J. Math. Kyoto Univ. (JMKYAZ)
33-4 (1993) 1125-1142

Generating functions and integral representations for the spherical functions on some classical Gelfand pairs

By

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Introduction

Let $\mathbf{F}$ be $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ and $a \mapsto \bar{a}$ the usual conjugation in $\mathbf{F}$. We define the following quadratic form in $\mathbf{F}^{n+1}$.

$$(x, y)_- = -\bar{x}_0 y_0 + \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n .$$

Let $U(1, n; \mathbf{F})$ be the group of the linear transformations g in $\mathbf{F}^{n+1}$ which satisfy $(gx, gy)_- = (x, y)_-$ for all x, $y \in \mathbf{F}^{n+1}$. We define the group G as follows.

1. If $\mathbf{F} = \mathbf{R}$, G is the connected component of the unit element in $U(1, n; \mathbf{R})$, i.e. $G = SO_0(1, n)$.
2. If $\mathbf{F} = \mathbf{C}$, G is the group of all the elements $g \in U(1, n; \mathbf{C})$ of determinant one, i.e. $G = SU(1, n)$.
3. If $\mathbf{F} = \mathbf{H}$, $G = U(1, n; \mathbf{H})$, i.e. $G = Sp(1, n)$.

Let $B(\mathbf{F}^n)$ be the unit ball in $\mathbf{F}^n$ and $S(\mathbf{F}^n)$ be the unit sphere in $\mathbf{F}^n$. The group G acts transitively on $B(\mathbf{F}^n)$ and $S(\mathbf{F}^n)$ as follows: for $\xi = {}^t(\xi_1, \dots, \xi_n) \in \mathbf{F}^n$ and $g = (g_{pq})_{0 \le p,q \le n} \in G$, we define

$$\xi' = g\xi ,$$

where $\xi' = {}^t(\xi'_1, \dots, \xi'_n)$, with

$$\xi'_p = \left(g_{p0} + \sum_{q=1}^{n} g_{pq}\xi_q\right)\left(g_{00} + \sum_{q=1}^{n} g_{0q}\xi_q\right)^{-1}, \qquad 1 \le p \le n .$$

Let K be the isotropy group of $O \in B(\mathbf{F}^n)$ in G. Then K is a maximal compact subgroup of G and $G/K \cong B(\mathbf{F}^n)$. Let $G = KAN$ be the corresponding Iwasawa decomposition and M be the centralizer of A in K. Then M is the isotropy group of $e_1 = {}^t(1, 0, \dots, 0) \in S(\mathbf{F}^n)$ in K and $K/M \cong S(\mathbf{F}^n)$ is the Martin boundary on $G/K \cong B(\mathbf{F}^n)$. Except for the case of real numbers, K/M is not a symmetric space, but it is known that (K, M) is a Gelfand pair, i.e. the convolution algebra of functions on K bi-invariant by M is commutative. As is well known, the spherical functions on K/M play an important role in the harmonic analysis on G/K.

Received November 21, 1992

$$K \cong \begin{cases} SO(n) & \mathbf{F}=\mathbf{R} \\ U(n) & \mathbf{F}=\mathbf{C} \\ Sp(1)\times Sp(n) & \mathbf{F}=\mathbf{H} \end{cases}$$

$$M \cong \begin{cases} SO(n-1) & \mathbf{F}=\mathbf{R} \\ U(n-1) & \mathbf{F}=\mathbf{C} \\ Sp(1)\times Sp(n-1) & \mathbf{F}=\mathbf{H} \end{cases}$$

Let φ be a zonal spherical function of the real case $SO(n)/SO(n-1) \cong S(\mathbf{R}^n)$. Then φ depends only on η_1 ($\eta = {}^t(\eta_1, \dots, \eta_n) \in S(\mathbf{R}^n)$) and there exists a unique nonnegative integer p such that

$$\varphi(\eta) = C_p^{(n-2)/2}(\eta_1)/C_p^{(n-2)/2}(1), \qquad \eta = {}^t(\eta_1, \dots, \eta_n) \in S(\mathbf{R}^n),$$

where $C_p^{(n-2)/2}$ is the Gegenbauer polynomial. It is well known that a generating function for the Gegenbauer polynomials $C_p^{(n-2)/2}$, $p = 0, 1, 2, \dots$, is given as follows.

$$(1 - 2tz + t^2)^{-(n-2)/2} = \sum_{p=0}^{\infty} C_p^{(n-2)/2}(z)t^p, \qquad -1 \le z \le 1, \qquad -1 < t < 1.$$

This formula also gives a generating function for the zonal spherical functions of $SO(n)/SO(n-1)$.

The first purpose of this paper is to show that we can also give generating functions for the zonal spherical functions in the complex and the quaternion cases (The conclusions were announced in [6]). In case of $\mathbf{F} = \mathbf{C}, \mathbf{H}$, however, the zonal spherical functions on K/M are determined by two parameters while those of the real case are determined by one parameter p. So we have to extend the definition of generating function to the case of a system of functions which has two parameters. If $\mathbf{F} = \mathbf{C}$, we consider the function $F(z, w)$ defined in the following as a generating function for functions $G_{pq}(z)$ ($z \in D$, D is a subset of $\mathbf{C}$).

$$F(z, w) = \sum_{p,q=0}^{\infty} G_{pq}(z) w^p \bar{w}^q,$$

where the right hand side absolutely converges for some $w \in \mathbf{C}$, $w \neq 0$. Then we see that

$$G_{pq}(z) = \frac{1}{p!q!}\left[\frac{\partial^{p+q}}{\partial w^p \partial \bar{w}^q} F(z, w)\right]_{w=0}.$$

This is one reason why we consider the function $F(z, w)$ as a generating function. In the generating function expansion, if we put $w = re^{i\theta}$, then we have

$$F(z, re^{i\theta}) = \sum_{p,q=0}^{\infty} G_{pq}(z) r^{p+q} e^{i(p-q)\theta}.$$

We now remark that the function $e^{ik\theta}$ (k is an integer) is a spherical function on $U(1) \cong S(\mathbf{R}^2)$. Therefore the expansion of $F(z, re^{i\theta})$ by the powers of r and the spherical functions of $U(1) \cong S(\mathbf{R}^2)$ can be considered to give a generating function expansion for the functions G_{pq}. This interpretation for generating function will be adapted to the quaternion case, i.e. let $E(z, w)$ be a function defined on a subset of $\mathbf{H} \times \mathbf{H}$ and we set $w = ru$ $(r > 0, u \in Sp(1))$. When we expand the function $E(z, ru)$ by the powers of r and the spherical functions of $Sp(1) \cong S(\mathbf{R}^4) \cong (Sp(1) \times Sp(1))/\Delta(Sp(1) \times Sp(1))$, $E(z, ru)$ is considered as a generating function for the functions of z which appear as the coefficients in that expansion.

The second purpose of this paper is to give integral representations for the zonal spherical functions on K/M in the complex and the quaternion cases. In particular, those in the complex case will give formulas which are analogous to Rodrigues' formulas.

Suppose that $n \geq 2$ throughout this paper.

1. Complex case

1.1. Generating function for the spherical functions on K/M. Let $H^{(n)}_{p,q}$ denote the space of restrictions to $S(\mathbf{C}^n)$ of harmonic polynomials $f(\xi, \bar{\xi})$ on $\mathbf{C}^n$ which are homogeneous of degree p in ξ and degree q in $\bar{\xi}$. Then it is known that (cf. [2], [4]) $H^{(n)}_{p,q}$ is $U(n)$-irreducible and moreover $L^2(S(\mathbf{C}^n)) = \bigoplus_{p,q=0}^{\infty} H^{(n)}_{p,q}$. Let $\varphi^{(n)}_{p,q}$ be the zonal spherical function which belongs to $H^{(n)}_{p,q}$. Then a generating function for the functions $\varphi^{(n)}_{p,q}$ is given in the following theorem.

Theorem 1.1. *If $w, z \in \mathbf{C}$, $|w| < 1$, $|z| \leq 1$, then*

$$(1 - 2\,\mathrm{Re}\,(wz) + |w|^2)^{1-n} = \sum_{p,q=0}^{\infty} a^{(n)}_{pq} Q^{(n)}_{pq}(z) w^p \bar{w}^q\,, \tag{1.1}$$

where

$$Q^{(n)}_{pq}(\eta_1) = \varphi^{(n)}_{pq}(\eta)\,, \qquad \eta \in S(\mathbf{C}^n)\,,$$

and

$$a^{(n)}_{pq} = \frac{\Gamma(n+p-1)}{\Gamma(n-1)\Gamma(p+1)} \frac{\Gamma(n+q-1)}{\Gamma(n-1)\Gamma(q+1)}\,.$$

The series on the right hand side converges absolutely and uniformly for $|z| \leq 1$ and $|w| \leq \rho$ for each $\rho < 1$.

Proof. The function f *defined by*

$$f(\xi) = \|\xi - e_1\|^{2-2n}\,,$$

is harmonic and M-invariant in $B(\mathbf{C}^n)$. Thus we have the following expansion which uniformly converges on every compact subset of $B(\mathbf{C}^n)$,

$$f(\xi) = \sum_{\ell=0}^{\infty} h_\ell(\xi)\,,$$

where the function h_ℓ is a harmonic polynomial on $\mathbf{R}^{2n}$ which is homogeneous of degree ℓ and M-invariant on $B(\mathbf{C}^n)$. On the other hand, the function h_ℓ is expressed on $B(\mathbf{C}^n)$ as follows,

$$h_\ell(\xi) = \sum_{p+q=\ell} h_{pq}(\xi, \bar{\xi}),$$

where the function h_{pq} is a harmonic polynomial on $\mathbf{C}^n$ which is homogeneous of degree p in ξ and degree q in $\bar{\xi}$.

For $\eta \in S(\mathbf{C}^n)$ and $w \in \mathbf{C}$, $|w| < 1$, if we put $\xi = w\eta$, then we obtain

$$\begin{aligned} f(w\eta) &= \sum_{\ell=0}^{\infty} h_\ell(w\eta) \\ &= \sum_{\ell=0}^{\infty} \sum_{p+q=\ell} h_{pq}(w\eta, \overline{w\eta}) \\ &= \sum_{\ell=0}^{\infty} \sum_{p+q=\ell} w^p \bar{w}^q h_{pq}(\eta, \bar{\eta}). \end{aligned}$$

The function $f(w\eta)$ is M-invariant as a function of $\eta \in S(\mathbf{C}^n)$, and so are the functions $h_{pq}(\eta, \bar{\eta})$. Thus there exist constants $a_{pq}^{(n)} \in \mathbf{C}$ such that

$$h_{pq}(\eta, \bar{\eta}) = a_{pq}^{(n)} \varphi_{pq}^{(n)}(\eta), \qquad \text{for all } \eta \in S(\mathbf{C}^n).$$

Therefore we have

$$f(w\eta) = \sum_{\ell=0}^{\infty} \left(\sum_{p+q=\ell} a_{pq}^{(n)} \varphi_{pq}^{(n)}(\eta) w^p \bar{w}^q \right).$$

Putting $\eta = e_1$, we can determine the constants $a_{pq}^{(n)}$.

$$|1 - w|^{2-2n} = \sum_{\ell=0}^{\infty} \left(\sum_{p+q=\ell} a_{pq}^{(n)} w^p \bar{w}^q \right),$$

so we see that

$$a_{\alpha\beta}^{(n)} = \frac{1}{\alpha!\beta!} \left[\frac{\partial^\alpha}{\partial w^\alpha} \frac{\partial^\beta}{\partial \bar{w}^\beta} |1 - w|^{2-2n} \right]_{w=0}.$$

On the other hand,

$$\frac{\partial^\alpha}{\partial w^\alpha} \frac{\partial^\beta}{\partial \bar{w}^\beta} |1 - w|^{2-2n} = \frac{\partial^\alpha}{\partial w^\alpha} (1 - w)^{1-n} \frac{\partial^\beta}{\partial \bar{w}^\beta} (1 - \bar{w})^{1-n}.$$

Thus we obtain

$$a_{\alpha\beta}^{(n)} = \frac{\Gamma(n + \alpha - 1)}{\Gamma(n - 1)\Gamma(\alpha + 1)} \frac{\Gamma(n + \beta - 1)}{\Gamma(n - 1)\Gamma(\beta + 1)}.$$

If we put $\eta = {}^t(z, 0, \dots, 0, \sqrt{1 - |z|^2}) \in S(\mathbf{C}^n)$ for a fixed $z \in \mathbf{C}$, $|z| \le 1$, we have the following:

$$(1 - 2\,\mathrm{Re}\,(wz) + |w|^2)^{1-n} = \sum_{\ell=0}^{\infty} \left(\sum_{p+q=\ell} a_{pq}^{(n)} Q_{pq}^{(n)}(z) w^p \overline{w}^q \right).$$

Finally, it follows from the definitions of $\varphi_{pq}^{(n)}$ that $|Q_{pq}^{(n)}(z)| \le 1$ for $|z| \le 1$, which implies the second assertion.

In the formula (1.1), if we put $w = re^{i\theta}$, then we have

$$(1 - 2r\,\mathrm{Re}\,(e^{i\theta}z) + r^2)^{1-n} = \sum_{p,q=0}^{\infty} a_{pq}^{(n)} Q_{pq}^{(n)}(z) e^{i(p-q)\theta} r^{p+q}. \tag{1.2}$$

This formula can be interpreted as follows. The zonal spherical functions $\varphi_{pq}^{(n)}$ appear as the coefficients in the expansion of the left hand side of (1.2) by the powers of r and the spherical functions of $U(1) \cong S(\mathbf{R}^2)$. This interpretation for generating function will be adapted to the quaternion case. See the formula (2.1).

1.2. Generalization of the formula (1.1). Let ν be a positive number. Suppose that $z, w \in \mathbf{C}$ and $|w|^2 + 2|zw| < 1$. Then we have the following expansion which absolutely converges:

$$(1 - 2\,\mathrm{Re}\,(wz) + |w|^2)^{-\nu} = \sum_{p,q=0}^{\infty} G_{pq}^{\nu}(z) w^p \overline{w}^q,$$

where the functions G_{pq}^{ν} are polynomials of z and $\bar{z}$.

Lemma 1.1. *The function G_{pq}^{ν} has the following expression*:

$$G_{pq}^{\nu}(z) = \sum_{k=0}^{\min(p,q)} \frac{\Gamma(\nu + p + q - k)}{(p-k)!(q-k)!k!\Gamma(\nu)} (-1)^k z^{p-k} \bar{z}^{q-k}.$$

Proof. We denote $(1 - wz - \overline{wz} + |w|^2)$ by $\alpha(z, w)$ and $(\alpha(z, w))^{-\nu}$ by $A(z, w)$. It is easy to see that

$$G_{pq}^{\nu}(z) = \frac{1}{p!q!} \left[\frac{\partial^{p+q}}{\partial \overline{w}^q \partial w^p} A(z, w) \right]_{w=0}.$$

By the way

$$\frac{\partial^p}{\partial w^p} A(z, w) = \nu(\nu+1)\dots(\nu+p-1)(z - \overline{w})^p (\alpha(z, w))^{-\nu-p},$$

and

$$\frac{\partial^{p+q}}{\partial \overline{w}^q \partial w^p} A(z, w) = \nu(\nu+1)\dots(\nu+p-1) \sum_{k=0}^{q} \binom{q}{k} \frac{\partial^k}{\partial \overline{w}^k} (z - \overline{w})^p \frac{\partial^{q-k}}{\partial \overline{w}^{q-k}} (\alpha(z, w))^{-\nu-p}$$

$$= \sum_{k=0}^{\min(p,q)} A_{p,q,k} (z - \overline{w})^{p-k} (\bar{z} - w)^{q-k} (\alpha(z, w))^{-\nu-p-q+k},$$

where

$$A_{p,q,k} = (-1)^k \binom{q}{k} \frac{p!}{(p-k)!} \frac{\Gamma(\nu + p + q - k)}{\Gamma(\nu)}.$$

Therefore

$$\left[\frac{\partial^{p+q}}{\partial\bar{w}^q\partial w^p}A(z,w)\right]_{w=0}=\sum_{k=0}^{\min(p,q)}\frac{p!q!\Gamma(\nu+p+q-k)}{(p-k)!(q-k)!k!\Gamma(\nu)}(-1)^k z^{p-k}\bar{z}^{q-k},$$

which implies our assertion.

Lemma 1.2. *Suppose that $p\geq q$. Then for $-1\leq x\leq 1$ we have*

$$G_{pq}^{\nu}(x)=\frac{(-1)^q\Gamma(\nu+p)}{\Gamma(\nu)\Gamma(p-q+1)q!}x^{p-q}{}_2F_1(-q,\nu+p;p-q+1;x^2).$$

Proof. From Lemma 1.1, we see that

$$\begin{aligned}G_{pq}^{\nu}(x)&=\frac{1}{\Gamma(\nu)}\sum_{k=0}^{q}\frac{\Gamma(\nu+p+q-k)}{(p-k)!(q-k)!k!}(-1)^k x^{p+q-2k}\\&=\frac{1}{\Gamma(\nu)}\sum_{\ell=0}^{q}\frac{\Gamma(\nu+p+\ell)}{(p-q+\ell)!\ell!(q-\ell)!}(-1)^{q-\ell}x^{p-q+2\ell}\\&=\frac{(-1)^q x^{p-q}}{\Gamma(\nu)}\sum_{\ell=0}^{q}\frac{\Gamma(\nu+p+\ell)}{\Gamma(p-q+\ell+1)\Gamma(q-\ell+1)\ell!}(-1)^\ell x^{2\ell}\\&=\frac{(-1)^q x^{p-q}}{\Gamma(\nu)}\sum_{\ell=0}^{q}\frac{\Gamma(\nu+p+\ell)(-q)_\ell}{\Gamma(p-q+\ell+1)q!\ell!}x^{2\ell},\end{aligned}$$

which implies the assertion.

We remark that

1. $G_{pq}^{\nu}(e^{i\theta}z)=e^{i(p-q)\theta}G_{pq}^{\nu}(z)$ for $\theta\in\mathbf{R}$ and $z\in\mathbf{C}$, $|z|\leq 1$.
2. $G_{pq}^{\nu}(x)=G_{qp}^{\nu}(x)$ for $-1\leq x\leq 1$.

Proposition 1.1. *The functions G_{pq}^{ν} have the following orthogonality relation:*

$$\int_{|z|\leq 1}G_{pq}^{\nu}(z)\overline{G_{p'q'}^{\nu}(z)}(1-|z|^2)^{\nu-1}dxdy=\delta_{pp'}\delta_{qq'}\frac{\pi\Gamma(p+\nu)\Gamma(q+\nu)}{(p+q+\nu)p!q![\Gamma(\nu)]^2},$$

with $z=x+iy$.

Proof. First of all,

$$\begin{aligned}&\int_{|z|\leq 1}G_{pq}^{\nu}(z)\overline{G_{p'q'}^{\nu}(z)}(1-|z|^2)^{\nu-1}dxdy\\&\qquad=\int_0^{2\pi}e^{i(p-q)\theta}e^{-i(p'-q')\theta}d\theta\int_0^1 G_{pq}^{\nu}(r)G_{p'q'}^{\nu}(r)r(1-r^2)^{\nu-1}dr.\end{aligned}$$

This integral is obviously equal to zero when $p-q\neq p'-q'$.

We now suppose that $p-q=p'-q'$ and set $m=p-q=p'-q'$, $\alpha=m+\nu$ and $\gamma=m+1$. If $m\geq 0$, by Lemma 1.2, we have

$$G^{\nu}_{pq}(r) = \frac{(-1)^q \Gamma(\nu + p)}{\Gamma(\nu)\Gamma(\gamma)q!} r^{\gamma-1} G_q(\alpha, \gamma; r^2),$$

$$G^{\nu}_{p'q'}(r) = \frac{(-1)^{q'} \Gamma(\nu + p')}{\Gamma(\nu)\Gamma(\gamma)q'!} r^{\gamma-1} G_{q'}(\alpha, \gamma; r^2),$$

where the functions $G_q(\alpha, \gamma; \xi)$ are the Jacobi polynomials defined by

$$\begin{aligned} G_q(\alpha, \gamma; \xi) &= {}_2F_1(-q, \alpha + q; \gamma; \xi) \\ &= \frac{\Gamma(\gamma)\xi^{1-\gamma}(1-\xi)^{\gamma-\alpha}}{\Gamma(\gamma + q)} \frac{d^q}{d\xi^q}[\xi^{\gamma+q-1}(1-\xi)^{\alpha+q-\gamma}], \end{aligned}$$

and have the following orthogonality relation:

$$\begin{aligned} &\int_0^1 \xi^{\gamma-1}(1-\xi)^{\alpha-\gamma} G_q(\alpha, \gamma; \xi) G_{q'}(\alpha, \gamma; \xi) d\xi \\ &= \delta_{qq'} \frac{q!\Gamma(q + \alpha - \gamma + 1)[\Gamma(\gamma)]^2}{(\alpha + 2q)\Gamma(\alpha + q)\Gamma(\gamma + q)}. \end{aligned}$$

Thus we see that

$$\begin{aligned} \int_0^1 G^{\nu}_{pq}(r) G^{\nu}_{p'q'}(r) r(1-r^2)^{\nu-1} dr &= \frac{(-1)^{q+q'}\Gamma(\nu+p)\Gamma(\nu+p')}{[\Gamma(\nu)]^2[\Gamma(\gamma)]^2 q!q'!} \\ &\quad \times \int_0^1 G_q(\alpha, \gamma; r^2) G_{q'}(\alpha, \gamma; r^2) r^{2\gamma-1}(1-r^2)^{\nu-1} dr \\ &= \delta_{qq'} \frac{\Gamma(p+\nu)\Gamma(q+\nu)}{2(p+q+\nu)p!q![\Gamma(\nu)]^2}. \end{aligned}$$

This indicates that our assertion is true for $m \geq 0$. For $m < 0$, our assertion is proved by the formula $G^{\nu}_{pq}(r) = G^{\nu}_{qp}(r)$.

1.3. Integral representations and Rodrigues' formulas for the functions $Q^{(n)}_{pq}$. Let ν be a positive number. For $z, \xi, \eta \in \mathbf{C}$, we define the function $F_\nu(\xi, \eta, z)$ by

$$F_\nu(\xi, \eta, z) = (1 - \xi z - \eta \bar{z} + \xi\eta)^{-\nu}.$$

If $|z| \leq 1$, then we have

$$|\xi z + \eta \bar{z} - \xi\eta| \leq (|\xi| + 1)(|\eta| + 1) - 1.$$

Thus for a fixed $z \in \mathbf{C}$, $|z| \leq 1$, the function $F_\nu(\xi, \eta, z)$ is holomorphic on $U = \{(\xi, \eta) \in \mathbf{C}^2; |\xi| < 1/3, |\eta| < 1/3\}$ with respect to ξ, η and has the following Taylor expansion on U:

$$F_\nu(\xi, \eta, z) = \sum_{p,q=0}^{\infty} F^{\nu}_{pq}(z)\xi^p\eta^q,$$

where

$$F_{pq}^{\nu}(z)=\frac{1}{p!q!}\left[\frac{\partial^{p+q}}{\partial\xi^p\partial\eta^q}F_\nu(\xi,\eta,z)\right]_{\xi=\eta=0}.$$

On the other hand, from the proof of Lemma 1.1, we see that

$$\left[\frac{\partial^{p+q}}{\partial\xi^p\partial\eta^q}F_\nu(\xi,\eta,z)\right]_{\xi=\eta=0}=\left[\frac{\partial^{p+q}}{\partial\bar{w}^q\partial w^p}A(z,w)\right]_{w=0},$$

with $A(z,w)=(1-wz-\overline{wz}+|w|^2)^{-\nu}$. So we obtain that

$$F_{pq}^{\nu}(z)=G_{pq}^{\nu}(z).$$

And from the Cauchy integral formula, if $0<r_1,\ r_2<1/3$, we have

$$\begin{aligned}F_{pq}^{\nu}(z)&=\frac{1}{(2\pi i)^2}\int_{|\xi|=r_1}\int_{|\eta|=r_2}\frac{F_\nu(\xi,\eta,z)}{\xi^{p+1}\eta^{q+1}}d\xi d\eta\\&=\frac{(\nu)_p}{2\pi ip!}\int_{|\eta|=r_2}\frac{(z-\eta)^p(1-\eta\bar{z})^{-\nu-p}}{\eta^{q+1}}d\eta.\end{aligned}$$

We suppose that $\nu=n-1$. Then we have

$$F_{pq}^{n-1}(z)=G_{pq}^{n-1}(z)=a_{pq}^{(n)}Q_{pq}^{(n)}(z),$$

which implies

$$a_{pq}^{(n)}Q_{pq}^{(n)}(z)=\frac{(n-1)_p}{2\pi ip!}\int_{|\eta|=r_2}\frac{(z-\eta)^p}{\eta^{q+1}(1-\eta\bar{z})^{n+p-1}}d\eta. \tag{1.3}$$

Let $|z|<1$. If we make the substitution

$$\zeta=\frac{z-\eta}{1-\eta\bar{z}},$$

then (1.3) transforms into the following integral:

$$a_{pq}^{(n)}Q_{pq}^{(n)}(z)=\frac{(n-1)_p}{2\pi ip!}(|z|^2-1)^{2-n}\int_{|\zeta-z|=r}\frac{\zeta^p(\zeta\bar{z}-1)^{n+q-2}}{(\zeta-z)^{q+1}}d\zeta, \tag{1.4}$$

with $r>0$. The formula (1.4) gives an integral representation for $Q_{pq}^{(n)}$. And moreover, from (1.4), we see that

$$\begin{aligned}a_{pq}^{(n)}Q_{pq}^{(n)}(z)&=\frac{(n-1)_p}{p!q!}(|z|^2-1)^{2-n}\left[\frac{d^q}{d\zeta^q}[\zeta^p(\zeta\bar{z}-1)^{n+q-2}]\right]_{\zeta=z}\\&=\frac{(n-1)_p}{p!q!}(|z|^2-1)^{2-n}\frac{\partial^q}{\partial z^q}[z^p(z\bar{z}-1)^{n+q-2}],\end{aligned}$$

where

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\qquad z=x+iy.$$

This formula gives Rodrigues' formula for $Q_{pq}^{(n)}$.

Theorem 1.2. *The function $Q_{pq}^{(n)}$ has the following integral representation and Rodrigues' formula.*

$$a_{pq}^{(n)}Q_{pq}^{(n)}(z) = \frac{(n-1)_p}{2\pi i p!}(|z|^2-1)^{2-n}\int_{|\zeta-z|=r}\frac{\zeta^p(\zeta\bar{z}-1)^{n+q-2}}{(\zeta-z)^{q+1}}d\zeta$$
$$= \frac{(n-1)_p}{p!q!}(|z|^2-1)^{2-n}\frac{\partial^q}{\partial z^q}[z^p(|z|^2-1)^{n+q-2}],$$

where $|z|<1$.

2. Quaternion case

2.1. Generating function for the spherical functions on K/M. A zonal spherical function φ of K/M depends only on η_1, more precisely on $\mathrm{Re}(\eta_1)$ and $|\eta_1|$ ($\eta = {}^t(\eta_1, \ldots, \eta_n) \in S(\mathbf{H}^n)$), and there uniquely exists a pair of nonnegative integers (p, q) such that

$$\varphi(\eta) = c_{pq}C_p^1\left(\frac{\mathrm{Re}(\eta_1)}{|\eta_1|}\right)|\eta_1|^p{}_2F_1(-q, p+q+2n-1; p+2; |\eta_1|^2),$$

where

$$c_{pq} = \frac{(-1)^q(p+2)_q}{(2(n-1))_q}[C_p^1(1)]^{-1}.$$

See Theorem 3.1 in [4], p. 144 and the formula (16) in [1], p. 170 and we follow the notations in [5]. From now on, we denote φ by $\varphi_{pq}^{(n)}$. When we denote $\{f*\varphi_{pq}^{(n)} | f \in L^2(K/M)\}$ by $H_{p,q}^{(n)}$, $H_{p,q}^{(n)}$ is K-irreducible and moreover $H_k^{(4n)} \cong \bigoplus_{p+2q=k} H_{p,q}^{(n)}$, $L^2(S(\mathbf{H}^n)) = \bigoplus_{p,q=0}^{\infty} H_{p,q}^{(n)}$, where $H_k^{(4n)}$ is the space of restrictions to $S(\mathbf{R}^{4n})$ of harmonic polynomials on $\mathbf{R}^{4n}$ which are homogeneous of degree k. A generating function for the functions $\varphi_{pq}^{(n)}$ is given in the following theorem.

Theorem 2.1. *If $z \in \mathbf{H}$, $|z| \leq 1$, $u \in Sp(1)$ and $0 \leq r < 1$, then*

$$\int_{Sp(1)}[1-2r\,\mathrm{Re}(vzv^{-1}u)+r^2]^{1-2n}dv = \sum_{p,q=0}^{\infty}\beta_{pq}^{(n)}R_{pq}^{(n)}(z)C_p^1(\mathrm{Re}(u))r^{p+2q}, \quad (2.1)$$

where dv is the normalized Haar measure on $Sp(1)$ and

$$R_{pq}^{(n)}(\eta_1) = \varphi_{pq}^{(n)}(\eta), \qquad \eta \in S(\mathbf{H}^n),$$

and

$$\beta_{pq}^{(n)} = \frac{p+1}{\Gamma(p+q+2)}\frac{(2n-1)_{p+q}(2n-2)_q}{q!}.$$

The series on the right hand side converges absolutely and uniformly for $|z| \leq 1$, $u \in Sp(1)$ and $r \leq \rho$ for each $\rho < 1$.

Proof. The function f defined by

$$f(\xi) = \|\xi - e_1\|^{2-4n},$$

is harmonic in $B(\mathbf{H}^n)$. Therefore we have the following expansion which uniformly converges on every compact subset of $B(\mathbf{H}^n)$,

$$f(\xi) = \sum_{\ell=0}^{\infty} h_\ell(\xi),$$

where the function h_ℓ is a harmonic polynomial on $\mathbf{R}^{4n}$ which is homogeneous of degree ℓ.

For $\eta \in S(\mathbf{H}^n)$, $u \in Sp(1)$ and $0 \leq r < 1$, if we set $\xi = r\eta u$, then we obtain that

$$f(r\eta u) = \sum_{\ell=0}^{\infty} r^\ell h_\ell(\eta u).$$

First of all, we put $u = 1$,

$$f(r\eta) = \sum_{\ell=0}^{\infty} r^\ell h_\ell(\eta).$$

The function $\eta \mapsto f(r\eta)$ is M-invariant, and so are the functions $\eta \mapsto h_\ell(\eta)$. Thus there exist constants $\alpha_{pq}^{(n)} \in \mathbf{R}$ such that

$$h_\ell(\eta) = \sum_{p+2q=\ell} \alpha_{pq}^{(n)} \varphi_{pq}^{(n)}(\eta), \qquad \text{for all } \eta \in S(\mathbf{H}^n).$$

So we see that

$$f(r\eta u) = \sum_{\ell=0}^{\infty} r^\ell \sum_{p+2q=\ell} \alpha_{pq}^{(n)} \varphi_{pq}^{(n)}(\eta u). \tag{2.2}$$

We now determine the coefficients $\alpha_{pq}^{(n)}$ using the following formula. See [3].

If $\nu > \lambda > 0$, then we have

$$C_\ell^\nu(t) = \sum_{q=0}^{[\ell/2]} \gamma_q^{(\ell)}(\nu, \lambda) C_{\ell-2q}^\lambda(t),$$

where

$$\gamma_q^{(\ell)}(\nu, \lambda) = \frac{(\lambda + \ell - 2q)}{(\lambda + \ell - q)} \frac{(\nu)_{\ell-q}}{(\lambda)_{\ell-q}} \frac{(\nu - \lambda)_q}{q!},$$

Putting $\eta = e_1$ in (2.2), we obtain

$$\begin{aligned}(1 - 2r \operatorname{Re}(u) + r^2)^{1-2n} &= \sum_{\ell=0}^{\infty} r^\ell \sum_{p+2q=\ell} \alpha_{pq}^{(n)} [C_p^1(1)]^{-1} C_p^1(\operatorname{Re}(u)) \\ &= \sum_{\ell=0}^{\infty} r^\ell \sum_{p+2q=\ell} \beta_{pq}^{(n)} C_p^1(\operatorname{Re}(u)),\end{aligned}$$

with $\beta_{pq}^{(n)} = \alpha_{pq}^{(n)}[C_p^1(1)]^{-1}$. On the other hand, we have

$$(1 - 2r\,\mathrm{Re}\,(u) + r^2)^{1-2n} = \sum_{\ell=0}^{\infty} C_\ell^{2n-1}(\mathrm{Re}\,(u))r^\ell\,.$$

So we can conclude that

$$C_\ell^{2n-1}(\mathrm{Re}\,(u)) = \sum_{p+2q=\ell} \beta_{pq}^{(n)} C_p^1(\mathrm{Re}\,(u))\,,$$

and moreover

$$\beta_{pq}^{(n)} = \frac{p+1}{\Gamma(p+q+2)} \frac{(2n-1)_{p+q}(2n-2)_q}{q!}\,.$$

Next we think of the integral

$$\int_M \varphi_{pq}^{(n)}((m\eta)u)dm\,.$$

Put

$$k' = \left[\begin{array}{cc|c} u^{-1} & 0 & \multirow{2}{*}{0} \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & 1_{n-1} \end{array}\right] \in K\,,$$

and $\eta = ke_1$, $k \in K$. Then $(m\eta)u = k'(m\eta) = k'mke_1$. By the function equation for $\varphi_{pq}^{(n)}$, we obtain that

$$\begin{aligned}\int_M \varphi_{pq}^{(n)}((m\eta)u)dm &= \int_M \varphi_{pq}^{(n)}(k'mke_1)dm \\ &= \varphi_{pq}^{(n)}(k'e_1)\varphi_{pq}^{(n)}(\eta) \\ &= C_p^1(\mathrm{Re}\,(u))[C_p^1(1)]^{-1}\varphi_{pq}^{(n)}(\eta)\,.\end{aligned}$$

Taking the average of (2.2) over M about η, we see that

$$\int_M f(r(m\eta)u)dm = \sum_{\ell=0}^{\infty} r^\ell \sum_{p+2q=\ell} \beta_{pq}^{(n)} C_p^1(\mathrm{Re}\,(u))\varphi_{pq}^{(n)}(\eta)\,.$$

On the other hand, we have

$$\int_M f(r(m\eta)u)dm = \int_{Sp(1)} [1 - 2r\,\mathrm{Re}\,(v\eta_1 v^{-1}u) + r^2]^{1-2n}dv\,.$$

Thus we obtain that

$$\int_{Sp(1)} [1 - 2r\,\mathrm{Re}\,(vzv^{-1}u) + r^2]^{1-2n}dv = \sum_{\ell=0}^{\infty} r^\ell \sum_{p+2q=\ell} \beta_{pq}^{(n)} C_p^1(\mathrm{Re}\,(u))R_{pq}^{(n)}(z)\,.$$

Finally, it follows from the definitions of $\varphi_{pq}^{(n)}$ that $|R_{pq}^{(n)}(z)| \le R_{pq}^{(n)}(1) = 1$ for $|z| \le 1$, which implies the second assertion.

The formula (2.1) means that the zonal spherical functions $\varphi_{pq}^{(n)}$ appear as the coefficients in the expansion of the left hand side of (2.1) by the powers of r and the spherical functions of $Sp(1) \cong S(\mathbf{R}^4)$. So we can consider that (2.1) gives a generating function for the functions $\varphi_{pq}^{(n)}$.

2.2. Generalization of the formula (2.1). For a fixed $n \geq 2$, we suppose that $\nu > 2n-1$. First of all, for $u \in Sp(1)$, $0 \leq r < 1$ and $z \in \mathbf{H}$, $|z| \leq 1$,

$$\int_{Sp(1)} [1 - 2r \operatorname{Re}(vzv^{-1}u) + r^2]^{-\nu} dv = \sum_{m=0}^{\infty} r^m \int_{Sp(1)} C_m^{\nu}(\operatorname{Re}(vzv^{-1}u))dv . \quad (2.3)$$

Using the following formula,

$$C_m^{\nu}(x) = \sum_{q=0}^{[m/2]} \gamma_q^{(m)}(\nu, 2n-1) C_{m-2q}^{2n-1}(x) ,$$

we have

$$\int_{Sp(1)} C_m^{\nu}(\operatorname{Re}(vzv^{-1}u))dv = \sum_{q=0}^{[m/2]} \gamma_q^{(m)}(\nu, 2n-1) \int_{Sp(1)} C_{m-2q}^{2n-1}(\operatorname{Re}(vzv^{-1}u))dv . \quad (2.4)$$

And it follows from the formula (2.1) that

$$\int_{Sp(1)} C_m^{2n-1}(\operatorname{Re}(vzv^{-1}u))dv = \sum_{k+2\ell=m} \beta_{k\ell}^{(n)} C_k^1(\operatorname{Re}(u)) R_{k\ell}^{(n)}(z) . \quad (2.5)$$

From (2.4) and (2.5), we obtain that

$$\begin{aligned}\int_{Sp(1)} C_m^{\nu}(\operatorname{Re}(vzv^{-1}u))dv &= \sum_{p+2q=m} \gamma_q^{(m)}(\nu, 2n-1) \sum_{k+2\ell=p} \beta_{k\ell}^{(n)} C_k^1(\operatorname{Re}(u)) R_{k\ell}^{(n)}(z) \\ &= \sum_{k+2s=m} \sum_{r=0}^{s} \gamma_r^{(m)}(\nu, 2n-1) \beta_{k,s-r}^{(n)} C_k^1(\operatorname{Re}(u)) R_{k,s-r}^{(n)}(z) \\ &= \sum_{k+2s=m} \left[\sum_{r=0}^{s} \gamma_{s-r}^{(k+2s)}(\nu, 2n-1) \beta_{kr}^{(n)} R_{kr}^{(n)}(z) \right] C_k^1(\operatorname{Re}(u)) . \quad (2.6)\end{aligned}$$

Here we have

$$\begin{aligned}&\sum_{r=0}^{s} \gamma_{s-r}^{(k+2s)}(\nu, 2n-1) \beta_{kr}^{(n)} R_{kr}^{(n)}(z) \\ &= (-1)^s \frac{1}{k!} [C_k^1(1)]^{-1} (\nu)_{k+s} C_k^1\left(\frac{\operatorname{Re}(z)}{|z|}\right) |z|^k \\ &\quad \times \sum_{r=0}^{s} \frac{1}{s!} \rho_r^{(s)}(k, n, \nu)\, {}_2F_1(-r, k+r+2n-1; k+2; |z|^2) , \quad (2.7)\end{aligned}$$

where

$$\rho_r^{(s)}(k, n, \nu) = \frac{(-1)^{r-s} s! (2n-1+k+2r) \Gamma(k+r+2n-1)(\nu-2n+1)_{s-r}(k+s+\nu)_r}{(s-r)! r! \Gamma(k+s+r+2n)} .$$

Lemma 2.1. *If α, β, $\gamma > 0$, $\beta - \gamma + 1 > 0$, then we have*

$$G_s(\alpha, \gamma; t) = \sum_{r=0}^{s} \sigma_r^{(s)}(\alpha, \beta, \gamma) G_r(\beta, \gamma; t),$$

where

$$\sigma_r^{(s)}(\alpha, \beta, \gamma) = \frac{s!(\beta + 2r)\Gamma(\beta + r)(\alpha + s)_r(\beta - \alpha + r - s + 1)_{s-r}}{(s-r)!r!\Gamma(\beta + r + s + 1)}.$$

In particular, putting $\alpha = k + \lambda$ ($\lambda > 0$), $\beta = k + 2n - 1$, $\gamma = k + 2$, we have

$$G_s(k + \lambda, k + 2; t) = \sum_{r=0}^{s} \rho_r^{(s)}(k, n, \lambda) G_r(k + 2n - 1, k + 2; t),$$

where

$$\rho_r^{(s)}(k, n, \lambda) = \frac{(-1)^{r-s}s!(2n - 1 + k + 2r)\Gamma(k + r + 2n - 1)(\lambda - 2n + 1)_{s-r}(k + s + \lambda)_r}{(s-r)!r!\Gamma(k + s + r + 2n)}.$$

Proof. We define $c(\ell, r)$ by

$$t^\ell = \sum_{r=0}^{\ell} c(\ell, r) G_r(\beta, \gamma; t).$$

We have

$$c(\ell, r) = \int_0^1 t^\ell G_r(\beta, \gamma; t) t^{\gamma-1}(1-t)^{\beta-\gamma} dt \times \left[\int_0^1 [G_r(\beta, \gamma; t)]^2 t^{\gamma-1}(1-t)^{\beta-\gamma} dt\right]^{-1}.$$

First of all,

$$\int_0^1 [G_r(\beta, \gamma; t)]^2 t^{\gamma-1}(1-t)^{\beta-\gamma} dt = \frac{r!\Gamma(r + \beta - \gamma + 1)[\Gamma(\gamma)]^2}{(\beta + 2r)\Gamma(\beta + r)\Gamma(\gamma + r)}$$

(see the proof of Proposition 1.1). Secondly

$$\begin{aligned}\int_0^1 t^\ell G_r(\beta, \gamma; t) t^{\gamma-1}(1-t)^{\beta-\gamma} dt &= \frac{\Gamma(\gamma)}{\Gamma(\gamma + r)} \int_0^1 t^\ell \frac{d^r}{dt^r}[t^{\gamma+r-1}(1-t)^{\beta+r-\gamma}] dt \\ &= \frac{(-1)^r\Gamma(\gamma)\Gamma(\ell + 1)}{\Gamma(\gamma + r)\Gamma(\ell - r + 1)} \int_0^1 t^{\ell+\gamma-1}(1-t)^{\beta+r-\gamma} dt \\ &= \frac{(-1)^r\Gamma(\gamma)\Gamma(\ell + 1)\Gamma(\ell + \gamma)\Gamma(\beta + r - \gamma + 1)}{\Gamma(\gamma + r)\Gamma(\ell - r + 1)\Gamma(\ell + \beta + r + 1)}.\end{aligned}$$

So we obtain that

$$c(\ell, r) = \frac{(-1)^r\Gamma(\ell + 1)\Gamma(\ell + \gamma)(\beta + 2r)\Gamma(\beta + r)}{\Gamma(\gamma)\Gamma(\ell - r + 1)\Gamma(\ell + \beta + r + 1)r!}.$$

Thus

$$G_s(\alpha, \gamma; t) = {}_2F_1(-s, \alpha + s; \gamma; t)$$
$$= \sum_{r=0}^{s} \sum_{\ell=r}^{s} \frac{(-s)_\ell(\alpha + s)_\ell}{(\gamma)_\ell \ell!} c(\ell, r) G_r(\beta, \gamma; t).$$

Here we have

$$\sum_{\ell=r}^{s} \frac{(-s)_\ell(\alpha + s)_\ell}{(\gamma)_\ell \ell!} c(\ell, r) = \frac{(-1)^r(\beta + 2r)\Gamma(\beta + r)(-s)_r(\alpha + s)_r}{r!\Gamma(\beta + 2r + 1)}$$
$$\times {}_2F_1(-s + r, \alpha + s + r; \beta + 2r + 1; 1)$$
$$= \frac{s!(\beta + 2r)\Gamma(\beta + r)(\alpha + s)_r(\beta - \alpha + r - s + 1)_{s-r}}{(s - r)!r!\Gamma(\beta + r + s + 1)}.$$

This implies our first assertion.

Next, putting $\alpha = k + \lambda$, $\beta = k + 2n - 1$, $\gamma = k + 2$, it follows from $(\beta - \alpha + r - s + 1)_{s-r} = (-1)^{s-r}(\lambda - 2n + 1)_{s-r}$ that our second assertion.

From (2.7) and Lemma 2.1, we obtain that

$$\sum_{r=0}^{s} \gamma_{s-r}^{(k+2s)}(v, 2n - 1)\beta_{kr}^{(n)} R_{kr}^{(n)}(z)$$
$$= (-1)^s \frac{(v)_{k+s}}{k!s!} [C_k^1(1)]^{-1} C_k^1\left(\frac{\mathrm{Re}\,(z)}{|z|}\right)|z|^k {}_2F_1(-s, k + s + v; k + 2; |z|^2).$$

We now define $c_{pq}(v)$, $\beta_{pq}(v)$ and Φ_{pq}^v by

$$c_{pq}(v) = \frac{(-1)^q(p + 2)_q}{(v - 1)_q} [C_p^1(1)]^{-1},$$

$$\beta_{pq}(v) = \frac{p + 1}{\Gamma(p + q + 2)} \frac{(v)_{p+q}(v - 1)_q}{q!},$$

$$\Phi_{pq}^v(z) = c_{pq}(v) C_p^1\left(\frac{\mathrm{Re}\,(z)}{|z|}\right)|z|^p {}_2F_1(-q, p + q + v; p + 2; |z|^2).$$

Then we have

$$\sum_{r=0}^{s} \gamma_{s-r}^{(k+2s)}(v, 2n - 1)\beta_{kr}^{(n)} R_{kr}^{(n)}(z) = \beta_{ks}(v)\Phi_{ks}^v(z).$$

Therefore from (2.6),

$$\int_{Sp(1)} C_m^v(\mathrm{Re}\,(vzv^{-1}u))dv = \sum_{k+2s=m} \beta_{ks}(v) C_k^1(\mathrm{Re}\,(u))\Phi_{ks}^v(z).$$

This equality holds also for $\mathrm{Re}\,(v) > 1$ because of the analyticity with respect to $v \in \mathbf{C}$.

Thus, from (2.3), we obtain the following theorem.

Theorem 2.2. *If* $v > 1$, *then we have*

$$\int_{Sp(1)} [1 - 2r \operatorname{Re}(vzv^{-1}u) + r^2]^{-v} dv = \sum_{m=0}^{\infty} r^m \sum_{k+2s=m} \beta_{ks}(v) C_k^1(\operatorname{Re}(u)) \Phi_{ks}^v(z),$$

for $u \in Sp(1)$, $0 \le r < 1$ *and* $z \in \mathbf{H}$, $|z| \le 1$.

Here we give the orthogonality relation of the functions Φ_{pq}^v. We set $\Psi_{pq}^v = \beta_{pq}(v)\Phi_{pq}^v$.

Proposition 2.1. *For* $v > 1$, *we have*

$$\int_{|z|\le 1} \Psi_{pq}^v(z) \Psi_{k\ell}^v(z)(1 - |z|^2)^{v-2} dz = \delta_{pk}\delta_{q\ell} \frac{\pi^2 \Gamma(p + q + v)\Gamma(q + v - 1)}{[\Gamma(v)]^2 (p + 2q + v) q! \Gamma(p + q + 2)},$$

where $dz = dz_1 dz_2 dz_3 dz_4$, $z = z_1 + z_2 i + z_3 j + z_4 k$.

Proof. By the definitions of Φ_{pq}^v,

$$\begin{aligned}
&\int_{|z|\le 1} \Psi_{pq}^v(z) \Psi_{k\ell}^v(z)(1 - |z|^2)^{v-2} dz \\
&= \delta_{pk} \frac{(-1)^q (v)_{p+q}}{(p + 1)! q!} \frac{(-1)^\ell (v)_{k+\ell}}{(k + 1)! \ell!} \pi^2 \\
&\quad \times \int_0^1 t^{p+1}(1 - t)^{v-2} G_q(p + v, p + 2; t) G_\ell(p + v, p + 2; t) dt .
\end{aligned}$$

This completes the proof.

2.3. Integral representations for the functions $R_{pq}^{(n)}$. We consider integral representations for the spherical functions $R_{pq}^{(n)}$. First of all, we shall give relations between the functions $R_{pq}^{(n)}$ and $Q_{k\ell}^{(2n)}$. In what follows, we shall use the following notation for $z \in \mathbf{H}$:

$$z = z_1 + iz_2 + jz_3 + kz_4 ,$$

where $z_v (1 \le v \le 4) \in \mathbf{R}$.

Proposition 2.2. *For* $z \in \mathbf{H}$, $|z| \le 1$, *we have*

$$\begin{aligned}
\beta_{k\ell}^{(n)} R_{k\ell}^{(n)}(z) &= a_{\ell,k+\ell}^{(2n)} \int_{Sp(1)} Q_{\ell,k+\ell}^{(2n)}(z_1 + i(vzv^{-1})_2) dv \\
&\quad - a_{\ell-1,k+\ell+1}^{(2n)} \int_{Sp(1)} Q_{\ell-1,k+\ell+1}^{(2n)}(z_1 + i(vzv^{-1})_2) dv
\end{aligned}$$

with $Q_{-1,k+1}^{(2n)} = 0$ *and* $a_{-1,k+1}^{(2n)} = 0$.

Proof. In Theorem 2.1, if we put $u = e^{i\theta} (\theta \in \mathbf{R})$, then we have

$$\int_{Sp(1)} [1 - 2r \operatorname{Re}(vzv^{-1}u) + r^2]^{1-2n} dv = \sum_{m=0}^{\infty} r^m \sum_{k+2\ell=m} \beta_{k\ell}^{(n)} R_{k\ell}^{(n)}(z) C_k^1(\cos\theta) . \quad (2.8)$$

On the other hand, by Theorem 1.1,

$$\begin{aligned}\int_{Sp(1)} & [1 - 2r \operatorname{Re}(vzv^{-1}u) + r^2]^{1-2n} dv \\ &= \int_{Sp(1)} [1 - 2r \operatorname{Re}\{(z_1 + i(vzv^{-1})_2)e^{i\theta}\} + r^2]^{1-2n} dv \\ &= \sum_{m=0}^{\infty} r^m \sum_{p+q=m} a_{pq}^{(2n)} e^{i(p-q)\theta} \int_{Sp(1)} Q_{pq}^{(2n)}(z_1 + i(vzv^{-1})_2) dv\,. \end{aligned} \tag{2.9}$$

Comparing the coefficients of r^m in (2.8) and (2.9), then we obtain

$$\begin{aligned}\sum_{p+q=m} & a_{pq}^{(2n)} e^{i(p-q)\theta} \int_{Sp(1)} Q_{pq}^{(2n)}(z_1 + i(vzv^{-1})_2) dv \\ &= \sum_{k+2\ell=m} \beta_{k\ell}^{(n)} R_{k\ell}^{(n)}(z) C_k^1(\cos\theta) \\ &= \sum_{\ell=0}^{[m/2]} \beta_{m-2\ell,\ell}^{(n)} R_{m-2\ell,\ell}^{(n)}(z) \sum_{r=0}^{m-2\ell} \frac{e^{i(2r+2\ell-m)\theta} + e^{-i(2r+2\ell-m)\theta}}{2}\,. \end{aligned}$$

Here we used the following formula:

$$C_k^1(\cos\theta) = \sum_{r=0}^{k} \cos(2r-k)\theta\,.$$

Thus we see that

$$\begin{aligned}\sum_{p=0}^{m} & a_{p,m-p}^{(2n)} e^{i(2p-m)\theta} \int_{Sp(1)} Q_{p,m-p}^{(2n)}(z_1 + i(vzv^{-1})_2) dv \\ &= \sum_{p=0}^{m} \left(\sum_{\ell=0}^{\min(p,m-p)} \beta_{m-2\ell,\ell}^{(n)} R_{m-2\ell,\ell}^{(n)}(z) \right) e^{i(2p-m)\theta}\,. \end{aligned}$$

Comparing the coefficients of $e^{i(2p-m)\theta}$ in both sides, then we obtain that

$$a_{p,m-p}^{(2n)} \int_{Sp(1)} Q_{p,m-p}^{(2n)}(z_1 + i(vzv^{-1})_2) dv = \sum_{\ell=0}^{\min(p,m-p)} \beta_{m-2\ell,\ell}^{(n)} R_{m-2\ell,\ell}^{(n)}(z)\,,$$

in particular, if $0 \le p \le \left[\dfrac{m}{2}\right]$,

$$a_{p,m-p}^{(2n)} \int_{Sp(1)} Q_{p,m-p}^{(2n)}(z_1 + i(vzv^{-1})_2) dv = \sum_{\ell=0}^{p} \beta_{m-2\ell,\ell}^{(n)} R_{m-2\ell,\ell}^{(n)}(z)\,,$$

which implies our assertion.

An integral representation for the function $R_{pq}^{(n)}$ is given in the following theorem.

Theorem 2.3. *For* $z \in \mathbf{H}$, $|z| \le 1$, *we have*

$$\beta_{k\ell}^{(n)} R_{k\ell}^{(n)}(z) = \frac{a_{\ell,k+\ell}^{(2n)}}{2} \int_0^\pi Q_{\ell,k+\ell}^{(2n)}(z_1 + i\sqrt{z_2^2 + z_3^2 + z_4^2} \cos\theta) \sin\theta d\theta$$

$$- \frac{a_{\ell-1,k+\ell+1}^{(2n)}}{2} \int_0^\pi Q_{\ell-1,k+\ell+1}^{(2n)}(z_1 + i\sqrt{z_2^2 + z_3^2 + z_4^2} \cos\theta) \sin\theta d\theta$$

$$= \int_0^\pi T_{k\ell}^{(n)}(z_1 + i\sqrt{z_2^2 + z_3^2 + z_4^2} \cos\theta) \sin\theta d\theta ,$$

where the functions $T_{k\ell}^{(n)}(\xi)$ *are defined for* $\xi \in \mathbf{C}$ *as follows*:

$$T_{k\ell}^{(n)}(\xi) = S_{k\ell}^{(n)}(\xi) - S_{k+2,\ell-1}^{(n)}(\xi) , \qquad (\ell \geq 1)$$

$$T_{k0}^{(n)}(\xi) = S_{k0}^{(n)}(\xi) ,$$

and

$$S_{k\ell}^{(n)}(\xi) = \frac{(-1)^\ell (2n-1)_{k+\ell}}{4k!\ell!} (\xi^k + \bar{\xi}^k) {}_2F_1(-\ell, k+\ell+2n-1; k+1; |\xi|^2) .$$

Proof. If $z = z_1 + iz_2$, i.e. $z_3 = z_4 = 0$, by Proposition 2.2,

$$\beta_{k\ell}^{(n)} R_{k\ell}^{(n)}(z_1 + iz_2) = a_{\ell,k+\ell}^{(2n)} \int_{SO(3)} Q_{\ell,k+\ell}^{(2n)}(z_1 + iz_2(ge_1)_1) dg$$

$$- a_{\ell-1,k+\ell+1}^{(2n)} \int_{SO(3)} Q_{\ell-1,k+\ell+1}^{(2n)}(z_1 + iz_2(ge_1)_1) dg$$

$$= a_{\ell,k+\ell}^{(2n)} \int_{S(\mathbf{R}^3)} Q_{\ell,k+\ell}^{(2n)}(z_1 + iz_2 b_1) d\sigma_3(b)$$

$$- a_{\ell-1,k+\ell+1}^{(2n)} \int_{S(\mathbf{R}^3)} Q_{\ell-1,k+\ell+1}^{(2n)}(z_1 + iz_2 b_1) d\sigma_3(b)$$

$$= \frac{a_{\ell,k+\ell}^{(2n)}}{2} \int_0^\pi Q_{\ell,k+\ell}^{(2n)}(z_1 + iz_2 \cos\theta) \sin\theta d\theta$$

$$- \frac{a_{\ell-1,k+\ell+1}^{(2n)}}{2} \int_0^\pi Q_{\ell-1,k+\ell+1}^{(2n)}(z_1 + iz_2 \cos\theta) \sin\theta d\theta ,$$

where dg is the normalized Haar measure on $SO(3)$ and $d\sigma_3(b)$ is the normalized element of surface area on $S(\mathbf{R}^3)$.

On the other hand, the function $R_{k\ell}^{(n)}$ is real-valued, so we see that

$$\beta_{k\ell}^{(n)} R_{k\ell}^{(n)}(z_1 + iz_2) = \frac{a_{\ell,k+\ell}^{(2n)}}{2} \int_0^\pi \mathrm{Re}\,[Q_{\ell,k+\ell}^{(2n)}(z_1 + iz_2 \cos\theta)] \sin\theta d\theta$$

$$- \frac{a_{\ell-1,k+\ell+1}^{(2n)}}{2} \int_0^\pi \mathrm{Re}\,[Q_{\ell-1,k+\ell+1}^{(2n)}(z_1 + iz_2 \cos\theta)] \sin\theta d\theta .$$

From Lemma 1.1, for $\xi \in \mathbf{C}$,

$$
\begin{aligned}
&2a^{(2n)}_{\ell,k+\ell}\,\mathrm{Re}\,[Q^{(2n)}_{\ell,k+\ell}(\xi)] \\
&= a^{(2n)}_{\ell,k+\ell}(Q^{(2n)}_{\ell,k+\ell}(\xi) + \overline{Q^{(2n)}_{\ell,k+\ell}(\xi)}) \\
&= \frac{\xi^k + \bar{\xi}^k}{\Gamma(2n-1)} \sum_{m=0}^{\ell} \frac{\Gamma(2n-1+k+2\ell-m)}{(\ell-m)!(k+\ell-m)!m!}(-1)^m|\xi|^{2\ell-2m} \\
&= \frac{\xi^k + \bar{\xi}^k}{\Gamma(2n-1)} \sum_{p=0}^{\ell} \frac{\Gamma(2n-1+k+\ell+p)}{p!(k+p)!(\ell-p)!}(-1)^{\ell-p}|\xi|^{2p} \\
&= \frac{(-1)^\ell(\xi^k + \bar{\xi}^k)}{\Gamma(2n-1)\ell!} \sum_{p=0}^{\ell} \frac{(-\ell)_p\Gamma(2n-1+k+\ell+p)}{(k+p)!p!}|\xi|^{2p} \\
&= \frac{(-1)^\ell(2n-1)_{k+\ell}}{k!\ell!}(\xi^k + \bar{\xi}^k)\,{}_2F_1(-\ell, 2n-1+k+\ell; k+1; |\xi|^2)\,.
\end{aligned}
$$

This completes the proof.

We should notice that our assertions for $\ell = 0$ are nothing but the integral representations for C^1_k which are well known. See the formula (31) in [1], p. 177.

Acknowledgement. The author would like to thank Professor R. Takahashi for his helpful suggestions.

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