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The semi-classical asymptotics of the total cross sections for elastic scattering for N-body systems

By

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1. Introduction

We consider an $N(\geq 3)$-body Schrödinger operator given by

$$\tilde{H}(h) = -\sum_{1\leq j\leq N} \frac{h^2}{2m_j}\Delta_{r_j} + \sum_{1\leq i<j\leq N} V_{ij}(r_i - r_j) \text{ in } L^2(\mathbf{R}^{3N}), \tag{1.1}$$

where $m_j > 0$ and $r_j \in \mathbf{R}^3$ are the mass and the position vector of the j-th particle, respectively. $h \in (0, 1]$ is a small parameter corresponding to the Planck constant. $V_{ij}(1 \leq i < j \leq N)$ is the pair potential between the i-th and j-th particles. Our main assumption on V_{ij} is the following:

$$V_{ij}(r) = O(|r|^{-\rho}), \qquad \text{as} \qquad |r| \to \infty,\ r \in \mathbf{R}^3, \tag{1.2}$$

for some $\rho > 2$.

We consider the following elastic scattering in the center of mass frame. In both the initial and final states the N particles are supposed to be divided into two clusters C_1 and C_2. Since $N \geq 3$, at least one cluster is not a one particle cluster. We assume that if C_j is not a one particle cluster, the particles in the cluster C_j form a bound state and the bound state energy $\lambda_j(h)$ belongs to the discrete spectrum of the cluster Hamiltonian $H^{(C_j)}(h)$. Furthermore, we assume that

$$\inf \sigma_{ess}(H^{(C_j)}(h)) - \lambda_j(h) \geq E_0 > 0 \tag{1.3}$$

for some h-independent constant E_0, where $\sigma_{ess}(A)$ denotes the essential spectrum of a self-adjoint operator A.

We denote by α the initial channel (= the final channel in this case). The purpose of this paper is to study the semi-classical asymptotics of the total cross section $\sigma_{\alpha\to\alpha}(\lambda, \omega; h)$ for elastic scattering $\alpha \to \alpha$ with energy λ and incident direction ω.

Let $J \subset (0, \infty)$ be any compact interval on which some semi-classical resolvent estimate is satisfied (see (3.2)). This condition is satisfied if J is included in the intersection of non-trapping energy ranges of the N-body classical system

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and all the subsystems. The following is our main result: If the total energy λ is in J, $\sigma_{\alpha\to\alpha}(\lambda, \omega; h)$ has the following asymptotics

$$\sigma_{\alpha\to\alpha}(\lambda, \omega; h) = \sigma_\alpha^0(\lambda, \omega; h) + o(h^{-2/(\rho-1)}) \tag{1.4}$$

as $h \to 0$ uniformly in $(\lambda, \omega) \in J \times S^2$ (S^2 is the unit sphere in $\mathbf{R}^3$), where

$$\sigma_\alpha^0(\lambda, \omega; h) = O(h^{-2/(\rho-1)}) \qquad (h \to 0)$$

and the explicit form of $\sigma_\alpha^0(\lambda, \omega; h)$ will be given later in (3.3).

It is known that there exists a large class of potentials satisfying (1.2) such that $\sigma_\alpha^0(\lambda, \omega; h) \neq o(h^{-2/(\rho-1)})(h \to 0)$ (see the remark below Theorem 3.1).

In [IT2] the semi-classical asymptotics of the total scattering cross section

$$\sigma_\alpha(\lambda, \omega; h) := \sum_\beta \sigma_{\alpha\to\beta}(\lambda, \omega; h) \tag{1.5}$$

for the N-body systems are studied, where the summation is taken over all the channels β and $\sigma_{\alpha\to\beta}(\lambda, \omega; h)$ is the total cross section for scattering $\alpha \to \beta$ with energy $\lambda > 0$ and incident direction ω. If (1.2) and (1.3) are satisfied, all the $\sigma_{\alpha\to\beta}(\lambda, \omega; h)$ are well-defined for a.e. $(\lambda, \omega) \in (0, \infty) \times S^2$ for each $h \in (0, 1]$ and $\sigma_\alpha(\lambda, \omega; h)$ has the semi-classical asymptotics ([IT2]):

$$\sigma_\alpha(\lambda, \omega; h) = \sigma_\alpha^0(\lambda, \omega; h) + o(h^{-2/(\rho-1)}) \qquad (h \to 0) \tag{1.6}$$

in the distribution sense as a function of $(\lambda, \omega) \in (0, \infty) \times S^2$. Thus we see that the elastic part $\sigma_{\alpha\to\alpha}(\lambda, \omega; h)$ contributes most to $\sigma_\alpha(\lambda, \omega; h)$:

$$\sigma_\alpha(\lambda, \omega; h) = \sigma_{\alpha\to\alpha}(\lambda, \omega; h) + o(h^{-2/(\rho-1)}) \qquad (h \to 0) \tag{1.7}$$

in the distribution sense as a function of $(\lambda, \omega) \in J \times S^2$.

The semi-classical aysmptotics of the total scattering cross sections for the 2-body case were studied by [Ya], [RT2], [Y], [ES] and for the 3-body case by [IT1], and for the N-body case by [IT2]. Our proof in this paper is based on the same ideas as [RT2] and [IT1].

The outline of this paper is the following. In Sect. 2 we prepare the notations and define the total scattering cross section for an N-body system. The main result (1.4) is stated in Sect. 3. The representation formula of the total scattering cross section is given in Sect. 4. The main result will be proved in Sect. 5–8.

2. The total cross section for elastic scattering

We consider an N-body Shrödinger operator given by

$$\tilde{H}(h) = -\sum_{1\le j\le N} \frac{h^2}{2m_j}\Delta_{r_j} + \sum_{1\le i<j\le N} V_{ij}(r_i - r_j) \text{ in } L^2(\mathbf{R}^{3N}),$$

where $m_j > 0$ and $r_j \in \mathbf{R}^3$ are the mass and the position vector of the j-th particle, respectively. The small parameter $h \in (0, 1]$ corresponds to the Planck constant, and the potential V_{ij} $(1 \le i < j \le N)$ is a real-valued function on $\mathbf{R}^3$ with the following decay condition:

$$(V)_\rho \begin{cases} V_{ij}(r) \in C^2(\mathbf{R}^3),\ r \in \mathbf{R}^3, \text{ and} \\ |\partial_r^\alpha V_{ij}(r)| \le C_\alpha \langle r \rangle^{-(\rho + d|\alpha|)},\ 0 \le |\alpha| \le 2 \text{ for some } d \in (0, 1], \end{cases}$$

for some $\rho > 2$, where $\langle r \rangle = (1 + |r|^2)^{1/2}$. Let $H(h)$ be the Hamiltonian with the center of mass removed from $\tilde{H}(h)$. $H(h)$ is a self-adjoint operator in $\mathscr{H} = L^2(\mathbf{R}^{3(N-1)})$. If we use the Jacobi coordinates $(x_1, \dots, x_{N-1}) \in \mathbf{R}^{3(N-1)}$.

$$x_j = r_{j+1} - \left(\sum_{1 \le k \le j} m_k \right)^{-1} \sum_{1 \le k \le j} m_k r_k, \qquad 1 \le j \le N-1,$$

$\tilde{H}(h)$ is written as follows

$$\tilde{H}(h) = H(h) \otimes Id + Id \otimes \left(-\frac{h^2}{2M} \Delta_R \right) \text{ in } L^2(\mathbf{R}^{3N}) = \mathscr{H} \otimes L^2(\mathbf{R}^3), \tag{2.1}$$

$$H(h) = -\sum_{1 \le j \le N-1} \frac{h^2}{2\nu_j} \Delta_{x_j} + \sum_{1 \le i < j \le N} V_{ij}(r_i - r_j) \text{ in } \mathscr{H}, \tag{2.2}$$

where $M = \sum_{1 \le j \le N} m_j$ and $R = M^{-1} \sum_{1 \le j \le N} m_j r_j$ are the total mass and the position of the center of mass, respectively, and ν_j is the reduced mass defined by $\nu_j^{-1} = m_{j+1}^{-1} + \left(\sum_{1 \le k \le j} m_k \right)^{-1}$.

A 2-cluster decomposition is the partition of the set $\{1, \dots, N\}$ into two nonempty subsets. In this paper we fix a 2-cluster decomposition $a = \{C_1, C_2\}$:

$$C_1 \cup C_2 = \{1, \dots, N\}, \qquad C_1 \cap C_2 = \phi, \qquad C_1 \ne \phi, \qquad C_2 \ne \phi.$$

Let N_j $(1 \le j \le 2)$ be the number of the elements in C_j $(N_1 + N_2 = N)$. We assume $N_1 \ge 2$ throughout this work. Each cluster C_j corresponds to a subsystem of the N-body system and the Hamiltonian for the subsystem is given by

$$\tilde{H}_j(h) = -\sum_{i \in C_j} \frac{h^2}{2m_i} \Delta_{r_i} + \sum_{i,k \in C_j} V_{ik}(r_i - r_k).$$

The cluster Hamiltonian $H_j(h)$ is defined by removing its center of mass. Let $M_j = \sum_{k \in C_j} m_k$, $R_j = M_j^{-1} \sum_{k \in C_j} m_k r_k \in \mathbf{R}^3$. Then we have, in the same way as (2.1),

$$\tilde{H}_j(h) = H_j(h) \otimes Id + Id \otimes \left(-\frac{h^2}{2M_j} \Delta_{R_j} \right) \text{ in } L^2(\mathbf{R}^{3(N_j-1)}) \otimes L^2(\mathbf{R}^3),$$

where we set $L^2(\mathbf{R}^{3(N_2-1)}) = \mathbf{C}$, $H_2(h) = 0$ if $N_2 = 1$. Let $z = R_2 - R_1$ and $n_a^{-1} = M_1^{-1} + M_2^{-1}$. Then we have

$$\sum_{1 \le j \le 2} -\frac{h^2}{2M_j} \Delta_{R_j} = -\frac{h^2}{2M} \Delta_R - \frac{h^2}{2n_a} \Delta_z.$$

We set

$$T_a(h) = -\frac{h^2}{2n_a}\Delta_z\,, \tag{2.3}$$

which acts in $L^2(\mathbf{R}_z^3)$, and define the intercluster potential I_a by

$$I_a = \sum_{1\le i<j\le N} V_{ij} - \sum_{i,j\in C_1} V_{ij} - \sum_{i,j\in C_2} V_{ij}\,, \tag{2.4}$$

where $V_{ij} = V_{ij}(r_i - r_j)$. Then we obtain the following relation:

$$H(h) - I_a = H_1(h)\otimes Id\otimes Id + Id\otimes H_2(h)\otimes Id + Id\otimes Id\otimes T_a(h) \tag{2.5}$$
$$\text{in } \mathscr{H} = L^2(\mathbf{R}^{3(N_1-1)})\otimes L^2(\mathbf{R}^{3(N_2-1)})\otimes L^2(\mathbf{R}^3)\,.$$

When $N_j \ge 2$, let $\lambda_j(h)$ be in $\sigma_{dis}(H_j(h))$, the discrete spectrum of $H_j(h)$, and let $\psi_j = \psi_j(y_j; h)$, $y_j \in \mathbf{R}^{3(N_j-1)}$ be the corresponding normalized eigenfunction for each $h \in (0, 1]$, however, we set $\lambda_2(h) = 0$, $\psi_2 = 1$ if $N_2 = 1$.

We set $\alpha = (a, \psi_1, \psi_2, h)$, which stands for a 2-body channel associated with the 2-cluster decomposition a for each h, and we define

$$\lambda_\alpha(h) = \lambda_1(h) + \lambda_2(h)\,, \tag{2.6}$$

$$\psi_\alpha = \psi_\alpha(y; h) = \psi_1\otimes\psi_2\,, \tag{2.7}$$

where $y = (y_1, y_2) \in \mathbf{R}^{3(N_1-1)}\otimes\mathbf{R}^{3(N_2-1)}$. Then we have

$$H^a(h)\psi_\alpha = \lambda_\alpha(h)\psi_\alpha\,, \tag{2.8}$$

where

$$H^a(h) = H_1(h)\otimes Id + Id\otimes H_2(h) \tag{2.9}$$

in $L^2(\mathbf{R}^{3(N-2)}) = L^2(\mathbf{R}^{3(N_1-1)})\otimes L^2(\mathbf{R}^{3(N_2-1)})$. The operator

$$H_\alpha(h) = \lambda_\alpha(h) + T_\alpha(h) \text{ in } L^2(\mathbf{R}_z^3) \tag{2.10}$$

is called the channel Hamiltonian and the channel identification operator $J_\alpha(h) \in B(L^2(\mathbf{R}_z^3), \mathscr{H})$ is defined by

$$J_\alpha(h)u = \psi_\alpha\otimes u\,, \tag{2.11}$$

where we have denoted by $\mathbf{B}(X, Y)$ the space of all bounded operators from X to Y. Here we note that

$$(J_\alpha(h)^*f)(z) = \int \overline{\psi_\alpha(y)}f(y, z)dy$$

for latter convenience. Under $(V)_\rho$, the channel wave operators

$$W_\alpha^\pm(h) = s - \lim_{t\to\pm\infty} \exp(ih^{-1}tH(h))J_\alpha(h)\exp(-ih^{-1}tH_\alpha(h)) \tag{2.12}$$

exist in $\mathbf{B}(L^2(\mathbf{R}_z^3), \mathscr{H})$ (cf. [RS] III, Theorem XI.34). The scattering operator for elastic scattering $\alpha\to\alpha$ is defined by

$$S_{\alpha\to\alpha}(h) = W_\alpha^+(h)^* W_\alpha^-(h) \in \mathbf{B}(L^2(\mathbf{R}_z^3)),$$

where $\mathbf{B}(L^2(\mathbf{R}_z^3)) = \mathbf{B}(L^2(\mathbf{R}_z^3), L^2(\mathbf{R}_z^3))$. For each $\lambda > \lambda_\alpha(h)$ and $\omega \in S^2$, we set

(2.13) $$\varphi_\alpha = \varphi_\alpha(z; \lambda, \omega, h) = \exp(ih^{-1}(2n_\alpha(\lambda - \lambda_\alpha(h)))^{1/2} z \cdot \omega)$$

and define $F_\alpha(h) \in \mathbf{B}(L^2(\mathbf{R}_z^3); L^2((\lambda_\alpha(h), \infty); L^2(S^2)))$ by

(2.14) $$(F_\alpha(h)f)(\lambda, \omega) = c_\alpha(\lambda, h) \int \overline{\varphi_\alpha}(z; \lambda, \omega, h) f(z) dz,$$

where $c_\alpha = c_\alpha(\lambda, h) = (2\pi h)^{-3/2} n_a^{1/2} (2n_a(\lambda - \lambda_\alpha(h)))^{1/4}$. $F_\alpha(h)$ is a unitary operator and give the spectral representation of $H_\alpha(h)$ i.e. $(F_\alpha(h)H_\alpha(h)f)(\lambda, *) = \lambda(F_\alpha(h)f)(\lambda, *)$ for *a.e.* λ if $f \in D(H_\alpha(h))$. Since the following property,

$$\exp(itH_\alpha(h)) S_{\alpha\to\alpha}(h) = S_{\alpha\to\alpha}(h) \exp(itH_\alpha(h)), \qquad t \in \mathbf{R},$$

holds, we can see that $F_\alpha(h) S_{\alpha\to\alpha}(h) F_\alpha(h)^*$ is decomposable by a family $\{S_{\alpha\to\alpha}(\lambda, h)\}$, $\lambda > \lambda_\alpha(h)$, of bounded operators on $L^2(S^2)$. In the similar way as in the 2-body case, we can see that $S_{\alpha\to\alpha}(\lambda, h) - Id$ is of Hilbert-Schmidt class for each $\lambda > 0$ and $T_{\alpha\to\alpha}(*, \omega; \lambda, h)$ is a $L^2(S^2)$-valued strongly continuous function of $(\lambda, \omega) \in (0, \infty) \times S^2$, where $T_{\alpha\to\alpha}(\theta, \omega; \lambda, h)$, $\theta \in S^2$, is the Hilbert-Schmidt kernel of $S_{\alpha\to\alpha}(\lambda, h) - Id$ (see Proposition 4.2). Thus the total cross section for elastic scattering $\alpha \to \alpha$ at $\lambda > 0$ and at incident direction $\omega \in S^2$ defined by

(2.15) $$\sigma_{\alpha\to\alpha}(\lambda, \omega; h) = \int_{S^2} |f_{\alpha\to\alpha}(\omega \to \theta; \lambda, h)|^2 d\theta$$

is continuous in $(\lambda, \omega) \in (0, \infty) \times S^2$, where

(2.16) $$f_{\alpha\to\alpha}(\omega \to \theta; \lambda, h) = -2\pi hi(2n_a(\lambda - \lambda_\alpha(h)))^{-1/2} T_{\alpha\to\alpha}(\theta, \omega; \lambda, h)$$

is the scattering amplitude for elastic scattering at energy λ.

3. The main result

We write $R(\zeta; A) = (A - \zeta)^{-1}$ for a self-adjoint operator A and $\zeta \in \mathbf{C}\backslash\mathbf{R}$. By $(V)_\rho$, $H(h)$ has no positive eigenvalue and no threshold ([FH1]), and the following norm limits exist

(3.1) $$X^{-s}\partial^\alpha R(\lambda \pm i0; H(h))X^{-s} = \lim_{\varepsilon \downarrow 0} X^{-s}\partial^\alpha R(\lambda \pm i\varepsilon; H(h))X^{-s}$$

in $\mathbf{B}(\mathscr{H})$ uniformly in λ in any compact set in $(0, \infty)$, where $X = (1 + |y|^2 + |z|^2)^{1/2}$, $s > 1/2$, and $\partial = (\partial_y, \partial_z)$, $|\alpha| \le 2$. This result was obtained in [PSS] by extending the results for 3-body systems in [M] (see also [FH2], [ABG] and [T]).

We fix a compact interval $J \subset (0, \infty)$ satisfying the following condition:

(N) For any $s > 1/2$ there exists a constant $C_s > 0$ such that

(3.2) $$\|X^{-s}R(\lambda \pm i0; H(h))X^{-s}\| \le C_s h^{-1}, \qquad \lambda \in J, \qquad h \in (0, 1].$$

If J is included in the intersection of non-trapping energy ranges of the N-body classical system and all the subsystems, the assumption (N) is satisfied ([W]). The semi-classical resolvent estimate (3.2) for non-trapping energies λ was first proved in [RT1] (see also [RT2], [GM]) for the 2-body case. Gérard ([G]) showed (3.2) by Mourre's method for the 3-body case and Wang ([W]) has extended his results for the N-body case.

For $\omega \in S^2$ we define a 2-dimensional plane $\Pi_\omega = \{u \in \mathbf{R}^3; u \cdot \omega = 0\}$. Then any $z \in \mathbf{R}^3$ can be written as $z = u + x\omega$, $u \in \Pi_\omega$, $x \in \mathbf{R}$ uniquely. Since the intercluster potential I_a is a function of (y, z) (see (2.4)), we set $I_a(y, z) = I_a$. We also define for $(\lambda, \omega) \in (0, \infty) \times S^2$

$$\sigma_\alpha^0(\lambda, \omega; h) = 4 \int_{\Pi_\omega} \sin^2 \left\{ \frac{1}{2\mu_\alpha(\lambda) h} \int_{-\infty}^{\infty} I_a(0, u + x\omega) dx \right\} du, \tag{3.3}$$

$$\mu_\alpha(\lambda) = \sqrt{\frac{2(\lambda - \lambda_\alpha(h))}{n_a}}. \tag{3.4}$$

Now we state our main result.

Theorem 3.1. *Let the notations be as above and assume $(V)_\rho$, $\rho > 2$. We fix $E_0 > 0$ and assume that the 2-body channel $\alpha = (a, \psi_1, \psi_2, h)$ satisfies*

$$\lambda_j(h) \leq \Sigma_j(h) - E_0 \tag{3.5}$$

for $j = 1, 2$ if $N_2 \geq 2$ (for $j = 1$ if $N_2 = 1$), where $\Sigma_j(h) = \inf \sigma_{ess}(H_j(h))$. Then, as a function of $(\lambda, \omega) \in J \times S^2$, $\sigma_{\alpha\to\alpha}(\lambda, \omega; h)$ behaves like

$$\sigma_{\alpha\to\alpha}(\lambda, \omega; h) = \sigma_\alpha^0(\lambda, \omega; h) + o(h^{-2/(\rho-1)}) \tag{3.6}$$

as $h \to 0$ uniformly in $(\lambda, \omega) \in J \times S^2$.

Remark. (i) Our proof really shows that the remainder term $o(h^{-2/(\rho-1)})$ can be replaced by $O(h^{-(2/(\rho-1))+\varepsilon})$ for some $\varepsilon > 0$.

(ii) We can see that

$$\sigma_\alpha^0(\lambda, \omega; h) = O(h^{-2/(\rho-1)}) \qquad (h \to 0) \tag{3.7}$$

uniformly in $(\lambda, \omega) \in J \times S^2$. Moreover, if $I_a(0, z)$ behaves like

$$I_a(0, z) = \Phi(z/|z|)|z|^{-\rho} + o(|z|^{-\rho}) \text{ as } |z| \to \infty \tag{3.8}$$

for some $\Phi \in C^2(S^2)$ with $\Phi < 0$, $\sigma_\alpha^0(\lambda, \omega; h)$ has the following asymptotics:

$$\sigma_\alpha^0(\lambda, \omega; h) = \sigma_0 \mu_\alpha(\lambda)^{-2/(\rho-1)} h^{-2/(\rho-1)}(1 + o(1)), \text{ as } h \to 0 \tag{3.9}$$

with some $\sigma_0 > 0$ ([Y]).

4. Representation formula of the total cross section for elastic scattering

We recall that $\psi_j = \psi_j(y_j; h)$ is a normalized eigenfunction of $H_j(h)$ with eigenvalue $\lambda_j(h) \in \sigma_{dis}(H_j(h))$. The following lemma can be obtained from [Ag], Sect. 4.1.

Lemma 4.1. *For any* $L \geq 0$, *one has*

$$\|\langle y_j \rangle^L \psi_j\|_{L^2(\mathbf{R}^{3(N_j-1)})} \leq C_L < \infty \tag{4.1}$$

uniformly in $h \in (0, 1]$ *if* (3.5) *is satisfied for some* $E_0 > 0$.

Throughout this paper, we fix the constants

$$\gamma = \frac{1}{\rho - 1} \text{ and } \beta = (1 + \delta)\gamma , \tag{4.2}$$

where $\delta > 0$ will be taken sufficiently small in Sect. 7. Following [RT2] (see also [IT1]), we introduce a partition of unity $\{\chi_j\}_{j=1,2,3}$, $\chi_j = \chi_j(z; h)$, over $\mathbf{R}^3$ with the following properties;

(χ.0) $\sum_{j=1}^3 \chi_j \equiv 1$,

(χ.1) $\operatorname{supp} \chi_1 \subset \{z \in \mathbf{R}^3; |z| < 2h^{-\gamma}\}$ (supp = support) ,

$\chi_1 = 1$ on $\{z \in \mathbf{R}^3; |z| < h^{-\gamma}\}$,

(χ.2) $\operatorname{supp} \chi_2 \subset B_{\gamma\beta}$,

$\chi_2 = 1$ on $\{z \in \mathbf{R}^3; 2h^{-\gamma} < |z| < h^{-\beta}\}$, where

$B_{\gamma\beta} = \{z \in \mathbf{R}^3; h^{-\gamma} < |z| < 2h^{-\beta}\}$,

(χ.3) $\operatorname{supp} \chi_3 \subset \{z \in \mathbf{R}^3; |z| > h^{-\beta}\}$,

$\chi_3 = 1$ on $\{z \in \mathbf{R}^3; |z| \geq 2h^{-\beta}\}$,

(χ.4) $|\partial_z^\alpha \chi_j(z; h)| \leq C_\alpha \langle z \rangle^{-|\alpha|}$, $1 \leq j \leq 3$,

uniformly in $h \in (0, 1]$ for any multi-index α .

We define the cluster decomposition Hamiltonian $H_a(h)$ in $\mathscr{H}$ by

$$H_a(h) = H - I_a = H^a(h) \otimes Id + Id \otimes T_a(h) \tag{4.3}$$

and its generalized eigenfunctions $e_\alpha(y, z; \lambda, \omega, h)$ by

$$e_\alpha(\omega) = \psi_\alpha(y; h)\varphi_\alpha(z; \lambda, \omega, h) . \tag{4.4}$$

(See (2.7), (2.13).) Then

$$H_a(h)e_\alpha(\omega) = \lambda e_\alpha(\omega) . \tag{4.5}$$

We also set $\chi = 1 - \chi_1 = \chi_2 + \chi_3$ and define the operators L and L^* by

$$\begin{aligned} L &= H(h)\chi - \chi H_a(h) \\ &= -\frac{h^2}{2n_a}(\Delta_z \chi) - \frac{h^2}{n_a} \nabla_z \chi \cdot \nabla_z + \chi I_a , \end{aligned} \tag{4.6}$$

$$L^* = \chi H(h) - H_a(h)\chi . \tag{4.7}$$

Since the support of $1-\chi(z)$ is compact in $\mathbf{R}_z^3$, the channel wave operators can be represented as follows:

$$W_\alpha^\pm(h) = s - \lim_{t\to\pm\infty} \exp(ih^{-1}tH(h))\chi J_\alpha(h)\exp(-ih^{-1}tH_\alpha(h)).$$

From this and almost the same argument as in the 2-body case, we have the following proposition, which gives the representation formula of $T_{\alpha\to\alpha}(\theta, \omega; \lambda, h)$ (the Hilbert-Schmidt kernel of $S_{\alpha\to\alpha}(\lambda, h) - Id$). We denote by $(\cdot, \cdot)_0$ the L^2-inner product in $\mathscr{H}$.

Proposition 4.2. *Let the notations be as above and assume $(V)_\rho$, $\rho > 2$. Then $T_{\alpha\to\alpha}(\theta, \omega; \lambda, h)$, θ, $\omega \in S^2$, $\lambda > 0$ is represented as*

$$T_{\alpha\to\alpha}(\theta, \omega; \lambda, h) = c_{0\alpha}G_\alpha(\theta, \omega; \lambda, h), \tag{4.8}$$

where

$$c_{0\alpha} = (2\pi)^{-2}in_\alpha(2n_\alpha(\lambda - \lambda_\alpha(h)))^{1/2}h^{-3}, \tag{4.9}$$

$$G_\alpha = ((-\chi L + L^*R(\lambda + i0; H(h))L)e_\alpha(\omega), e_\alpha(\theta))_0 .$$

Thus the scattering amplitude $f_{\alpha\to\alpha}(\omega \to \theta; \lambda, h)$ for scattering $\alpha \to \alpha$ is represented as

$$f_{\alpha\to\alpha}(\omega \to \theta; \lambda, h) = (2\pi)^{-1}n_\alpha h^{-2}G_\alpha(\theta, \omega; \lambda, h). \tag{4.10}$$

Since $(-\chi L + L^*R(\lambda + i0; H(h))L)e_\alpha(\omega)$ is an $L_s^2(\mathbf{R}^{3(N-1)}) := L^2(\mathbf{R}^{3(N-1)}; X^{2s}dydx)$ valued strongly continuous function of $(\lambda, \omega) \in (0, \infty) \times S^2$ for some $s > 1/2$, the R.H.S. of (4.8) is well-defined and continuous as an $L^2(S^2)$-valued function of (λ, ω) by the trace theorem.

A similar representation formula for the two body case is given in [Y].

The proof of Theorem 3.1 is based on the following representation formula of $\sigma_{\alpha\to\alpha}(\lambda, \omega; h)$.

Proposition 4.3. *For each $(\lambda, \omega) \in (0, \infty) \times S^2$, $\sigma_{\alpha\to\alpha}(\lambda, \omega; h)$ is represented as*

$$\sigma_{\alpha\to\alpha}(\lambda, \omega; h) = 2h^{-1}\mu_\alpha(\lambda)^{-1}(Q_1 + Q_2) \tag{4.11}$$

with

$$\begin{aligned} Q_1 &= Q_1(\lambda, \omega; h) \\ &= \mathrm{Im}\,(R(\lambda + i0; H(h))Le_\alpha(\omega), \chi^2 E_\alpha Le_\alpha(\omega))_0 , \end{aligned} \tag{4.12}$$

$$\begin{aligned} Q_2 &= Q_2(\lambda, \omega; h) \\ &= \mathrm{Im}\,(E_\alpha\chi L^*R(\lambda + i0; H(h))Le_\alpha(\omega), R(\lambda + i0; H(h))Le_\alpha(\omega))_0 , \end{aligned} \tag{4.13}$$

where $E_\alpha = J_\alpha J_\alpha^* \in \mathbf{B}(\mathscr{H})$.

We denote by $(\cdot, \cdot)_z$ the L^2-inner product in $L^2(\mathbf{R}_z^3)$ and define the weighted L^2-space $L_s^2(\mathbf{R}_z^3)$ by $L_s^2(\mathbf{R}_z^3) = L^2(\mathbf{R}^3; \langle z\rangle^{2s}dz)$ for $s \in \mathbf{R}$.

Proof. For the sake of simlicity, we write $R(\lambda \pm i0) = R(\lambda \pm i0; H(h))$,

$R_a(\lambda \pm i0) = R(\lambda \pm i0; H_a(h))$, $R_\alpha(\lambda \pm i0) = R(\lambda \pm i0; H_\alpha(h))$, $J = J_\alpha(h)$ and $K(\lambda) = -\chi L + L^*R(\lambda + i0)L$. By the trace theorem we can define the operator $F(\lambda) \in \mathbf{B}(L^2_s(\mathbf{R}^3_z), L^2(S^2))$, $s > 1/2$, $\lambda > 0$ by

$$(F(\lambda)u)(\omega) = (F_\alpha(h)u)(\lambda, \omega) .$$

Since $(L^2_s)^* = L^2_{-s}$ and $J^*_\alpha K(\lambda)e_\alpha(\omega) \in L^2_s(\mathbf{R}^2_z)$ for some $s > 1/2$, we have by (2.15), (2.16) and Proposition 4.2

$$\begin{aligned}\sigma_{\alpha\to\alpha}(\lambda, \omega; h) &= 2\pi h^{-1}\mu_\alpha(\lambda)^{-1}(F(\lambda)^*F(\lambda)J^*_\alpha K(\lambda)e_\alpha(\omega), J^*_\alpha K(\lambda)e_\alpha(\omega))_z \\ &= -ih^{-1}\mu_\alpha(\lambda)^{-1}Q ,\end{aligned} \tag{4.14}$$

with $Q = ((R_\alpha(\lambda + i0) - R_\alpha(\lambda - i0))J^*_\alpha K(\lambda)e_\alpha(\omega), J^*_\alpha K(\lambda)e_\alpha(\omega))_z$, where we have used the relations

$$G_\alpha(\theta, \omega; \lambda, h) = c_\alpha(\lambda, h)^{-1}(F(\lambda)J^*_\alpha K(\lambda)e_\alpha(\omega))(\theta)$$

in the first step and

$$F(\lambda)^*F(\lambda) = (2\pi i)^{-1}(R_\alpha(\lambda + i0) - R_\alpha(\lambda - i0))$$

in the last step. Now note the following relations:

$$J_\alpha R_a(\zeta) = R_\alpha(\zeta)J_\alpha , \qquad R(\zeta)LR_a(\zeta) = \chi R_a(\zeta) - R(\zeta)\chi$$

$$R_\alpha(\zeta)J^*_\alpha = J^*_\alpha R_a(\zeta) , \qquad R_a(\zeta)L^*R(\zeta) = R_a(\zeta)\chi - \chi R(\zeta)$$

for $\zeta \in \mathbf{C}\backslash\mathbf{R}$. From these relations it follows that

$$R_a(\lambda + i0)E_\alpha K(\lambda) = -E_\alpha \chi R(\lambda + i0)L .$$

Thus we have

$$\begin{aligned}Q &= (R_a(\lambda + i0)E_\alpha K(\lambda)e_\alpha(\omega), K(\lambda)e_\alpha(\omega))_0 - (E_\alpha K(\lambda)e_\alpha(\omega), R_a(\lambda + i0)K(\lambda)e_\alpha(\omega))_0 \\ &= -(E_\alpha \chi R(\lambda + i0)Le_\alpha(\omega), K(\lambda)e_\alpha(\omega))_0 + (\chi E_\alpha K(\lambda)e_\alpha(\omega), R(\lambda + i0)Le_\alpha(\omega))_0 \\ &= 2i \operatorname{Im} (R(\lambda + i0)Le_\alpha(\omega), \chi^2 E_\alpha Le_\alpha(\omega))_0 \\ &\quad + 2i \operatorname{Im} (E_\alpha \chi L^*R(\lambda + i0)Le_\alpha(\omega), R(\lambda + i0)Le_\alpha(\omega))_0 ,\end{aligned}$$

which together with (4.14) implies the desired result.

5. Remainder estimates I

From this section to the last section we assume all the assumptions of Theorem 3.1 and devote ourselves to the proof of Theorem 3.1.

We begin by dividing $Le_\alpha(\omega)$ into two parts:

$$\begin{aligned}Le_\alpha(\omega) &= \theta_1 + \theta_2 , \\ \theta_1 &= [\chi_1, H_0(h)]e_\alpha + \chi_2 I^0_a e_\alpha , \\ \theta_2 &= \chi_2(I_a - I^0_a)e_\alpha + \chi_3 I_a e_\alpha ,\end{aligned} \tag{5.1}$$

where $e_\alpha = e_\alpha(\omega)$ and $I^0_a(z) = I_a(y, z)|_{y=0}$.

The next lemma shows that θ_2 does not contribute to the leading term of the asymptotics of $Q_1(\lambda, \omega; h)$ as $h \to 0$.

Lemma 5.1.

$$Q_1(\lambda, \omega; h) = \mathrm{Im}\,(R(\lambda + i0; H(h))\theta_1, \chi^2 E_\alpha \theta_1)_0 + o(h^{1-2\gamma}) \tag{5.2}$$

as $h \to 0$ *uniformly in* $(\lambda, \omega) \in J \times S^2$.

Proof. For the proof it suffices to show that

$$D_{ij} := (R(\lambda + i0; H(h))\theta_i, \chi^2 E_\alpha \theta_j)_0 = o(h^{1-2\gamma}) \qquad (h \to 0) \tag{5.3}$$

uniformly in $(\lambda, \omega) \in J \times S^2$ for $1 \le i, j \le 2$, $(i, j) \neq (1, 1)$.

By $(V)_\rho$ we have

$$|I_a(y, z) - I_a^0(z)| \le C\langle y\rangle^{1+\rho+d}\langle z\rangle^{-\rho-d}. \tag{5.4}$$

Thus, recalling $X = (1 + |y|^2 + |z|^2)^{1/2}$ and taking s with $1/2 < s < \rho - (3/2)$, we get, by $(V)_\rho$ and Lemma 4.1,

$$\begin{aligned} \|X^s\chi_2(I_a - I_a^0)e_\alpha\|_0^2 &\le C\int_{|z|>h^{-\gamma}} (h^{-\gamma} + |z|)^{2s-2\rho-2d}dz \\ &\le Ch^{\gamma(2\rho+2d-2s-3)}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \|X^s\chi_3 I_a e_\alpha\|_0^2 &\le C\int_{|z|>h^{-\beta}} (h^{-\beta} + |z|)^{2s-2\rho}dz \\ &\le Ch^{\beta(2\rho-2s-3)}, \end{aligned} \tag{5.6}$$

where $\|\cdot\|_0$ denotes the L^2-norm in $\mathscr{H}$. Therefore it follows that

$$\begin{aligned} \|X^s\theta_2\|_0 &= O(h^{\gamma(\rho+d-s-(3/2))}) + O(h^{\beta(\rho-s-(3/2))}) \\ &= o(h^{\gamma(\rho-s-(3/2))}) \qquad (h \to 0), \end{aligned} \tag{5.7}$$

uniformly in $(\lambda, \omega) \in J \times S^2$. Now we note that θ_1 has the form

$$\theta_1 = f(z; h)e_\alpha, \tag{5.8}$$

where f satisfies i) $\mathrm{supp}\, f \subset B_{\gamma\beta}$, ii)

$$|\partial_z^\alpha f(z; h)| \le C_\alpha(|z| + h^{-\gamma})^{-\rho}h^{|\alpha|\gamma d}, \qquad 0 \le |\alpha| \le 2. \tag{5.9}$$

Thus we have

$$\begin{aligned} \|X^s\theta_1\|_0^2 &\le C\int_{|z|>h^{-\gamma}} (h^{-\gamma} + |z|)^{2s-2\rho}dz \\ &= O(h^{2\gamma(\rho-s-(3/2))}). \end{aligned} \tag{5.10}$$

This together with (5.7) and (N) yields

$$D_{12} = O(h^{-1})(O(h^{\gamma(\rho+d-s-(3/2))}) + O(h^{\beta(\rho-s-(3/2))}))O(h^{\gamma(\rho-s-(3/2))})$$
$$= h^{1-2\gamma}(O(h^{\mu}) + O(h^{\mu'}))\,,$$

where $\mu = \gamma(1 - 2s + d) > 0$, $\mu' = \gamma(1 - 2s + \delta(\rho - s - (3/2))) > 0$ if we take s such that $0 < 2s - 1 < d$, $\delta > (2s - 1)/(\rho - s - (3/2)) > 0$. Hence, $D_{12} = o(h^{1-2\gamma})$. In the same way as above we can get $D_{ij} = o(h^{1-2\gamma})$ for $(i, j) = (2, 1)$, (2.2) and finish the proof.

Now we shall investigate

$$(5.11)\qquad Q_2(\lambda, \omega; h) = 2i^{-1}((\chi E_a L^* - LE_a\chi)R(\lambda + i0; H(h)) \times (\theta_1 + \theta_2), R(\lambda + i0; H(h))(\theta_1 + \theta_2))_0\,.$$

By direct calculations we have

$$(5.12)\qquad \chi E_a L^* - LE_a\chi = B_1(h) + B_2(h) + B_3(h)\,,$$
$$B_1(h) = \frac{h^2}{n_a}(\Delta_z\chi)\chi E_a + \frac{h^2}{n_a}E_a|\nabla_z\chi|^2\,,$$
$$B_2(h) = \frac{2h^2}{n_a}E_a\chi\nabla_z\chi\cdot\nabla_z\,,$$
$$B_3(h) = \chi^2(E_a I_a - I_a E_a)\,.$$

Lemma 5.2. *For $j = 1$, 3,*

$$(5.13)\qquad (B_j(h)R(\lambda + i0; H(h))(\theta_1 + \theta_2), R(\lambda + i0; H(h))(\theta_1 + \theta_2))_0 = o(h^{1-2\gamma})$$

as $h \to 0$ uniformly in $(\lambda, \omega) \in J \times S^2$.

Proof. We fix $s > 1/2$ sufficiently near 1/2. Since $|\Delta\chi| \le C\langle z\rangle^{-2}$, $|\nabla\chi| \le C\langle z\rangle^{-1}$ uniformly in $h \in (0, 1]$ and since $|z| \ge h^{-\gamma}$ for $z \in \operatorname{supp}\chi$, we have

$$(5.14)\qquad \|\langle y\rangle^s\langle z\rangle^s B_1(h)\langle y\rangle^s\langle z\rangle^s\| \le Ch^2(h^{-\gamma})^{2s-2} = Ch^{2+\gamma(2-2s)}\,,$$

where we have used the fact $\|\langle y\rangle^s E_a\langle y\rangle^s\| < +\infty$, which follows from Lemma 4.1. Hence it follows from (5.7), (5.10) and (N) that the L.H.S. of (5.13) for $j = 1$ is of order $O(h^{\mu})h^{1-2\gamma}$ with

$$(5.15)\qquad \mu = 2\gamma - 1 + 2 + \gamma(2 - 2s) - 2 + 2\gamma(\rho - s - (3/2)) = \gamma(\rho - 4s + 2) > 0\,.$$

This proves (5.13) for $j = 1$. Using (5.4), $[E_a, I_a^0] = 0$ and Lemma 4.1 ([,] denotes the commutator), we have

$$(5.16)\qquad \|\langle y\rangle^s\langle z\rangle^s B_3(h)\langle y\rangle^s\langle z\rangle^s\| \le C\sup_{|z|>h^{-\gamma}}\langle z\rangle^{2s-\rho-d} \le Ch^{\gamma(\rho+d-2s)}\,.$$

Therefore, by (5.7), (5.10) and (N), the L.H.S. of (5.13) for $j = 3$ is of order $O(h^{\mu'})h^{1-2\gamma}$ with

(5.17)
$$\mu' = 2\gamma - 1 + \gamma(\rho + d - 2s) - 2 + 2\gamma(\rho - s - (3/2))$$
$$= \gamma(d - 4s + 2) > 0\,,$$

since $s > 1/2$ is sufficiently near 1/2. This completes the proof.

Lemma 5.3. *Let $s > 1/2$. Then*

(5.18)
$$\|X^{-s}\nabla_z R(\lambda + i0; H(h))X^{-s}\| \le Ch^{-2}\,, \qquad h \in (0, 1]\,,$$

uniformly in $\lambda \in J$.

Proof. We set $V = \sum_{1\le i<j\le N} V_{ij}(r_i - r_j)$ and $H_0(h) = H(h) - V$. For any $\zeta \in \mathbf{C}\backslash\mathbf{R}$ we have

$$(H_0(h) + 1)R(\zeta; H(h)) = Id + (\zeta + 1 - V)R(\zeta; H(h))\,.$$

This together with the assumption (N) yields

(5.19)
$$\|X^{-s}(-\Delta_y - \Delta_z + 1)R(\lambda + i0; H(h))X^{-s}\| \le Ch^{-3}$$

uniformly in $\lambda \in J$. Thus, by (N) and interpolation, we have

$$\|X^{-s}(-\Delta_y - \Delta_z + 1)^{1/2}R(\lambda + i0; H(h))X^{-s}\| \le Ch^{-2}\,.$$

(5.18) follows from this and $X^{-s}\nabla_z(-\Delta_y - \Delta_z + 1)^{-1/2}X^s \in \mathbf{B}(\mathscr{H})$.

The following lemma together with Lemma 5.1 shows that θ_2 does not contribute to the leading term of the asymptotics of $\sigma_{\alpha\to\alpha}(\lambda, \omega; h)$ $(h \to 0)$.

Lemma 5.4.

$$Q_2(\lambda, \omega; h) = \frac{1}{2i}(B_2(h)R(\lambda + i0; H(h))\theta_1, R(\lambda + i0; H(h))\theta_1)_0 + o(h^{1-2\gamma})$$

as $h \to 0$ uniformly in $(\lambda, \omega) \in J \times S^2$.

Proof. We set $Q_{ij} = (B_2(h)R(\lambda + i0; H(h))\theta_i, R(\lambda + i0; H(h))\theta_j)$, $1 \le i, j \le 2$. By Lemma 5.2 it suffices to show that

(5.20)
$$Q_{ij} = o(h^{1-2\gamma}) \qquad (h \to 0)$$

uniformly in $(\lambda, \omega) \in J \times S^2$ for $(i, j) \ne (1, 1)$. We first fix $s > 1/2$ sufficiently near 1/2. Noting that $|\nabla\chi| \le C\langle z\rangle^{-1}$, $h \in (0, 1]$, we have

(5.21)
$$\|X^s h^2 E_\alpha \chi(\nabla_z\chi)X^s\| \le Ch^2 \sup_{h^{-\gamma}\le|z|\le 2h^{-\gamma}} \langle z\rangle^{2s-1}$$
$$\le Ch^{2+\gamma(1-2s)}\,.$$

Hence, by (5.7), (5.10) and lemma 5.3, we get

$$Q_{12} = (O(h^\mu) + O(h^{\mu'}))h^{1-2\gamma} \qquad (h \to 0)$$

uniformly in $(\lambda, \omega) \in J \times S^2$, where

$$\mu = 2\gamma - 1 + 2 + \gamma(1 - 2s) - 3 + \gamma(\rho - s - (3/2)) + \gamma(\rho + d - s - (3/2))$$
$$= \gamma(d + 2 - 4s) > 0\,,$$
$$\mu' = 2\gamma - 1 + 2 + \gamma(1 - 2s) - 3 + \gamma(\rho - s - (3/2)) + \beta(\rho - s - (3/2))$$
$$= \gamma(2 - 4s + \delta(\rho - s - (3/2))) > 0$$

if $s > 1/2$ is sufficiently near 1/2. Q_{21}, Q_{22} can be treated similarly. This proves the lemma.

6. Remainder estimates II

Recall that θ_1 has the form $\theta_1 = f(z; h)e_\alpha$ (see (5.8)),

$$\begin{cases} \operatorname{supp} f \subset B_{\gamma\beta} \\ |\partial_z^\alpha f(z; h)| \le C_\alpha h^{|\alpha|\gamma d}(|z| + h^{-\gamma})^{-\rho}\,, \qquad 0 \le |\alpha| \le 2\,, \end{cases}$$

and that $z \in \mathbf{R}^3$ is written as $z = u + t\omega$, $u \in \Pi_\omega$, $t \in \mathbf{R}$ uniquely. We set $\kappa = (1 - \delta)\gamma$ for the same $\delta > 0$ as in (4.2) and define $f_s(z; h) = \chi_0(u/h^{-\kappa})f(z; h)$ and $f_l(z; h) = (1 - \chi_0(u/h^{-\kappa}))f(z; h)$ where χ_0 is a C^∞-function on Π_ω with $0 \le \chi_0 \le 1$, $\chi_0 = 0$ for $|u| > 2$ and $\chi_0 = 1$ for $|u| \le 1$. f_s and f_l have the following properties:

$$\text{(6.1)} \qquad \begin{cases} \operatorname{supp} f_s \subset \{z = u + t\omega;\, |u| < 2h^{-\kappa}\} \cap B_{\gamma\beta}\,, \\ \operatorname{supp} f_l \subset \{z = u + t\omega;\, |u| > h^{-\kappa}\} \cap B_{\gamma\beta}\,, \\ |\partial_z^\alpha f_l(z; h)|,\, |\partial_z^\alpha f_s(z; h)| \le C_\alpha h^{|\alpha|\kappa d}(|z| + h^{-\gamma})^{-\rho}\,, \qquad 0 \le |\alpha| \le 2\,. \end{cases}$$

We write $\theta_1 = \theta_{1s} + \theta_{1l}$, $\theta_{1s} = f_s(z; h)e_\alpha$, $\theta_{1l} = f_l(z; h)e_\alpha$ and put it in the leading term of Q_1 and Q_2 (see Lemmas 5.1, 5.4). The aim of this section is to show that the terms containing θ_{1s} are negligible in our analysis.

Lemma 6.1.

$$Q_1(\lambda, \omega; h) = \operatorname{Im}(R(\lambda + i0; H(h))\theta_{1l}, E_\alpha \chi^2 \theta_{1l})_0 + o(h^{1-2\gamma})$$

as $h \to 0$ *uniformly in* $(\lambda, \omega) \in J \times S^2$.

Proof. According to Lemma 5.1, we have only to prove

$$\text{(6.2)} \qquad \operatorname{Im}(R(\lambda + i0; H(h))\Psi, E_\alpha \chi^2 \Phi)_0 = o(h^{1-2\gamma}) \qquad (h \to 0)$$

uniformly in $(\lambda, \omega) \in J \times S^2$ for $(\Psi, \Phi) = (\theta_{1s}, \theta_{1s})$, $(\theta_{1s}, \theta_{1l})$, $(\theta_{1l}, \theta_{1s})$. We shall prove (6.2) only for $(\Psi, \Phi) = (\theta_{1l}, \theta_{1s})$ because the other cases can be treated similarly. Taking $s > 1/2$ sufficiently near 1/2, we have by Lemma 4.1

$$\text{(6.3)} \qquad \|X^s\theta_{1s}\|_0^2 \le C \int_{-\infty}^{\infty} dx \int_{|u| < 2h^{-\kappa}} (h^{-\gamma} + |u| + |x|)^{-2\rho + 2s} du$$
$$\le Ch^{-2\kappa} \int_{-\infty}^{\infty} (h^{-\gamma} + |x|)^{-2\rho + 2s} dx$$
$$= O(h^{\gamma(2\rho - 2s - 3) + 2\gamma - 2\kappa}) \qquad (h \to 0)\,.$$

By (5.10), the following estimates holds:

(6.4) $$\|X^s\theta_{1l}\|_0 = O(h^{\gamma(\rho-s-(3/2))}) \qquad (h\to 0)$$

for the same s as above. Therefore we obtain by (N)

$$(R(\lambda+i0; H(h))\theta_{1l}, \chi^2 E_\alpha \theta_{1s})_0 = h^{1-2\gamma}O(h^\mu) \qquad (h\to 0),$$

where $\mu = 2\gamma - 1 - 1 + 2\gamma(\rho - s - (3/2)) + \gamma - \kappa = \gamma(\delta + 1 - 2s) > 0$ if we take s sufficiently near 1/2.

Lemma 6.2.

$$Q_2(\lambda, \omega; h) = \frac{1}{2i}(B_2(h)R(\lambda+i0; H(h))\theta_{1l}, R(\lambda+i0; H(h))\theta_{1l})_0 + o(h^{1-2\gamma})$$

as $h\to 0$ *uniformly in* $(\lambda, \omega)\in J\times S^2$.

Proof. In the same way as the proof of Lemma 6.1 we shall only prove

(6.5) $$(B_2(h)R(\lambda+i0; H(h))\theta_{1l}, R(\lambda+i0; H(h))\theta_{1s})_0 = o(h^{1-2\gamma}).$$

From (5.21), (6.3), (6.4) and Lemma 5.3 it follows that the L.H.S. of (6.5) is of order $O(h^\mu)h^{1-2\gamma}$, where

$$\begin{aligned}\mu &= 2\gamma - 1 + 2 + \gamma(1-2s) - 2 + 2\gamma(\rho - s - (3/2)) - 1 + \gamma - \kappa \\ &= \gamma(\delta + 2 - 4s) > 0\end{aligned}$$

if we take $s > 1/2$ sufficiently near 1/2.

7. Approximation of $R(\lambda+i0; H(h))f_l e_\alpha$

We define $v(t) = v(t, z; \lambda, \omega, h)$ by

$$v(t) = f_l(z - \mu_\alpha \omega t; h)\exp\left(-ih^{-1}\int_0^t I_\alpha^0(z - \mu_\alpha\omega s)ds\right),$$

where $\mu_\alpha = \mu_\alpha(\lambda)$ (see (3.4)). It is easy to verify that $v(t)e_\alpha$ satisfies the following equation (see (4.4) for $e_\alpha = e_\alpha(\omega)$)

(7.1) $$(ih\partial_t - H(h) + \lambda)v(t)e_\alpha = r_1(t) + r_2(t),$$

where

$$r_1(t) = r_1(t, y, z; \lambda, \omega, h) = \frac{h^2}{2n_a}(\Delta_z v(t))e_\alpha,$$

$$r_2(t) = r_2(t, y, z; \lambda, \omega, h) = (I_a^0(z) - I_a(y, z))v(t)e_\alpha.$$

Taking a large constant $N_0 > 0$ independent of $h > 0$ and setting $\tau = N_0 h^{-\beta}$, we define $g_0 = g_0(z; \lambda, \omega, h)$ by

$$g_0 = \int_0^\tau v(t, z; \lambda, \omega, h)dt .$$

Then the following relation follows from (7.1):

$$(H(h) - \lambda)g_0 e_\alpha = ih(v(\tau)e_\alpha - v(0)e_\alpha) - \int_0^\tau (r_1(t) + r_2(t))dt .$$

Since $g_0 e_\alpha$ and all the term in the R.H.S. belong to $L_s^2(\mathbf{R}^{3(N-1)}) = L^2(\mathbf{R}^{3(N-1)}; X^{2s}dydz)$ for some $s > 1/2$ (see (6.1)), we have

$$R(\lambda + i0; H(h))f_l e_\alpha = ih^{-1}g_0 e_\alpha + R(\lambda + i0; H(h))v(\tau)e_\alpha + ih^{-1}R(\lambda + i0; H(h)) \int_0^\tau (r_1(t) + r_2(t))dt , \tag{7.2}$$

where $f_l = f_l(z; h)$.

Lemma 7.1. *Let* $s > 0$. *Then*

$$h^{-1} \left\| X^s \int_0^\tau r_1(t)dt \right\|_0 = O(h^{2\kappa(\rho+d-1)+\beta(-s-(3/2))})$$

as $h \to 0$ *uniformly in* $(\lambda, \omega) \in J \times S^2$.

Proof. The following estimates hold on the region $\{z = u + x\omega \in \mathbf{R}^3; u \in \Pi_\omega, x \in \mathbf{R}, |u| > h^{-\kappa}\}$;

$$\int_{-\infty}^{\infty} |\nabla_z I_a^0(z - \mu_\alpha \omega s)| ds \le Ch^{\kappa(\rho+d-1)} , \tag{7.3}$$

$$\int_{-\infty}^{\infty} |\Delta_z I_a^0(z - \mu_\alpha \omega s)| ds \le Ch^{\kappa(\rho+2d-1)} , \tag{7.4}$$

and, moreover, we have by (6.1)

$$|\partial_z^\alpha f_l(z - \mu_\alpha \omega t)| \le Ch^{|\alpha|\kappa d}(h^{-\gamma} + |u| + |x - \mu_\alpha t|)^{-\rho} , \tag{7.5}$$

for $0 \le |\alpha| \le 2$, where $z = u + t\omega$. Therefore we see that

$$\begin{aligned} |\Delta v(t)| &\le C(h^{2\kappa d} + h^{-1+\kappa(\rho+2d-1)} + h^{-2+2\kappa(\rho+d-1)})(h^{-\gamma} + |u| + |x - \mu_\alpha t|)^{-\rho} \\ &\le Ch^{-2+2\kappa(\rho+d-1)}(h^{-\gamma} + |u| + |x - \mu_\alpha t|)^{-\rho} , \end{aligned} \tag{7.6}$$

which yields

$$\int_0^\tau |\Delta v(t)| dt \le \int_{-\infty}^{\infty} |\Delta v(t)| dt \le Ch^{2\kappa(\rho+d-1)-1} \text{ on } \mathbf{R}^3 .$$

Hence, taking account of the fact that $\operatorname{supp} r_1(t) \subset \{z \in \mathbf{R}^3, |z| \le (N_0 + 2)h^{-\beta}\}$ for $0 \le t \le \tau$, we obtain

$$h^{-1}\left\| X^s \int_0^\tau r_1(t)dt \right\|_0 \le Ch\cdot h^{2\kappa(\rho+d-1)-1}\left(\int_{|z|<(N_0+2)h^{-\beta}} \langle z\rangle^{2s}dz\right)^{1/2}$$

$$\le Ch^{2\kappa(\rho+d-1)}(h^{-\beta})^{s+(3/2)}.$$

This proves the lemma.

Lemma 7.2. *Let* $1/2 < s < \rho + d - 1/2$. *Then*

$$h^{-1}\left\| X^s \int_0^\tau r_2(t)dt \right\|_0 = O(h^{-(\beta/2)+\kappa(2\rho+d-s-2)-1})$$

as $h \to 0$ *uniformly in* $(\lambda, \omega) \in J \times S^2$.

Proof. By (5.4) and Lemma 4.1, it suffices to estimate

$$h^{-1}\left\| \langle z\rangle^{s-\rho-d} \int_0^\tau |v(t)|\,dt \right\|_z ,$$

where $\|\cdot\|_z$ denotes the L^2-norm in $L^2(\mathbf{R}_z^3)$. Since the support of $v(t)$ is contained in $\{z = u + x\omega;\, u \in \Pi_\omega,\, x \in \mathbf{R},\, |u| > h^{-\kappa}\}$ and the estimate

(7.7) $$|v(t)| \le C(h^{-\gamma} + |u| + |x - \mu_\alpha t|)^{-\rho}$$

holds, we have

$$\langle z\rangle^{s-\rho-d} \int_0^\tau |v(t)|\,dt \le C(h^{-\kappa} + |u| + |x|)^{-\rho-d+s}(h^{-\gamma} + |u|)^{-\rho+1},$$

which yields

$$\left\| \langle z\rangle^{s-\rho-d} \int_0^\tau |v(t)|\,dt \right\|_z^2$$

$$\le c\int du \int_{|x|<(N_0+2)h^{-\beta}} (h^{-\kappa} + |u| + |x|)^{-2\rho-2d+2s}(h^{-\gamma} + |u|)^{-2\rho+2}dx$$

$$\le Ch^{-\beta}h^{\kappa(4\rho+2d-2s-4)}.$$

This completes the proof.

We denote by $\tilde{\chi} = \tilde{\chi}(z)$ the characteristic function of the ball $\{z \in \mathbf{R}^3;\, |z| < 3h^{-\beta}\}$.

Lemma 7.3. *Let* m *be a multi-index with* $|m| \le 1$ *and* $D_z = -i\nabla_z$. *If* $s > 1/2$ *and* $\delta > 0$ *is sufficiently small, one has*

(7.8) $$\|\tilde{\chi}\langle z\rangle^{-s}D_z^m R(\lambda + i0;\, H_\alpha(h))v(\tau)\varphi_\alpha\|_z = o(h^{\beta(\rho-3s)+\gamma(\rho-(3/2))-|m|})$$

as $h \to 0$ *uniformly in* $(\lambda, \omega) \in J \times S^2$.

Proof. We introduce two functions $\phi_1, \phi_2 \in C^\infty(\mathbf{R}^3)$ satisfying $\phi_1 + \phi_2 = 1$, $\operatorname{supp}\phi_1 \subset \{\xi \in \mathbf{R}^3;\, |\xi - n_a\mu_\alpha\omega| < \delta_0\}$ and $\phi_1 = 1$ on $\{\xi \in \mathbf{R}^3;\, |\xi - n_a\mu_\alpha\omega| < \delta_0/2\}$ for a small constant $\delta_0 > 0$. We first show that

(7.9) $$\|\tilde{\chi}\langle z\rangle^{-s}R(\lambda+i0;H_\alpha(h))D_z^m\phi_1(hD_z)v(\tau)\varphi_\alpha\|_z=O(h^L)\qquad(h\to 0)$$

for any $L\geq 0$. To verify this, by the formula

$$R(\lambda+i0;H_\alpha(h))=ih^{-1}\int_0^\infty \exp(ih^{-1}t(\lambda-\lambda_\alpha(h)))\exp(-ih^{-1}tT_a(h))dt\,,$$

we have only to prove

(7.10) $$\|\tilde{\chi}\exp(-ih^{-1}tT_a(h))D_z^m\phi_1(hD_z)v(\tau)\varphi_\alpha\|_z\leq C_Lh^L(1+t)^{-L}\,,\quad h\in(0,1]\,,\quad t>0\,,$$

for any $L\geq 0$. By the Fourier transform, $\exp(-ih^{-1}tT_a(h))D_z^m\phi_1(hD_z)u(z)$, $u=v(\tau)e_\alpha$, is expressed as

$$(2\pi)^{-3}\iint \exp\left(i\xi\cdot(z-z')-i\frac{\xi^2}{2n_a}ht\right)\xi^m\phi_1(h\xi)u(z')dz'd\xi\,.$$

Since $\left|\nabla_\xi\left(\xi\cdot(z-z')-\frac{\xi^2}{2n_a}ht\right)\right|\geq C(|z'|+t)\geq C'(|z'|+h^{-\beta}+t)$ for $z\in\operatorname{supp}\tilde{\chi}$, $z'\in\operatorname{supp}v(\tau)$ and $h\xi\in\operatorname{supp}\phi_1$, we can obtain (7.10) by integrating by parts in ξ. Next we prove

(7.11) $$\|\tilde{\chi}\langle z\rangle^{-s}R(\lambda+i0;H_\alpha(h))D_z^m\phi_2(hD_z)v(\tau)\varphi_\alpha\|_z=o(h^{\beta(\rho-3s)+\gamma(\rho-(3/2))-|m|})\quad(h\to 0)\,.$$

To see this we set $\phi_3(\xi)=\xi^m\phi_2(\xi)|\xi-n_a\mu_\alpha\omega|^2$ and observe that ϕ_3 is bounded smooth function with bounded derivatives and satisfies

(7.12) $$D_z^m\phi_2(hD_z)\varphi_\alpha v(\tau)=-h^{2-|m|}\phi_3(hD_z)\varphi_\alpha\Delta v(\tau)\,,$$

(7.13) $$\|\langle z\rangle^s\phi_3(hD_z)\langle z\rangle^{-s}\|=O(1)\qquad(h\to 0)\,.$$

From (7.13) and the well known estimate

(7.14) $$\|\langle z\rangle^{-s}R(\lambda+i0;H_\alpha(h))\langle z\rangle^{-s}\|=O(h^{-1})\qquad(h\to 0)\,,$$

it follows that

(7.15) $$\|\langle z\rangle^{-s}R(\lambda+i0;H_\alpha(h))\phi_3(hD_z)\langle z\rangle^{-s}\|=O(h^{-1})\,.$$

Since $\operatorname{supp}\Delta v(\tau)\subset\{z\in\mathbf{R}^3;|z|\leq(N_0+2)h^{-\beta}\}$, we have by (7.6)

(7.16) $$\|\langle z\rangle^s\Delta v(\tau)\|_z=O(h^{-2+2\kappa(\rho+d-1)-\beta s+\gamma(\rho-(3/2))})\qquad(h\to 0)\,.$$

Thus by (7.12), (7.15) and (7.16) we obtain

$$\begin{aligned}\text{the L.H.S. of (7.11)}&=O(h^{-1+2\kappa(\rho+d-1)-\beta s+\gamma(\rho-(3/2))-|m|})\\&=O(h^{\beta(\rho-3s)+\gamma(\rho-(3/2))-|m|}h^\mu)\,,\end{aligned}$$

where $\mu=\gamma(2d-1+2s-\delta(3\rho+2d-2s-2))$. If $\delta>0$ is so small, we have $\mu>0$, and hence prove (7.11). This together with (7.9) yields (7.8).

Lemma 7.4. *If* $s-1/2>0$ *and* $\delta>0$ *are sufficiently small, one has*

$$\|\tilde{\chi}X^{-s}D_z^m R(\lambda+i0;H(h))v(\tau)e_\alpha\|_0 = O(h^{\beta(\rho-3s)+\gamma(\rho-(3/2))-2-|m|})$$

as $h\to 0$ *uniformly in* $(\lambda,\omega)\in J\times S^2$ *for* $0\le|m|\le 1$.

Proof. We divide $R(\lambda+i0;H(h))$ into three parts:

$$R(\lambda+i0;H_a(h)) - R(\lambda+i0;H(h))\tilde{\chi}I_a R(\lambda+i0;H_a(h)) - R(\lambda+i0;H(h)) \times(1-\tilde{\chi})I_a R(\lambda+i0;H_a(h)),$$

and insert them into the left side. Since $R(\lambda+i0;H_a(h))v(\tau)e_\alpha = R(\lambda+i0;H_\alpha(h))v(\tau)\varphi_\alpha$, we have only to consider the contribution from the last two terms by Lemmas 7.3 and 4.1. By Lemmas 5.3 and 7.3 we have

$$\begin{aligned}&\|\tilde{\chi}X^{-s}D_z^m R(\lambda+i0;H(h))\tilde{\chi}I_a R(\lambda+i0;H_a(h))v(\tau)e_\alpha\|_0\\ &\le Ch^{-1-|m|}\|X^s\tilde{\chi}I_a\langle y\rangle^{-\rho-s}\langle z\rangle^s\|\,\|\langle z\rangle^{-s}\tilde{\chi}R(\lambda+i0;H_\alpha(h))v(\tau)\varphi_\alpha\|_z\\ &= O(h^{\beta(\rho-3s)+\gamma(\rho-(3/2))-|m|-1})\qquad(h\to 0),\end{aligned}$$

where we have used Lemma 4.1 in the first step and

$$\sup_{h,y,z}|X^s\tilde{\chi}I_a\langle y\rangle^{-\rho-s}\langle z\rangle^s|<\infty$$

in the last step. The estimate $\|X^s(1-\tilde{\chi})I_a\langle z\rangle^s\langle y\rangle^{-\rho-s}\| = O(h^{\beta(\rho-2s)})$ is easily verified, and the fact that $\operatorname{supp} v(\tau)\subset\{z\in\mathbf{R}^3;|z|<(N_0+2)h^{-\beta}\}$ yields

$$\begin{aligned}(7.17)\qquad \|\langle z\rangle^s v(\tau)\|_z &\le C(h^{-\beta})^s\left(\int(h^{-\gamma}+|z-\mu_\alpha\omega\tau|)^{-2\rho}dz\right)^{1/2}\\ &= O(h^{-s\beta+\gamma(\rho-(3/2))})\qquad(h\to 0).\end{aligned}$$

Thus, using Lemmas 4.1, 5.3, (N) and (7.14), we obtain

$$\begin{aligned}&\|\tilde{\chi}X^{-s}D_z^m R(\lambda+i0;H(h))(1-\tilde{\chi})I_a R(\lambda+i0;H_a(h))v(\tau)e_\alpha\|_0\\ &= O(h^{\beta(\rho-3s)+\gamma(\rho-(3/2))-2-|m|})\qquad(h\to 0).\end{aligned}$$

This completes the proof.

Now we return to (7.2). From Lemmas 7.1, 7.2, 7.4 and (N) it follows that

$$\begin{aligned}&\|\tilde{\chi}X^{-s}R(\lambda+i0;H(h))f_l e_\alpha - ih^{-1}\tilde{\chi}X^{-s}g_0 e_\alpha\|_0\\ &= h^{\gamma(\rho-s-(3/2))-1}(O(h^{\mu_1})+O(h^{\mu_2})+O(h^{\mu_3})),\end{aligned}$$

where

$$\begin{aligned}\mu_1 &= \beta(\rho-3s)+\gamma(\rho-(3/2))-2-\gamma(\rho-s-(3/2))+1\\ &= \gamma(1-2s+\delta(\rho-3s)),\\ \mu_2 &= 2\kappa(\rho+d-1)+\beta(-s-(3/2))-1-\gamma(\rho-s-(3/2))+1\\ &= \gamma(\rho+2d-2-\delta(2\rho+2d+s-(1/2))),\end{aligned}$$

$$\mu_3 = -(\beta/2) + \kappa(2\rho + d - s - 2) - 2 - \gamma(\rho - s - (3/2)) + 1$$
$$= \gamma(d - \delta(2\rho + d - s - (3/2)))\,.$$

First taking $\delta > 0$ small and then taking $s > 1/2$ sufficiently near 1/2, we see that $\mu_2 > \mu_3 > \mu_1 > 0$. Thus it follows that

$$\|\tilde{\chi}X^{-s}R(\lambda + i0; H(h))f_l e_\alpha - ih^{-1}\tilde{\chi}X^{-s}g_0 e_\alpha\|_0 = h^{\gamma(\rho-s-(3/2))-1}O(h^{\mu_1}) \qquad (h \to 0) \tag{7.18}$$

uniformly in $(\lambda, \omega) \in J \times S^2$. Similarly, by Lemmas 5.3 and 7.4, we have

$$\|\tilde{\chi}X^{-s}D_z^m R(\lambda + i0; H(h))f_l e_\alpha - ih^{-1}\tilde{\chi}X^{-s}D_z^m g_0 e_\alpha\|_0 = h^{\gamma(\rho-s-(3/2))-2}O(h^{\mu_1}) \qquad (h \to 0) \tag{7.19}$$

uniformly in $(\lambda, \omega) \in J \times S^2$ for $|m| = 1$.

Lemma 7.5. *If $\delta > 0$ is so small, one has*

$$(R(\lambda + i0; H(h))\theta_{1l}, \chi^2 E_\alpha \theta_{1l})_0 = ih^{-1}(g_0, \chi^2 f_l)_z + o(h^{1-2\gamma})$$

as $h \to 0$ uniformly in $(\lambda, \omega) \in J \times S^2$.

Proof. We fix $s > 1/2$ sufficiently near 1/2. By (6.1), (7.18) and

$$\|X^s \chi^2 E_\alpha \theta_{1l}\|_0 = O(h^{\gamma(\rho-s-(3/2))}) \qquad (h \to 0)\,,$$

which is obtained in the same way as (5.10), we have

$$(R(\lambda + i0; H(h))\theta_{1l}, \chi^2 E_\alpha \theta_{1l})_0 = ih^{-1}(g_0, \chi^2 f_l)_z + h^{1-2\gamma}O(h^{\mu})\,,$$

where

$$\mu = 2\gamma - 1 + 2\gamma(\rho - s - (3/2)) - 1 + \mu_1 = \gamma(1 - 2s) + \mu_1\,.$$

Since $s > 1/2$ is sufficiently near 1/2, we have $\mu > 0$.

Lemma 7.6. *If $\delta > 0$ is so small, one has*

$$(B_2(h)R(\lambda + i0; H(h))\theta_{1l}, R(\lambda + i0; H(h))\theta_{1l})_0 = (B_2(h)ih^{-1}g_0 e_\alpha, ih^{-1}g_0 e_\alpha)_0 + o(h^{1-2\gamma})$$

as $h \to 0$ uniformly in $(\lambda, \omega) \in J \times S^2$.

Proof. By (6.1), (7.3) we have

$$\begin{aligned} |v(t)| &\le C(h^{-\gamma} + |u| + |x - \mu_\alpha t|)^{-\rho}\,, \\ |\nabla v(t)| &\le C(h^{-\gamma} + |u| + |x - \mu_\alpha t|)^{-\rho}(h^{\kappa d} + h^{\kappa(\rho+d-1)-1}) \\ &\le C(h^{-\gamma} + |u| + |x - \mu_\alpha t|)^{-\rho}\,, \end{aligned} \tag{7.20}$$

where we have used $\kappa d > \kappa(\rho + d - 1) - 1 = \gamma(d - \delta(\rho + d - 1)) > 0$, which follows from $0 < \delta \ll 1$. Thus it follows that

$$|D_z^m g_0| \le C(h^{-\gamma} + |u|)^{-\rho+1}\,, \qquad 0 \le |m| \le 1\,. \tag{7.21}$$

Let $\chi_4(z) \in C^\infty(\mathbf{R}^3)$ with $\chi_4 = 1$ on $\{z; h^{-\gamma} \le |z| \le 2h^{-\gamma}\}$ and $\operatorname{supp} \chi_4 \subset \{z; h^{-\gamma}/2 \le |z| \le 4h^{-\gamma}\}$. Then by (7.21) we get

$$\|h^{-1}\chi_4 X^{-s} D_z^m g_0 e_\alpha\|_0 = O(h^{-1+\gamma(\rho+s-(5/2))-|m|}), \tag{7.22}$$

as $h \to 0$ for $0 \le |m| \le 1$, $s > 1/2$. On the other hand, by (6.4) and (N), we have

$$\|X^{-s}R(\lambda + i0; H(h))\theta_{1l}\|_0 \le Ch^{-1+\gamma(\rho-s-(3/2))}.$$

Therefore, by (7.18), (7.19). (7.22) and (5.21) we can see that

$$\begin{aligned}(B_2(h)R(\lambda + i0; H(h))\theta_{1l}, R(\lambda + i0; H(h))\theta_{1l})_0 \\ = (B_2(h)ih^{-1}g_0 e_\alpha, ih^{-1}g_0 e_\alpha)_0 + h^{1-2\gamma}(O(h^\mu) + O(h^{\mu'})),\end{aligned}$$

where

$$\begin{aligned}\mu &= 2\gamma - 1 + 2 + \gamma(1 - 2s) + \gamma(\rho - s - (3/2)) - 2 + \mu_1 - 1 + \gamma(\rho - s - (3/2)) \\ &= \gamma(2 - 4s) + \mu_1, \\ \mu' &= 2\gamma - 1 + 2 + \gamma(1 - 2s) - 1 + \gamma(\rho + s - (5/2)) - 1 + \gamma(\rho - s - (3/2)) - 1 + \mu_1 \\ &= \gamma(1 - 2s) + \mu_1.\end{aligned}$$

We can take $s > 1/2$ sufficiently near $1/2$ so that $\mu > 0$, $\mu' > 0$. This proves the lemma.

8. The proof of the theorem

By Lemmas 6.1, 6.2, 7.5 and 7.6 we obtain

$$Q_1(\lambda, \omega; h) = h^{-1} \operatorname{Re}(g_0, \chi^2 f_l)_z + o(h^{1-2\gamma}), \tag{8.1}$$

$$\begin{aligned}Q_2(\lambda, \omega; h) &= \frac{1}{n_a i}(\chi\nabla_z\chi \cdot \nabla_z g_0 e_\alpha, g_0 e_\alpha)_0 + o(h^{1-2\gamma}) \\ &= \frac{\mu_\alpha}{h}(\chi(\omega \cdot \nabla_z\chi)g_0, g_0)_z + \frac{1}{n_a i}(\chi\nabla_z\chi \cdot \nabla_z g_0, g_0)_z + o(h^{1-2\gamma})\end{aligned} \tag{8.2}$$

as $h \to 0$ uniformly in $(\lambda, \omega) \in J \times S^2$.

Lemma 8.1.

$$(\chi\nabla_z\chi \cdot \nabla_z g_0, g_0)_z = o(h^{1-2\gamma})$$

uniformly in $(\lambda, \omega) \in J \times S^2$.

Proof. Since $|\nabla_z\chi| \le C(h^{-\gamma} + |z|)^{-1}$ and $\operatorname{supp} \nabla_z\chi \subset \{z = u + x\omega; |x| < 2h^{-\gamma}\}$, we have by (7.21)

$$|(\chi\nabla_z\chi \cdot \nabla_z g_0, g_0)_z| \le C\int (h^{-\gamma} + |u|)^{-2\rho+2} du = O(h^{2-2\gamma}).$$

Lemma 8.2.

$$h^{-1}\mu_a(\chi(\omega\cdot\nabla_z\chi)g_0, g_0)_z = h^{-1}\operatorname{Re}(g_0, (1-\chi^2)f_l)_z .$$

Proof. Since $v(t)$ satisfies

$$i\partial_t v(t) + i\mu_a\omega\cdot\nabla_z v(t) - h^{-1}I_a^0 v(t) = 0 , \tag{8.3}$$

we see that

$$ih^{-1}\mu_a\omega\cdot\nabla_z g_0 = -ih^{-1}(v(\tau)-v(0)) + h^{-2}I_a^0 g_0 . \tag{8.4}$$

Thus, an integration by parts yields

$$\begin{aligned} h^{-1}\mu_a(\chi(\omega\cdot\nabla_z\chi)g_0, g_0)_z &= (2h)^{-1}\mu_a((\omega\cdot\nabla_z(-1+\chi^2))g_0, g_0)_z \\ &= \operatorname{Re}((1-\chi^2)h^{-1}\mu_a(\omega\cdot\nabla_z g_0), g_0)_z \\ &= \operatorname{Re}((1-\chi^2)[h^{-1}(v(0)-v(\tau)) - ih^{-2}I_a^0 g_0], g_0)_z \\ &= h^{-1}\operatorname{Re}((1-\chi^2)(f_l - v(\tau)), g_0)_z \\ &= h^{-1}\operatorname{Re}(g_0, (1-\chi^2)f_l)_z . \end{aligned}$$

Here we have used the fact that $\operatorname{supp}(1-\chi^2)\cap\operatorname{supp} v(\tau) = \phi$, which follows from $N_0 \gg 1$, in the last step. This completes the proof.

Consequently, we obtain by Lemmas 8.1 and 8.2

$$Q_1(\lambda, \omega; h) + Q_2(\lambda, \omega; h) = h^{-1}\operatorname{Re}(g_0, f_l)_z + o(h^{1-2\gamma})$$

as $h \to 0$ uniformly in $(\lambda, \omega) \in J \times S^2$. Since N_0 is large enough and $\mu_a \geq 2\sqrt{E_0/n_a}$ by (3.4) and (3.5), we observe that $\operatorname{supp} v(t)\cap\operatorname{supp} f_l = \phi$ for $t \geq \tau$, and that

$$(g_0, f_l)_z = \int_0^\infty (v(t), f_l)_z dt .$$

Hence, by the following lemma the proof of Theorem 3.1 is accomplished.

Lemma 8.3.

$$2h^{-2}\mu_a^{-1}\int_0^\infty \operatorname{Re}(v(t), f_l)_z\, dt = \sigma_a^0(\lambda, \omega; h) + o(h^{-2\gamma})$$

as $h \to 0$ *uniformly in* $(\lambda, \omega) \in J \times S^2$.

For the proof, see the proof of Lemma 7.1 in [IT1].

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