defined by

$$(\mathcal{F}^{-1})_{x_Z\to x'_Z}u(x')=\int e^{ix_Z\cdot x'_Z}u(x_Z, x'_{Z^c})đx_Z \quad (đx_Z=(2\pi)^{-|Z|}dx_Z)$$

for $u(x)\in L^2(R_x{}^n)$.

Let define a mapping $(x, \xi)=\Phi(x', \xi')$ from $R^{2n}_{x',\xi'}$ onto $R^{2n}_{x,\xi}$ by

$$(x_Z, x_{Z^c})=(\xi'_Z, x'_{Z^c}), \quad (\xi_Z, \xi_{Z^c})=(-x'_Z, \xi'_{Z^c}) \tag{1.7}$$

and denote by $(x', \xi')=\Phi^{-1}(x, \xi)$ its inverse mapping, where Z is the subset given in Theorem 1.1. We define the symbols $k_j(t, x', \xi')(j=1, 2)$ from $h_j(t, x, \xi)(j=1, 2)$ in Theorem 1.1 by

$$k_j(t, x', \xi')=h_j(t, \Phi(x', \xi')) \tag{1.8}$$

and denote the solution of the Hamilton canonical equations for $k_2(t, x', \xi')$ issuing from (y', ξ') at $t=s$ by (q', p') $(t, s; y', \xi')$. Then, we obtain

Proposition 1.3. *We suppose the same assumptions as in Theorem* 1.1 *and consider* L *defined in* (0.1). *Then,*

$$(1.9)\qquad (\mathcal{F}^{-1})_{x_Z\to x'_Z}\circ L\circ \mathcal{F}_{x'_Z\to x_Z}v(t,x')$$

$$=\frac{1}{i}\partial_t v(t,x')+K_2(t,x',D_{x'})v+K_1(t,x',D_{x'})v+R(t,x',D_{x'})v$$

is valid for any $v(t,x')\in\mathcal{E}^1_t([0,T];L^2)$, *where the symbol* $r(t,x',\xi')$ *of* $R(t,x',D_{x'})$ *satisfies for all* α *and* β

$$(1.10)\qquad |r^{(\alpha)}_{(\beta)}(t,x',\xi')|\le C'_{\alpha,\beta}$$

with constants $C'_{\alpha,\beta}$ *independent of* $(t,x',\xi')\in[0,T]\times R^{2n}$. *Here,* $\cdot\circ\cdot$ *implies the product of operators. Also, we get for any* $(y',\xi')\in R^{2n}$

$$(1.11)\qquad \exp\left\{-i\int_s^t k_1(\theta,q'(\theta,s;y',\xi'),p'(\theta,s;y',\xi'))d\theta\right\}$$

$$=\exp\left\{-i\int_s^t h_1(\theta,q(\theta,s;y,\xi),p(\theta,s;y,\xi))d\theta\right\}$$

$$((y,\xi)=\Phi(y',\xi'))\,.$$

Proof. Let $h(t,x,\xi)$ be $h_1(t,x,\xi)$ or $h_2(t,x,\xi)$. Then, we can easily see for $v(x')\in\mathcal{S}$

$$H(t,x,D_x)\circ\mathcal{F}_{x'\to x}v(x)=\iint e^{i(x-y)\cdot\xi}h(t,x,\xi)dy đ\xi\int e^{-iy\cdot x'}v(x')dx'$$

$$=\int e^{-ix\cdot x'}h(t,x,-x')v(x')dx'\,.$$

So using the pseudo-differential operator with a double symbol,

$$(\mathcal{F}^{-1})_{x\to x'}\circ H(t,x,D_x)\circ\mathcal{F}_{x'\to x}v(x')=\iint e^{ix\cdot(x'-\tilde{x}')}h(t,x,-\tilde{x}')v(\tilde{x}')d\tilde{x}' đx$$

$$=H(t,D_{x'},-\tilde{x}')v(x')$$

is valid. In the same way, we get for $j=1,2$

$$(1.12)\qquad (\mathcal{F}^{-1})_{x_Z\to x'_Z}\circ H_j(t,x,D_x)\circ\mathcal{F}_{x'_Z\to x_Z}v(x')$$

$$=H_j(t,D_{x'_Z},x'_{Z^c},-\tilde{x}'_Z,D_{x'_{Z^c}})v(x')\quad(v(x')\in\mathcal{S})\,.$$

Let apply the asymptotic expansion formula (Theorem 3.1 in chapter 2 of [12]) to the right-hand side of (1.12). Then, noting the assumption (0.2) on $h_2(t,x,\xi)$ and (1.8), we can see

$$H_2(t,D_{x'_Z},x'_{Z^c},-\tilde{x}'_Z,D_{x'_{Z^c}})=K_2(t,x',D_{x'})+R_2(t,x',D_{x'})$$

is valid, where $r_2(t, x', \xi')$ satisfies the same inequalities as (1.10) for all α and β. For $H_1(t, D_{x'_Z}, x'_{Z^c}, -\hat{x}'_Z, D_{x'_{Z^c}})$ we also get the similar equality to the above from the assumption (1.2). Hence, we obtain

$$(1.13)\qquad (\mathcal{F}^{-1})_{x_Z\to x'_Z}\circ H_j(t, x, D_x)\circ \mathcal{F}_{x'_Z\to x_Z}v(x')$$
$$=\{K_j(t, x', D_{x'})+R_j(t, x', D_{x'})\}v(x') \quad (j=1, 2),$$

where $r_j(t, x', \xi')$ satisfies the same inequalities as (1.10) for all α and β. Hence, we obtain (1.9) and (1.10).

Next, we know well that a mapping Ψ from $R^{2n}_{x',\xi'}$ to $R^{2n}_{x,\xi}$ is called a canonical transformation, if

$$(1.14)\qquad \Psi^*\sum_{j=1}^n dx_j\wedge d\xi_j=\sum_{j=1}^n dx'_j\wedge d\xi'_j$$

holds, where Ψ^* denotes the pull back of differential forms (see section 38 in [1]). It is easily seen that the mapping Φ defined by (1.7) is a canonical transformation. So, since $k_2(t, x', \xi')=h_2(t, \Phi(x', \xi'))$, it follows from the well known classical theory on the canonical transformation that

$$(1.15)\qquad \Phi(q'(t, s; y', \xi'), p'(t, s; y', \xi'))$$
$$=(q(t, s; y, \xi), p(t, s; y, \xi)) \quad ((y, \xi)=\Phi(y', \xi'))$$

is valid for any (y', ξ') (see section 45 in [1]), which shows from $k_1(t, x', \xi')= h_1(t, \Phi(x', \xi'))$

$$k_1(t, q'(t, s; y', \xi'), p'(t, s; y', \xi'))$$
$$=h_1(t, q(t, s; y, \xi), p(t, s; y, \xi)) \quad ((y, \xi)=\Phi(y', \xi')).$$

Hence, we get (1.11). Q.E.D.

Example 1.1. Set $h_2(t, x, \xi)=\frac{1}{2}|x|^2$ and $h_1(t, x, \xi)=\sum_{j=1}^n x_j b^j(-\xi)$, where we assume $b^j(\xi)\in\mathcal{B}^\infty(R^n)(j=1, 2, \cdots, n)$. Then, we can easily get $(q, p)(t, s; y, \xi)=(y, -(t-s)y+\xi)$ and $w(t, s; y, \xi)=\exp\left\{-i\sum_j\int_0^{t-s} b^j(\theta y-\xi)y_j d\theta\right\}$. Apply Proposition 1.3 as $Z=\{1, 2, \cdots, n\}$. Then, we get

$$(\mathcal{F}^{-1})_{x\to x'}\circ L\circ\mathcal{F}_{x'\to x}v(t, x')$$
$$=\frac{1}{i}\partial_t v(t, x')-\frac{1}{2}\Delta v+\sum_{j=1}^n b^j(x')D_{x'_j}v+R(t, x', D_{x'})v.$$

This equation was considered in Remark 1.1. Then, we can easily see that our sufficient condition obtained in Theorem 1.1 for $Lu(t, x)=f(t, x)$ equals that for $(\mathcal{F}^{-1})_{x\to x'}\circ L\circ\mathcal{F}_{x'\to x}v(t, x')=g(t, x')$. This fact is valid for more general transformation (see Remark 3.2 for its general proof).

Example 1.2. Let $h_2(t, x, \xi)$ be $\frac{1}{2m}|\xi|^2+\frac{m\omega^2}{2}|x|^2$, which is the Hamilton function of the harmonic oscillator, where $m>0$ and $\omega\geq 0$ are constants. Then, the solution of canonical equations (1.1) is given by

$$(q, p)(t, s; y, \xi)$$
$$=\left(y\cos\omega(t-s)+\frac{\xi}{m\omega}\sin\omega(t-s),\ -my\omega\sin\omega(t-s)+\xi\cos\omega(t-s)\right).$$

Take $\sum_{j=1}^{n}\frac{\partial\Omega}{\partial x_j}(x)\xi_j+\sum_{j=1}^{n}b^j\xi_j$ as $h_1(t, x, \xi)$, where $\Omega(x)\in\mathscr{B}^\infty(R^n)$ and b^j are real constants. Then, it follows from $p=m\frac{dq}{dt}$ that

$$\sum_{j=1}^{n}\int_0^t\frac{\partial\Omega}{\partial x_j}(q(\theta, s; y, \xi))p_j(\theta, s; y, \xi))d\theta+\sum_{j=1}^{n}b^j\int_0^t p_j(\theta, s; y, \xi)d\theta$$
$$=m[\Omega(q(t, s; y, \xi))-\Omega(y)+\sum_{j=1}^{n}b^j\{q_j(t, s; y, \xi)-y\}]$$

is valid. Hence, applying Theorem 1.1, we can see that the Cauchy problem (0.1) is L^2 well posed on $[0, T]$ for any $T>0$.

Example 1.3. Let $h_2(t, x, \xi)$ be $\frac{1}{2m}\{(\xi_1-ex_2)^2+\xi_2^2+\xi_3^2+\cdots+\xi_n^2\}$, where $m>0$ and $e\geq 0$ are constants. When $n=3$, $h_2(t, x, \xi)$ is the Hamilton function which represents the movement of a charged particle under the constant magnetic field (chapter 21 in volume 3 of [4]). Then, canonical equations are given by

$$\begin{cases}\frac{dq_1}{dt}=\frac{1}{m}(p_1-eq_2),\quad \frac{dq_2}{dt}=\frac{1}{m}p_2,\quad \frac{dp_1}{dt}=0,\quad \frac{dp_2}{dt}=\frac{e}{m}(p_1-eq_2),\\ \frac{dq_j}{dt}=\frac{1}{m}p_j(j\geq 3),\quad \frac{dp_j}{dt}=0(j\geq 3).\end{cases}$$

So, setting $\omega=e/m$, we obtain the solution of (1.1)

$$\begin{cases}eq_1=\xi_2\cos\omega(t-s)+(\xi_1-ey_2)\sin\omega(t-s)+(ey_1-\xi_2),\\ eq_2=\xi_2\sin\omega(t-s)-(\xi_1-ey_2)\cos\omega(t-s)+\xi_1,\\ q_j=\frac{(t-s)}{m}\xi_j+y_j\ (j\geq 3),\\ p_2=\xi_2\cos\omega(t-s)+(\xi_1-ey_2)\sin\omega(t-s),\quad p_j=\xi_j(j\neq 2).\end{cases}$$

Example 1.4. Let $h_2(x, \xi)$ be a polynomial of degree 2 in only x and ξ with real coefficients satisfying $h_2(x, \xi)\geq 0$ on R^{2n}. We define $h_1(x, \xi)$ by

$$h_1(x,\xi)=c\{1+h_2(x,\xi)\}^{1/2},$$

where c is a complex constant. We take these $h_j(x,\xi)$ in (0.1). It follows from the assumption on $h_2(x,\xi)$ that for any variable $z=x_j$ and $\xi_k(1\leq j,k\leq n)$ $h_2(x,\xi)$ is written in the form

$$h_2(x,\xi)=l_0(z-l_1)^2+l_2,$$

where l_0 is a non-negative constant, l_1 is a polynomial of degree 1 not including z variable and l_2 is a polynomial of degree 2 not including z variable which satisfies $l_2\geq 0$ on R^{2n}. So, it follows that $\frac{\partial h_1}{\partial z}(x,\xi)$ is bounded on R^{2n}. In the same way, we can see that $h_1(x,\xi)$ belongs to the symbol class $\mathcal{B}^{1,\infty}(R^{2n})$. The function $w(t,s;y,\xi)$ defined by (1.3) becomes

$$\exp[-ic(t-s)\{1+h_2(y,\xi)\}^{1/2}],$$

because of the energy equality $\frac{d}{dt}h_2(q(t,s;y,\xi),p(t,s;y,\xi))=0$. Hence, noting that $h_1(x,\xi)\in\mathcal{B}^{1,\infty}(R^{2n})$, we can see from Theorem 1.1 or Corollary 1.2 that the Cauchy problem (0.1) is L^2 well posed on $[0,T]$ for any $T>0$, if the imaginary part of c is non-positive.

Example 1.5. We write

$$L_1u(t,x)=\frac{1}{i}\partial_t u-\frac{1}{2}\Delta u+\sum_{j=1}^n b^j(x)D_{x_j}u$$

and

$$L_2u(t,x)=\frac{1}{i}\partial_t u+\frac{1}{2}|x|^2u+h_1(x,D_x)u,$$

where $b^j(x)\in\mathcal{B}^\infty(R^n)(j=1,2,\cdots,n)$ and we set

$$h_1(x,\xi)=-\sum_{j=1}^n x_j Os-\iint e^{-iy\cdot\eta}b^j\Big(\xi-x+\eta-\frac{1}{2}y\Big)dy\,đ\eta,$$

where $\mathrm{Os}-\iint\cdots dy\,đ\eta$ denotes the oscillatory integral (see section 6 in chapter 1 of [12]). We assume that every $b^j(x)$ are not constant. Then, we remark that the above $h_1(x,\xi)$ doesn't satisfy (1.2) in Theorem 1.1 for any subset Z. This will be proven below. So, we can not apply Theorem 1.1 to the Cauchy problem for L_2. But, we shall be able to see from Theorem 3.4 and Example 3.3 in section 3 that the L^2 well-posedness on $[0,T]$ of the Cauchy problem for L_2 is equivalent to that for L_1.

We can easily get for $f(x)\in\mathcal{B}^\infty(R^n)$

$$Os-\iint e^{-iy\cdot\eta}f\Big(\xi-x+\eta-\frac{1}{2}y\Big)dy\,đ\eta\in\mathcal{B}^\infty(R^{2n}_{x,\xi}) \tag{1.16}$$

and

$$Os-\iint e^{-iy\cdot\eta}f\Big(-x-\eta+\frac{1}{2}y\Big)dy\,đ\eta=2^n\int e^{-2ix\cdot\eta}e^{-2i|\eta|^2}\hat{f}(2\eta)đ\eta\,. \tag{1.17}$$

So, since every $b^j(x)$ is not constant, it follows that there exist numbers j, k and a point $x^{(0)}\in R^n$ satifying

$$Os-\iint e^{-iy\cdot\eta}\frac{\partial b^j}{\partial x_k}\Big(x^{(0)}+\eta-\frac{1}{2}y\Big)dy\,đ\eta\neq 0\,. \tag{1.18}$$

Take sequences $x(m)=(x_1(m),\cdots,x_n(m))\in R^n(m=0,1,\cdots)$ such that $x_j(m)=m$ and $x_l(m)=0(l\neq j)$ and define $\xi(m)\in R^n$ by $\xi(m)=x(m)+x^{(0)}$. Then,

$$\frac{\partial h_1}{\partial x_k}(x(m),\xi(m))=mOs-\iint e^{-iy\cdot\eta}\frac{\partial b^j}{\partial x_k}\Big(x^{(0)}+\eta-\frac{1}{2}y\Big)dy\,đ\eta$$
$$-Os-\iint e^{-iy\cdot\eta}b^k\Big(x^{(0)}+\eta-\frac{1}{2}y\Big)dy\,đ\eta$$

and

$$\frac{\partial h_1}{\partial \xi_k}(x(m),\xi(m))=-mOs-\iint e^{-iy\cdot\eta}\frac{\partial b^j}{\partial x_k}\Big(x^{(0)}+\eta-\frac{1}{2}y\Big)dy\,đ\eta$$

are valid. It follows from (1.16) and (1.18) that both $\frac{\partial h_1}{\partial x_k}$ and $\frac{\partial h_1}{\partial \xi_k}$ are not bounded on R^{2n}.

Example 1.6. Let $b_k{}^j(x)\in\mathcal{B}^\infty(R^n)$ $(j=1,2,\cdots,n,\ k=1,2)$. When we take $\sum_{j=1}^n b_1{}^j(x)\xi_j$ and $\sum_{j=1}^n b_2{}^j(\xi)x_j$ as $h_1(t,x,\xi)$ in (0.1), each $h_1(t,x,\xi)$ satisfies the assumption stated in Theorem 1.1. But, if we set $h_1(t,x,\xi)=\sum_{j=1}^n b_1{}^j(x)\xi_j+\sum_{j=1}^n b_2{}^j(\xi)x_j$, this $h_1(t,x,\xi)$ doesn't satisfy the assumption in Theorem 1.1 generally. Moreover, it seems that we can not apply the results in section 3.

2. Proof of Theorem 1.1

At first, we shall introduce some results from [10], [11] and others. Since we will often use their results in the modified form and make the present paper self-contained, we shall give the outline of some of their proofs.

Lemma 2.1 (*Proposition 3.1 in* [10]). *Let $h_2(t,x,\xi)$ be a symbol defined in introduction and (q,p) $(t,s;y,\xi)=(q_1,\cdots,q_n,p_1,\cdots,p_n)$ the solution of the Hamilton canonical equations* (1.1). *Then, $q_j(t,s;y,\xi)\in\mathcal{B}^{1,\infty}(R^{2n})(j=1,2,\cdots,n)$ and $p_k(t,s;y,\xi)\in\mathcal{B}^{1,\infty}(R^{2n})(k=1,2,\cdots,n)$ are valid for each t and s $(0\leq$*

$t, s \leq T$). In more detail, we get

$$(2.1)\qquad \begin{cases} |\partial_\xi^\alpha D_y^\beta\{q_j(t,s;y,\xi)-y_j\}| \leq M_{|\alpha+\beta|}|t-s| & (|\alpha+\beta|\geq 1), \\ |\partial_\xi^\alpha D_y^\beta\{p_k(t,s;y,\xi)-\xi_k\}| \leq M_{|\alpha+\beta|}|t-s| & (|\alpha+\beta|\geq 1) \end{cases}$$

for j and $k=1,2,\cdots,n$ with constants $M_{|\alpha+\beta|}$ independent of t, $s(0\leq t, s\leq T)$ and $(x,\xi)\in R^{2n}$.

Proof. It follows from (1.1) that

$$(1.1)'\qquad \begin{cases} \dfrac{d}{dt}\dfrac{{}^t\partial q}{\partial z} = \dfrac{\partial^2 h_2}{\partial x\partial\xi}(t,q,p)\dfrac{{}^t\partial q}{\partial z} + \dfrac{\partial^2 h_2}{\partial\xi\partial\xi}\dfrac{{}^t\partial p}{\partial z}, \\ \dfrac{d}{dt}\dfrac{{}^t\partial p}{\partial z} = -\dfrac{\partial^2 h_2}{\partial x\partial x}(t,q,p)\dfrac{{}^t\partial q}{\partial z} - \dfrac{\partial^2 h_2}{\partial\xi\partial x}\dfrac{{}^t\partial p}{\partial z}, \end{cases}$$

are valid for $z=x$ and ξ, where $\dfrac{{}^t\partial q}{\partial z}$ and $\dfrac{{}^t\partial p}{\partial z}$ denote the transposed vectors of $\dfrac{\partial q}{\partial z}$ and $\dfrac{\partial p}{\partial z}$ respectively. If we take account of the assumption (0.2) on $h_2(t, x, \xi)$, we can easily get for t and s $(0\leq t, s\leq T)$

$$\left|\frac{d}{dt}\left\{\left|\frac{\partial q}{\partial z}\right|^2 + \left|\frac{\partial p}{\partial z}\right|^2\right\}\right| \leq C_1\left\{\left|\frac{\partial q}{\partial z}\right|^2 + \left|\frac{\partial p}{\partial z}\right|^2\right\}$$

and so

$$\left|\frac{\partial q}{\partial z}\right|^2 + \left|\frac{\partial p}{\partial z}\right|^2 \leq C_2,$$

where C_1 and C_2 are constants independent of t, s and (y,ξ). Consequently, we obtain from (1.1)′

$$\left|\frac{d}{dt}\frac{\partial}{\partial z}\{q(t,s;y,\xi)-y\}\right| + \left|\frac{d}{dt}\frac{\partial}{\partial z}\{p(t,s;y,\xi)-\xi\}\right| \leq C_3$$

with a constants C_3 independent of t, s and (y,ξ). Hence, we can prove (2.1) in the case $|\alpha+\beta|=1$. In the same way we can complete the proof. Q.E.D.

In the present paper we will use the lemma below on the global homeomorphism without its proof, instead of the contraction principle used in [10] and [11]. Then, we can construct the fundamental solution on the wider interval in t variable than that obtained in [10] and [11], as will be seen in the proof of Theorem 1.1. We will also use this lemma for showing the well-definedness of the canonical transformation determined by generating function in section 3 of the present paper and [8].

Lemma 2.2. (*Theorem* 1.22 *in* [16]). *Let f be a C^1 mapping: $R^n \ni x \to$*

$f(x)=(f_1(x), \cdots, f_n(x))\in R^n$. *If there exists the inverse matrix* $\frac{\partial f}{\partial x}(x)^{-1}$ *of* $\frac{\partial f}{\partial x}(x)$ *for each* $x\in R^n$ *and we have*

$$\sup_{x\in R^n}\left\|\frac{\partial f}{\partial x}(x)^{-1}\right\|<\infty,$$

then f is a homeomorphism of R^n onto R^n. Here, $\|Q\|$ for a matrix Q indicates the operator norm of Q as the mapping from R^n to R^n.

Lemma 2.3 (*c.f. Proposition* 3.3 *in* [10]). *We consider $q(t, s; y, \xi)$ determined in Lemma* 2.1. *Then, we can take a $T_0(0<T_0\leq T)$ such that for each t, s $(0\leq t, s\leq T_0)$ and $\xi\in R^n$ there exists the inverse C^∞ diffeomorphism: $R^n\ni x\to y=y(t, s; x, \xi)=(y_1, y_2, \cdots, y_n)\in R^n$ of the mapping: $R^n\ni y\to x=q(t, s; y, \xi)\in R^n$. Moreover, we get*

$$|\partial_\xi^\alpha D_x^\beta\{y_j(t, s; x, \xi)-x_j\}|\leq M'_{|\alpha+\beta|}|t-s| \quad (|\alpha+\beta|\geq 1), \tag{2.2}$$

with constants $M'_{|\alpha+\beta|}$ independent of t, s $(0\leq t, s\leq T_0)$ and $(x, \xi)\in R^{2n}$.

Proof. It follows from Lemma 2.1 that we can take a T_0 satisfying

$$\left|\det\frac{\partial q}{\partial y}(t, s; y, \xi)\right|\geq\delta>0$$

for t,s $(0\leq t, s\leq T_0)$ and $y, \xi\in R^n$, where δ is a constant. Then, the existence of the inverse C^∞ diffeomorphism stated in this lemma follows from (2.1) and Lemma 2.2. Inequalities (2.2) can be derived by differentiating $q(t, s; y(t, s; x, \xi), \xi)=x$. Q.E.D.

Lemma 2.4 (*c.f. Proposition* 3.5 *in* [10]). *Suppose the same assumptions as in Lemma 2.1 and take a T_0 determined in Lemma* 2.3. *Then, the solution $\Phi(t, s; x, \xi)$ $(0\leq t, s\leq T_0, (x, \xi)\in R^{2n})$ of the eiconal equation*

$$\partial_t\Phi+h_2\left(t, x, \frac{\partial\Phi}{\partial x}\right)=0, \quad \Phi|_{t=s}=x\cdot\xi \tag{2.3}$$

is determined uniquely and satisfies

$$\left\{\begin{array}{l}\frac{\partial\Phi}{\partial x}(t, s; q(t, s; y, \xi), \xi)=p(t, s; y, \xi), \\ \frac{\partial\Phi}{\partial\xi}(t, s; q(t, s; y, \xi), \xi)=y\end{array}\right. \tag{2.4}$$

and

$$\frac{\partial^2\Phi}{\partial x\partial\xi}(t, s; q(t, s; y, \xi), \xi)=\frac{\partial q}{\partial y}(t, s; y, \xi)^{-1}. \tag{2.5}$$

Proof. We know well the classical result that setting

$$S(t, s; y, \xi)=y\cdot\xi+\int_s^t \sum_{j=1}^n p_j(\theta, s; y, \xi)dq_j(\theta, s; y, \xi)$$

$$-h_2(\theta, q(\theta, s; y, \xi), p(\theta, s; y, \xi))d\theta\,,$$

an equality on differential forms

$$dS=\sum_{j=1}^n p_j dq_j-h_2(t, q, p)dt \quad \text{on } [0, T]\times R_y^n \tag{2.6}$$

holds. In fact, it follows from (1.1) and the definition of $S(t, s; y, \xi)$ that

$$\begin{cases} \dfrac{\partial S}{\partial t}(t, s; y, \xi)=\sum_j p_j(t, s; y, \xi)\dfrac{\partial q_j}{\partial t}(t, s; y, \xi)-h_2(t, q, p)\,, \\ \dfrac{\partial S}{\partial y_k}(t, s; y, \xi)=\sum_j p_j(t, s; y, \xi)\dfrac{\partial q_j}{\partial y_k}(t, s; y, \xi) \end{cases}$$

is valid. Hence, we get (2.6). If we determine $\Phi(t, s; x, \xi)$ by $\Phi=S(t, s; y(t, s; x, \xi), \xi)$ $(0\leq t, s\leq T_0)$, we get from (2.6) at once

$$\begin{cases} \partial_t\Phi+h_2(t, x, p(t, s; y(t, s; x, \xi), \xi))=0\,, \\ \dfrac{\partial \Phi}{\partial x}(t, s; x, \xi)=p(t, s; y(t, s; x, \xi), \xi)\,. \end{cases}$$

So, we obtain (2.3) and the first equality of (2.4). Using these two equalities obtained now, we can easily see

$$\frac{d}{dt}\frac{\partial\Phi}{\partial\xi}(t, s; q(t, s; y, \xi), \xi)=0$$

for any t, s, y and ξ. So, the second equality of (2.4) is derived. (2.5) can be also yielded from the second equality of (2.4). Q.E.D.

Now, we shall introduce Fourier integral operators. Let $\phi(x, \xi)\in\mathscr{B}^{2,\infty}(R^{2n})$ be a real valued function. Then, the Fourier integral operator $P_\phi(x, D_x)$ with a symbol $p(x, \xi)\in T^m(m\geq 0)$ and a phase function $\phi(x, \xi)$ is defined by

$$P_\phi(x, D_x)u(x)=\int e^{i\phi(x,\xi)}p(x, \xi)\hat{u}(\xi)đ\xi \tag{2.7}$$

for $u(x)\in\mathcal{S}$.

Lemma 2.5 (*c.f. Theorem 3.7 in* [11]). *For any $p(x, \xi)\in T^{m_1}$ $(m_1\geq 0)$ and $q(x, \xi)\in T^{m_2}(m_2\geq 0)$ there exists an $r(x, \xi)\in T^{m_1+m_2}$ such that*

$$P(x, D_x)\circ Q_\phi(x, D_x)=R_\phi(x, D_x)$$

holds. Concretely, $r(t, x, \xi)$ is given by

(2·8)
$$r(t, x, \xi)$$
$$= Os-\iint e^{-iy\cdot\eta}p\Big(x, \eta+\int_0^1\frac{\partial\phi}{\partial x}(x+\theta y, \xi)d\theta\Big)q(x+y, \xi)dy đ\eta$$
$$=p\Big(x, \frac{\partial\phi}{\partial x}(x, \xi)\Big)q(x, \xi)+\Sigma_{|\alpha|=1}D_y^\alpha\Big\{p^{(\alpha)}\Big(x, \int_0^1\frac{\partial\phi}{\partial x}(x+\theta y, \xi)d\theta\Big)$$
$$\times q(x+y, \xi)\Big\}\Big|_{y=0}+2\int_0^1(1-\theta')d\theta'\Sigma_{|\alpha|=2}\frac{1}{\alpha!}Os$$
$$-\iint e^{-iy\cdot\eta}D_y^\alpha\Big\{p^{(\alpha)}\Big(x, \theta'\eta+\int_0^1\frac{\partial\phi}{\partial x}(x+\theta y, \xi)d\theta\Big)q(x+y, \xi)\Big\}$$
$$\times dy đ\eta .$$

Proof. We can prove in the same way to the proof of Theorem 3.7 in [11]. It follows from the definition of Fourier integral operators that

$$P\circ Q_\phi u(x)=\iiint e^{i\phi(x,\xi')+i\psi}p(x, \xi)q(x', \xi')\hat{u}(\xi')đ\xi'\mathrm{d}x'đ\xi \tag{2.9}$$

is valid for $u(x)\in\mathcal{S}$, where

$$\psi=(x-x')\cdot\xi+\phi(x', \xi')-\phi(x, \xi') .$$

The well definedness of the right-hand side of (2.9) is assured, if we use the integrals by parts in x' and ξ. Changing the variables ξ to $\eta=\xi-\int_0^1\frac{\partial\phi}{\partial x}(x'+\theta(x-x'), \xi')\mathrm{d}\theta$, the right-hand side of (2.9) is written as

$$\iiint e^{i\phi(x,\xi')+i(x-x')\cdot\eta}p\Big(x, \int_0^1\frac{\partial\phi}{\partial x}(x'+\theta(x-x'), \xi')d\theta+\eta\Big)$$
$$\times q(x', \xi')\hat{u}(\xi')đ\xi'\mathrm{d}x'đ\eta .$$

Once again, make the change of variables x' to $y=x'-x$. Then, we get the first equality of (2.8). The second equality of (2.8) is easily obtained from the first equality by using the Taylor expansion formula.

As for $r(x, \xi)\in T^{m_1+m_2}$ we can see from (2.8)

$$r(x, \xi)=Os-\iint e^{-iy\cdot\eta}\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}$$
$$\times p\Big(x, \eta+\int_0^1\frac{\partial\phi}{\partial x}(x+\theta y, \xi)d\theta\Big)q(x+y, \xi)\mathrm{d}y đ\eta .$$

Setting $l\geq(n+1+m_1+m_2)/2$ and noting that $p\in T^{m_1}$, $q\in T^{m_2}$ and $\phi\in\mathcal{B}^{2,\infty}$, we get

$$|r(x, \xi)|\leq\text{Const. }(1+|x|^2+|\xi|^2)^{(m_1+m_2)/2} .$$

In the same way we obtain $r(x, \xi)\in T^{m_1+m_2}$. Q.E.D.

Lemma 2.6 (*c.f. Proposition 3.2 in* [*11*]). *We assume that a real valued function* $\phi(x, \xi)$ *belongs to the class* $\mathscr{B}^{2,\infty}(R^{2n})$ *and satisfies*

$$\left|det\frac{\partial^2\phi}{\partial x\partial\xi}(x, \xi)\right|\geq\delta>0, \tag{2.10}$$

where δ *is a constant independent of* $(x, \xi)\in R^{2n}$. *Then, the Fourier integral operator* $P_\phi(x, D_x)$ *with a symbol* $p(x, \xi)\in T^m$ *for an* $m\geq 0$ *defines a linear continuous operator from* $\mathcal{S}$ *to* $\mathcal{S}$.

Proof. Let define J_1 and J_2 by

$$J_1=\langle\xi\rangle^{-2}\left(1+i\xi\cdot\frac{\partial}{\partial x'}\right)$$

and

$$J_2=\left[1+\left\{\frac{\partial\phi}{\partial\xi}(x, \xi)-\frac{\partial\phi}{\partial\xi}(0, \xi)\right\}^2\right]^{-1}\left[1-i\left\{\frac{\partial\phi}{\partial\xi}(x, \xi)-\frac{\partial\phi}{\partial\xi}(0, \xi)\right\}\cdot\frac{\partial}{\partial\xi}\right].$$

Here, we note that

$$\left|\frac{\partial\phi}{\partial\xi}(x, \xi)-\frac{\partial\phi}{\partial\xi}(0, \xi)\right|\geq\delta'|x|$$

follows from $(2\cdot10)$ with a constant $\delta'>0$. We can easily have

$$P_\phi(x, D_x)u(x)=\iint e^{i\phi(x,\xi)-i\phi(0,\xi)}({}^tJ_2)^{l_2}e^{i\phi(0,\xi)-ix'\cdot\xi}$$

$$\times({}^tJ_1)^{l_1}p(x, \xi)u(x')dx'đ\xi,$$

where ${}^tJ_j(j=1, 2)$ implies the transposed operator of J_j. So,

$$\langle x\rangle^k|P_\phi(x, D_x)u(x)|$$

$$\leq\text{Const.}\sum_{|\alpha|\leq l_1}\langle x\rangle^{k-l_2}\iint(1+|x'|+|\xi|)^{l_2}\langle\xi\rangle^{-l_1}$$

$$\times(1+|x|+|\xi|)^m|D_x{}^\alpha u(x')|dx'đ\xi$$

is valid. Hence, taking $l_2=k+m$ and $l_1=l_2+m+n+1$, we have

$$\langle x\rangle^k|P_\phi(x, D_x)u(x)|\leq\text{Const.}\sum_{|\alpha|\leq l_1}\sup_x(\langle x\rangle^{k+m+n+1}|D_x{}^\alpha u(x)|).$$

In the same way we can complete the proof. Q.E.D.

We know about the L^2-bondedness of Fourier integral operators. We introduce from [2] without its proof.

Lemma 2.7 (*Theorem* 1 *in* [2]). *Suppose the same assumptions as in Lemma* 2.6. *Then, the Fourier integral operator* $P_\phi(x, D_x)$ *with a symbol* $p(x, \xi)\in T^0(R^{2n})$ *is an* L^2*-bounded operator. That is, we have for* $u\in L^2(R^n)$

$$\|P_\phi(x, D_x)u(\cdot)\|\leq C\|u(\cdot)\|,$$

where C *is a constant determined from* $\max_{|\alpha+\beta|\leq l}\sup_{x,\xi}|p_{(\beta)}^{(\alpha)}(x, \xi)|$ *and* $\max_{2\leq|\alpha+\beta|\leq l}\sup_{x,\xi}|\phi_{(\beta)}^{(\alpha)}(x, \xi)|$ *for an integer* $l\geq 2$.

Proof of Theorem 1.1. We can assume from Proposition 1.3 that the subset Z in Theorem 1.1 is an empty set. That is, we assume

$$|\partial_{\xi_j}h_{1(\beta)}^{(\alpha)}(t, x, \xi)|\leq C_{\alpha,\beta}\quad (j=1, 2, \cdots, n) \tag{2.11}$$

for all α and β with constants $C_{\alpha,\beta}$ independent of $(t, x, \xi)\in[0, T]\times R^{2n}$.

Let $\Phi(t, s; x, \xi)$ be the solution of eiconal equation (2.3). $\Phi(t, s; x, \xi)$ $(0\leq t, s\leq T_0, (x, \xi)\in R^{2n})$ is obtained from Lemma 2.4, where T_0 is a constant determined in Lemma 2.3. We can see from (2.1), (2.2) and (2.4) that

$$|\Phi_{(\beta)}^{(\alpha)}(t, s; x, \xi)|\leq C_{\alpha,\beta}\quad (|\alpha+\beta|\geq 2) \tag{2.12}$$

are valid, where $C_{\alpha,\beta}$ are constants independent of t, s $(0\leq t, s\leq T_0)$ and $(x, \xi)\in R^{2n}$. We remark that $C_{\alpha,\beta}$ in (2.12) are different from constants in (2.11). If there is no confusion, we shall use the same symbol $C_{\alpha,\beta}$. We consider the Fourier integral operator $A_\Phi(t, s; x, D_x)$ $(0\leq s\leq t\leq T_0)$. Take account of assumptions (0.2) and (2.11). Then, if symbol $a(t, s; x, \xi)$ belongs to $T^0(R^{2n})$, it follows from Lemma 2.5 and (2.12) that

$$\begin{cases} L\circ A_\Phi(t, s; x, D_x)=B_\Phi(t, s; x, D_x), \\ b(t, s; x, \xi)=\left\{\dfrac{\partial\Phi}{\partial t}(t, s; x, \xi)+h_2\left(t, x, \dfrac{\partial\Phi}{\partial x}(t, s; x, \xi)\right)\right\} \\ \qquad\times a(t, s; x, \xi)+\dfrac{1}{i}\left\{\dfrac{\partial a}{\partial t}(t, s; x, \xi)\right. \\ \qquad+\sum_{j=1}^n\dfrac{\partial h_2}{\partial\xi_j}\left(t, x, \dfrac{\partial\Phi}{\partial x}\right)\dfrac{\partial a}{\partial x_j}(t, s; x, \xi) \\ \qquad\left.+ih_1\left(t, x, \dfrac{\partial\Phi}{\partial x}\right)a(t, s; x, \xi)\right\}+r(t, s; x, \xi), \end{cases} \tag{2.13}$$

where

$$|r_{(\beta)}^{(\alpha)}(t, s; x, \xi)|\leq C_{\alpha,\beta} \tag{2.14}$$

are valid for all α and β with constants $C_{\alpha,\beta}$ independent of t, s $(0\leq s\leq t\leq T_0)$ and $(x, \xi)\in R^{2n}$.

We determine $a(t, s; x, \xi)$ as the solution of

$$\begin{cases} \frac{\partial a}{\partial t}(t, s; x, \xi)+\sum_{j=1}^{n}\frac{\partial h_2}{\partial \xi_j}\left(t, x, \frac{\partial \Phi}{\partial x}(t, s; x, \xi)\right)\frac{\partial a}{\partial x_j}(t, s; x, \xi) \\ \qquad +ih_1\left(t, x, \frac{\partial \Phi}{\partial x}\right)a(t, s; x, \xi)=0\,, \\ a(s, s; x, \xi)=1\,. \end{cases}$$

Let $(q, p)(t, s; y, \xi)$ be the solution of (1.1). Then, since $\frac{\partial \Phi}{\partial x}(t, s; q(t, s; y, \xi), \xi)=p(t, s; y, \xi)$ is valid from (2.4), the above equation becomes

$$\frac{d}{dt}a(t, s; q(t, s; y, \xi), \xi)+ih_1(t, q, p)a(t, s; q, \xi)=0\,.$$

So, we get

$$a(t, s; q(t, s; y, \xi), \xi)=w(t, s; y, \xi)\,, \tag{2.15}$$

where $w(t, s; y, \xi)$ was defined by (1.3). We can easily see from the assumption (1.4) on $w(t, s; y, \xi)$ and Lemma 2.3 that $a(t, s; x, \xi)$ determined by (2.15) belongs to $T^0(R^{2n})$ for t and s $(0\leq s\leq t\leq T_0)$. Therefore, we obtain together with (2.3)

$$L\circ A_\Phi(t, s; x, D_x)=R_\Phi(t, s; x, D_x),\ A_\Phi(s, s; x, D_x)=\text{Identity}, \tag{2.16}$$

where $r(t, s; x, \xi)$ satisfies (2.14) for all α and β.

Now, we denote by $\mathcal{L}$ the set of all linear L^2-bounded operators and by $\mathcal{E}^0_{t,s}([0, T_0]; \mathcal{L})$ the set of all $\mathcal{L}$-valued continuous functions in t and s such that $0\leq s\leq t\leq T_0$. Following the usual method (see section 4 in chapter 7 of [12]), we shall construct the fundamental solution $E(t, s)\in \mathcal{E}^0_{t,s}([0, T_0]; \mathcal{L})$ of (0.1) (i.e. $LE(t, s)=0$, $E(s, s)=\text{Identity}$) in the form

$$E(t, s)=A_\Phi(t, s; x, D_x)+\int_s^t A_\Phi(t, \tau; x, D_x)\circ\Omega(\tau, s)d\tau\,, \tag{2.17}$$

where $\Omega(t, s)\in \mathcal{E}^0_{t,s}([0, T_0]; \mathcal{L})$. Then, $\Omega(t, s)$ is determined as the solution of the integral equation of the Volterra type

$$0=\Omega(t, s)+iR_\Phi(t, s; x, D_x)+i\int_s^t R_\Phi(t, \tau; x, D_x)\circ\Omega(\tau, s)d\tau\,.$$

We see from (2.14), Lemmas 2.4 and 2.7 that $R_\Phi(t, s; x, D_x)$ belongs to $\mathcal{E}^0_{t,s}([0, T_0]; \mathcal{L})$. So, this $\Omega(t, s)$ can be obtained succesively in the form

$$\Omega(t, s)=\sum_{j=0}^{\infty}\Omega_j(t, s)\,,$$

where

$$\begin{cases} \Omega_0(t, s) = -iR_\Phi(t, s; x, D_x), \\ \Omega_j(t, s) = -i\int_s^t R_\Phi(t, \tau; x, D_x)\circ\Omega_{j-1}(\tau, s)d\tau \quad (j=1, 2, \cdots). \end{cases}$$

Thus, we obtain $E(t, s)\in\mathcal{E}^0_{t,s}([0, T_0]; \mathcal{L})$.

For data $u_0(x)\in L^2$ and $f(t, x)\in L^1_t([s, T_0]; L^2)$ we define $u(t, x)$ $(s\leq t\leq T_0)$ by

$$u(t, x)=E(t, s)u_0(x)+i\int_s^t E(t, \tau)f(\tau, x)d\tau. \tag{2.18}$$

We will show that this $u(t, x)$ satisfies the equation in (0.1) in a distribution sense. We can see from the proof of Theorem 6.1 in [11] that if T_1 is small $(0<T_1\leq T_0)$, $\Omega(t, s)$ in (2.17) is written by a Fourier integral operator as

$$\Omega(t, s)=D_\Phi(t, s; x, D_x) \quad (0\leq s\leq t\leq T_1),$$

where $d(t, s; x, \xi)\in T^0(R^{2n})$. So, we can see from Lemma 2.6 that if $u_0(x)\in\mathcal{S}$ and $f(t, x)\in L^1_t([s, T]; \mathcal{S})$, $u(t, x)$ defined by (2.18) is a genuine solution of (0.1) on $[s, T_1]\times R_x^n$. In the same way $u(t, x)$ becomes a genuine solution on $[T_1, T_1']\times R_x^n$ $(T_1'=\min(2T_1, T_0))$. Consequently, this $u(t, x)$ becomes a genuine solution on $[s, T_0]\times R_x^n$. Hence, approximating $u_0(x)\in L^2$ and $f(t, x)\in L^1_t([s, T]; L^2)$ by the elements of $\mathcal{S}$ and $L^1_t([s, T]; \mathcal{S})$ respectively, we can see that $u(t, x)$ defined by (2.18) is a solution of (0.1) on $[s, T_0]\times R_x^n$ in a distribution sense, because $E(t, s)$ belongs to $\mathcal{E}^0_{t,s}([0, T_0]; \mathcal{L})$. Here, we used the delicate result in [11] to show that $u(t, x)$ defined by (2.18) where $u_0(x)\in\mathcal{S}$ and $f(t, x)\in L^1_t([s, T]; \mathcal{S})$ is a genuine solution of (0. 1). But, we can prove it in a much easier way from a theorem on the boundedness of our Fourier integral operators. This theorem will be publised eleswhere.

We can easily extend the existence interval $[s, T_0]$ of the solution $u(t, x)$ constructed above to $[s, T]$ as follows. Consider the Cauchy problem

$$Lv(t, x)=f(t, x) \text{ on } [T_0, T]\times R_x^n,\ v(t_0, x)=u(T_0, x).$$

Then, we get the solution $v(t, x)\in\mathcal{E}^0_t([T_0, T_2]; L^2)$ where $T_2=\min(2T_0, T)$ in the same way as in the construction of $u(t, x)\in\mathcal{E}^0_t([s, T_0]; L^2)$. Consequently, we obtain the solution $u(t, x)\in\mathcal{E}^0_t([s, T_2]; L^2)$ of (0.1) for $u_0(x)\in L^2$ and $f(t, x)\in\mathcal{E}^0_t([s, T]; L^2)$. Repeating this process, the solution $u(t, x)$ of (0.1) is obtained on $[s, T]$. It is easy from (2.18) to show that the energy inequality stated in Definition 1.1 is valid for the solution $u(t, x)$ obtained now.

Next, we will show that the solution $u(t, x)\in\mathcal{E}^0_t([s, T]; L^2)$ is only one. Here, we may suppose $s=0$ without the loss of generality. The formally adjoint operator $^*L$ for L in (0.1) is given in the form

$$^*L=\frac{1}{i}\partial_t+\bar{H}_2(t, \hat{x}, D_x)+\bar{H}_1(t, \hat{x}, D_x) \tag{2.19}$$

from Theorem 1.7 in chapter 2 of [12], where $\bar{H}_j(t, \tilde{x}, D_x)(j=1, 2)$ denotes the pseudo-differential operator with a double symbol $\overline{h_j(\tilde{x}, \xi)}$. $\overline{h_j(x, \xi)}$ is the complex conjugate of $h_j(x, \xi)$. That is,

$$\int_0^T \int \{L\varphi(t, x)\}\overline{\psi(t, x)}\mathrm{d}x\mathrm{d}t = \int_0^T \int \varphi(t, x)\overline{\{{}^*L\psi(t, x)\}}\mathrm{d}x\mathrm{d}t \tag{2.20}$$

is valid for any $\varphi(t, x)$ and $\psi(t, x)$ belonging to $C_0^\infty((0, T)\times R_x^n)$. Apply the asymptotic expansion formula of double symbols (Theorem 3.1 in chapter 2 of [12]) to the each term in (2.19). Then, since $h_2(t, x, \xi)$ is real valued, satisfies (0.2) and $h_1(t, x, \xi)$ satisfies (2.11), so ${}^*L$ is written as

$$ {}^*L = \frac{1}{i}\partial_t + H_2(t, x, D_x) + \bar{H}_1(t, x, D_x) + R_1(t, x, D_x)\,, \tag{2.19$'$}$$

where $r_1(t, x, \xi)$ satisfies the same inequalities as (2.14).

Take a $g(t, x)\in C_0^\infty((0, T)\times R_x^n)$ and consider the backward Cauchy problem

$$ {}^*Lv(t, x) = g(t, x) \quad \text{on} \quad [0, T]\times R_x^n\,, \quad v(T, x) = 0\,. \tag{2.21}$$

We take t and s such that $0\leq t\leq s\leq T$ and set for any $(y, \xi)\in R^{2n}$

$$(y', \xi') = (q(t, s;\ y, \xi), p(t, s;\ y, \xi))\,. \tag{2.22}$$

Then,

$$(q, p)(\theta, s;\ y, \xi) = (q, p)(\theta, t;\ y', \xi') \quad (t\leq\theta\leq s)$$

follows from the uniqueness of the solution of ordinary equations (1.1). So,

$$\begin{aligned} &\exp\left\{-i\int_s^t \overline{h_1(\theta, q(\theta, s;\ y, \xi), p(\theta, s;\ y, \xi))}\mathrm{d}\theta\right\} \\ &= \overline{\exp\left\{-i\int_t^s h_1(\theta, q(\theta, t;\ y', \xi'), p(\theta, t;\ y', \xi'))\mathrm{d}\theta\right\}} \\ &= \overline{w(s, t;\ y', \xi')} \end{aligned} \tag{2.23}$$

is valid from (1.3) for all t and s such that $0\leq t\leq s\leq T$. Hence, we can see from assumption (1.4) and Lemma 2.1 that

$$\begin{aligned} &\sup_{0\leq t\leq s\leq T}\ \sup_{(y, \xi)\in R^{2n}} |\partial_\xi^\alpha D_y^\beta \exp\left\{-i\int_s^t \overline{h_1(\theta, q(\theta, s;\ y, \xi),} \right. \\ &\qquad \left. \overline{p(\theta, s;\ y, \xi)}\mathrm{d}\theta\right\}| < \infty \end{aligned} \tag{2.24}$$

is valid for all α and β. From this result we can construct a solution $v(t, x)\in \mathcal{E}_t^0([0, T];\ L^2)$ of (2.21) in the similar form to (2.18).

Let $u(t, x)\in \mathcal{E}_t^0([0, T]; L^2)$ be the solution of (0.1) where $s=0$, $u_0(x)=0$ and $f(t, x)=0$. We have already proved about the solution $v(t, x)$ determined above that $v(t, x)$ and $\frac{\partial v}{\partial t}(t, x)$ are continuous as an $\mathcal{S}$-valued function on $[0, T]$. So, we get

$$(2.25)\qquad \begin{aligned} 0&=\int_0^T\int u(t, x)^*\overline{Lv(t, x)}\mathrm{d}x\mathrm{d}t \\ &=\int_0^T\int u(t, x)\overline{g(t, x)}\mathrm{d}x\mathrm{d}t\,, \end{aligned}$$

because $u(t, x)$ is a solution in a distribution sense. Here, $g(t, x)$ was an arbitrary function belonging to $C_0^\infty((0, T)\times R_x^n)$. Consequently, the above equality (2.25) shows that $u(t, x)$ vanishes on $[0, T]\times R_x^n$. Therefore, we can see that the solution of (0.1) belonging to $\mathcal{E}_t^0([0, T]; L^2)$ is only one. Q.E.D.

3. Canonical transformations and the Egorov theorem

In this section we shall extend the result obtained in Theorem 1.1 to other equations by applying the Egorov theorem in the revised form. Ju. V. Egorov in [3] treated real valued functions $\phi(x, \xi)$ locally defined on an open set $U\times R_\xi^n$ in $R_x^n\times R_\xi^n$ satisfying $\phi(x, \lambda\xi)=\lambda\phi(x, \xi)$ for all $\lambda>0$. We stated an example in section 1 (Example 1.5).

Throughout this section we assume that a phase function $\phi(x', \xi)$ is a polynomial of degree 2 in x' and ξ with real coefficients. Though we can relax this assumption on $\phi(x', \xi)$ a little more, we limit our phase funtions to those mentioned above for the simplicity. We denote by $I_\phi=I_{\phi(x',\xi)}$ the Fourier integral operator with a symbol 1 and a phase function $\phi(x', \xi)$, that is,

$$(3.1)\qquad I_\phi u(x')=\iint e^{i\phi(x',\xi)-ix\cdot\xi}u(x)\mathrm{d}x\,đ\xi \quad (u(x)\in \mathcal{S})\,.$$

Then, the formally adjoint operator $I_\phi{}^*$ of I_ϕ in the L^2-inner product can be defined by

$$(3.2)\qquad I_\phi^* v(x)=\iint e^{ix\cdot\xi-i\phi(x',\xi)}v(x')\mathrm{d}x'đ\xi$$

for $v(x')\in \mathcal{S}$.

We set

$$(3.3)\qquad \begin{cases} \tilde{\nabla}_{x'}\phi(x', z', \xi)=\int_0^1\frac{\partial\phi}{\partial x'}(z'+\theta(x'-z'), \xi)\mathrm{d}\theta\,, \\ \tilde{\nabla}_\xi\phi(x', \xi, \eta)=\int_0^1\frac{\partial\phi}{\partial\xi}(x', \eta+\theta(\xi-\eta))\mathrm{d}\theta \end{cases}$$

as in [11] for x', z', ξ and $\eta\in R^n$. If we assume that

$$\det\frac{\partial^2\phi}{\partial x'\partial\xi}(x',\xi)\neq 0 \tag{3.4}$$

is valid, we can see from Lemma 2.2 that the mapping for any fixed x' and z': $R^n\ni\xi\to\xi'=\tilde{\nabla}_{x'}\phi(x',z',\xi)\in R^n$ makes a diffeomorphism from R_ξ^n onto $R_{\xi'}^n$. We denote by $\xi=\tilde{\nabla}_{x'}\phi^{-1}(x',z',\xi')$ the inverse mapping. Under the same assumption (3.4) the mapping for any fixed ξ and η: $R^n\ni x'\to x=\tilde{\nabla}_\xi\phi(x',\xi,\eta)\in R^n$ makes a diffeomorphism from $R_{x'}^n$ onto R_x^n. We also denote by $x'=\tilde{\nabla}_\xi\phi^{-1}(x,\xi,\eta)$ its inverse mapping. In the same way, we can define a diffeomorphic mapping $\Phi=\Phi(x',\xi')$ from $R_{x',\xi'}^{2n}$ onto $R_{x,\xi}^{2n}$ by

$$x=\frac{\partial\phi}{\partial\xi}(x',\xi),\quad \xi'=\frac{\partial\phi}{\partial x'}(x',\xi), \tag{3.5}$$

if we assume (3.4). It is well known that this mapping Φ is a canonical transformation, that is, Φ satisfies (1.14) (see section 48 in [1]). Such Φ is called the canonical transformation generated by $\phi(x',\xi)$.

Lemma 3.1. *Let $\phi(x',\xi)$ be a polynomial of degree 2 in x' and ξ with real coefficients such that* (3.4) *is valid. We suppose that $h(x,\xi)$ belongs to $T^m(R^{2n})$ for an $m\geq 0$. Then,*

(i) *we get*

$$I_\phi\circ H(x,D_x)\cdot I_\phi^*=P(x',D_{x'},\bar{x}'), \tag{3.6}$$

where double symbol $p(x',\xi',\bar{x}')$ is defined by

$$p(x',\xi',\bar{x}')=Os-\iint e^{-iy\cdot\eta}h(\tilde{\nabla}_\xi\phi(\bar{x}',\xi,\xi+\eta)-y,\xi+\eta)dy\,đ\eta \times\left|\det\frac{\partial^2\phi}{\partial x'\partial\xi}\right|^{-1}\quad(\xi=\tilde{\nabla}_{x'}\phi^{-1}(x',\bar{x}',\xi')). \tag{3.7}$$

(ii) *We get*

$$I_\phi^*\circ K(x',D_{x'})\circ I_\phi=S(x,D_x), \tag{3.8}$$

if we define $s(x,\xi)$ by

$$\begin{cases} s(x,\xi)=Os-\iint e^{-iy\cdot\eta}s_1(\tilde{\nabla}_\xi\phi^{-1}(x+y,\xi,\xi+\eta),\xi)dy\,đ\eta \\ \qquad\times\left|\det\dfrac{\partial^2\phi}{\partial x'\partial\xi}\right|^{-1}, \\ s_1(x',\xi)=Os-\iint e^{-iy'\cdot\eta'}k(x',\tilde{\nabla}_{x'}\phi(x',x'+y',\xi)+\eta')dy'\,đ\eta'. \end{cases} \tag{3.9}$$

Proof. We can prove this lemma in the same way to the proof of Theorem 3.5 in [11].

(i) It follows from the definitions of I_ϕ and I_ϕ^* that setting

$$w(\xi, z')=Os-\iint e^{i\phi(z',\xi)-i\phi(z',\eta)+iy\cdot(\eta-\xi)}h(y, \eta)dy đ\eta, \tag{3·10}$$

then,

$$I_\phi \circ H(x, D_x)\circ I_\phi^* v(x')$$
$$=\iint e^{i\phi(x',\xi)-iy\cdot\xi}dy đ\xi \iint e^{iy\cdot\eta-iz\cdot\eta}h(y, \eta)dz đ\eta \iint e^{iz\cdot\omega-i\phi(z',\omega)}v(z')dz' đ\omega$$
$$=\iint e^{i\phi(x',\xi)-iy\cdot\xi}dy đ\xi \iint e^{iy\cdot\eta-i\phi(z',\eta)}h(y, \eta)v(z')dz' đ\eta$$
$$=\iint e^{i\phi(x',\xi)-i\phi(z',\xi)}w(\xi, z')v(z')dz' đ\xi$$

is valid. So, making the change of variables ξ to $\xi'=\tilde{\nabla}_{x'}\phi(x', z', \xi)$, we get

$$I_\phi \circ H(x, D_x)\circ I_\phi^* v(x')=\iint e^{i(x'-z')\cdot\xi'}w(\xi, z')v(z')dz' đ\xi'\left|\det\frac{\partial^2\phi}{\partial x'\partial\xi}\right|^{-1}$$
$$(\xi=\tilde{\nabla}_{x'}\phi^{-1}(x', z', \xi')).$$

Consequently, if we set

$$w_1(x', \xi', \tilde{x}')=w(\xi, \tilde{x}')\left|\det\frac{\partial^2\phi}{\partial x'\partial\xi}\right|^{-1} \quad (\xi=\tilde{\nabla}_{x'}\phi^{-1}(x', \tilde{x}', \xi')), \tag{3.11}$$

we obtain

$$I_\phi \circ H(x, D_x)\circ I_\phi^* = w_1(x, D_{x'}, \tilde{x}'). \tag{3.12}$$

Now, make the change of variables (y, η) to $(y', \eta')=(-y+\tilde{\nabla}_\xi\phi(z', \xi, \eta), \eta-\xi)$ in (3.10). Then, $w(\xi, z')$ is written as

$$Os-\iint e^{-iy'\cdot\eta'}h(\tilde{\nabla}_\xi\phi(z', \xi, \xi+\eta')-y', \xi+\eta')dy' đ\eta'.$$

So, we can see that $w_1(x', \xi', \tilde{x}')$ in (3.11) is equal to $p(x', \xi', \tilde{x}')$ defined by (3.7). Hence, we can complete the proof of (i).

(ii) $I_\phi^* \circ K(x', D_{x'})\circ I_\phi u(x)$ for $u(x)\in \mathcal{S}$ is written in the form

$$\iint e^{ix\cdot\eta-i\phi(x',\eta)}dx' đ\eta \iint e^{i(x'-y')\cdot\xi'}k(x', \xi')dy' đ\xi' \int e^{i\phi(y',\xi)}\hat{u}(\xi)đ\xi \tag{3.13}$$
$$=\int e^{ix\cdot\xi}\hat{u}(\xi)đ\xi Os-\iint e^{i\{x\cdot\eta-x\cdot\xi+\phi(x',\xi)-\phi(x',\eta)\}}w_2(x', \xi)dx' đ\eta,$$

where

$$w_2(x',\xi)=Os-\iint e^{i\{x'\cdot\xi'-y'\cdot\xi'+\phi(y',\xi)-\phi(x',\xi)\}}k(x',\xi')dy'đ\xi'.$$

If we make the change of variables (y',ξ') to $(\tilde{y}',\tilde{\eta}')=(y'-x',\xi'-\tilde{\nabla}_{x'}\phi(x',y',\xi))$, we have

$$w_2(x',\xi)=Os-\iint e^{-i\tilde{y}'\cdot\tilde{\eta}'}k(x',\tilde{\nabla}_{x'}\phi(x',x'+\tilde{y}',\xi)+\tilde{\eta}')d\tilde{y}'đ\tilde{\eta}', \tag{3.14}$$

which is equal to $s_1(x',\xi)$. Next, change the variables (x',η) to $(\tilde{y},\tilde{\eta})=(\tilde{\nabla}_\xi\phi(x',\xi,\eta)-x,\eta-\xi)$ in the right hand side of (3.13). Then, it follows that

$$\begin{aligned}&I_\phi^*\circ K(x',D_{x'})\circ I_\phi u(x)\\&=\int e^{ix\cdot\xi}\hat{u}(\xi)đ\xi Os-\iint e^{-i\tilde{y}\cdot\tilde{\eta}}s_1(\tilde{\nabla}_\xi\phi^{-1}(x+\tilde{y},\xi,\xi+\tilde{\eta}),\xi)d\tilde{y}đ\tilde{\eta}\\&\quad\times\left|\det\frac{\partial^2\phi}{\partial x'\partial\xi}\right|^{-1}\end{aligned}$$

is valid. Hence, we obtain (ii). Q.E.D.

Lemma 3.2. *We suppose the same assumptions on $\phi(x',\xi)$ as in Lemma 3.1. We set $\mu=\left|\det\frac{\partial^2\phi}{\partial x'\partial\xi}\right|$. We see from the assumption that μ is a non-zero constant. Then, $(\mu)^{1/2}I_\phi$ makes a unitary operator on L^2 space. That is,*

$$\mu I_\phi\circ I_\phi^*=\mu I_\phi^*\circ I_\phi=Identity.$$

Proof. Apply Lemma 3.1 to $I_\phi\circ I_\phi^*$ and $I_\phi^*\circ I_\phi$. Then, we can easily prove this lemma. Q.E.D.

Remark 3.1. We can see the following from Lemma 3.2. Operations $I_\phi\circ\cdot\circ I_\phi^*$ and $I_\phi^*\circ\cdot\circ I_\phi$ in Lemma 3.1 are opposite ones each other in the sense that

$$\begin{cases}I_\phi^*\circ\{I_\phi\circ H(x,D_x)\circ I_\phi^*\}\circ I_\phi=\dfrac{1}{\mu^2}H(x,D_x),\\ I_\phi\circ\{I_\phi^*\circ K(x',D_{x'})\circ I_\phi^*\}\circ I_\phi^*=\dfrac{1}{\mu^2}K(x',D_{x'})\end{cases}$$

are valid where $\mu=\left|\det\frac{\partial^2\phi}{\partial x'\partial\xi}\right|$.

Consider a phase function $\phi(x',\xi)$ satisfying the properties assumed in Lemma 3.1 and denote by

$$(x,\xi)=\Phi(x',\xi')\equiv(x(x',\xi'),\xi(x',\xi')) \tag{3.15}$$

the canonical transformation generated by $\phi(x', \xi)$, which was defined by (3.5). Then, we get

Theorem 3.3. *Suppose that the assumptions in Theorem* 1.1 *where* $Z=\{1, 2, \cdots, n\}$ *holds. We set*

$$(3.16)\qquad \begin{cases} k_2(t, x', \xi')=h_2(t, \Phi(x', \xi')), \\ k_1(t, x', \xi')=Os-\iint e^{-iy'\cdot\eta'}h_1\Big(t, x(x', \xi'), \xi(x', \xi') \\ \qquad +\Big(\eta'-\frac{1}{2}y'\frac{\partial^2\phi}{\partial x'\partial x'}\Big)\frac{\partial^2\phi}{\partial x'\partial\xi}^{-1}\Big)dy'đ\eta'. \end{cases}$$

Then, we can see that the L^2 *well-posedness on* $[0, T]$ *of the Cauchy problem* (0.1) *is equivalent to that of the Cauchy problem for the equation*

$$(3.17)\qquad L'v(t, x')\equiv\frac{1}{i}\partial_t v(t, x')+K_2(t, x', D_{x'})v+K_1(t, x', D_{x'})v$$
$$=g(t, x') \quad on \quad [0, T]\times R^n_{x'}.$$

Proof. It follows from Lemmas 2.7 and 3.2 that the L^2 well-posedness on $[0, T]$ of the Cauchy problem (0.1) is equivalent to that of the Cauchy problem

$$I_\phi\circ L\circ I_\phi^{-1}v(t, x')=g(t, x').$$

We will write $I_\phi\circ L\circ I_\phi^{-1}$ concretely by using Lemmas 3.1 and 3.2.

Since $\phi(x', \xi)$ is a polynomial of degree 2,

$$(3.18)\qquad \tilde{\nabla}_\xi\phi(\tilde{x}', \xi, \xi+\eta)=\int_0^1\frac{\partial\phi}{\partial\xi}(\tilde{x}', \xi+\eta-\theta\eta)d\theta$$
$$=\frac{\partial\phi}{\partial\xi}(\tilde{x}', \xi)+\frac{\eta}{2}\frac{\partial^2\phi}{\partial\xi\partial\xi}$$

and

$$(3.19)\qquad \tilde{\nabla}_{x'}\phi(x', x'+y', \xi)=\int_0^1\frac{\partial\phi}{\partial x'}(x'+y'-\theta y', \xi)d\theta$$
$$=\frac{\partial\phi}{\partial x'}(x', \xi)+\frac{1}{2}y'\frac{\partial^2\phi}{\partial x'\partial x'}$$

are valid from (3.3). So, because $\xi=\tilde{\nabla}_{x'}\phi^{-1}(x', x'+y', \xi'+\eta')$ is determined as the solution of $\xi'+\eta'=\tilde{\nabla}_{x'}\phi(x', x'+y', \xi)=\frac{\partial\phi}{\partial x'}(x', \xi)+\frac{1}{2}y'\frac{\partial^2\phi}{\partial x'\partial x'}$, we have

$$\tilde{\nabla}_{x'}\phi^{-1}(x', x'+y', \xi'+\eta')=\xi\Big(x', \xi'+\eta'-\frac{1}{2}y'\frac{\partial^2\phi}{\partial x'\partial x'}\Big).$$

Here, we used the notation (3.15). Hence, noting that

$$\frac{\partial \xi}{\partial \xi'}(x', \xi') = \frac{\partial^2 \phi}{\partial \xi \partial x'}^{-1}$$

is yielded from (3.5), we obtain

(3.20) $$\tilde{\nabla}_{x'}\phi^{-1}(x', x'+y', \xi'+\eta') = \xi(x', \xi') + \left(\eta' - \frac{1}{2}y' \frac{\partial^2 \phi}{\partial x' \partial x'}\right) \frac{\partial^2 \phi}{\partial x' \partial \xi}^{-1},$$

where we used $\frac{{}^t\partial^2 \phi}{\partial \xi \partial x'} = \frac{\partial^2 \phi}{\partial x' \partial \xi}$.

At first, we consider $I_\phi \circ H_2(t, x, D_x) \circ I_\phi^{-1}$. We can see from Lemma 3.2

$$I_\phi \circ H_2(t, x, D_x) \circ I_\phi^{-1} = \mu I_\phi \circ H_2(t, x, D_x) \circ I_\phi^* \quad \left(\mu = \left|\det \frac{\partial^2 \phi}{\partial x' \partial \xi}\right|\right).$$

So, applying (i) of Lemma 3.1,

(3.21) $$I_\phi \circ H_2(t, x, D_x) \circ I_\phi^{-1} = P_2(t, x', D_{x'}, \tilde{x}')$$

is valid, where

(3.22) $$P_2(t, x', \xi', \tilde{x}')$$
$$= \mathrm{Os}-\iint e^{-iy\cdot\eta} h_2(t, \tilde{\nabla}_\xi \phi(\tilde{x}', \xi, \xi+\eta) - y, \xi+\eta) dy đ\eta$$
$$(\xi = \tilde{\nabla}_{x'}\phi^{-1}(x', \tilde{x}', \xi')).$$

We know well that setting

(3.23) $$P_{2s}(t, x', \xi') = Os-\iint e^{-iy'\cdot\eta'} p_2(t, x', \xi'+\eta', x'+y') dy' đ\eta',$$

we have $P_{2s}(t, x', D_{x'}) = P_2(t, x', D_{x'}, \tilde{x}')$ (Theorem 2.5 in chapter 2 of [12]). The double symbol $P_2(t, x', \xi', \tilde{x}')$ is written from (3.18) as

$$Os-\iint e^{-iy\cdot\eta} h_2\left(t, \frac{\partial \phi}{\partial \xi}(\tilde{x}', \xi) + \frac{\eta}{2} \frac{\partial^2 \phi}{\partial \xi \partial \xi} - y, \xi+\eta\right) dy đ\eta$$
$$(\xi = \tilde{\nabla}_{x'}\phi^{-1}(x', \tilde{x}', \xi')).$$

Here, if we use the assumption (0.2) on $h_2(t, x, \xi)$ and the Taylor expansion formula of the integrand, $p_2(t, x', \xi', \tilde{x}')$ is rewritten in the form

(3.24) $$p_2(t, x', \xi', \tilde{x}') = h_2\left(t, \frac{\partial \phi}{\partial \xi}(\tilde{x}', \xi), \xi\right) + \tilde{r}_2(t, \tilde{x}', \xi)$$
$$(\xi = \tilde{\nabla}_{x'}\phi^{-1}(x', \tilde{x}', \xi')),$$

where $\tilde{r}_2(t, \tilde{x}', \xi)$ satisfies for all multi-indices α and β

(3.25) $$|\partial_\xi^\alpha D_{\tilde{x}'}^\beta \tilde{r}_2(t, \tilde{x}', \xi)| \le \tilde{C}_{\alpha,\beta}$$

with constants $\tilde{C}_{\alpha,\beta}$ independent of $(t, \tilde{x}', \xi) \in [0, T] \times R^{2n}$. Consequently, $p_{2s}(t, x', \xi')$ defined by (3.23) is given from (3.20) by

(3.26) $$p_{2s}(t, x', \xi') = Os-\iint e^{-iy'\cdot\eta'} h_2\Big(t, \frac{\partial\phi}{\partial\xi}(x'+y', \xi), \xi\Big) dy' đ\eta' + Os-\iint e^{-iy'\cdot\eta'} \tilde{r}_2(t, x'+y', \xi) dy' đ\eta'$$

$$\Big(\xi = \xi(x', \xi') + \Big(\eta' - \frac{1}{2} y' \frac{\partial^2\phi}{\partial x'\partial x'}\Big) \frac{\partial^2\phi}{\partial x'\partial\xi}^{-1}\Big).$$

Hence, we can see from the assumptions on $h_2(t, x, \xi)$ and $\phi(x', \xi)$ that $p_{2s}(t, x', \xi')$ is written in the form

(3.27) $$p_{2s}(t, x', \xi') = h_2\Big(t, \frac{\partial\phi}{\partial\xi}(x', \xi(x', \xi')), \xi(x', \xi')\Big) + r_2(t, x', \xi') = k_2(t, x', \xi') + r_2(t, x', \xi'),$$

where $k_2(t, x', \xi')$ is the symbol defined by (3.16) and $r_2(t, x', \xi')$ satisfies

(3.28) $$|r_{2(\beta)}^{(\alpha)}(t, x', \xi')| \le C_{\alpha,\beta}$$

for all α and β with constants $C_{\alpha,\beta}$ independent of $(t, x', \xi') \in [0, T] \times R^{2n}$. Therefore,

(3.29) $$I_\phi \circ H_2(t, x, D_x) \circ I_\phi^{-1} = K_2(t, x', D_{x'}) + R_2(t, x', D_{x'})$$

is valid because of $P_{2s}(t, x', D_{x'}) = P_2(t, x', D_{x'}, \tilde{x}')$, where $R_2(t, x, D_x)$ belongs to $\mathcal{E}_t^0([0, T]; \mathcal{L})$ from (3.28) and Lemma 2.7. $\mathcal{L}$ denoted the space of all L^2 bounded operators.

Next, consider $I_\phi \circ H_1(t, x, D_x) \circ I_\phi^{-1}$. In the same way to the proof of (3.29)

(3.30) $$I_\phi \circ H_1(t, x, D_x) \circ I_\phi^{-1} = P_1(t, x', D_{x'}, \tilde{x}')$$

is valid, where

(3.31) $$p_1(t, x', \xi', \tilde{x}') = Os-\iint e^{-iy\cdot\eta} h_1\Big(t, \frac{\partial\phi}{\partial\xi}(\tilde{x}', \xi) + \frac{\eta}{2}\frac{\partial^2\phi}{\partial\xi\partial\xi} - y, \xi + \eta\Big) dy đ\eta$$

$$(\xi = \tilde{\nabla}_{x'}\phi^{-1}(x', \tilde{x}', \xi')).$$

It follows from the assumption on $h_1(t, x, \xi)$ that $p_1(t, x', \xi', \tilde{x}')$ is written in the form

$$h_1\left(t, \frac{\partial\phi}{\partial\xi}(\tilde{x}', \xi), \xi\right) + \tilde{r}_1(t, \tilde{x}', \xi),$$

where $\tilde{r}_1(t, \tilde{x}', \xi)$ satisfies the same inequalities as (3.25). We set

$$(3.32) \qquad p_{1s}(t, x', \xi') = Os - \iint e^{-iy'\cdot\eta'} p_1(t, x', \xi'+\eta', x'+y') dy' đ\eta'.$$

Then, we have correspondingly to (3.26)

$$(3.33) \qquad p_{1s}(t, x', \xi') = Os - \iint e^{-iy'\cdot\eta'} h_1\left(t, \frac{\partial\phi}{\partial\xi}(x'+y', \xi), \xi\right) dy' đ\eta'$$

$$+ Os - \iint e^{-iy'\cdot\eta'} \tilde{r}_1(t, x'+y', \xi) dy' đ\eta'$$

$$\left(\xi = \xi(x', \xi') + \left(\eta' - \frac{1}{2} y' \frac{\partial^2\phi}{\partial x' \partial x'}\right) \frac{\partial^2\phi}{\partial x' \partial\xi}^{-1}\right).$$

Using the assumptions on $h_1(t, x, \xi)$ and $\phi(x', \xi)$,

$$p_{1s}(t, x', \xi') = k_1(t, x', \xi') + r_1(t, x', \xi')$$

is valid, where $r_1(t, x', \xi')$ satisfies the same inequalities as (3.28). Therefore, we obtain

$$(3.34) \qquad I_\phi \circ H_1(t, x, D_x) \circ I_\phi^{-1} = K_1(t, x', D_{x'}) + R_1(t, x', D_{x'}),$$

where $R_1(t, x', D_{x'}) \in \mathcal{E}_t^0([0, T]; \mathcal{L})$.

We can see from (3.29) and (3.34) that

$$I_\phi \circ L \circ I_\phi^{-1} = \frac{1}{i}\partial_t + K_2(t, x', D_{x'}) + K_1(t, x', D_{x'}) + R(t, x', D_{x'})$$

$$= L' + R(t, x', D_{x'})$$

holds, where $R(t, x', D_{x'}) \in \mathcal{E}_t^0([0, T]; \mathcal{L})$. So, we can complete the proof of Theorem 3.3 from Lemma A.1 in the appendix. Q.E.D.

Now, we consider a more general generating function $\psi(x', \xi_Z, x_{Z^c})$ than $\phi(x', \xi)$ in Theorem 3.3, where Z is a subset in $\{1, 2, \cdots, n\}$ and we used the notations ξ_Z and x_{Z^c} in section 1. We assume that ψ is a polynomial of degree 2 in x', ξ_Z and x_{Z^c}, and that

$$(3.35) \qquad \det \frac{\partial^2\psi}{\partial x' \partial\omega} \neq 0 \quad (\omega = (\xi_Z, x_{Z^c})).$$

Then, the canonical transformation from $R^{2n}_{x',\xi'}$ onto $R^{2n}_{x,\xi}$ generated by $\psi(x', \xi_Z, x_{Z^c})$ is determined from

$$\xi' = \frac{\partial\psi}{\partial x'}, \quad x_Z = \frac{\partial\psi}{\partial\xi_Z}, \quad \xi_{Z^c} = -\frac{\partial\psi}{\partial x_{Z^c}}. \tag{3.36}$$

We denote the above canonical transformation by

$$(x, \xi) = \Psi(x', \xi') = (x(x', \xi'), \xi(x', \xi')). \tag{3.37}$$

For the above Z we consider $h_1(t, x, \xi)$ satisfying (1.2). We can state the following theorem generally. But, we will state only the case where Z is an empty set, because the representation for the general case becomes much complicated. We note that the result in the case of $Z=\{1, 2, \cdots, n\}$ has been already stated in Theorem 3.3. As will be seen in the proof below, it is easy to prove the general case.

Theorem 3.4. *We consider a generating function $\psi(x', x)$ stated above. Suppose the assumptions in Theorem* 1.1 *where Z is an empty set. We set*

$$\begin{cases} k_2(t, x', \xi') = h_2(t, \Psi(x', \xi')), \\ k_1(t, x', \xi') = Os-\iint e^{-iy'\cdot\eta'} h_1(t, x(x', \xi') \\ \qquad + \left(\eta' - \frac{1}{2} y' \frac{\partial^2\psi}{\partial x'\partial x'}\right) \frac{\partial^2\psi}{\partial x'\partial x}^{-1}, \xi(x', \xi')\Big) dy' đ\eta', \end{cases} \tag{3.38}$$

where Ψ is defined by (3.37). *Then, the same statement as in Theorem* 3.3 *is valid.*

Proof. We define a canonical transformation Φ_1 from $R^{2n}_{\tilde{x},\tilde{\xi}}$ onto $R^{2n}_{x,\xi}$ by

$$(x, \xi) = \Phi_1(\tilde{x}, \tilde{\xi}) = (\tilde{\xi}, -\tilde{x}). \tag{3.39}$$

Then, we can see from Proposition 1.3 that setting

$$\tilde{h}_j(t, \tilde{x}, \tilde{\xi}) = h_j(t, \Phi_1(\tilde{x}, \tilde{\xi})), \tag{3.40}$$

we have

$$\begin{aligned} &(\mathcal{F}^{-1})_{x\to\tilde{x}} \circ L \circ \mathcal{F}_{\tilde{x}\to x} \\ &\qquad = \frac{1}{i}\partial_t + \tilde{H}_2(t, \tilde{x}, D_{\tilde{x}}) + \tilde{H}_1(t, \tilde{x}, D_{\tilde{x}}) + \tilde{R}(t, \tilde{x}, D_{\tilde{x}}), \end{aligned} \tag{3.41}$$

where $\tilde{R}(t, \tilde{x}, D_{\tilde{x}}) \in \mathcal{E}^0_t([0, T]; \mathcal{L})$.

We define $\phi(x', \tilde{\xi})$ by

$$\phi(x', \tilde{\xi}) = \psi(x', \tilde{\xi}). \tag{3.42}$$

Then, we can see from the assumption on $\psi(x', x)$ that $\det\frac{\partial^2\phi}{\partial x'\partial\tilde{\xi}}$ is a non-zero contant. So, we can determine a canonical transformation Φ_2 from $R^{2n}_{x',\xi'}$ onto

$R^{2n}_{\tilde{x},\tilde{\xi}}$ generated by $\phi(x', \tilde{\xi})$. That is, we define $\Phi_2(x', \xi')=(\tilde{x}(x', \xi'), \tilde{\xi}(x', \xi'))$ from

$$\text{(3.43)} \qquad \tilde{x}=\frac{\partial\phi}{\partial\tilde{\xi}}(x', \tilde{\xi}), \quad \xi'=\frac{\partial\phi}{\partial x'}(x', \tilde{\xi}).$$

Consider the product of $\Phi_1\circ\Phi_2(x', \xi')$ of Φ_1 and Φ_2. Then, since we have from (3.39), (3.42) and (3.43)

$$\xi'=\frac{\partial\psi}{\partial x'}(x', x), \quad \xi=-\frac{\partial\psi}{\partial x}(x', x),$$

we obtain

$$\text{(3.44)} \qquad \Psi(x', \xi')=\Phi_1\circ\Phi_2(x', \xi')$$

and so,

$$\text{(3.44)}' \qquad (x(x', \xi'), \xi(x', \xi'))=(\tilde{\xi}(x', \xi'), -\tilde{x}(x', \xi')).$$

Now, it is easy to see that $\tilde{h}_j(t, \tilde{x}, \tilde{\xi})(j=1, 2)$ defined by (3.40) and $\phi(x', \tilde{\xi})$ defined by (3.42) satisfy the assumptions in Theorem 3.3. So, we can apply Theorem 3.3 to (3.41). Hence, the L^2 well posedness of the Cauchy problem (0.1) on $[0, T]$ is equivalent to that of the Cauchy problem for the equation

$$\frac{1}{i}\partial_t v(t, x')+K_2'(t, x', D_{x'})v+K_1'(t, x', D_{x'})v=g(t, x').$$

Here, we defined correspondingly to (3.16)

$$\begin{cases} k_2'(t, x', \xi')=\tilde{h}_2(t, \Phi_2(x', \xi')), \\ k_1'(t, x', \xi')=Os-\iint e^{-iy'\cdot\eta'}\tilde{h}_1\Big(t, \tilde{x}(x', \xi'), \tilde{\xi}(x', \xi') \\ \qquad\qquad +\Big(\eta'-\frac{1}{2}y'\frac{\partial^2\phi}{\partial x'\partial x'}\Big)\frac{\partial^2\phi}{\partial x'\partial\tilde{\xi}}^{-1}\Big)dy'đ\eta'. \end{cases}$$

Then, we can easily see from (3.40), (3.42), (3.44) and (3.44)′ that $k_j'(t, x', \xi')(j=1, 2)$ are equal to $k_j(t, x', \xi')$ defined by (3.38). Thus, we can complete the proof. Q.E.D.

Remark 3.2. Suppose the assumptions in Theorem 3.4 and take $k_j(t, x', \xi')(j=1, 2)$ defined by (3.38). We consider the equation

$$L'v(t, x)\equiv\frac{1}{i}\partial_t v(t, x')+K_2(t, x', D_{x'})v+K_1(t, x', D_{x'})v$$
$$=g(t, x').$$

It is easy to see from Examples 1.5 and 3.3 below that we can not apply

Theorem 1.1 directly to the above equation in general. But, we can see from Theorems 1.1 and 3.4 that if (1.4) are valid for all α and β, the Cauchy problem for the equation $L'v(t, x')=g(t, x')$ is L^2 well posed on $[0, T]$. If we use the Taylor expansion formula for the integrand in the second equality of (3.38), $k_1(t, x', \xi')$ is written in the form

$$k_1(t, x', \xi')=h_1(\Psi(x', \xi'))+\text{“a remainder term”} \tag{3.45}$$
$$\equiv k_{1p}(t, x', \xi')+k_{1s}(t, x', \xi'),$$

where $\Psi(x', \xi')$ was defined by (3.37). Denote by $(q'(t, s; y', \xi'), p'(t, s; y', \xi'))$ the solution of the Hamilton canonical equations

$$\begin{cases} \dfrac{dq'}{dt}=\dfrac{\partial k_2}{\partial \xi'}(t, q', p'), \quad \dfrac{dp'}{dt}=-\dfrac{\partial k_2}{\partial x'}(t, q', p'), \\ (q', p')|_{t=s}=(y', \xi'). \end{cases}$$

Then, we obtain

$$h_1(t, q(t, s; y, \xi), p(t, s; y, \xi))$$
$$=k_{1p}(t, q'(t, s; y', \xi'), p'(t, s; y', \xi')) \quad ((y, \xi)=\Psi(y', \xi')),$$

from the first equality of (3.38) and (3.45) as in the proof of Proposition 1.3. Consequently, noting that the each component of $\Psi(y', \xi')$ is a polynomial of degree 1, we see that if and only if (1.4) are valid for all α and β,

$$\sup_{0\le s\le t\le T}\ \sup_{(y', \xi')\in R^{2n}}\Big|\partial_{\xi'}^{\alpha'}D_{y'}^{\beta'}\exp\Big\{-i \times \int_s^t k_{1p}(\theta, q'(\theta, s; y', \xi'), p'(\theta, s; y', \xi'))d\theta\Big\}\Big|<\infty \tag{3.46}$$

are valid for all α' and β'. Hence, if (3.46) are valid for all α' and β', the Cauchy problem for $L'v(t, x')=g(t, x')$ is L^2 well posed on $[0, T]$. We remark that this sufficient condition stated just above does not depend on $k_{1s}(t, x', \xi')$ and that $k_{1s}(t, x', \xi')$ is not a bounded function on $[0, T]\times R^{2n}$ in general (see Examples 1.5 and 3.3). When we consider the general generating functions $\phi(x', \xi_Z, x_{Z^c})$, we also obtain the similar results as in the above.

Now, we suppose in Theorem 3.4 a stronger assumption that $h_1(t, x, \xi)$ satisfies (1.2)' in Corollary 1.2. Then, we can easily see that $k_{1s}(t, x', \xi')$ in (3.45) satisfies the same inequalities as (3.28) for all α and β. Consequently, it follows that the assumptions in Theorem 1.1 hold for the $k_j(t, x', \xi')(j=1, 2)$ defined by (3.38), because the each component of $\Psi(x', \xi')$ is a polynomial of degree 1. Hence, we obtain a sufficient condition from Theorem 1.1 directly for the Cauchy problem $L'v(t, x')=g(t, x')$, $v(s, x')=v_0(x')$ to be L^2 well posed on $[0, T]$. We can eaasily see that this condition equals one that (3.46) are valid for all α' and β', which was equivalent to the condition that (1.4) are valid for all α and β.

Example 3.1. We take $h_j(t, x, \xi)(j=1, 2)$ stated in Theorem 3.3 and $x'\cdot\xi+\frac{1}{2}|x'|^2$ as $\phi(x', \xi)$. Then, canonical transformation Φ generated by (3.15) becomes $(x, \xi)=\Phi(x', \xi')=(x', \xi'-x')$. So, symbols defined by (3.16) are

$$\begin{cases} k_2(t, x', \xi')=h_2(t, x', \xi'-x'), \\ k_1(t, x', \xi')=Os-\iint e^{-iy'\cdot\eta'}h_1\left(t, x', \xi'-x'+\eta'-\frac{1}{2}y'\right)dy'đ\eta'. \end{cases}$$

Example 3.2. We also take $h_j(t, x, \xi)(j=1, 2)$ stated in Theorem 3.3 and $x'\cdot\xi+\frac{1}{2}|\xi|^2$ as $\phi(x', \xi)$. Then, the canonical transformation Φ generated by $\phi(x', \xi)$ is $(x, \xi)=\Phi(x', \xi')=(x'+\xi', \xi')$. So, the symbols determined by (3.16) become

$$\begin{cases} k_2(t, x', \xi')=h_2(t, x'+\xi', \xi'), \\ k_1(t, x', \xi')=h_1(t, x'+\xi', \xi'), \end{cases}$$

because $Os-\iint e^{-iy'\cdot\eta'}h_1(t, x'+\xi', \xi'+\eta')dy'đ\eta'$ is equal to $h_1(t, x'+\xi', \xi')$.

Example 3.3. Let $h_j(t, x, \xi)(j=1, 2)$ be the symbols satisfying the assumptions in Theorem 3.4. We take $\psi(x', x)=x'\cdot x+\frac{1}{2}|x'|^2$. Then, the canonical transformation generated by ψ becomes $(x, \xi)=\Psi(x', \xi')=(\xi'-x', -x')$. So, $k_j(t, x', \xi')(j=1, 2)$ in (3.38) are determined by

$$\begin{cases} k_2(t, x', \xi')=h_2(t, \xi'-x', -x'), \\ k_1(t, x', \xi')=Os-\iint e^{-iy'\cdot\eta'}h_1\left(t, \xi'-x'+\left(\eta'-\frac{1}{2}y'\right), -x'\right)dy'đ\eta'. \end{cases}$$

Here, if we take $\sum_{j=1}^{n}b^j(x)\xi_j$ as $h_1(t, x, \xi)$, we get the result in Example 1.5.

4. Derivations of Schrödinger type equations

In this section we will derive the Schrödinger type equations from the Maxwell equations under a special assumption imposed on the polarization $\boldsymbol{P}=(P_x, P_y, P_z)\in R^3$. We will follow mainly P. L. Kelly [9] where the cubic non-linear Schrödinger equations were derived. One derivation will be done from electromagnetic waves in a dielectric material. Another derivation will start from the equation treated in [9].

Derivation 1. Consider a dielectric material in which there are no extra charges other than those bound in atoms. Then, the Maxwell equations become

$$
(4.1)\qquad \begin{cases} \text{(a)}\ \ \nabla\cdot \boldsymbol{E} = -\dfrac{1}{\varepsilon_0}\nabla\cdot \boldsymbol{P} & \text{(b)}\ \ c^2\nabla\times \boldsymbol{B} = \dfrac{\partial}{\partial t}\left(\dfrac{\boldsymbol{P}}{\varepsilon_0}+\boldsymbol{E}\right) \\ \text{(c)}\ \ \nabla\times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} & \text{(d)}\ \ \nabla\cdot \boldsymbol{B}=0\,, \end{cases}
$$

where c is the velocity of light and ε_0 is the permittivity of empty space (see chapter 32 in volume 2 of *R. P.* Feynman [4]). $\boldsymbol{E}=\boldsymbol{E}(t, x, y, z)=(E_x, E_y, E_z) \in R^3$, $\boldsymbol{B}=\boldsymbol{B}(t, x, y, z)=(B_x, B_y, B_z)\in R^3$ denote the electric field and the magnetic field, respectively. Also, $\nabla\cdot\boldsymbol{E}$ and $\nabla\times\boldsymbol{E}$ denote the divergence and the rotation of $\boldsymbol{E}$ respectively. It follows from (b) and (c) in (4.1) that

$$
(4.2)\qquad c^{-2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}-\Delta\boldsymbol{E}=-\nabla(\nabla\cdot\boldsymbol{E})-\frac{1}{e_0}c^{-2}\frac{\partial^2\boldsymbol{P}}{\partial t^2}\quad (\Delta\boldsymbol{E}=(\Delta E_x, \Delta E_y, \Delta E_z))
$$

is valid, where Δ denotes the Laplacian in R^3 and $\nabla(\nabla\cdot\boldsymbol{E})$ the gradient of $\nabla\cdot\boldsymbol{E}$. Here, we used the formula $\nabla\times(\nabla\times\boldsymbol{E})=-\Delta\boldsymbol{E}+\nabla(\nabla\cdot\boldsymbol{E})$.

We consider a solution $\boldsymbol{E}$ polarized in the x-direction, that is,

$$
(4.3)\qquad \boldsymbol{E}=(E_x(t, x, y, z), 0, 0)\,.
$$

Here, we assume the following. There exist real valued functions $\chi(y, z)$ and $b_j(y, z)(j=1, 2)$ such that if $\boldsymbol{E}$ is given by (4.3), $\boldsymbol{P}$ is determined by

$$
(4.4)\qquad \boldsymbol{P}=e_0\left(\chi(y, z)E_x+b_1(y, z)\frac{\partial E_x}{\partial x}+b_2(y, z)\frac{\partial E_x}{\partial y}, 0, 0\right).
$$

Then, we can consider $\boldsymbol{E}$ and $\boldsymbol{P}$ as three dimensional complex valued functions, because their real parts give the actual electric field and polarization. Now, we will find a solution $\boldsymbol{E}$ of (4.2) under the assumption (4.4) in the form

$$
(4.5)\qquad \boldsymbol{E}=(E'_x(y, z), 0, 0)e^{i(kz-\omega t)}\quad (\omega/k=c)
$$

following [9], where k is assumed to be large. The factor $e^{i(kz-\omega t)}$ represents the propagating part of the wave and $(E'_x(y, z), 0, 0)$ is the slowly varying part. We note that $\boldsymbol{E}$ given by (4.5) satisfies the equation (a) in (4.1). Inserting (4.5) into (4.2) and (4.4),

$$
(4.6)\qquad 2ik\frac{\partial E'_x}{\partial z}+\Delta E'_x+k^2\left(\chi E'_x+b_2\frac{\partial E'_x}{\partial y}\right)=0
$$

can be derived. Here, we will neglect the term $\left(\dfrac{\partial}{\partial z}\right)^2 E'_x$ as in [9], assuming that $\left|\left(\dfrac{\partial}{\partial z}\right)^2 E'_x\right|$ is smaller than $\left|k\dfrac{\partial E'_x}{\partial z}\right|$. Then, we obtain

$$(4.7)\qquad 2ik\frac{\partial E'_x}{\partial z}+\left\{\frac{\partial^2}{\partial y^2}+k^2\chi(y,z)\right\}E'_x+k^2b_2(y,z)\frac{\partial E'_x}{\partial y}=0\,.$$

The above equation (4.7) is written in the form

$$(4.7)'\qquad 2ik\frac{\partial E'_x}{\partial z}+H_2(y,z,D_y)E'_x+H_1(y,z,D_y)E'_x=0\,,$$

where $h_2(y,z,\xi_y)=-\xi_y{}^2+k^2\chi(y,z)$ and $h_1(y,z,\xi_y)=ik^2b_2(y,z)\xi_y$ (ξ_y denotes the dual variable of y). Recall that $b_2(y,z)$ is real valued. This fact is important, because we know from [8], [14] and Remark 1.2 in the present paper that the Cauchy problem for (4.7) is not necessarily L^2 well posed.

Derivation 2. We start as in [9] from the equation

$$(4.8)\qquad c^{-2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}-\Delta\boldsymbol{E}=-\frac{1}{\varepsilon_0}c^{-2}\frac{\partial^2\boldsymbol{P}}{\partial t^2}\,.$$

We note that we can get (4.8), if the term $-\nabla(\nabla\cdot\boldsymbol{E})$ is neglected in (4.2). We also assume as in the first derivation that if $\boldsymbol{E}$ is given by (4.3), $\boldsymbol{P}$ is determined by

$$(4.4)'\qquad \boldsymbol{P}=\varepsilon_0\left(\chi(x,y,z)E_x+b_1(x,y,z)\frac{\partial E_x}{\partial x}+b_2(x,y,z)\frac{\partial E_x}{\partial y},0,0\right),$$

where χ, b_1 and b_2 are real valued functions. Find a solution of (4.8) in the form

$$(4.9)\qquad \boldsymbol{E}=(E'_x(x,y,z),0,0)e^{i(kz-\omega t)}\qquad(\omega/k=c)\,.$$

Insert (4.9) into (4.4)′ and (4.8). Then, we get

$$(4.10)\qquad 2ik\frac{\partial E'_x}{\partial z}+\Delta E'_x+k^2\left(\chi E'_x+b_1\frac{\partial E'_x}{\partial x}+b_2\frac{\partial E'_x}{\partial y}\right)=0\,.$$

If we neglect the term $\left(\frac{\partial}{\partial z}\right)^2E'_x$ in (4.10) under the same consideration as in the first derivation, we obtain the Schrödinger type equation

$$(4.11)\qquad 2ik\frac{\partial E'_x}{\partial z}+\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\chi(x,y,z)\right\}E'_x$$
$$+k^2\left\{b_1(x,y,z)\frac{\partial E'_x}{\partial x}+b_2(x,y,z)\frac{\partial E'_x}{\partial y}\right\}=0\,.$$

Remark 4.1. Suppose that $\boldsymbol{E}$ is given by the real part of the right hand side of (4.9) and that

$$(4.12)\qquad \boldsymbol{P}=\varepsilon_0\chi(x,y,z)\boldsymbol{E}$$

is valid, where $\chi(x, y, z)$ is a complex valued function. (4.12) is a natural assumption (see chapter 32 in volume 2 of [4]). If we assume that $\boldsymbol{E}$ satisfies (4.8), we get

$$\left\{2ik\frac{\partial E_x'}{\partial z}+\Delta E_x'+k^2\chi E_x'\right\}e^{i(kz-\omega t)}$$
$$+\left\{-2ik\frac{\partial \bar{E}_x'}{\partial z}+\Delta \bar{E}_x'+k^2\chi \bar{E}_x'\right\}e^{-i(kz-\omega t)}=0\,.$$

Since we supposed that E_x' was slowly varying, we consider

$$2ik\frac{\partial E_x'}{\partial z}+\Delta E_x'+k^2\chi E_x'=0\,,\quad -2ik\frac{\partial \bar{E}_x'}{\partial z}+\Delta \bar{E}_x'+k^2\chi \bar{E}_x'=0\,. \tag{4.13}$$

So, if χ is not real valued, the existence of the solution E_x' of (4.13) will not be able to be expected.

Now, assume formally that $\boldsymbol{E}$ is given by the right-hand side of (4.9) and that (4.12) is valid. Then, we get from (4.8) $2ik\frac{\partial E_x'}{\partial z}+\Delta E_x'+k^2\chi E_x'=0$. So, if we use the same neglect as in Derivations 1 and 2, we obtain the Schrödinger type equations

$$2ik\frac{\partial E_x'}{\partial z}+\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2(\mathrm{Re}\ \chi)\right\}E_x'+ik^2(\mathrm{Im}\chi)E_x'=0\,,$$

where $\mathrm{Re}\chi$ and $\mathrm{Im}\chi$ denote the real and imaginary parts of χ.

Remark 4.2. We will find a solution of (4.8) by the real part of the right-hand side of (4.9). We assume

$$\boldsymbol{P}=\varepsilon_2|\boldsymbol{E}|^2\boldsymbol{E}$$

in place of the assumption (4.4)′, where ε_2 is a real constant. Then, we can see from the same argument as in Remark 4.1 that $E_x'(x, y, z)$ satisfies the cubic non-linear Schrödinger equation

$$2ik\frac{\partial E_x'}{\partial z}+\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{3}{4}\varepsilon_2 k^2|E_x'|^2\right\}E_x'=0\,,$$

neglecting the term $\left(\frac{\partial}{\partial z}\right)^2 E_x'$ and the terms with $e^{3i(kz-\omega t)}$ and $e^{-3i(kz-\omega t)}$. This was the derivation done in [9].

Appendix

We shall prove the lemma below for the rigorous proofs of Theorems 3.3 and 3.4.

Lemma A.1. *Let L be a general evolutionary operator. We assume that the Cauchy problem for the equation*

$$Lu(t, x)=f(t, x) \tag{A.1}$$

is L^2 well posed on $[0, T]$ in the sense of Definition 1.1. *Let $R(t)\in \mathcal{E}_t^0([0, T]; \mathcal{L})$. Then, the Cauchy problem for the equation*

$$\{L+R(t)\}u(t, x)=g(t, x) \tag{A.2}$$

is also L^2 well posed on $[0, T]$.

Proof. We have only to prove without the loss of generality that one and only one solution $u(t, x)$ in $\mathcal{E}_t^0([0, T]; L^2)$ of

$$\{L+R(t)\}u(t, x)=g(t, x), \quad u(0, x)=u_0(x) \tag{A.2$'$}$$

exists on $[0, T]$ for any $u_0(x)\in L^2$ and $g(t, x)\in L_t^1([0, T]; L^2)$.

We set

$$M=\max_{0\le t\le T}\|R(t)\|, \tag{A.3}$$

where $\|R(t)\|$ denotes the operator norm. Let $u(t, x)\in \mathcal{E}_t^0([0, T]; L^2)$ be a solution of (A.2)$'$ where $g(t, x)=0$ and $u_0(x)=0$. Then, it follows from the energy inequality in the Cauchy problem for (A.1) that

$$\|u(t, \cdot)\|\le C(T)\left\{\int_0^t\|-R(\theta)u(\theta, \cdot)\|d\theta\right\} \tag{A.4}$$

is valid for $t\in[0, T]$. We take a $T_0(0< T_0\le T)$ satisfying

$$C(T)MT_0<1 \tag{A.5}$$

for $C(T)$ in (A.4). Then, we have from (A.3)

$$\max_{0\le t\le T_0}\|u(t, \cdot)\|\le C(T)MT_0\max_{0\le t\le T_0}\|u(t, \cdot)\|,$$

so

$$u(t, x)=0 \quad \text{on} \quad [0, T_0]\times R_x^n. \tag{A.6}$$

Consequently, we also have for this $u(t, x)$

$$\|u(t, \cdot)\|\le C(T)\int_{T_0}^t\|-R(\theta)u(\theta, \cdot)\|d\theta \quad (T_0\le t\le T).$$

So, $u(t, x)=0$ on $[T_0, T_0']\times R_x^n$ $(T_0'=\min(2T_0, T))$ also holds in the same way as in the above. Repeating this process, we can see that this solution $u(t, x)$

vanishes on $[0, T]\times R_x^n$. Thus, we can prove the uniqueness of the solution.

Next, we show the existence of the solution of (A.2)′. Since the Cauchy problem for (A.1) is L^2 well posed on $[0, T]$, we can define $v_n(t, x)\in \mathcal{E}_t^0([0, T_0]; L^2)(n=1, 2, \cdots)$ inductively by the solution of

$$Lv_n(t, x)=-R(t)v_{n-1}(t, x)+g(t, x),\ v_n(0, x)=u_0(x)\,, \tag{A.7}$$

where we set $v_0(t, x)=0$. We can easily see

$$L(v_{n+1}-v_n)=-R(t)(v_n-v_{n-1})\,,\quad (v_{n+1}-v_n)|_{t=0}=0$$

for $n=1, 2, \cdots$. So, it follows from the energy inequality in the Cauchy problem for (A.1) that

$$\max_{0\le t\le T_0}\|v_{n+1}(t, \cdot)-v_n(t, \cdot)\|\le C(T)MT_0\max_{0\le t\le T_0}\|v_n(t, \cdot)-v_{n-1}(t, \cdot)\| \tag{A.8}$$

are valid for $n=1, 2, \cdots$. Consequently, we can see from $C(T)MT_0<1$ that $v_n(t, x)=(v_n-v_{n-1})+(v_{n-1}-v_{n-2})+\cdots+(v_1-v_0)$ converges to an element $u(t, x)$ in $\mathcal{E}_t^0([0, T_0]; L^2)$. It is easy to show that $u(t, x)$ obtained now satisfies (A.2)′. In the same way as in the proof of the uniquess, we can easily get the solution $u(t, x)\in\mathcal{E}_t^0([0, T]; L^2)$. Q.E.D.

Department of Applied Mathematics
Ehime University

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