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On the structure of the solutions of the first initial boundary value problem for the Sobolev's equation

By

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0. Introduction

In this paper we investigate the first mixed problem for the so-called Sobolev's equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}\right)+\frac{\partial^2 u}{\partial x_3^2}=0 \tag{0.1}$$

in the model case of two spatial variables.

Investigations of the equation (0.1) were initiated by S. L. Sobolev [1], [2], although this equation arose in the work [3] (pp. 355–356) by H. Poincare in 1885. Equation (0.1) is closely connected with the system of equations describing small oscillations of a rotating fluid

$$\frac{\partial \vec{v}}{\partial t}=\vec{v}\times\vec{e}_3-\nabla p\,,\qquad div\,\vec{v}=0 \tag{0.2}$$

where $\vec{v}$ is a relative velocity, p is a relative pressure.

System (0.2) is a linearization of the Euler equations in the uniformly rotating coordinate system near an equilibrium solution describing the motion of the ideal fluid rotating with a fixed angular velocity about axis x_3. This system was considered by S. L. Sobolev in [2] in connection with the studying of the stability to the first approximation of the top with symmetric cavity filled with an ideal fluid. He showed that if $\vec{v}$, p satisfy (0.2), initial conditions

$$\vec{v}\,|_{t=0}=\vec{v}_0(x)\,,\qquad x=(x_1,x_2,x_3)\in\Omega \tag{0.3}$$

and one of the following boundary conditions

$$p|_{\partial\Omega}=0 \tag{0.4}$$

$$(\vec{v}\cdot\vec{n})|_{\partial\Omega}=0 \tag{0.5}$$

then p satisfies (0.1), initial conditions

$$p|_{t=0}=p_0\,,\qquad p_t|_{t=0}=p_1 \tag{0.6}$$

and the boundary condition (0.4) or

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial p}{\partial \vec{n}}\right)+\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial x_2}\cos(\vec{n}, x_1)-\frac{\partial p}{\partial x_1}\cos(\vec{n}, x_2)\right)+\frac{\partial p}{\partial x_3}\cos(\vec{n}, x_3)|_{\partial\Omega}=0 \tag{0.7}$$

respectively.

Problems (0.1), (0.4), (0.6) and (0.1), (0.6), (0.7) are called the first and the second initial boundary value problems for the Sobolev's equation and can be written in the form of abstract Cauchy problem

$$p_{tt}=\mathbf{A}p\,,\qquad p|_{t=0}=p_0\,,\qquad p_t|_{t=0}=p_1 \tag{0.8}$$

$$\vec{u}_t=i\mathrm{B}\vec{u},\qquad \vec{u}\,|_{t=0}=\vec{u}_0 \tag{0.9}$$

where $\mathbf{A}$ is a bounded selfadjoint operator in $\mathring{\mathbf{W}}_2^1(\Omega)$ with spectrum $\sigma(\mathbf{A})=[-1, 0]$; $\mathbf{B}$ is a bounded selfadjoint operator in $H=\{\vec{u}\in \mathbf{L}_2(\Omega)|\operatorname{div}\vec{u}=0,\ \vec{u}\cdot\vec{n}\,|_{\partial\Omega}=0\}$ with spectrum $\sigma(\mathbf{B})\subseteq[-1, 1]$.

One of the basic questions in theory of small oscillations of the rotating fluid is that of the behavior of the relative pressure p for the large time. A complete solution of this question is closely connected with a detailed study of the spectral properties of the operators $\mathbf{A}$, $\mathbf{B}$ which represents one of the most interesting and complex problems in this theory. This is connected with the fact that the qualitative properties of the solutions of (0.8), (0.9) as $t\to\infty$ differ in a number of cases from the properties of the solutions of the majority of problems of mathematical physics. In connection with this S. L. Sobolev posed the problem of studying of the asymptotic properties of the solutions of mixed problems for the equation (0.1) in various domains.

A number of papers by R. A. Aleksandryan, T. I. Zelenyak, V. N. Maslennikova, M. V. Fokin, B. V. Kapitonov, V. V. Skazka and others have been devoted to this theme; the history and bibliography can be found in [4], [5].

The present paper deals with the model case of two space variables. T. I. Zelenyak showed that in two-dimensional case solutions of the first and the second initial boundary value problems had the same qualitative properties when $t\to\infty$. So we consider the first mixed problem

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)+\frac{\partial^2 u}{\partial x_2^2}=0\,,\qquad x\in\Omega,\ t>0 \tag{0.10}$$

$$p|_{\partial\Omega}=0\,,\qquad t>0 \tag{0.11}$$

$$p|_{t=0}=p_0\,,\qquad p_t|_{t=0}=p_1\,,\qquad x\in\Omega \tag{0.12}$$

which can be rewritten in the operator form (0.8). In the paper the following results are obtained: a class of bounded domains $\Omega\subset\mathbf{R}^2$ with corner points is selected such that the operator $\mathbf{A}$ has no eigenfunctions in $\mathring{\mathbf{W}}_2^1(\Omega)$ but for every $\lambda\in[-1, 0]$ there exists a class of generalized eigenfunctions (G.E.) from $\mathbf{L}_2(\Omega)$ corresponding to λ; the structure of the G.E. is completely described; the completeness of the G.E. in $\mathring{\mathbf{W}}_2^1(\Omega)$ is proved; general integral representation of the solutions of the problem (0.8) is obtained; inversion formula and Parseval's equality for the integral representation are derived.

1. Choice of the domains

Let Ω be a bounded convex domain; $\partial\Omega = \Gamma = \bigcup_{j=1}^{n} \Gamma_j$, $\Gamma_j \in \mathbf{C}^4$, Γ_j is either piece of line or has positive curvature at any point including endpoints (curvature at the endpoints is regarded as the limit of curvature at the interior points).

Let the set $\{\alpha_1, \dots, \alpha_m\} \subset \left[0, \frac{\pi}{2}\right]$ consist of angles between axis x_1 and all one-sided tangents to Γ at the endpoints of Γ_j, $j = 1, \dots, n$. We suppose $0 \le \alpha_1 < \cdots < \alpha_m \le \frac{\pi}{2}$.

Problem (0.10)–(0.12) can be rewritten as an abstract Cauchy problem in $\mathring{\mathbf{W}}_2^1(\Omega)$

$$p_{tt} = \mathbf{A}p\,, \qquad p|_{t=0} = p_0\,, \qquad p_t|_{t=0} = p_1 \tag{1.1}$$

where $\mathbf{A} = \overline{\mathbf{A}_0}$-closure in $\mathring{\mathbf{W}}_2^1(\Omega)$ of an operator $\mathbf{A}_0$, $D(\mathbf{A}_0) = \mathring{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^2(\Omega)$ and for any $u \in D(\mathbf{A}_0)$ image $v = \mathbf{A}_0 u$ is a solution of the following Dirichlet problem

$$\Delta v = -\frac{\partial^2 u}{\partial x_2^2}\,, \qquad v|_\Gamma = 0. \tag{1.2}$$

From (1.2) it follows that for any $u,\ v \in \mathring{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^2(\Omega)$

$$(\mathbf{A}u, v)_1 = -(\Delta \mathbf{A}u, v)_{\mathbf{L}_2(\Omega)} = (u_{x_2x_2}, v)_{\mathbf{L}_2(\Omega)} = -(u_{x_2}, v_{x_2})_{\mathbf{L}_2(\Omega)}$$

where

$$(u, v)_1 = (u, v)_{\mathring{\mathbf{W}}_2^1(\Omega)} = \int_\Omega u_{x_1} \cdot \bar{v}_{x_1} + u_{x_2} \cdot \bar{v}_{x_2} d\Omega. \tag{1.3}$$

So $\mathbf{A}$ is a bounded selfadjoint operator in $\mathring{\mathbf{W}}_2^1(\Omega)$ and

$$(\mathbf{A}u, v)_1 = -(u_{x_2}, v_{x_2})_{\mathbf{L}_2}\,, \qquad u,\ v \in \mathring{\mathbf{W}}_2^1(\Omega). \tag{1.4}$$

Hence for any $u \in \mathring{\mathbf{W}}_2^1(\Omega)$

$$0 > (\mathbf{A}u, u)_1 = -\|u_{x_2}\|_{\mathbf{L}_2} \ge -\|u\|_1 \tag{1.5}$$

and the spectrum $\sigma(\mathbf{A}) \subset [-1, 0]$. It is easy to show [6] that $\sigma(\mathbf{A}) = [-1, 0]$.

We shall say that a function $u \in \mathbf{L}_2(\Omega)$ is a generalized eigenfunction of the operator $\mathbf{A}$ corresponding to $\lambda \in [-1, 0]$ if

$$\int_\Omega u \cdot (\lambda \phi_{x_1x_1} + (1 + \lambda)\phi_{x_2x_2}) d\Omega = 0\,, \qquad \phi \in \mathring{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^2(\Omega)\,. \tag{1.6}$$

It is easy to see that if $u \in \mathring{\mathbf{W}}_2^1(\Omega)$ and (1.6) holds then u is an eigenfunction of $\mathbf{A}$ corresponding to λ. Denote

$$\alpha = \alpha(\lambda) = \arccos(\sqrt{-\lambda}) \in \left[0, \frac{\pi}{2}\right], \qquad \lambda = -\cos^2\alpha,$$

$$\xi = \xi(x, \alpha) = x_1 \sin\alpha + x_2 \cos\alpha, \qquad \eta = \eta(x, \alpha) = x_1 \sin\alpha - x_2 \cos\alpha.$$

Then (1.6) can be rewritten in the following form

$$(1.7) \qquad \int_\Omega u \cdot \phi_{\xi\eta} d\Omega = 0, \qquad \phi \in \mathring{\mathbf{W}}_2^1(\Omega) \cap \mathbf{W}_2^2(\Omega).$$

It is known [7] that if Ω is strictly convex relative to the lines $\xi = \text{const}$, $\eta = \text{const}$ and (1.7) holds for some $u \in \mathbf{L}_2(\Omega)$ then

$$u(x) = P(\xi(x, \alpha)) + Q(\eta(x, \alpha)), \qquad x \in \Omega,$$

$$P(\xi(x, \alpha)) + Q(\eta(x, \alpha)) = 0, \qquad a.e.\ x \in \Gamma$$

where $P(\xi) \in \mathbf{L}_{2,\rho_1}(\xi_0, \xi_1)$, $Q(\eta) \in \mathbf{L}_{2,\rho_2}(\eta_0, \eta_1)$; $\xi_0 = \xi_0(\alpha) = \min_{x \in \Gamma} \xi(x, \alpha)$, $\xi_1 = \xi_1(\alpha) = \max_{x \in \Gamma} \xi(x, \alpha)$, $\eta_0 = \eta_0(\alpha) = \min_{x \in \Gamma} \eta(x, \alpha)$, $\eta_1 = \eta_1(\alpha) = \max_{x \in \Gamma} \eta(x, \alpha)$; $\rho_1(\xi)$, $\rho_2(\eta)$ are some weight functions; $\rho_1 \in \mathbf{C}[\xi_0, \xi_1]$, $\rho_2 \in \mathbf{C}[\eta_0, \eta_1]$; $\rho_1(\xi) > 0$, $\xi \in (\xi_0, \xi_1)$; $\rho_2(\eta) > 0$, $\eta \in (\eta_0, \eta_1)$; $\rho_1(\xi_0) = \rho_1(\xi_1) = \rho_2(\eta_0) = \rho_2(\eta_1) = 0$.

Thus if we denote $\lambda_j = -\cos^2\alpha_j$, $j = 1, \ldots, m$ then for all $\lambda \in (-1, 0) \setminus \{\lambda_1, \ldots, \lambda_m\}$ each G.E. $u(x, \alpha)$ corresponding to $\lambda = -\cos^2\alpha$ can be written in the form

$$(1.8) \qquad u(x, \alpha) = P(\xi(x, \alpha), \alpha) + Q(\eta(x, \alpha), \alpha), \qquad x \in \Omega,$$

$$(1.9) \qquad P(\xi(x, \alpha), \alpha) + Q(\eta(x, \alpha), \alpha) = 0, \qquad a.e.\ x \in \Gamma.$$

Following [8], we define for any $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ homeomorphisms $T^+(\alpha)$, $T^-(\alpha)$ of the boundary Γ: $T^+(\alpha)$, $T^-(\alpha)$ assign to a point $x \in \Gamma$ other boundary points $T^+(\alpha)x$, $T^-(\alpha)x$ such that the following equations hold

$$(1.10) \qquad \eta(x, \alpha) = \eta(T^+(\alpha)x, \alpha), \qquad \xi(x, \alpha) = \xi(T^-(\alpha)x, \alpha).$$

In other words, $T^+(\alpha)(T^-(\alpha))$ assigns to a point of the boundary another boundary point with the same η (ξ) coordinate (see Figure 1). We set

$$F(\alpha) = T^-(\alpha) \circ T^+(\alpha).$$

The homeomorphism $F(\alpha)$ preserves the orientation of Γ (see Figure 1).

The spectral properties of the operator $\mathbf{A}$ depends essentially on the properties of the homeomorphism $F(\alpha)$. The point is that from (1.9), (1.10) it follows

$$(1.11) \qquad \begin{cases} P(\xi(x, \alpha), \alpha) = P(\xi(F^n(\alpha)x, \alpha), \alpha) = P(\xi(T^\pm(\alpha) \circ F^n(\alpha)x, \alpha), \alpha), & n \in \mathbf{Z} \\ Q(\eta(x, \alpha), \alpha) = Q(\eta(F^n(\alpha)x, \alpha), \alpha) = Q(\eta(T^\pm(\alpha) \circ F^n(\alpha)x, \alpha), \alpha), & n \in \mathbf{Z} \end{cases}$$

for a.e. $x \in \Gamma$.

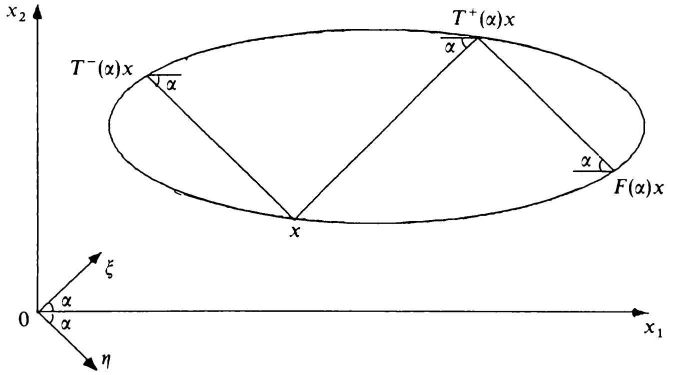

Figure 1

Theorem 1.1 ([9]). *If for some* $\alpha \in \left[0, \frac{\pi}{2}\right] \setminus \{\alpha_1, \ldots, \alpha_m\}$ *the homeomorphism* $F(\alpha)$ *possesses a fixed point then* $\lambda = -\cos^2 \alpha$ *is not an eigenvalue of* $\mathbf{A}$.

There exist domains with corner points such that for every $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ the homeomorphism $F(\alpha)$ has a fixed point.

Theorem 1.2 ([9]). *The homeomorphism* $F(\alpha)$ *possesses a fixed point for any* $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ *if and only if there exist* $\alpha^* \in \{\alpha_1, \ldots, \alpha_m\}$, x^1, $x^2 \in \Gamma$ *such that* (*see Figure* 2)

$$\text{(1.12)} \qquad \begin{cases} \xi(x^1, \alpha^*) = \xi(x^2, \alpha^*) \quad \text{or} \quad \eta(x^1, \alpha^*) = \eta(x^2, \alpha^*)\,, \\ \{x \in \mathbf{R}^2 | \xi(x, \alpha^*) = \xi(x^j, \alpha^*)\} \cap \Omega = \emptyset\,, \qquad j = 1,\ 2\,, \\ \{x \in \mathbf{R}^2 | \eta(x, \alpha^*) = \eta(x^j, \alpha^*)\} \cap \emptyset\,, \qquad j = 1,\ 2\,. \end{cases}$$

Domain Ω will be said to have Property 1 if (1.12) holds for some $\alpha^* \in \left(0, \frac{\pi}{2}\right)$, x^1, $x^2 \in \partial\Omega$.

From Theorem 1.1, Theorem 1.2 it follows that if Ω has Property 1 then any $\lambda \in (-1, 0) \setminus \{\lambda_1, \ldots, \lambda_m\}$ is not an eigenvalue of $\mathbf{A}$. It is easy to show that values -1, λ_1, $\ldots$, λ_m, 0 are not eigenvalues of the operator $\mathbf{A}$ as well. So for any domain Ω which has Property 1 the operator $\mathbf{A}$ has purely continuous spectrum. Henceforth we shall consider only domains which have Property 1.

2. The structure of the G.E.

Let Ω have Property 1. Then the operator $\mathbf{A}$ has no eigenfunctions in $\mathring{\mathbf{W}}_2^1(\Omega)$. At the same time for any $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ there exists a system of

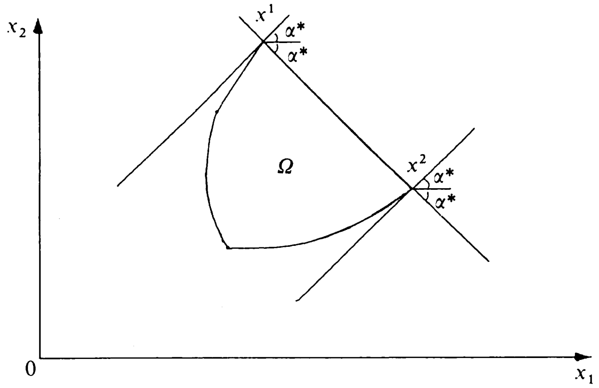

Figure 2

G.E. $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ corresponding to $\lambda = -\cos^2 \alpha$ which can be written in the form (1.8), (1.9). In the present section we shall describe the structure of the G.E..

Since (1.12) holds for some $\alpha^* \in \{\alpha_1, \ldots, \alpha_m\}$, x^1, $x^2 \in \Gamma$, $x^1 \neq x^2$ then one of the following two cases holds:

(1) x^1 is the fixed point for $F(\alpha)$ for any $\alpha \in (0, \alpha^*) \backslash \{\alpha_1, \ldots, \alpha_m\}$, x^2 is the fixed point for $F(\alpha)$ for any $\alpha \in \left(\alpha^*, \frac{\pi}{2}\right) \Big\backslash \{\alpha_1, \ldots, \alpha_m\}$; or

(2) x^2 is the fixed point for $F(\alpha)$ for any $\alpha \in (0, \alpha^*) \backslash \{\alpha_1, \ldots, \alpha_m\}$, x^1 is the fixed point for $F(\alpha)$ for any $\alpha \in \left(\alpha^*, \frac{\pi}{2}\right) \Big\backslash \{\alpha_1, \ldots, \alpha_m\}$.

It may be assumed with no loss of generality that the case (1) holds. For each case $\alpha \lessgtr \alpha^*$ we choose the natural parametrization of Γ

$$\Gamma = \{x(s) = (x_1(s), x_2(s)) | 0 \leq s < l\}$$

such that

$$x(0) = x^1, \qquad \alpha < \alpha^*,$$

$$x(0) = x^2, \qquad \alpha > \alpha^*,$$

and the domain Ω remains on the left when traversing the boundary in the positive direction.

It is easy to see that for any $\alpha \in \left(0, \frac{\pi}{2}\right) \Big\backslash \{\alpha_1, \ldots, \alpha_m\}$ the homeomorphism $F(\alpha)$ can possess one or two fixed points (each fixed point of $F(\alpha)$ is a corner point).

Lemma 2.1 ([9]). *If Ω has Property 1 then there exist numbers α^1, $\alpha^2 \in \left[0, \frac{\pi}{2}\right]$, $\alpha^1 \leq \alpha^2$ such that $F(\alpha)$ has exactly one fixed point for any $\alpha \in (\alpha^1, \alpha^2) \backslash$ $\{\alpha_1, \ldots, \alpha_m\}$; $F(\alpha)$ has exactly two fixed points for any $\alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \Big\backslash$*

$\{\alpha_1, \ldots, \alpha_m\}$. *Besides* $\alpha^1, \alpha^2 \in \left\{0, \alpha_1, \ldots, \alpha_m, \frac{\pi}{2}\right\}$ *and*

$$0 \le \alpha^1 \le \alpha^* \le \alpha^2 \le \frac{\pi}{2}. \tag{2.1}$$

The structure of G.E. $u(x, \alpha)$ corresponding to $\lambda = -\cos^2 \alpha$ depends essentially on the number of fixed points of the homeomorphism $F(\alpha)$. Following [10], we define for any $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ functions $f_1(s, \pm\alpha)\colon (0, l) \overset{\text{on}}{\to} (0, l)$ by the following equations

$$f_1(s, \alpha) = S(T^+(\alpha)x(s)), \qquad f_1(s, -\alpha) = S(T^-(\alpha)x(s)), \qquad s \in (0, l) \tag{2.2}$$

where by $S(x) \in [0, l)$ we denote the coordinate s of $x \in \Gamma$. The functions $f_1(s, \pm\alpha)$ can be defined also as continuous strictly decreasing solutions of the following implicit equations

$$\eta(x(s), \alpha) = \eta(x(f_1(s, \alpha)), \alpha), \qquad \xi(x(s), \alpha) = \xi(x(f_1(s, -\alpha)), \alpha). \tag{2.3}$$

In other words $f_1(s, \pm\alpha)$ are "representations" of the homeomorphisms $T^\pm(\alpha)$ in the variable s.

Define for any $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$, $k \in \mathbf{Z}$ functions $f_k(s, \pm\alpha)\colon (0, l) \overset{\text{on}}{\to} (0, l)$

$$\begin{aligned} &f_0(s, \pm\alpha) \equiv s, \quad f_{-1}(s, \pm\alpha) = f_1(s, \mp\alpha), \\ &f_k(s, \pm\alpha) = f_1(f_{k-1}(s, \pm\alpha), (-1)^{k+1}(\pm\alpha)). \end{aligned} \tag{2.4}$$

Then for any $k \in \mathbf{Z}$ the functions $f_{2k}(s, \pm\alpha)$ are "representations" of the homeomorphisms $F^{\pm k}(\alpha)$ in the variable s. It means that

$$f_{2k}(s, \pm\alpha) = S(F^{\pm k}(\alpha)x(s)), \qquad s \in (0, l), \qquad \alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}. \tag{2.5}$$

Because of the properties of Γ and choice of the parametrization the functions $f_k(s, \pm\alpha)$ satisfy the following properties [9]:

(1) For any $k \in \mathbf{Z}$ the functions $f_{2k}(s, \alpha)$, $-f_{2k+1}(s, \alpha)$ are strictly increasing functions in $s \in (0, l)$ and strictly decreasing functions in $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0, \pm\alpha_1, \ldots, \pm\alpha_m\}$;

(2) For any $k \in \mathbf{Z}$, $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0, \pm\alpha_1, \ldots, \pm\alpha_m\}$

$$f_k(f_k(s, \alpha), (-1)^{k+1}\alpha) \equiv s;$$

(3) For any $k \in \mathbf{Z}$, $f_k(s, \alpha) \in \mathbf{C}\left((0, l) \times \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0, \pm\alpha_1, \ldots, \pm\alpha_m\}\right)\right)$;

(4) $f_{2k}(s, \alpha) \neq s$ for any $|\alpha| \in (\alpha^1, \alpha^2) \setminus \{\alpha_1, \ldots, \alpha_m\}$, $s \in (0, l)$, $|k| \in \mathbf{N}$; $f_{2k}(s, \alpha) \neq$

s, $f_k(s^1, \alpha) = s^1$ for any $|\alpha| \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$, $s \in (0, s^1) \cup (s^1, l)$, $|k| \in \mathbf{N}$ where

(2.6) $$s^1 \in (0, l), \qquad F(|\alpha|)x(s^1) = x(s^1).$$

Such number $s^1 \in (0, l)$ exists and is uniquely determined since $F(|\alpha|)$ has exactly two fixed points for any $|\alpha| \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ and $x(0)$ is fixed point for $F(|\alpha|)$ for any $|\alpha| \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$;

(5) For any $|\alpha| \in (\alpha^1, \alpha^2) \setminus \{\alpha_1, \ldots, \alpha_m\}$, $s \in (0, l)$

(2.7) $$\{f_{2k}(s, \alpha)\}_{-\infty}^{\infty}, \qquad \{f_{2k+1}(s, \alpha)\}_{-\infty}^{\infty}$$

are strictly monotone sequences having the sets of the limit points $\{0, l\}$;

For any $|\alpha| \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$, $s \in (0, s^1) \cup (s^1, l)$ the sequences (2.7) are strictly monotone and the sets of the limit points coincide with $\{0, s^1, l\}$;

(6) For any $|\alpha| \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$, $k \in \mathbf{Z}$

$$f_{2k}(s, \alpha) \xrightarrow[s \to +0]{} +0, \qquad f_{2k}(s, \alpha) \xrightarrow[s \to l-0]{} l-0,$$
$$f_{2k+1}(s, \alpha) \xrightarrow[s \to +0]{} l-0, \qquad f_{2k+1}(s, \alpha) \xrightarrow[s \to l-0]{} +0.$$

For any $|\alpha| \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$, $k \in \mathbf{Z}$

$$f_{2k}(s, \alpha) \xrightarrow[s \to s^1 \pm 0]{} s^1 \pm 0, \qquad f_{2k+1}(s, \alpha) \xrightarrow[s \to s^1 \pm 0]{} s^1 \mp 0.$$

(7) For any $k \in \mathbf{Z}$, $\alpha^0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0, \pm\alpha_1, \ldots, \pm\alpha_m\}$, $s^0 \in (0, l) \setminus E_k(\alpha^0)$ there exists $\varepsilon > 0$ such that $f_k(s, \alpha) \in \mathbf{C}^4((s^0 - \varepsilon, s^0 + \varepsilon) \times (\alpha^0 - \varepsilon, \alpha^0 + \varepsilon))$ where

(2.8) $$E_k(\alpha) = \{f_j(s_p, \alpha) \mid j = -k, \ldots, k, p = 1, \ldots, n\}$$

$\{x(s_1), \ldots, x(s_n)\}$ is the set of endpoints of $\{\Gamma_j\}_1^n$.

If $i, j = 0, \ldots, 4$, $i + j \leq 4$ then there exist $\dfrac{\partial^{i+j} f_k}{\partial s^i \partial \alpha^j}(s^* \pm 0, \alpha^0)$ for any $s^* \in E_k(\alpha^0)$, $\alpha^0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0, \pm\alpha_1, \ldots, \pm\alpha_m\}$.

Let $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ be a G.E. of $\mathbf{A}$ corresponding to $\lambda = -\cos^2 \alpha$, $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$. Then $u(x, \alpha)$ can be written in the form (1.8), (1.9)

(1.8) $$u(x, a) = P(\xi(x, \alpha), \alpha) + Q(\eta(x, \alpha), \alpha), \qquad x \in \Omega,$$

(1.9) $$P(\xi(x, \alpha), \alpha) + Q(\eta(x, \alpha), \alpha) = 0\,, \qquad a.e.\ x \in \Gamma\,.$$

We set

(2.9) $$p(s, \alpha) = P(\xi(x(s), \alpha), \alpha)\,, \qquad q(s, \alpha) = Q(\eta(x(s), \alpha), \alpha)\,, \qquad s \in [0, l)\,.$$

Then from (1.9) it follows

(2.10) $$q(s, \alpha) = -p(s, \alpha)\,, \qquad \text{a.e. } s \in [0, l)\,.$$

Because of (2.3), (2.9) we obtain

(2.11) $$p(s, \alpha) = p(f_1(s, -\alpha), \alpha)\,, \qquad q(s, \alpha) = q(f_1(s, \alpha), \alpha)\,, \qquad s \in [0, l)\,.$$

From (2.10), (2.11) it follows

(2.12) $$p(s, \alpha) = p(f_k(s, \pm\alpha), \alpha)\,, \qquad \text{a.e. } s \in (0, l),\ \forall k \in \mathbf{Z}\,.$$

Thus any G.E. $u(x, \alpha)$ corresponding to $\lambda = -\cos^2 \alpha$, $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ generates some function $p(s, \alpha)$ satisfying (2.12).

Let $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$. Then for any $\xi^0 \in (\xi_0(\alpha), \xi_1(\alpha))$, $\eta^0 \in (\eta_0(\alpha), \eta_1(\alpha))$ each of the lines $\xi(x, \alpha) = \xi^0$, $\eta(x, \alpha) = \eta^0$ has exactly two different points of intersection with the boundary Γ. So there exist $\hat{s}^\pm(\xi^0, \alpha)$, $\check{s}^\pm(\eta^0, \alpha) \in (0, l)$ such that

$$\{x(\hat{s}^+(\xi^0, \alpha)), x(\hat{s}^-(\xi^0, \alpha))\} = \{x \in \mathbf{R}^2 | \xi(x, \alpha) = \xi^0\} \cap \Gamma\,,$$

$$\{x(\check{s}^+(\eta^0, \alpha)), x(\check{s}^-(\eta^0, \alpha))\} = \{x \in \mathbf{R}^2 | \eta(x, \alpha) = \eta^0\} \cap \Gamma\,,$$

(2.13) $$\eta(\hat{s}^-(\xi^0, \alpha), \alpha) < \eta(\hat{s}^+(\xi^0, \alpha), \alpha)\,, \qquad \xi(\check{s}^-(\eta^0, \alpha), \alpha) < \xi(\check{s}^+(\eta^0, \alpha), \alpha)\,.$$

In other words the functions $\hat{s}^\pm(\xi, \alpha)$, $\check{s}^\pm(\eta, \alpha)$ are solutions of the following implicit equations

(2.14) $$\begin{cases} x_1(\hat{s}^\pm(\xi, \alpha)) \cdot \sin \alpha + x_2(\hat{s}^\pm(\xi, \alpha)) \cdot \cos \alpha \equiv \xi\,, & \xi \in [\xi_0(\alpha), \xi_1(\alpha)]\,, \\ x_1(\check{s}^\pm(\eta, \alpha)) \cdot \sin \alpha - x_2(\check{s}^\pm(\eta, \alpha)) \cdot \cos \alpha \equiv \eta\,, & \eta \in [\eta_0(\alpha), \eta_1(\alpha)]\,, \end{cases}$$

satisfying (2.13). Obviously, $\hat{s}^\pm(\xi, \alpha) \in \mathbf{C}[\xi_0(\alpha), \xi_1(\alpha)]$, $\check{s}^\pm(\eta, \alpha) \in \mathbf{C}[\eta_0(\alpha), \eta_1(\alpha)]$ for any $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$.

Let $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ and $p(s, \alpha)$ satisfies (2.12). We set

(2.15) $$P(\xi, \alpha) = p(\hat{s}^-(\xi, \alpha), \alpha)\,, \qquad Q(\eta, \alpha) = -p(\check{s}^-(\eta, \alpha), \alpha)\,.$$

Then from (2.12), (2.14), (2.15) it follows that (1.9) holds. Hence, the following function

$$u(x, \alpha) = P(\xi(x, \alpha), \alpha) + Q(\eta(x, \alpha), \alpha)$$

is G.E. from $\mathbf{L}_2(\Omega)$ if

$$(2.16)\qquad \begin{cases} P(\xi, \alpha) = p(\hat{s}^{-}(\xi, \alpha), \alpha) \in \mathbf{L}_{2,\rho_1}(\xi_0(\alpha), \xi_1(\alpha)), \\ Q(\eta, \alpha) = -p(\check{s}^{-}(\eta, \alpha), \alpha) \in \mathbf{L}_{2,\rho_2}(\eta_0(\alpha), \eta_1(\alpha)). \end{cases}$$

Thus any measurable function $p(s, \alpha)$ satisfying (2.12), (2.16) generates in $\mathbf{L}_2(\Omega)$ the following G.E.

$$(2.17)\qquad u(x, \alpha) = p(\hat{s}^{-}(\xi(x, \alpha), \alpha) - p(\check{s}^{-}(\eta(x, \alpha), \alpha), \alpha)$$

corresponding to $\lambda = -\cos^2 \alpha$, $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \dots, \alpha_m\}$.

Consider the system of functional equations (2.12). Because of (2.4) the system (2.12) is equivalent to the system

$$(2.18)\qquad \begin{cases} p(s, \alpha) = p(f_1(s, \alpha), \alpha), & \text{a.e. } s \in (0, l), \\ p(s, \alpha) = p(f_2(s, \alpha), \alpha), & \text{a.e. } s \in (0, l). \end{cases}$$

Such functional equations are studied in [11], [12].

Consider the following two cases.

I. Let $\alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \setminus \{\alpha_1, \dots, \alpha_m\}$. Then the homomorphism $F(\alpha)$ possesses exactly two fixed points $x(0)$ and $x(s^1)$ where $s^1 \in (0, l)$ is uniquely determined by (2.6). From (2.3), (2.4) and the properties of $f_k(s, \alpha)$ it follows

$$(2.19)\qquad f_k(s^1, \pm\alpha) = s^1, \quad f_k(s, \pm\alpha) \neq s, \quad |k| \in \mathbf{N},\ s \in (0, s^1) \cup (s^1, l).$$

Let $s^0 \in (0, s^1)$ be an arbitrary point. We set

$$(2.20)\qquad M_0(\alpha) = \begin{cases} (s^0, f_2(s^0, \alpha)], & s^0 < f_2(s^0, \alpha), \\ (f_2(s^0, \alpha), s^0], & s^0 > f_2(s^0, \alpha). \end{cases}$$

Following [10] we shall call $M_0(\alpha)$ the generating set for $f_2(s, \alpha)$. We denote

$$(2.21)\qquad M_k(\alpha) = \{f_k(s, \alpha) | s \in M_0\}, \quad k \in \mathbf{Z}.$$

The following lemma follows from the properties of the functions $f_k(s, \alpha)$.

Lemma 2.1 ([9]).
(1) $M_k(\alpha) \cap M_j(\alpha) = \emptyset$, $k \neq j$, $k, j \in \mathbf{Z}$;
(2) $\bigcup_{k \in \mathbf{Z}} M_{2k}(\alpha) = (0, s^1)$, $\bigcup_{k \in \mathbf{Z}} M_{2k+1} = (s^1, l)$.

Corollary 1. *Let* $p_1(s, \alpha)$, $p_2(s, \alpha)$ *satisfy* (2.12). *Then*

$$p_1(s, \alpha) = p_2(s, \alpha), \quad \textit{a.e. } s \in (0, l)$$

if and only if

$$p_1(s, \alpha) = p_2(s, \alpha), \quad \textit{a.e. } s \in M_0(\alpha)$$

Corollary 2. *Let* $\tilde{p}(s, \alpha)$, $s \in M_0(\alpha)$ *be an arbitrary measurable function defined in* $M_0(\alpha)$. *Because of Lemma* 2.1 *for any* $s \in (0, l)$ *there exists a unique number* $k(s, \alpha) \in \mathbf{Z}$ *such that*

$$f_{k(s,\alpha)}(s, \alpha) \in M_0(\alpha)\,. \tag{2.22}$$

Then the function

$$p(s, \alpha) = \tilde{p}(f_{k(s,\alpha)}(s, \alpha), \alpha)\,, \qquad s \in (0, l) \tag{2.23}$$

is a solution of (2.12).

In other words, any measurable solution $p(s, \alpha)$ of (2.12) is uniquely determined by the values $p(s, \alpha)$, $s \in M_0(\alpha)$ and any measurable function $\tilde{p}(s, \alpha)$, $s \in M_0(\alpha)$ generates by (2.23) a measurable solution of (2.12).

Now we can completely describe the structure of G.E. in the case $\alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$.

Theorem 2.1 ([9]). *Any G.E.* $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ *corresponding to* $\lambda = -\cos^2 \alpha$ *generates a function* $p(s, \alpha) \in \mathbf{L}_2(M_0(\alpha))$ *by the formulas* (1.8), (2.9).

Any function $\tilde{p}(s, \alpha) \in \mathbf{L}_2(M_0(\alpha))$ *uniquely determines some G.E.* $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ *by the formulas* (2.17), (2.23).

II. Let $\alpha \in (\alpha^1, \alpha^2) \setminus \{\alpha_1, \ldots, \alpha_m\}$. Then $F(\alpha)$ has exactly one fixed point $x(0)$ and $f_2(s, \pm\alpha) \neq s$ for any $s \in (0, l)$. As far as $f_1(s, \pm\alpha)\colon (0, l) \overset{\text{on}}{\to} (0, l)$ are strictly decreasing functions then there exist points $s^{\pm}(\alpha) \in (0, l)$ such that

$$f_1(s^+(\alpha), \alpha) = s^+(\alpha)\,, \qquad f_1(s^-(\alpha), -\alpha) = s^-(\alpha)\,. \tag{2.24}$$

If $s^+(\alpha) = s^-(\alpha)$ then $f_2(s^+(\alpha), \alpha) = s^+(\alpha)$ contradicts to $f_2(s, \pm\alpha) \neq s$, $s \in (0, l)$. Hence $s^+(\alpha) \neq s^-(\alpha)$.

We set

$$M_0(\alpha) = \begin{cases} (s^-(\alpha), s^+(\alpha))\,, & s^-(\alpha) < s^+(\alpha)\,, \\ (s^+(\alpha), s^-(\alpha))\,, & s^-(\alpha) > s^+(\alpha) \end{cases} \tag{2.25}$$

the generating set in the case $\alpha \in (\alpha^1, \alpha^2) \setminus \{\alpha_1, \ldots, \alpha_m\}$. Define $M_k(\alpha)$ by (2.21).

Lemma 2.2 ([9]).
(1) $M_k(\alpha) \cap M_j(\alpha) = \emptyset$, $k \neq j$, $k, j \in \mathbf{Z}$;
(2) $\bigcup_{k \in \mathbf{Z}} \overline{M_k(\alpha)} = (0, l)$.

Therefore in the case $\alpha \in (\alpha^1, \alpha^2) \setminus \{\alpha_1, \ldots, \alpha_m\}$ Corollaries 1, 2 hold and the structure of G.E. $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ is compltely described by Theorem 2.1.

3. Reformulation of Theorem 2.1

Using the intervals $M_k(\alpha)$ and the functions $f_k(s, \pm\alpha)$ we have described completely the structure of the G.E. of the operator $\mathbf{A}$. However the system of the intervals $M_k(\alpha)$ is not convenient because the length of $M_k(\alpha)$ tends to zero as $|k| \to \infty$ but, at the same time, for any $k, j \in \mathbf{Z}$

$$\{p(s, \alpha) | s \in M_k(\alpha) \setminus O\} = \{p(s, \alpha) | s \in M_j(\alpha) \setminus O\}$$

for some set $O \subset [0, l)$, $\mathrm{mes}(O) = 0$ due to (2.12). T. I. Zelenyak suggested to apply a change of variables $r = G(s, \alpha)\colon (0, l) \overset{\text{on}}{\to} \mathbf{R}$ such that the condition

$$p(s, \alpha) = p(f_k(s, \pm\alpha), \alpha), \qquad \text{a.e. } s \in (0, l)$$

could be written in the form

$$\hat{p}(r, \alpha) = \hat{p}(r + 2\pi, \alpha), \qquad \text{a.e. } r \in \mathbf{R} \tag{3.1}$$

where $\hat{p}(r, \alpha) = p(G^{-1}(r, \alpha), \alpha)$. We shall use this idea to reformulate Theorem 2.1 in a more convenient form.

We need the following statement.

Lemma 3.1 ([9]). *Let* $v(s, \alpha)\colon [0, 1] \times [0, 1] \to [0, 1]$ *satisfy the following conditions*:

(1) $v(s, \alpha) \in \mathbf{C}^4([0, 1] \times [0, 1])$;

(2) $v(0, \alpha) \equiv 0$, $0 < v(s, \alpha) < s$, $s \in (0, 1)$, $\alpha \in [0, 1]$;

(3) $v_s(+0, \alpha) = q(\alpha)$, $0 < q_1 \le q(\alpha) \le q_2 < 1$, $\alpha \in [0, 1]$.

We set

$$v_0(s, \alpha) \equiv s\,, \qquad v_k(s, \alpha) = v(v_{k-1}(s, \alpha), \alpha)\,, \qquad k \in \mathbf{N}\,, \tag{3.2}$$

$$h_k(s, \alpha) = \frac{v_k(s, \alpha)}{q^k(\alpha)}\,, \qquad k \in \mathbf{N}\,. \tag{3.3}$$

There then exists a function $H(s, \alpha) \in \mathbf{C}^2([0, 1] \times [0, 1])$ *such that for any* $\varepsilon \in (0, 1)$

$$h_k \xrightarrow[k \to \infty]{\mathbf{C}^2([0, \varepsilon] \times [0, 1])} H$$

$$\left| \frac{\partial^{i+j} h_k(s, \alpha)}{\partial s^i \partial \alpha^j} - \frac{\partial^{i+j} H(s, \alpha)}{\partial s^i \partial \alpha^j} \right| \le \mathrm{const} \cdot k^j q^k(\alpha)\,, \qquad s \in [0, \varepsilon],\ \alpha \in [0, 1] \tag{3.4}$$

where $i, j = 0, 1, 2$, $i + j \le 2$. *Besides, if* $v_s(s, \alpha) > 0$, $s \in [0, 1]$, $\alpha \in [0, 1]$ *then* $H_s(s, \alpha) > 0$, $s \in [0, 1)$, $\alpha \in [0, 1]$.

Such functions $H(s, \alpha)$ are considered in [11], [12]. The function $H(s, \alpha)$ satisfies the functional equation of Schröder [11], [12]

$$H(v(s, \alpha), \alpha) = q(\alpha) \cdot H(s, \alpha)\,, \qquad s \in [0, 1),\ \alpha \in [0, 1]\,. \tag{3.5}$$

Consider

$$G(s, \alpha) = 2\pi \cdot \log_{q(\alpha)} H(s, \alpha)\,, \qquad s \in (0, 1),\ \alpha \in [0, 1]\,. \tag{3.6}$$

The function $G(s, \alpha)$ satisfies the functional equation of Abel [11], [12]

$$G(v(s, \alpha), \alpha) = G(s, \alpha) + 2\pi\,, \qquad s \in (0, 1),\ \alpha \in [0, 1]\,. \tag{3.7}$$

Assume $v(s, \alpha)$ is strictly increasing function in s for any $\alpha \in [0, 1]$ and $v(1, \alpha) \equiv 1$, $\alpha \in [0, 1]$. Then $G(s, \alpha)$ is strictly decreasing function, $G(s, \alpha)\colon (0, 1) \times [0, 1] \overset{\text{on}}{\to} \mathbf{R}$ and $G(s, \alpha) \in \mathbf{C}^2((0, 1) \times [0, 1])$.

If some function $p(s, \alpha)$ satisfies

$$p(v(s, \alpha), \alpha) = p(s, \alpha), \qquad \text{a.e. } s \in (0, 1), \ \alpha \in [0, 1], \tag{3.8}$$

then the following function

$$\hat{p}(r, \alpha) = p(G^{-1}(r, \alpha), \alpha), \qquad r \in \mathbf{R} \tag{3.9}$$

satisfies (3.1).

Let $[a, b] \subset \left(0, \frac{\pi}{2}\right) \backslash \{\alpha_1, \ldots, \alpha_m\}$ be an arbitrary interval. Consider the following two cases.

I. Let $[a, b] \subset \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \backslash \{\alpha_1, \ldots, \alpha_m\}$. Then for any $\alpha \in [a, b]$ there exists $s^1 \in (0, l)$ defined by (2.6). It is easy to check that s^1 does not depend on $\alpha \in [a, b]$. Then $f_k(s^1, \alpha) \equiv s^1$, $f_k(s, \alpha) \neq s$, $\alpha \in [a, b]$, $s \in (0, s^1) \cup (s^1, l)$. Define for any $\alpha \in [a, b]$, $s \in (0, s^1)$

$$v(s, \alpha) = \begin{cases} f_2(s, \alpha), & f_2\left(\frac{s^1}{2}, \alpha\right) < \frac{s^1}{2}, \\ f_2(s, -\alpha), & f_2\left(\frac{s^1}{2}, \alpha\right) > \frac{s^1}{2}, \end{cases} \tag{3.10}$$

$$v(0, \alpha) \equiv 0, \qquad v(s^1, \alpha) \equiv s^1. \tag{3.11}$$

Then $v(s, \alpha)\colon [0, s^1] \times [a, b] \overset{\text{on}}{\to} [0, s^1]$ is a strictly increasing function in s; $0 < v(s, \alpha) < s$, $s \in (0, s^1)$, $\alpha \in [a, b]$. It is easy to verify that for any $s \in (0, l) \backslash E_2(\alpha)$ the following equality holds [9]:

$$\frac{\partial f_2}{\partial s}(s, \pm\alpha) = \frac{\sin(\pm\alpha - \gamma(s))}{\sin(\pm\alpha - \gamma(f_1(s, \pm\alpha)))} \cdot \frac{\sin(\pm\alpha + \gamma(f_1(s, \pm\alpha)))}{\sin(\pm\alpha + \gamma(f_2(s, \pm\alpha)))} \tag{3.12}$$

where $E_2(\alpha)$ is defined by (2.8) and $\gamma(s) \in [0, 2\pi)$ is uniquely defined for every $s \in (0, l) \backslash E_0(\alpha)$ by the following equation

$$(\cos \gamma(s), \sin \gamma(s)) = \frac{dx(s)}{ds}. \tag{3.13}$$

Equation (3.13) means that $\gamma(s)$ is equal to the angle between $\overrightarrow{(1, 0)}$ and $\overrightarrow{(x_1'(s), x_2'(s))}$.

From the property (6) of the functions f_k and (3.12) it follows that

$$f_{2_s}(+0, \pm\alpha) = \frac{\sin(\pm\alpha - \gamma(+0))}{\sin(\pm\alpha - \gamma(l - 0))} \cdot \frac{\sin(\pm\alpha + \gamma(l - 0))}{\sin(\pm\alpha + \gamma(+0))} = \frac{1}{f_{2_s}(+0, \mp\alpha)}. \tag{3.14}$$

As far as $[a, b] \cap \{\alpha_1, \ldots, \alpha_m\} = \emptyset$ then

$$f_{2_s}(+0, \pm\alpha) \notin \{0, 1, \infty\}, \qquad \alpha \in [a, b]. \tag{3.15}$$

Because of (3.14), (3.15) we obtain

$$\begin{cases} f_{2_s}(+0, \alpha) \in (0, 1), \ \alpha \in [a, b] & \text{if } f_2(s, \alpha) < s, \ s \in (0, s^1), \\ f_{2_s}(+0, -\alpha) \in (0, 1), \ \alpha \in [a, b] & f_2(s, \alpha) > s, \ s \in (0, s^1). \end{cases}$$

Hence $v_s(+0, \alpha) \in (0, 1)$, $\alpha \in [a, b]$ and there exist $q_1, q_2 \in (0, 1)$ that

$$0 < q_1 \leq q(\alpha) = v_s(+0, \alpha) \leq q_2 < 1, \qquad \alpha \in [a, b]. \tag{3.16}$$

From the property (7) of the functions f_k and (2.8) it follows that there exists $\varepsilon \in (0, s^1)$ such that $f_2(s, \pm\alpha) \in \mathbf{C}^4([0, \varepsilon] \times [a, b])$. Using Lemma 3.1 we obtain that the function

$$\tilde{G}_1(s, \alpha) = 2\pi \cdot \log_{q(\alpha)}\left(\lim_{k \to \infty} \frac{v_k(s, \alpha)}{q^k(\alpha)}\right), \qquad s \in (0, \varepsilon], \ \alpha \in [a, b] \tag{3.17}$$

satisfies (3.7) for any $s \in (0, \varepsilon]$, $\alpha \in [a, b]$ and

$$\tilde{G}_1(s, \alpha) \in \mathbf{C}^2((0, \varepsilon] \times [a, b]). \tag{3.18}$$

Let $M_0(\alpha) = (v(\varepsilon, \alpha), \varepsilon]$ be the generating set for $v(s, \alpha)$. Hence for every $s \in (0, s^1)$, $\alpha \in [a, b]$ there exists $k(s, \alpha) \in \mathbf{Z}$ such that

$$v_{k(s,\alpha)}(s, \alpha) \in M_0(\alpha) \tag{3.19}$$

where we set

$$v_{-k}(s, \alpha) = v_k(s, -\alpha) = \begin{cases} f_{2k}(s, -\alpha), & v(s, \alpha) = f_2(s, \alpha), \\ f_{2k}(s, \alpha), & v(s, \alpha) = f_2(s, -\alpha). \end{cases} \tag{3.20}$$

Define

$$G_1(s, \alpha) = \tilde{G}_1(v_{k(s,\alpha)}(s, \alpha), \alpha) - 2\pi \cdot k(s, \alpha), \qquad s \in (0, s^1), \ \alpha \in [a, b]. \tag{3.21}$$

Because of (2.4), (3.17), (3.21) and continuity of $f_k(s, \pm\alpha)$ the following equality holds

$$G_1(s, \alpha) = 2\pi \cdot \log_{q(\alpha)}\left(\lim_{k \to \infty} \frac{v_k(s, \alpha)}{q^k(\alpha)}\right), \qquad s \in (0, s^1), \ \alpha \in [a, b] \tag{3.22}$$

and $G_1(s, \alpha)\colon (0, s^1) \overset{\text{on}}{\to} \mathbf{R}$ is strictly decreasing function satisfying (3.7) for any $s \in (0, s^1)$, $\alpha \in [a, b]$. Using (3.18), (3.21) and the property (7) of the functions f_k we obtain

$$G_1(s, \alpha) \in \mathbf{C}((0, s^1) \times [a, b]) \cap \mathbf{C}^2(((0, s^1) \backslash E(\alpha)) \times [a, b])$$

and for any $i, j = 0, 1, 2$, $i + j \leq 2$, $\alpha \in [a, b]$, $s^* \in E(\alpha)$ there exists

$$\frac{\partial^{i+j} G_1}{\partial s^i \partial \alpha^j}(s^* \pm 0, \alpha) = \lim_{s \to s^* \pm 0} \frac{\partial^{i+j} G_1}{\partial s^i \partial \alpha^j}(s, \alpha)$$

where

$$E(\alpha) = \bigcup_{k \in \mathbf{N}} E_k(\alpha) = \{f_k(s_j, \pm\alpha) | k \in \mathbf{Z}, \ j = 1, \ldots, n\}. \tag{3.23}$$

Hence the function

$$\hat{p}_1(r, \alpha) = p(G_1^{-1}(r, \alpha), \alpha), \qquad r \in \mathbf{R},\ \alpha \in [a, b] \tag{3.24}$$

satisfies (3.1).

Consider the interval (s^1, l). Denote for $\alpha \in [a, b]$, $s \in [0, l - s^1]$

$$w(s, \alpha) = \begin{cases} f_2(s + s^1, \alpha) - s^1, & f_2\left(\dfrac{s^1 + l}{2}, \alpha\right) < \dfrac{s^1 + l}{2}, \\ f_2(s + s^1, -\alpha) - s^1, & f_2\left(\dfrac{s^1 + l}{2}, \alpha\right) > \dfrac{s^1 + l}{2}, \end{cases} \tag{3.25}$$

$$w(0, \alpha) \equiv 0, \qquad w(l - s^1, \alpha) \equiv l - s^1, \qquad \alpha \in [a, b]. \tag{3.26}$$

Then $w(s, \alpha)$: $[0, l - s^1] \times [a, b] \overset{\text{on}}{\to} [0, l - s^1]$ is a strictly increasing function in s; $0 < w(s, \alpha) < s$, $s \in (0, l - s^1)$, $\alpha \in [a, b]$. Using (3.12) and the same arguments as for the function $v(s, \alpha)$ we obtain that there exist $d_1, d_2 \in (0, 1)$ such that

$$0 < d_1 \le d(\alpha) = w_s(+0, \alpha) \le d_2 < 1, \qquad \alpha \in [a, b] \tag{3.27}$$

and the following function

$$\tilde{G}_2(s, \alpha) = 2\pi \cdot \log_{d(\alpha)}\left(\lim_{k \to \infty} \frac{w_k(s, \alpha)}{d^k(\alpha)}\right), \qquad s \in (0, l - s^1),\ \alpha \in [a, b] \tag{3.28}$$

satisfies

$$\tilde{G}_2(w(s, \alpha), \alpha) = \tilde{G}_2(s, \alpha) + 2\pi, \qquad s \in (0, l - s^1),\ \alpha \in [a, b]. \tag{3.29}$$

Define

$$G_2(s, \alpha) = \tilde{G}_2(s - s^1, \alpha) + s^1, \qquad s \in (s^1, l),\ \alpha \in]a, b]. \tag{3.30}$$

Then $G_2(s, \alpha)$: $(s^1, l) \overset{\text{on}}{\to} \mathbf{R}$ is strictly decreasing function satisfying

$$G_2(w(s - s^1, \alpha) + s^1, \alpha) = G_2(s, \alpha) + 2\pi, \qquad s \in (s^1, l),\ \alpha \in [a, b]. \tag{3.31}$$

Besides

$$G_2(s, \alpha) \in \mathbf{C}((s^1, l) \times [a, b]) \cap \mathbf{C}^2(((s^1, l) \backslash E(\alpha)) \times [a, b])$$

and for any $i, j = 0, 1, 2$, $i + j \le 2$, $\alpha \in [a, b]$, $s^* \in E(\alpha)$ there exist

$$\frac{\partial^{i+j} G_2}{\partial s^i \partial \alpha^j}(s^* \pm 0, \alpha) = \lim_{s \to s^* \pm 0} \frac{\partial^{i+j} G_2}{\partial s^i \partial \alpha^j}(s, \alpha).$$

Let $p(s, \alpha)$ satisfy (2.12). From (3.25) it follows

$$p(w(s - s^1, \alpha) + s^1, \alpha) = p(s, \alpha), \qquad \text{a.e. } s \in (s^1, l). \tag{3.32}$$

Then the following function

$$\hat{p}_2(r, \alpha) = p(G_2^{-1}(r, \alpha), \alpha), \qquad r \in \mathbf{R} \tag{3.33}$$

satisfies

$$(3.34)\qquad \hat{p}_2(r, \alpha) = \hat{p}_2(r + 2\pi, \alpha)\,, \qquad \text{a.e. } r \in \mathbf{R},\ \alpha \in [a, b]\,.$$

Because of (2.11), (3.24), (3.33) and $f_1(s, -\alpha)\colon (0, s^1) \overset{\text{on}}{\to} (s^1, l)$ we obtain

$$(3.35)\qquad \hat{p}_2(\beta(r, \alpha), \alpha) = \hat{p}_1(r, \alpha)\,, \qquad r \in \mathbf{R},\ \alpha \in [a, b]$$

where

$$(3.36)\qquad \beta(r, \alpha) = G_2(f_1(G_1^{-1}(r, \alpha), -\alpha), \alpha)\,, \qquad r \in \mathbf{R},\ \alpha \in [a, b]$$

is a continuous strictly decreasing function, $\beta(r, \alpha)\colon \mathbf{R} \overset{\text{on}}{\to} \mathbf{R}$ and

$$(3.37)\qquad \beta(r + 2\pi, \alpha) = \beta(r, \alpha) - 2\pi\,, \qquad r \in \mathbf{R},\ \alpha \in [a, b]\,.$$

Thus Theorem 3.1 in the case $\alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \backslash \{\alpha_1, \ldots, \alpha_m\}$ can be written in the following form.

Theorem 3.1. *Let* $\alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right)\ \{\alpha_1, \ldots, \alpha_m\}$. *There then exist continuous piecewise smooth strictly decreasing functions* $G_1(s, \alpha)\colon (0, s^1) \overset{\text{on}}{\to} \mathbf{R}$, $G_2(s, \alpha)\colon (s^1, l) \overset{\text{on}}{\to} \mathbf{R}$, $\beta(r, \alpha)\colon \mathbf{R} \overset{\text{on}}{\to} \mathbf{R}$ *satisfying* (3.7), (3.31), (3.37) *which are uniquely determined by the shape of the boundary* Γ *and such that*:

(1) *If* $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ *is G.E. corresponding to* $\lambda = -\cos^2 \alpha$ *then there exist* $\hat{p}_1(r, \alpha)$, $\hat{p}_2(r, \alpha)$, $r \in \mathbf{R}$ *satisfying* (3.35),

$$(3.38)\qquad \hat{p}_j(r, \alpha) = \hat{p}_j(r + 2\pi, \alpha)\,, \qquad \text{a.e. } r \in \mathbf{R},\ j = 1,\ 2\,,$$

$$(3.39)\qquad \hat{p}_j(r, \alpha) \in \mathbf{L}_2(0, 2\pi)\,, \qquad j = 1,\ 2$$

and such that $u(x, \alpha)$ *satisfies* (2.17) *where*

$$(3.40)\qquad p(s, \alpha) = \begin{cases} \hat{p}_1(G_1(s, \alpha), \alpha)\,, & s \in (0, s^1)\,, \\ \hat{p}_2(G_2(s, \alpha), \alpha)\,, & s \in (s^1, l)\,. \end{cases}$$

(2) *Any functions* $\hat{p}_1(r, \alpha)$, $\hat{p}_2(r, \alpha)$, $r \in \mathbf{R}$ *satisfying* (3.35), (3.38), (3.39) *determine uniquely some G.E.* $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ *corresponding to* $\lambda = -\cos^2 \alpha$ *by the formulas* (2.17), (3.40).

II. Let $[a, b] \subset (\alpha^1, \alpha^2) \backslash \{\alpha_1, \ldots, \alpha_m\}$. Then $f_2(s, \alpha) \neq s$, $s \in (0, l)$, $a \in [a, b]$. We set for any $s \in (0, l)$, $\alpha \in [a, b]$

$$(3.41)\qquad v(s, \alpha) = \begin{cases} f_2(s, \alpha)\,, & f_2\left(\frac{l}{2}, \alpha\right) < \frac{l}{2}\,, \\ f_2(s, -\alpha)\,, & f_2\left(\frac{l}{2}, \alpha\right) > \frac{l}{2}\,, \end{cases}$$

$$(3.42)\qquad v(0, \alpha) \equiv 0\,, \qquad v(l, \alpha) \equiv l\,.$$

Then $v(s, \alpha)$: $[0, l] \times [a, b] \overset{\text{on}}{\to} [0, l]$ is strictly increasing function in s; $0 < v(s, \alpha) < s$, $s \in (0, l)$, $\alpha \in [a, b]$. Applying the same arguments as in the case **I** we obtain

$$0 < q_1 \le q(\alpha) = v_s(+0, \alpha) \le q_2 < 1 , \qquad \alpha \in [a, b] ,$$

the following function

$$G(s, \alpha) = 2\pi \cdot \log_{q(\alpha)}\left(\lim_{k\to\infty} \frac{v_k(s, \alpha)}{q^k(\alpha)}\right), \qquad s \in (0, l),\ \alpha \in [a, b] \tag{3.43}$$

satisfies (3.7), $G(s, \alpha)$: $(0, l) \times [a, b] \overset{\text{on}}{\to} \mathbf{R}$ is strictly decreasing in s,

$$G(s, \alpha) \in \mathbf{C}((0, l) \times [a, b]) \cap \mathbf{C}^2((0, l)\backslash E(\alpha)) \times [a, b])$$

and for any $i, j = 0, 1, 2, i + j \le 2, \alpha \in [a, b], s^* \in E(\alpha)$ there exists

$$\frac{\partial^{i+j} G}{\partial s^i \partial \alpha^j}(s^* \pm 0, \alpha) = \lim_{s\to s^*\pm 0} \frac{\partial^{i+j} G}{\partial s^i \partial \alpha^j}(s, \alpha) .$$

Since

$$p(s, \alpha) = p(f_2(s, \pm\alpha), \alpha) , \qquad \text{a.e. } s \in (0, l)$$

the following function

$$\hat{p}(r, \alpha) = p(G^{-1}(r, \alpha), \alpha) , \qquad r \in \mathbf{R} \tag{3.45}$$

satisfies

$$\hat{p}(r + 2\pi, \alpha) = \hat{p}(r, \alpha) , \qquad \text{a.e. } r \in \mathbf{R} . \tag{3.46}$$

Because of (2.11)

$$p(s, \alpha) = p(f_1(s, -\alpha), \alpha) , \qquad s \in (0, l) .$$

Hence

$$\hat{p}(\mu(r, \alpha), \alpha) = \hat{p}(r, \alpha) , \qquad r \in \mathbf{R} \tag{3.47}$$

where

$$\mu(r, \alpha) = G(f_1(G^{-1}(r, \alpha), -\alpha), \alpha) , \qquad r \in \mathbf{R} \tag{3.48}$$

is continuous strictly decreasing function which maps **R** onto **R** and

$$\mu(r + 2\pi, \alpha) = \mu(r, \alpha) - 2\pi , \qquad r \in \mathbf{R} , \tag{3.49}$$

$$\mu(\mu(r, \alpha), \alpha) \equiv r , \qquad r \in \mathbf{R} . \tag{3.50}$$

Thus in the case $\alpha \in (\alpha^1, \alpha^2)\backslash\{\alpha_1, \ldots, \alpha_m\}\}$ Theorem 2.1 can be written in the following form.

Theorem 3.2. *Let* $\alpha \in (\alpha^1, \alpha^2)\backslash\{\alpha_1, \ldots, \alpha_m\}$. *Then there exist continuous piecewise smooth strictly decreasing functions* $G(s, \alpha)$: $(0, l) \overset{\text{on}}{\to} \mathbf{R}$, $\mu(r, \alpha)$: $\mathbf{R} \overset{\text{on}}{\to} \mathbf{R}$ *satisfying* (3.7), (3.49), (3.50) *which are uniquely determined by the shape of the boundary* Γ *and such that*:

(1) *If* $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ *is G.E. corresponding to* $\lambda = -\cos^2 \alpha$ *then there exists* $\hat{p}(r, \alpha)$, $r \in \mathbf{R}$ *satisfying* (3.46), (3.47),

$$\hat{p}(r, \alpha) \in \mathbf{L}_2(0, 2\pi) \tag{3.51}$$

and such that $u(x, \alpha)$ *satisfies* (2.17) *where*

$$p(s, \alpha) = \hat{p}(G(s, \alpha), \alpha), \qquad s \in (0, l). \tag{3.52}$$

(2) *Any function* $\hat{p}(r, \alpha)$, $r \in \mathbf{R}$ *satisfying* (3.46), (3.47), (3.51) *determines uniquely some G.E.* $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ *corresponding to* $\lambda = -\cos^2 \alpha$ *by the formulas* (2.17), (3.52).

4. Integral representation of solutions

Consider the operator equation in $\overset{\circ}{\mathbf{W}}{}^1_2(\Omega)$

$$u_{tt} = \mathbf{A}u, \qquad u \in \overset{\circ}{\mathbf{W}}{}^1_2(\Omega). \tag{4.1}$$

S. L. Sobolev suggested to look for solutions of (4.1) in the form

$$u(t) = \int_{-1}^{0} \chi_\lambda \cdot \exp(i\sqrt{-\lambda} \cdot t) d\mu(\lambda) \tag{4.2}$$

where χ_λ is a family of G.E. of the operator $\mathbf{A}$, $\mu(\lambda)$ is some nondecreasing function. Consider the following function

$$v(t, x) = \int_0^{\pi/2} \exp(i \cdot \cos \alpha \cdot t) \cdot u(x, \alpha) d\alpha \tag{4.3}$$

where $u(x, \alpha)$ is some family of G.E. in $\mathbf{L}_2(\Omega)$ corresponding to $\lambda = -\cos^2 \alpha$. If $v(t, x) \in \overset{\circ}{\mathbf{W}}{}^1_2(\Omega)$ for any $t > 0$ then $v(t, x)$ satisfies (4.1). So we shall discuss under what conditions on the family $u(x, \alpha)$ the following function

$$u(x) = \int_0^{\pi/2} u(x, \alpha) d\alpha \tag{4.4}$$

belongs to $\overset{\circ}{\mathbf{W}}{}^1_2(\Omega)$. In section 6 we derive the Parseval's equality for the integral representation (4.4) which gives complete answer to this question. In the present section we construct a class of smooth G.E. $u(x, \alpha) \in \mathbf{C}^2\left(\Omega \times \left[0, \frac{\pi}{2}\right]\right)$ such that $u(x) \in \overset{\circ}{\mathbf{W}}{}^1_2(\Omega) \cap \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^2(\Omega)$.

We consider again two cases.

I. Let $[a, b] \subset \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ be an arbitrary interval. Then for any $\alpha \in [a, b]$ each G.E. $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ corresponding to $\lambda = -\cos^2 \alpha$ is uniquely determined by some functions $\hat{p}_j(r, \alpha)$, $j = 1, 2$ satisfying (3.35), (3.38), (3.39). From (3.38), (3.39) it follows that $\hat{p}_j(r, \alpha)$ can be regarded as a function defined on the

(4.5) $$S^1 \times [a, b] = \{\exp(ir) | r \in \mathbf{R}\} \times [a, b] = \{t \in \mathbf{C} \,|\, |t| = 1\} \times [a, b]$$

the lateral surface of the unit cylinder $\Pi_a^b = \{(x, \alpha) \,|\, |x| \le 1, \alpha \in [a, b]\}$. So instead of $\hat{p}_j(r, \alpha)$ we can consider the functions

(4.6) $$\begin{cases} \check{p}_j(t, \alpha) = \hat{p}_j(\arg t, \alpha), & |t| = 1, \ \alpha \in [a, b], \\ \check{p}_j(t, \alpha) \in \mathbf{L}_2(S^1), & \alpha \in [a, b]. \end{cases}$$

The function $\beta(r, \alpha)$ defined by (3.36) generates

(4.7) $$\check{\beta}(t, \alpha) = \exp(i \cdot \beta(\arg t, \alpha)), \qquad |t| = 1, \ \alpha \in [a, b]$$

an orientation-reversing homeomorphism of S^1.

Consider the set $E(\alpha)$ defined by (3.23). From (2.4), (3.23) it follows that

(4.8) $$E(\alpha) = \{f_{2k}(s, \alpha) | k \in \mathbf{Z}, s \in E_1(\alpha)\}$$

where from (2.8)

(4.9) $$E_1(\alpha) = \{f_j(s_p, \alpha) | j = -1, 0, 1; p = 1, \ldots, n\},$$

$\{x(s_1), \ldots, x(s_n)\}$ is the set of the endpoints of $\{\Gamma_j\}_1^n$.

So $E_1(\alpha)$ is a finite set. Denote

(4.10) $$\begin{cases} O_1(\alpha) = \{\exp(iG_1(s, \alpha)) | s \in E_1(\alpha) \cap (0, s^1)\} \subset S^1, \\ O_2(\alpha) = \{\exp(iG_2(s, \alpha)) | s \in E_1(\alpha) \cap (s^1, l)\} \subset S^1. \end{cases}$$

Since $G_j(s, \alpha)$ satisfy (3.7), (3.31) then using (3.10), (3.25), (4.8), (4.10) we obtain

(4.11) $$\begin{cases} O_1(\alpha) = \{\exp(iG_1(s, \alpha)) | s \in E(\alpha) \cap (0, s^1)\}, \\ O_2(\alpha) = \{\exp(iG_2(s, \alpha)) | s \in E(\alpha) \cap (s^1, l)\}. \end{cases}$$

Because of (4.9), (4.10) and continuity of $f_k(s, \pm\alpha)$

(4.12) $$O_1(\alpha) = \{\tau_j^1(\alpha) | j = 1, \ldots, k_1\}, \qquad O_2(\alpha) = \{\tau_j^2(\alpha) | j = 1, \ldots, k_2\}$$

where $\tau_j^1(\alpha)$, $\tau_j^2(\alpha)$: $[a, b] \to S^1$ are some continuous functions uniquely determined by the shape of the boundary.

Since

$$\{f_1(s, -\alpha) | s \in (0, s^1) \cap E(\alpha)\} = (s^1, l) \cap E(\alpha),$$

$$\{f_1(s, -\alpha) | s \in (s^1, l) \cap E(\alpha)\} = (0, s^1) \cap E(\alpha)$$

then from (4.7), (4.11), (4.12)

(4.13) $$O_2(\alpha) = \check{\beta}(O_1(\alpha), \alpha) = \{\check{\beta}(t, \alpha) | t \in O_1(\alpha)\}.$$

Hence

(4.14) $$\begin{cases} k_1 = k_2, \\ \tau_j^2(\alpha) = \check{\beta}(\tau_j^1(\alpha), \alpha), & j = 1, \ldots, k_1. \end{cases}$$

For any $\varepsilon > 0$ we denote by $\check{B}^{\varepsilon}_{[a,b]} \subset \mathbf{C}^2(S^1 \times [a, b])$ such a set that $p(t, \alpha) \in \check{B}^{\varepsilon}_{[a,b]}$ if $p(t, \alpha) \in \mathbf{C}^2(S^1 \times [a, b])$ and the following condition holds

(i) $p_t(t, \alpha) = 0$ if $|t - \tau^1_j(\alpha)| < \varepsilon$ for some $j \in \{1, \dots, k_1\}$.

Let $\check{p}(t, \alpha) \in \check{B}^{\varepsilon}_{[a,b]}$ be an arbitrary function. Then using the properties of G_1, G_2, β and formulas (4.11)–(4.14) we obtain that

$$p(s, \alpha) = \begin{cases} \check{p}(\exp(iG_1(s, \alpha)), \alpha), & s \in (0, s^1), \\ \check{p}(\exp(i\beta^{-1}(G_2(s, \alpha), \alpha)), \alpha), & s \in (s^1, l) \end{cases} \tag{4.15}$$

satisfies (2.12),

$$p(s, \alpha) \in \mathbf{C}^2((0, s^1) \times [a, b]) \cap \mathbf{C}^2((s^1, l) \times [a, b]) \tag{4.16}$$

and for any $\alpha^* \in [a, b]$, $s^* \in E(\alpha^*)$ there exists $\delta > 0$ such that

$$p_s(s, \alpha) = 0, \qquad s \in (s^* - \delta, s^* - \delta),\ \alpha \in (\alpha^* - \delta, \alpha^* + \delta). \tag{4.17}$$

From the properties of the boundary Γ and formulas (2.14), (4.15)–(4.17) it follows that G.E. $u(x, \alpha)$ defined by (2.17), (4.15) satisfies

$$u(x, \alpha) \in \mathbf{C}^2(\Omega \times [a, b]); \qquad u(x, \alpha) \in \mathbf{L}_2(\Omega),\ \alpha \in [a, b].$$

We denote by $B_{[a,b]} \subset \mathbf{C}^2(\Omega \times [a, b])$ such set of G.E. $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ that $u(x, \alpha) \in B_{[a,b]}$ if there exist $\varepsilon > 0$, $\check{p}(t, \alpha) \in \check{B}^{\varepsilon}_{[a,b]}$ that (2.17), (4.15) hold.

II. Let $[a, b] \subset (\alpha^1, \alpha^2) \setminus \{\alpha_1, \dots, \alpha_m\}$ be an arbitrary interval. Then for any $\alpha \in [a, b]$ each G.E. $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ corresponding to $\lambda = -\cos^2 \alpha$ is uniquely determined by some function $\check{p}(t, \alpha) \in \mathbf{L}_2(S^1)$ satisfying

$$\check{p}(\check{\mu}(t, \alpha), \alpha) = \check{p}(t, \alpha) \tag{4.18}$$

where

$$\check{\mu}(t, \alpha) = \exp(i \cdot \mu(\arg t, \alpha)) \tag{4.19}$$

is an orientation-reversing homeomorphism of the unit circle S^1 satisfying the Carleman's condition

$$\check{\mu}(\check{\mu}(t, \alpha), \alpha) \equiv t, \qquad t \in S^1. \tag{4.20}$$

Denote

$$O(\alpha) = \{\exp(iG(s, \alpha)) | s \in E_1(\alpha)\} \subset S^1. \tag{4.21}$$

Then there exist $k_0 \in \mathbf{N}$ and continuous functions $\tau_1(\alpha), \dots, \tau_{k_0}(\alpha)\colon [a, b] \to S^1$ such that

$$O(\alpha) = \{\tau_j(\alpha) | j = 1, \dots, k_0\}. \tag{4.22}$$

From (4.8) it follows that

$$O(\alpha) = \{\exp(iG(s, \alpha)) | s \in E(\alpha)\}. \tag{4.23}$$

Since

$$\{f_1(s, -\alpha) | s \in E(\alpha)\} = E(\alpha)$$

then

$$\{\check{\mu}(s, \alpha) | s \in O(\alpha)\} = O(\alpha)\,. \tag{4.24}$$

For any $\varepsilon > 0$ we denote by $\check{B}^{\varepsilon}_{[a,b]} \subset \mathbf{C}^2(S^1 \times [a, b])$ such set that $p(t, \alpha) \in \check{B}^{\varepsilon}_{[a,b]}$ if $p \in \mathbf{C}^2(S^1 \times [a, b])$ and the following condition holds:

(ii) $p_t(t, \alpha) = 0$ if $|t - \tau_j(\alpha)| < \varepsilon$ for some $j \in \{1, \ldots, k_0\}$.

Let $\check{p}(t, \alpha) \in \check{B}^{\varepsilon}_{[a,b]}$ be an arbitrary function satisfying (4.18). Then

$$p(s, \alpha) = \check{p}(\exp(iG(s, \alpha)), \alpha)\,, \qquad s \in (0, l) \tag{4.25}$$

satisfies (2.12),

$$p(s, \alpha) \in \mathbf{C}^2((0, l) \times [a, b]) \tag{4.26}$$

and for any $\alpha^* \in [a, b]$, $s^* \in E(\alpha^*)$ there exists $\delta > 0$ such that (4.17) holds.

So from the properties of the boundary Γ and formulas (2.14), (4.17), (4.25), (4.26) it follows that G.E. $u(x, \alpha)$ defined by (2.17), (4.25) satisfies

$$u(x, \alpha) \in \mathbf{C}^2(\Omega \times [a, b])\,; \qquad u(x, \alpha) \in \mathbf{L}_2(\Omega),\ \alpha \in [a, b]\,.$$

We denote by $B_{[a,b]} \subset \mathbf{C}^2(\Omega \times [a, b])$ such set of G.E. $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ that $u(x, \alpha) \in B_{[a,b]}$ if there exist $\varepsilon > 0$, $\check{p}(t, \alpha) \in \check{B}^{\varepsilon}_{[a,b]}$ that (2.17), (4.25) hold.

Thus for any $[a, b] \subset \left(0, \frac{\pi}{2}\right) \Big\backslash \{\alpha_1, \ldots, \alpha_m\}$ we have defined the set of functions $u(x, \alpha) \in B_{[a,b]}$ such that $u(x, \alpha) \in \mathbf{C}^2(\Omega \times [a, b])$ and for any $\alpha \in [a, b]$, $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ is G.E. of the operator $\mathbf{A}$ corresponding to $\lambda = -\cos^2 \alpha$.

We denote by $\mathbf{D}$ set of the functions $u(x, \alpha)$, $x \in \Omega$, $\alpha \in \left[0, \frac{\pi}{2}\right]$ satisfying the following conditions:

(1) $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ is G.E. of $\mathbf{A}$ corresponding to $\lambda = -\cos^2 \alpha$ for any $\alpha \in \left(0, \frac{\pi}{2}\right)$;

(2) $u(x, \alpha) \in \mathbf{C}^2\left(\Omega \times \left[0, \frac{\pi}{2}\right]\right)$;

(3) there exists $\delta > 0$ such that $u(x, \alpha) \equiv 0$, $\alpha \in [0, \delta] \bigcup \left[\frac{\pi}{2} - \delta, \frac{\pi}{2}\right] \bigcup_{j=1}^{m} [\alpha_j - \delta, \alpha_j + \delta]$;

(4) $u(x, \alpha) \in B_{[a,b]}$ for any $[a, b] \subset \left(0, \frac{\pi}{2}\right) \Big\backslash \{\alpha_1, \ldots, \alpha_m\}$.

Theorem 4.1 ([9]). *If* $u(x, \alpha) \in \mathbf{D}$ *then*

$$u(x) = \int_0^{\pi/2} u(x, \alpha) d\alpha \in \mathring{\mathbf{W}}{}^1_2(\Omega) \cap \mathbf{C}(\bar{\Omega}) \cap \mathbf{C}^2(\bar{\Omega} \backslash \{P_1, \ldots, P_k\}) \tag{4.27}$$

$$(4.28)\qquad |u_{x_1}(x)|,\ |u_{x_2}(x)| \le \mathrm{const}\cdot \sum_{j=1}^{k} \frac{1}{|P_j - x|}$$

where $P_1, \ldots, P_k \in \Gamma$, $k \le 4$ *are corner points of* Γ *such that each* P_j *is fixed point of* $F(\alpha)$ *for some* $\alpha \in \left(0, \frac{\pi}{2}\right)\Big\backslash\{\alpha_1, \ldots, \alpha_m\}$.

Corollary. *If* $u(x, \alpha) \in \mathbf{D}$ *then the function* $v(x, t)$ *defined by* (4.3) *is a continuous solution of the equation* (4.1).

5. Piecewise constant G.E.

Piecewise constant G.E. (P.C.G.E.) of the operator **A** were first studied by R. A. Aleksandryan [13]. We consider again two cases.

I. Let $\alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right)\Big\backslash\{\alpha_1, \ldots, \alpha_m\}$. Let $\theta \in [0, 2\pi)$ be an arbitrary point. We set

$$(5.1)\qquad \hat{p}_1(r, \alpha, \theta) = \begin{cases} 1, & \exp(ir) \in [1, \exp(i\theta)]_{S^1}, \\ 0, & \exp(ir) \notin [1, \exp(i\theta)]_{S^1} \end{cases}$$

where for any $t_1, t_2 \in S^1$ we denote

$$[t_1, t_2]_{S^1} = \begin{cases} \{\exp(ir) | \arg t_1 \le r \le \arg t_2\}, & \arg t_1 \le \arg t_2, \\ \{\exp(ir) | \arg t_1 \le r \le \arg t_2 + 2\pi\}, & \arg t_1 > \arg t_2. \end{cases}$$

Following (2.17), (4.15) we define

$$(5.2)\qquad p(s, \alpha, \theta) = \begin{cases} \hat{p}_1(G_1(s, \alpha), \alpha, \theta), & s \in (0, s^1), \\ \hat{p}_1(\beta^{-1}(G_2(s, \alpha), \alpha, \theta), & s \in (s^1, l), \end{cases}$$

$$(5.3)\qquad u(x, \alpha, \theta) = p(\hat{s}^-(\xi(x, \alpha), \alpha), \alpha, \theta) - p(\check{s}^-(\eta(x, \alpha), \alpha), \alpha, \theta).$$

We shall call $u(x, \alpha, \theta)$ the P.C.G.E. generated by $\theta \in [0, 2\pi)$. From (5.1)–(5.3) it follows that $u(x, \alpha, \theta) \in \{-1, 0, 1\}$ for any $x \in \Omega$.

The structure of $u(x, \alpha, \theta)$ is rather simple: values θ, $0 \in [0, 2\pi)$ generate points

$$(5.4)\qquad s(\theta, \alpha) = G_1^{-1}(\theta, \alpha), \qquad s(0, \alpha) = G_1^{-1}(0, \alpha)$$

which belong to $(0, s^1)$; the points $s(\theta, \alpha)$, $s(0, \alpha)$ generate two infinite polygonal lines with the segments parallel to the lines $\xi(x, \alpha) = \mathrm{const}$, $\eta(x, \alpha) = \mathrm{const}$ (see Figure 3); these polygonal lines form infinite set of parallelograms $\Pi_k \subset \Omega$, $k \in \mathbf{Z}$ such that (see Figure 3)

$$(5.5)\qquad u(x, \alpha, \theta) = \begin{cases} (-1)^k, & x \in \Pi_k, \\ 0, & x \in \Omega \backslash \bigcup_{k \in \mathbf{Z}} \Pi_k. \end{cases}$$

II. Let $\alpha \in (\alpha^1, \alpha^2) \backslash \{\alpha_1, \ldots, \alpha_m\}$. Let $\theta \in [0, 2\pi)$ be an arbitrary point. We set

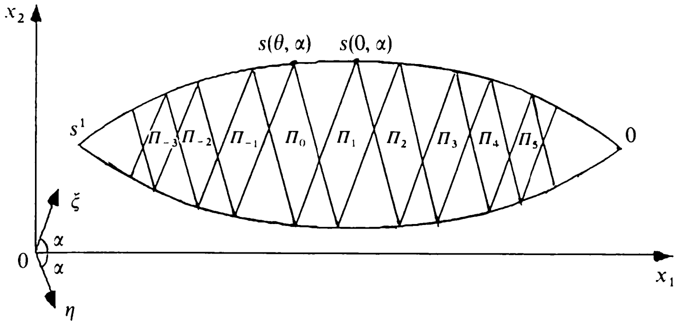

Figure 3

$$(5.6) \qquad \hat{p}(r, \alpha, \theta) = \begin{cases} 1, & \exp(ir) \in [\exp(i\theta), \exp(i\mu(\theta, \alpha))]_{S^1}, \\ 0, & \exp(ir) \notin [\exp(i\theta), \exp(i\mu(\theta, \alpha))]_{S^1}. \end{cases}$$

Then $\hat{p}(r, \alpha, \theta)$ satisfies (3.47). Following (2.17), (4.25) we define

$$(5.7) \qquad p(s, \alpha, \theta) = \hat{p}(G(s, \alpha), \alpha, \theta), \qquad s \in (0, l),$$

$$(5.8) \qquad u(x, \alpha, \theta) = p(\hat{s}^-(\xi(x, \alpha), \alpha), \alpha, \theta) - p(\check{s}^-(\eta(x, \alpha), \alpha), \alpha, \theta).$$

Using (5.6)–(5.8) we obtain that $u(x, \alpha, \theta) \in \{-1, 0, 1\}$ for any $x \in \Omega$. The structure of $u(x, \alpha, \theta)$ is rather simple: the point $\theta \in [0, 2\pi)$ generates

$$(5.9) \qquad s(\theta, \alpha) = G^{-1}(\theta, \alpha)$$

which belongs to $(0, l)$; the point $s(\theta, \alpha)$ generates infinite polygonal line with the segments parallel to the lines $\xi(x, \alpha) = \text{const}$, $\eta(x, \alpha) = \text{const}$ (see Figure 4); this polygonal line forms an infinite set of parallelograms $\Pi_k \subset \Omega$, $k \in \mathbf{N}$ such that (see Figure 4)

$$(5.10) \quad u(x, \alpha, \theta) = \begin{cases} (-1)^k \cdot \operatorname{sgn}(\theta - \arg(\exp(i\mu(\theta, \alpha)))), & x \in \Pi_k, \\ 0, & x \in \Omega \setminus \bigcup_{k \in \mathbf{N}} \Pi_k. \end{cases}$$

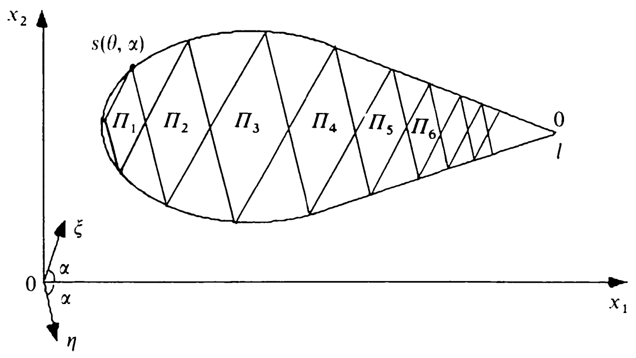

Figure 4

Theorem 5.1 ([9]). *Let $u(x,\alpha)\in \mathbf{D}$. Then the function $u(x)$ defined by* (4.4) *satisfies*:

$$(5.11)\qquad (\Delta u(x), u(x,\alpha,\theta))_{\mathbf{L}_2(\Omega)} = \frac{1}{2}\int_0^{2\pi} \hat{p}_1^u(r,\alpha)\left[\cot\frac{r-\theta}{2} - \cot\frac{r}{2}\right] - \hat{p}_2^u(r,\alpha)\left[\cot\frac{r-\beta(\theta,\alpha)}{2} - \cot\frac{r}{2}\right]dr$$

if $\alpha \in \left((0,\alpha^1)\cup\left(\alpha^2,\frac{\pi}{2}\right)\right)\setminus\{\alpha_1,\dots,\alpha_m\}$;

$$(5.12)\qquad (\Delta u(x), u(x,\alpha,\theta))_{\mathbf{L}_2(\Omega)} = -\frac{1}{2}\int_0^{2\pi} \hat{p}^u(r,\alpha)\left[\cot\frac{r-\theta}{2} - \cot\frac{r-\mu(\theta,\alpha)}{2}\right]dr$$

if $\alpha\in(\alpha^1,\alpha^2)\setminus\{\alpha_1,\dots,\alpha_m\}$.

The functions $\hat{p}_1^u$, $\hat{p}_2^u$, $\hat{p}^u$ correspond to the family of G.E. $u(x,\alpha)$ and are determined by (1.8), (2.9), (3.24), (3.33), (3.45).

Remark 1. Since $u(x,\alpha)\in\mathbf{D}$ then $u(x)\in \mathring{\mathbf{W}}_2^1(\Omega)\cap \mathbf{C}^2(\bar{\Omega})\cap \mathbf{C}^2(\bar{\Omega}\setminus\{P_1,\dots,P_k\})$. So regard

$$(5.13)\qquad \begin{gathered}(\Delta u(x), u(x,\alpha,\theta))_{\mathbf{L}_2(\Omega)} = \lim_{\varepsilon\to+0}(\Delta u(x), u(x,\alpha,\theta))_{\mathbf{L}_2(\Omega_\varepsilon)}\\ \Omega_\varepsilon = \{x\in\Omega \mid \operatorname{dist}(P_j,x)>\varepsilon, j=1,\dots,k\}\,.\end{gathered}$$

Remark 2. The integrals on the right-hand side of (5.11), (5.12) are understood in the sense of the Cauchy principal values.

6. The Parseval's equality

In the present section we derive the Parseval's equality for the integral representation (4.4).

I. Let $\alpha \in \left((0,\alpha^1)\cup\left(\alpha^2,\frac{\pi}{2}\right)\right)\setminus\{\alpha_1,\dots,\alpha_m\}$. We denote

$$(6.1)\qquad \hat{\mathbf{L}}_2 = \{p(r),\ r\in\mathbf{R} \mid p\in\mathbf{L}_2(0,2\pi);\ p(r+2\pi)=p(r),\ \text{a.e. } r\in\mathbf{R}\}$$

Let $\hat{p}(r)\in\hat{\mathbf{L}}_2$ be an arbitrary function. We define

$$(6.2)\qquad \Lambda(\alpha)\hat{p}(r) = u(x,\alpha) = p(\check{s}^-(\xi(x,\alpha),\alpha),\alpha) - p(\check{s}^-(\eta(x,\alpha),\alpha),\alpha)$$

where

$$(6.3)\qquad p(s,\alpha) = \begin{cases}\hat{p}(G_1(s,\alpha))\,, & s\in(0,s^1)\,,\\ \hat{p}(\beta^{-1}(G_2(s,\alpha),\alpha))\,, & s\in(s^1,l)\,.\end{cases}$$

From Theorem 3.1 it follows that $\Lambda(\alpha)$ is linear operator which maps $\hat{\mathbf{L}}_2$ onto the space of G.E. $u(x,\alpha)\in\mathbf{L}_2(\Omega)$ corresponding to $\lambda=-\cos^2\alpha$. The following statement can be derived from the results obtained in [9].

Lemma 6.1. *For any interval* $[a, b] \subset \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \setminus \{\alpha_1, \dots, \alpha_m\}$ *the operator* $\Lambda(\alpha)$ *is bounded uniformly in* $\alpha \in [a, b]$. *It means that there exists a constant* $C > 0$ *such that*

(6.4) $$\|\Lambda(\alpha)\hat{p}\|_{\mathbf{L}_2(\Omega)} \leq C \cdot \|\hat{p}\|_{\mathbf{L}_2(0, 2\pi)}, \qquad \hat{p} \in \hat{\mathbf{L}}_2,\ \alpha \in [a, b].$$

Let $v \in [0, 2\pi)$ be an arbitrary point. From (5.1)–(5.3) it follows that

(6.5) $$\Lambda(\alpha)\Theta(v - r) = u(x, \alpha, v)$$

where

$$\Theta(\varphi) = \begin{cases} 1, & \varphi \geq 0, \\ 0, & \varphi < 0 \end{cases}$$

is the Heaviside's function. It is easy to see that (6.2), (6.3) imply

(6.6) $$\Lambda(\alpha)C = 0$$

for any constant C.

Let $\hat{p}(r, \alpha) \in \mathbf{C}^1[0, 2\pi]$. Then

(6.7) $$u(x, \alpha) = \Lambda(\alpha)\hat{p}(r, \alpha) = \Lambda(\alpha)\left(\int_0^{2\pi} \hat{p}_\varphi(\varphi, \alpha) \cdot \Theta(r - \varphi)d\varphi + \hat{p}(0, \alpha)\right)$$
$$= \int_0^{2\pi} \hat{p}_\varphi(\varphi, \alpha) \cdot \Lambda(\alpha)\Theta(r - \varphi)d\varphi$$
$$= \int_0^{2\pi} \hat{p}_\varphi(\varphi, \alpha) \cdot \Lambda(\alpha)(1 - \Theta(\varphi - r))d\varphi$$
$$= -\int_0^{2\pi} \hat{p}_\varphi(\varphi, \alpha) \cdot u(x, \alpha, \varphi)d\varphi .$$

II. Let $\alpha \in (\alpha^1, \alpha^2) \setminus \{\alpha_1, \dots, \alpha_m\}$. Consider the function $\mu(r, \alpha)$ defined by (3.48). Since $\mu(r, \alpha)\colon \mathbf{R} \overset{\text{on}}{\to} \mathbf{R}$ is a strictly decreasing function then there exists a unique number $r^*(\alpha) \in \mathbf{R}$ such that

(6.8) $$\mu(r^*(\alpha), \alpha) = r^*(\alpha).$$

Define

(6.9) $$\mu_0(r, \alpha) = 2\pi - r^*(\alpha) + \mu(r + r^*(\alpha), \alpha), \qquad r \in \mathbf{R}.$$

Then from (3.49), (3.50), (6.8), (6.9) we obtain

(6.10) $$\mu_0(r + 2\pi, \alpha) = \mu_0(r, \alpha) - 2\pi, \qquad r \in \mathbf{R},$$

(6.11) $$\mu_0(\mu_0(r, \alpha), \alpha) \equiv r, \qquad r \in \mathbf{R},$$

(6.12) $$\mu_0(0, \alpha) = 2\pi, \qquad \mu_0(2\pi, \alpha) = 0.$$

Hence $\mu(r, \alpha)$: $[0, 2\pi] \overset{\text{on}}{\to} [0, 2\pi]$ is a strictly decreasing function satisfying (6.11). If $\hat{p}(r, \alpha)$ satisfies (3.46), (3.47) then the following function

$$\hat{p}_0(r, \alpha) = \hat{p}(r + r^*(\alpha)) \tag{6.13}$$

satisfies (3.46) and

$$\hat{p}_0(\mu_0(r, \alpha), \alpha) = \hat{p}_0(r, \alpha)\,, \qquad \text{a.e. } r \in \mathbf{R}\,. \tag{6.14}$$

We denote

$$G_0(s, \alpha) = G(s, \alpha) - r^*(\alpha)\,, \qquad s \in (0, l)\,. \tag{6.15}$$

Thus Theorem 3.2 can be written in the following form.

Theorem 6.1. *Let* $\alpha \in (\alpha^1, \alpha^2)\backslash\{\alpha_1, \ldots, \alpha_m\}$. *Then there exist continuous piecewise smooth strictly decreasing functions* $G_0(s, \alpha)$: $(0, l) \overset{\text{on}}{\to} \mathbf{R}$, $\mu_0(r, \alpha)$: $\mathbf{R} \overset{\text{on}}{\to} \mathbf{R}$ *satisfying* (3.7), (6.10)–(6.12) *which are uniquely determined by the shape of the boundary* Γ *and such that*:

(1) *If* $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ *is a G.E. corresponding to* $\lambda = \cos^2 \alpha$ *then there exists* $\hat{p}_0(r, \alpha)$, $r \in \mathbf{R}$ *satisfying* (3.46), (6.14),

$$\hat{p}_0(r, \alpha) \in \mathbf{L}_2(0, 2\pi) \tag{6.16}$$

and such that $u(x, \alpha)$ *satisfies* (2.17) *where*

$$p(s, \alpha) = \hat{p}_0(G_0(s, \alpha), \alpha)\,, \qquad s \in (0, l) \tag{6.17}$$

(2) *Any function* $\hat{p}_0(r, \alpha)$, $r \in \mathbf{R}$ *satisfying* (3.46), (6.14), (6.16) *determines some G.E.* $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ *corresponding to* $\lambda = -\cos^2 \alpha$ *by the formulas* (2.17), (6.17).

It is also easy to check that (5.12) can be written in the form

$$(\Delta u(x), u(x, \alpha, \theta))_{\mathbf{L}_2(\Omega)} = -\frac{1}{2}\int_0^{2\pi} \hat{p}_0^u(r, \alpha)\left[\cot\frac{r - \theta}{2} - \cot\frac{r - \mu_0(\theta, \alpha)}{2}\right] dr\,. \tag{6.18}$$

As far as $\mu_0(r, \alpha)$: $[0, 2\pi] \overset{\text{on}}{\to} [0, 2\pi]$ is strictly decreasing function then there exists unique $\varphi^*(\alpha) \in (0, 2\pi)$ such that

$$\mu_0(\varphi^*(\alpha), \alpha) = \varphi^*(\alpha)\,.$$

Denote

$$\hat{\mathbf{L}}_{2,\mu_0} = \left\{p(r, \alpha),\, r \in \mathbf{R},\, \alpha \in \left(0, \frac{\pi}{2}\right) \middle| p(r, \alpha) \in \hat{\mathbf{L}}_2,\, \alpha \in \left(0, \frac{\pi}{2}\right); \right. \tag{6.19}$$

$$\left. p(\mu_p(r, \alpha), \alpha) = p(r, \alpha),\ \text{a.e. } r \in (0, 2\pi),\ \alpha \in (\alpha^1, \alpha^2)\backslash\{\alpha_1, \ldots, \alpha_m\} \right\}.$$

For any $\hat{p}(r, \alpha) \in \hat{\mathbf{L}}_{2,\mu_0}$ we define $u(x, \alpha) = \Lambda(\alpha)\hat{p} \in \hat{\mathbf{L}}_2(\Omega)$ by (6.2) where

$$p(s, \alpha) = \hat{p}(G_0(s, \alpha), \alpha)\,. \tag{6.20}$$

From Theorem 6.1 it follows that $\Lambda(\alpha)$ is a linear operator which map $\hat{\mathbf{L}}_{2,\mu_0}$ onto the space of G.E. $u(x,\alpha) \in \mathbf{L}_2(\Omega)$ corresponding to $\lambda = -\cos^2 \alpha$. The following statement can be derived from the results obtained in [9].

Lemma 6.2. *For any interval* $[a, b] \subset \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ *the operator* $\Lambda(\alpha)$ *is bounded uniformly in* $\alpha \in [a, b]$. *It means that there exists a constant* $C > 0$ *such that*

$$\|\Lambda(\alpha)\hat{p}\|_{\mathbf{L}_2(\Omega)} \leq C \cdot \|\hat{p}\|_{\mathbf{L}_2(0,2\pi)}, \qquad \hat{p} \in \hat{\mathbf{L}}_{2,\mu_0}, \ \alpha \in [a, b]. \tag{6.21}$$

Let $v \in [0, \varphi^*(\alpha)]$ be an arbitrary point. Because of (5.6)–(5.8)

$$\Lambda(\alpha)(\Theta(\mu_0(v, \alpha) - r) - \Theta(v - r)) = u(x, \alpha, v) \tag{6.22}$$

Let $\hat{p}(r, \alpha) \in \mathbf{C}^1[0, 2\pi] \cap \hat{\mathbf{L}}_{2,\mu_0}$. Then

$$\begin{aligned} u(x, \alpha) &= \Lambda(\alpha)\hat{p}(r, \alpha) = \Lambda(\alpha)\left(\frac{\hat{p}(r, \alpha) + \hat{p}(\mu_0(r, \alpha), \alpha)}{2}\right) \\ &= \Lambda(\alpha)\frac{1}{2}\left(\int_0^{2\pi} \hat{p}_\varphi(\varphi, \alpha)[\Theta(r - \varphi) + \Theta(\mu_0(r, \alpha) - \varphi)]d\varphi + 2\hat{p}(0, \alpha)\right) \\ &= \Lambda(\alpha)\frac{1}{2}\left(\int_0^{2\pi} \hat{p}_\varphi(\varphi, \alpha)[1 - \Theta(\varphi - r) + \Theta(\mu_0(\varphi, \alpha) - r)]d\varphi + 2\hat{p}(0, \alpha)\right) \\ &= \frac{1}{2}\int_0^{2\pi} \hat{p}_\varphi(\varphi, \alpha)\Lambda(\alpha)[\Theta(\mu_0(\varphi, \alpha) - r) - \Theta(\varphi - r)]d\varphi \\ &= \frac{1}{2}\int_0^{2\pi} \hat{p}_\varphi(\varphi, \alpha)\cdot u(x, \alpha, \varphi)d\varphi . \end{aligned} \tag{6.23}$$

Let

$$v(x) = \int_0^{2\pi} u^v(x, \alpha)d\alpha, \qquad w(x) = \int_0^{2\pi} u^w(x, \alpha)d\alpha \tag{6.24}$$

where $u^v(x, \alpha)$, $u^w(x, \alpha) \in \mathbf{D}$. Then using (4.27), (5.13) we obtain

$$\begin{aligned} (v, w)_{\dot{\mathbf{W}}_2^1(\Omega)} &= \int_\Omega v_{x_1}\cdot w_{x_1} + v_{x_2}\cdot w_{x_2}dx = \lim_{\varepsilon \to 0}\int_{\Omega_\varepsilon} v_{x_1}\cdot w_{x_1} + v_{x_2}\cdot w_{x_2}dx \\ &= -\lim_{\varepsilon \to 0}\int_{\Omega_\varepsilon} v\cdot \Delta w dx + \sum_{j=1}^k \lim_{\varepsilon \to 0}\int_{S_j(\varepsilon)} v\cdot\frac{\partial w}{\partial \vec{n}}ds \end{aligned}$$

where

$$S_j(\varepsilon) = \{x \in \Omega \mid |x - P_j| = \varepsilon\}, \qquad j = 1, \ldots, k.$$

From (4.27), (4.28) it follows that

$$\sum_{j=1}^k \lim_{\varepsilon \to 0}\int_{S_j(\varepsilon)} v\cdot\frac{\partial w}{\partial \vec{n}}ds = 0$$

Using (5.11), (6.7), (6.18), (6.23) we obtain

$$
\begin{aligned}
(v, w)_{\mathring{\mathbf{W}}_2^1(\Omega)}
&= -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\int_0^{2\pi} u^v(x,\alpha)\cdot \Delta w d\alpha dx \\
&= -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\Bigg[-\int_0^{\alpha^1}\int_0^{2\pi}\hat{p}_\varphi(\varphi,\alpha)\cdot u(x,\alpha,\varphi)d\varphi d\alpha \\
&\quad +\frac{1}{2}\int_{\alpha^1}^{\alpha^2}\int_0^{2\pi}\hat{p}_\varphi(\varphi,\alpha)\cdot u(x,\alpha,\varphi)d\varphi d\alpha - \int_{\alpha^2}^{\pi/2}\int_0^{2\pi}\hat{p}_\varphi(\varphi,\alpha)\cdot u(x,\alpha,\varphi)d\varphi d\alpha\Bigg] \\
&\quad \cdot \Delta w(x)dx \\
&= \frac{1}{2}\int_0^{\alpha^1}\int_0^{2\pi}\int_0^{2\pi}\hat{p}^v_\varphi(\varphi,\alpha)\Bigg[\hat{p}^w_1(r,\alpha)\left(\cot\frac{r-\varphi}{2}-\cot\frac{r}{2}\right) \\
&\qquad - \hat{p}^w_2(r,\alpha)\left(\cot\frac{r-\beta(\varphi,\alpha)}{2}-\cot\frac{r}{2}\right)\Bigg]drd\varphi d\alpha \\
&\quad +\frac{1}{4}\int_0^{\alpha^2}\int_0^{2\pi}\int_0^{2\pi}\hat{p}^v_\varphi(\varphi,\alpha)\left[\hat{p}^w_0(r,\alpha)\left(\cot\frac{r-\varphi}{2}-\cot\frac{r-\mu_0(\varphi,\alpha)}{2}\right)\right]drd\varphi d\alpha \\
&\quad +\frac{1}{2}\int_{\alpha^2}^{\pi/2}\int_0^{2\pi}\int_0^{2\pi}\hat{p}^v_\varphi(\varphi,\alpha)\Bigg[\hat{p}^w_1(r,\alpha)\left(\cot\frac{r-\varphi}{2}-\cot\frac{r}{2}\right) \\
&\qquad - \hat{p}^w_2(r,\alpha)\left(\cot\frac{r-\beta(\varphi,\alpha)}{2}-\cot\frac{r}{2}\right)\Bigg]drd\varphi d\alpha
\end{aligned}
$$

where

$$
u^v(x,\alpha) = \mathbf{\Lambda}(\alpha)\hat{p}^v(r,\alpha)\,, \qquad u^w(x,\alpha) = \mathbf{\Lambda}(\alpha)\hat{p}^w(r,\alpha) \tag{6.25}
$$

$$
\hat{p}^w_1(r,\alpha) = \hat{p}^w(r,\alpha)\,, \qquad \hat{p}^w_2(r,\alpha) = \hat{p}^w(\beta^{-1}(r,\alpha),\alpha)\,, \qquad \alpha\in\left((0,\alpha^1)\cup\left(\alpha^2,\frac{\pi}{2}\right)\right) \tag{6.26}
$$

$$
\hat{p}^w_0(r,\alpha) = \hat{p}^w(r,\alpha)\,, \qquad \alpha\in(\alpha^1,\alpha^2) \tag{6.27}
$$

and $\hat{p}^v$, $\hat{p}^w \in \mathbf{C}^2\left(\mathbf{R}\times\left[\mathbf{0},\frac{\pi}{2}\right]\right)\cap B_{[a,b]}$ for any $[a,b]\subset\left(0,\frac{\pi}{2}\right)\Big\backslash\{\alpha_1,\dots,\alpha_m\}$ as u^v, $u^w\in\mathbf{D}$.

Denote

$$
\hat{p}^v_1(r,\alpha) = \hat{p}^v(r,\alpha)\,, \qquad \hat{p}^v_2(r,\alpha) = \hat{p}^v(\beta^{-1}(r,\alpha),\alpha)\,, \qquad \alpha\in\left((0,\alpha^1)\cup\left(\alpha^2,\frac{\pi}{2}\right)\right),
$$

$$
\hat{p}^v_0(r,\alpha) = \hat{p}^v(r,\alpha)\,, \qquad \alpha\in(\alpha^1,\alpha^2)\,.
$$

Using the properties of $\beta(r,\alpha)$, $\mu_0(r,\alpha)$, $\hat{p}^v(r,\alpha)$ we obtain

$$-\int_0^{2\pi} \hat{p}_\varphi^v(\varphi, \alpha)\cdot\cot\frac{r-\beta(\varphi,\alpha)}{2}d\varphi = -\int_{2\pi}^0 (\hat{p}^v(\beta^{-1}(\gamma,\alpha),\alpha))_\gamma\cdot\cot\frac{r-\gamma}{2}d\gamma$$

$$= \int_0^{2\pi} (\hat{p}_2^v(\gamma,\alpha))_\gamma\cdot\cot\frac{r-\gamma}{2}d\gamma$$

$$-\int_0^{2\pi} \hat{p}_\varphi^v(\varphi, \alpha)\cdot\cot\frac{r-\mu_0(\varphi,\alpha)}{2}d\varphi = \int_0^{2\pi} (\hat{p}_0^v(\gamma,\alpha))_\gamma\cdot\cot\frac{r-\gamma}{2}d\gamma\,.$$

Hence

(6.28)

$$(v, w)_{\mathring{W}_2^1(\Omega)}$$

$$= \frac{1}{2}\int_0^{\alpha^1}\int_0^{2\pi}\int_0^{2\pi} [(\hat{p}_1^v(\varphi,\alpha))_\varphi \hat{p}_1^w(r,\alpha) + (\hat{p}_2^v(\varphi,\alpha))_\varphi \hat{p}_2^w(r,\alpha)]\cot\frac{r-\varphi}{2}drd\varphi d\alpha$$

$$+ \frac{1}{2}\int_{\alpha^1}^{\alpha^2}\int_0^{2\pi}\int_0^{2\pi} (\hat{p}_0^v(\varphi,\alpha))_\varphi \hat{p}_0^w(r,\alpha)\cot\frac{r-\varphi}{2}drd\varphi d\alpha$$

$$+ \frac{1}{2}\int_{\alpha^2}^{\pi/2}\int_0^{2\pi}\int_0^{2\pi} [(\hat{p}_1^v(\varphi,\alpha))_\varphi \hat{p}_1^w(r,\alpha) + (\hat{p}_2^v(\varphi,\alpha))_\varphi \hat{p}_2^w(r,\alpha)]\cot\frac{r-\varphi}{2}drd\varphi d\alpha\,.$$

We write $\hat{p}_j^v$, $\hat{p}_j^w \in \hat{\mathbf{L}}_2$, $j = 0, 1, 2$ in the form of Fourier series

$$(6.29)\quad \begin{cases} \hat{p}_j^v(r,\alpha) = a_0^{j,v}(\alpha) + \sum_{k=1}^{\infty}\left(a_k^{j,v}(\alpha)\cos kr + b_k^{j,v}(\alpha)\sin kr\right), & r\in\mathbf{R}\,,\\ \hat{p}_j^w(r,\alpha) = a_0^{j,w}(\alpha) + \sum_{k=1}^{\infty}\left(a_k^{j,w}(\alpha)\cos kr + b_k^{j,w}(\alpha)\sin kr\right), & r\in\mathbf{R}\,. \end{cases}$$

From the Hilbert's formulas [14] it follows that

$$(6.30)\quad \begin{cases} \cos kr = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\sin k\gamma\cdot\cot\frac{\gamma-r}{2}\,d\gamma\,,\\ \sin kr = -\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\cos k\gamma\cdot\cot\frac{\gamma-r}{2}d\gamma\,. \end{cases}$$

Using (6.28)–(6.30) we obtain

$$(6.31)\quad (v, w)_{\mathring{W}_2^1(\Omega)} = \pi^2\sum_{k=1}^{\infty} k\cdot\Bigg[\int_0^{\alpha^1}\sum_{j=1,2}(a_k^{j,v}(\alpha)a_k^{j,w}(\alpha) + b_k^{j,v}(\alpha)b_k^{j,w}(\alpha))d\alpha$$

$$+ \int_{\alpha^1}^{\alpha^2}(a_k^{0,v}(\alpha)a_k^{0,w}(\alpha) + b_k^{0,v}(\alpha)b_k^{0,w}(\alpha))d\alpha$$

$$+ \int_{\alpha^2}^{\pi/2}\sum_{j=1,2}(a_k^{j,v}(\alpha)a_k^{j,w}(\alpha) + b_k^{j,v}(\alpha)b_k^{j,w}(\alpha))d\alpha\Bigg]$$

We denote

$$\tilde{\mathbf{W}}_2^{1/2} = \left\{ p(r) \in \hat{\mathbf{L}}_2 \cap \mathbf{W}_2^{1/2}(0, 2\pi) \middle| \int_0^{2\pi} p(r)dr = 0 \right\}, \tag{6.32}$$

$$\|p\|^2_{\tilde{\mathbf{W}}_2^{1/2}} = \pi^2 \cdot \sum_{k=1}^{\infty} k(a_k^2 + b_k^2), \tag{6.33}$$

where

$$a_k = \frac{1}{\pi} \int_0^{2\pi} p(r) \cos krdr, \qquad b_k = \frac{1}{\pi} \int_0^{2\pi} p(r) \sin krdr. \tag{6.34}$$

Thus if $u(x, \alpha) \in \mathbf{D}$ then the function $u(x)$ defined by (4.4) satisfies

$$\begin{aligned} \|u\|^2_{\mathring{\mathbf{W}}_2^1(\Omega)} = & \int_0^{\alpha^1} (\|\hat{p}_1^u(\cdot, \alpha)\|^2_{\tilde{\mathbf{W}}_2^{1/2}} + \|\hat{p}_2^u(\cdot, \alpha)\|^2_{\tilde{\mathbf{W}}_2^{1/2}})d\alpha \\ & + \int_{\alpha^1}^{\alpha^2} \|\hat{p}_0^u(\cdot, \alpha)\|^2_{\tilde{\mathbf{W}}_2^{1/2}} d\alpha + \int_{\alpha^2}^{\pi/2} (\|\hat{p}_1^u(\cdot, \alpha)\|^2_{\tilde{\mathbf{W}}_2^{1/2}} + \|\hat{p}_2^u(\cdot, \alpha)\|^2_{\tilde{\mathbf{W}}_2^{1/2}})d\alpha \end{aligned} \tag{6.35}$$

where $\hat{p}_1^u, \hat{p}_2^u \in \hat{\mathbf{L}}_2 \cap \tilde{\mathbf{W}}_2^{1/2}$, $\hat{p}_0^u \in \hat{\mathbf{L}}_{2,\mu_0} \cap \tilde{\mathbf{W}}_2^{1/2}$ satisfy

$$u(x, \alpha) = \Lambda(\alpha)\hat{p}_1^u(r, \alpha) = \Lambda(\alpha)\hat{p}_2^u(\beta(r, \alpha), \alpha), \qquad \alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \Big\backslash \{\alpha_1, \dots, \alpha_m\},$$

$$u(x, \alpha) = \Lambda(\alpha)\hat{p}_0^u(r, \alpha), \qquad \alpha \in (\alpha^1, \alpha^2) \backslash \{\alpha_1, \dots, \alpha_m\}.$$

We denote

$$\tilde{\mathbf{L}}_{2,\mu_0} = \left\{ p(r, \alpha) \in \hat{\mathbf{L}}_{2,\mu_0} \middle| \int_0^{2\pi} p(r, \alpha)dr = 0 \right\}. \tag{6.36}$$

Then $u(x, \alpha) = \Lambda(\alpha)p(r, \alpha)$ is a family of G.E. for every $p(r, \alpha) \in \tilde{\mathbf{L}}_{2,\mu_0}$. From the equality (6.35) it follows that the following statement is valid.

Theorem 6.2. *For any* $p(r, \alpha) \in \tilde{\mathbf{L}}_{2,\mu_0}$ *the function*

$$u(x) = \int_0^{\pi/2} \Lambda(\alpha)p(r, \alpha)d\alpha \tag{6.37}$$

belongs to $\mathring{\mathbf{W}}_2^1(\Omega)$ *if and only if*

$$p(r, \alpha), \qquad p(\gamma(r, \alpha), \alpha) \in \mathbf{L}_2\left(\tilde{\mathbf{W}}_2^{1/2} \times \left(0, \frac{\pi}{2}\right)\right) \tag{6.38}$$

where

$$\gamma(r, \alpha) = \begin{cases} \beta^{-1}(r, \alpha), & \alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \Big\backslash \{\alpha_1, \dots, \alpha_m\}, \\ \mu_0(r, \alpha), & \alpha \in (\alpha^1, \alpha^2) \backslash \{\alpha_1, \dots, \alpha_m\}. \end{cases} \tag{6.39}$$

If (6.38) *holds then*

$$\|u(x)\|^2_{\mathring{\mathbf{W}}_2^1(\Omega)} = \|p(r, \alpha)\|^2_{\mathbf{L}_2(\tilde{\mathbf{W}}_2^{1/2} \times (0, \pi/2))} + \|p(\gamma(r, \alpha), \alpha)\|^2_{\mathbf{L}_2(\tilde{\mathbf{W}}_2^{1/2} \times ((0, \alpha^1) \cup (\alpha^2, \pi/2)))} \tag{6.40}$$

7. Inversion formula

In the present section we derive an inversion formula for the integral representation (4.4). It means that we construct an operator $\mathbf{\Pi}(\alpha)$: $\mathring{\mathbf{W}}_2^1(\Omega) \to \mathbf{L}_2(\Omega)$, $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$ such that if $u(x, \alpha) \in \mathbf{D}$ and

$$u(x) = \int_0^{\pi/2} u(x, \alpha) d\alpha \tag{7.1}$$

then

$$u(x, \alpha) = \mathbf{\Pi}(\alpha) u(x) . \tag{7.2}$$

I. Let $\alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \setminus \{\alpha_1, \ldots, \alpha_m\}$. Let $u(x)$ be defined by (7.1) where $u(x, \alpha) \in \mathbf{D}$. We define

$$h(\theta, \alpha) = (\Delta u(x), u(x, \alpha, \theta))_{\mathbf{L}_2(\Omega)} , \qquad \theta \in [0, 2\pi) . \tag{7.3}$$

As far as $u(x, \alpha) \in \mathbf{D}$ then there exist 2π-periodic functions $\hat{p}_1(r, \alpha)$, $\hat{p}_2(r, \alpha) \in \mathbf{C}^2(\mathbf{R})$ satisfying (3.35) and such that

$$u(x, \alpha) = \mathbf{\Lambda}(\alpha) \hat{p}_1(r, \alpha) . \tag{7.4}$$

From (5.11) it follows that

(7.5)
$$h(\theta, \alpha) = \frac{1}{2} \int_0^{2\pi} \hat{p}_1(r, \alpha) \left[\cot \frac{r - \theta}{2} - \cot \frac{r}{2} \right] - \hat{p}_2(r, \alpha) \left[\cot \frac{r - \beta(\theta, \alpha)}{2} - \cot \frac{r}{2} \right] dr .$$

We denote for $j = 1, 2$

$$R_j(t, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \hat{p}_j(r, \alpha) \left[\cot \frac{r - \arg t}{2} - \cot \frac{r}{2} \right] dr , \qquad t \in S^1 , \tag{7.6}$$

$$I_j(t, \alpha) = \hat{p}_j(\arg t, \alpha) , \qquad t \in S^1 \tag{7.7}$$

where $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. From the Hilbert's formulas [14] it follows that there exist some functions $A_j(z, \alpha)$, $j = 1, 2$ analytic in the domain $D^+ = \{z \in \mathbf{C} \mid |z| < 1\}$, continuous in $D^+ \cup S^1$ and such that

$$A_j^+(t, \alpha) = R_j(t, \alpha) + i \cdot I_j(t, \alpha) , \qquad t \in S^1 \tag{7.8}$$

where

$$A_j^+(t, \alpha) = \lim_{\substack{z \to t \\ |z| < 1}} A_j(z, \alpha) , \qquad |t| = 1 . \tag{7.9}$$

From (3.35), (7.5)–(7.8) it follows that $A_1(z, \alpha)$, $A_2(z, \alpha)$ satisfy the following equation

$$A_1^+(t, \alpha) - A_2^+(\check{\beta}(t, \alpha), \alpha) = H(t, \alpha) , \qquad t \in S^1 \tag{7.10}$$

where

$$H(t,\alpha) = \frac{1}{\pi} h(\arg t, \alpha), \qquad t \in S^1 \tag{7.11}$$

and $\breve{\beta}(t,\alpha)$ defined by (4.7) is the orientation-reversing homeomorphism of S^1.

We define a piecewise analytic function

$$A(z,\alpha) = \begin{cases} A_1(z,\alpha), & |z| < 1, \\ A_2\left(\frac{1}{z}, \alpha\right), & |z| > 1. \end{cases} \tag{7.12}$$

Then $A(z,\alpha)$ satisfies

$$A^+(t,\alpha) - A^-(\kappa(t,\alpha),\alpha) = H(t,\alpha), \qquad t \in S^1, \tag{7.13}$$

$$|A(\infty,\alpha)| < \infty \tag{7.14}$$

where

$$\kappa(t,\alpha) = \overline{\breve{\beta}(t,\alpha)} = \exp(-i\cdot\beta(\arg t,\alpha)), \qquad t \in S^1 \tag{7.15}$$

is an orientation-preserving homeomorphism of S^1. Following [14] we denote

$$A^-(t,\alpha) = \lim_{\substack{z\to t\\ |z|>1}} A(z,\alpha). \tag{7.16}$$

Thus in the case $\alpha \in \left((0,\alpha^1)\cup\left(\alpha^2,\frac{\pi}{2}\right)\right)\setminus\{\alpha_1,\ldots,\alpha_m\}$ we reduced the problem of constructing of the inversion formula to the Riemann boundary value problem (7.13), (7.14) with the shift $\kappa(t,\alpha)$ [14].

II. Let $\alpha \in (\alpha^1,\alpha^2)\setminus\{\alpha_1,\ldots,\alpha_m\}$. Let $u(x)$ be defined by (7.1) where $u(x,\alpha) \in$ **D**. We consider the function $h(\theta,\alpha)$ defined by (7.3). From (6.18) it follows that

$$h(\theta,\alpha) = -\frac{1}{2}\int_0^{2\pi} \hat{p}_0(r,\alpha)\left[\cot\frac{r-\theta}{2} - \cot\frac{r-\mu_0(\theta,\alpha)}{2}\right]dr \tag{7.17}$$

where $\hat{p}_0(r,\alpha) \in \mathbf{C}^2(\mathbf{R})$ is a 2π-periodic function satisfying (6.14) and such that

$$u(x,\alpha) = \Lambda(\alpha)\hat{p}_0(r,\alpha). \tag{7.18}$$

We denote

$$R_0(t,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \hat{p}_0(r,\alpha)\cdot\cot\frac{r-\arg t}{2}\,dr, \qquad t \in S^1, \tag{7.19}$$

$$I_0(t,\alpha) = \hat{p}_0(\arg t,\alpha), \qquad t \in S^1. \tag{7.20}$$

From the Hilbert's formulas [14] it follows that there exists a function $A_0(z,\alpha)$ analytic in the domain D^+, continuous in $D^+\cup S^1$ and such that

$$A_0^+(t,\alpha) = R_0(t,\alpha) + i\cdot I_0(t,\alpha), \qquad t \in S^1 \tag{7.21}$$

where

$$A_0^+(t, \alpha) = \lim_{\substack{z\to t\\|z|<1}} A_0(z, \alpha), \qquad t \in S^1. \tag{7.22}$$

Because of (6.14), (7.17), (7.19)–(7.21) the function $A_0(z, \alpha)$ satisfies

$$A_0^+(\kappa_0(t, \alpha), \alpha) - A_0^+(t, \alpha) = H(t, \alpha), \qquad t \in S^1 \tag{7.23}$$

where $H(t, \alpha)$ is defined by (7.11) and

$$\kappa_0(t, \alpha) = \exp(i \cdot \mu_0(\arg t, \alpha)), \qquad t \in S^1 \tag{7.24}$$

is an orientation-reversing homeomorphism of S^1 satisfying the Carleman's condition

$$\kappa_0(\kappa_0(t, \alpha), \alpha) \equiv t, \qquad t \in S^1. \tag{7.25}$$

Thus in the case $\alpha \in (\alpha^1, \alpha^2)\backslash\{\alpha_1, \ldots, \alpha_m\}$ we reduced the problem of constructing of the inversion formula to the Carleman boundary value problem (7.23) with the shift $\kappa_0(t, \alpha)$ satisfying the Carleman's condition (7.25) [15].

From (3.36), (3.48), (6.9), (7.15), (7.24) and properties of $G_j(s, \alpha)$, $j = 0, 1, 2$ it follows that

$$\kappa(t, \alpha) \in \mathbf{C}(S^1) \cap \mathbf{C}^1(S^1\backslash\{\tau_1^1(\alpha), \ldots, \tau_{k_1}^1(\alpha)\}),$$

$$\kappa_0(t, \alpha) \in \mathbf{C}(S^1) \cap \mathbf{C}^1(S^1\backslash\{\tau_1(\alpha), \ldots, \tau_{k_0}(\alpha)\})$$

and for any $j \in \{1, \ldots, k_1\}$, $k \in \{1, \ldots, k_0\}$ there exist

$$\kappa_t(\tau_j^1(\alpha) \pm 0, \alpha) = \lim_{\varphi \to \arg \tau_j^1 \pm 0} \kappa_t(\exp(i\varphi), \alpha),$$

$$\kappa_{0t}(\tau_k(\alpha) \pm 0, \alpha) = \lim_{\varphi \to \arg \tau_k \pm 0} \kappa_{0_t}(\exp(i\varphi), \alpha)$$

where $\tau_j^1(\alpha)$, $\tau_k(\alpha) \in S^1$ are defined by (4.12), (4.22). So the general theory of the equations of the type (7.13), (7.23) detailed in [14], [15] can not be applied. Therefore we shall use the results of the work [16] which deals with the solvability of the Riemann boundary value problems with nondifferentiable shift.

Consider the following problem. It is required to find a function $\Phi(z)$ analytic in $\mathbf{C}\backslash S^1$, the limiting values of which on the unit circle satisfy

$$\Phi^+(t) - \Phi^-(\beta(t)) = G(t), \qquad t \in S^1, \tag{7.26}$$

$$\Phi(\infty) = C, \qquad C \in \mathbf{C} \tag{7.27}$$

where $\beta(t)$ is an orientation-preserving homeomorphism of S^1.

Theorem 7.1 ([16]). *Let $G(t)$, $t \in S^1$ be extendable to a function $G(z) \in \mathbf{W}_p^1(D^-)$ for some $p > 2$ where $D^- = \{z \in \mathbf{C} \mid |z| > 1\}$. Let $\beta(t)$, $t \in S^1$ can be extended to a K-quasiconformal mapping $\beta(z)$ [17] which maps $D^+ = \{z \in \mathbf{C} \mid |z| < 1\}$ onto itself.*

Then for any $C \in \mathbf{C}$ there exists a unique solution $\Phi(z, C)$ of the problem (7.26), (7.27). *This solution can be written in the following form*

$$\begin{cases} \Phi(z, C) = -\dfrac{1}{\pi}\displaystyle\iint_{|t|<1}\dfrac{\varphi(t)}{t-z}dt - \dfrac{1}{2\pi i}\int_{|t|=1}\dfrac{G(t)}{t-z}dt + C\,, & |z| > 1\,, \\ \Phi(\beta(z), C) = -\dfrac{1}{\pi}\displaystyle\iint_{|t|<1}\dfrac{\varphi(t)}{t-z}dt - \dfrac{1}{2\pi i}\int_{|t|=1}\dfrac{G(t)}{t-z}dt + C\,, & |z| < 1 \end{cases} \tag{7.28}$$

where

$$\varphi(z) = (I - \mu(z)T)^{-1}\left(\frac{\mu(z)}{2\pi i}\int_{|t|=1}\frac{G'(t)}{t-z}dt\right), \qquad |z| < 1\,, \tag{7.29}$$

$$T\varphi(z) = \lim_{\varepsilon \to +0}\left\{-\frac{1}{\pi}\iint_{\substack{|z-t|>\varepsilon \\ |t|<1}}\frac{\varphi(t)}{(t-z)^2}dt\right\}, \qquad |z| < 1\,,$$

$$\mu(z) = \frac{\beta_{\bar z}(z)}{\beta_z(z)}, \qquad |z| < 1\,. \tag{7.31}$$

Theorem 7.2 ([17]). *Let $h(r)\colon \mathbf{R} \overset{\text{on}}{\to} \mathbf{R}$ be a strictly increasing continuous function. Then there exists K-quasiconformal function $H(z)$ which maps the half-plane $\{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ on itself and*

$$\lim_{\substack{z \to r \\ \operatorname{Im} z > 0}} H(z) = h(r)\,, \qquad r \in \mathbf{R}$$

if and only if $h(r)$ satisfies the following M-condition

$$M^{-1} \le \frac{h(r+s) - h(r)}{h(r) - h(r-s)} \le M\,, \qquad r,\ s \in \mathbf{R} \tag{7.32}$$

where M depends only on K.

If (7.32) *holds then $H(z)$ can be taken in the following form*

$$\begin{aligned} H(z) = H(x + i\cdot y) &= \frac{1}{2y}\int_0^y (h(x+s) + h(x-s))ds \\ &+ i\cdot\frac{1}{2y}\int_0^y (h(x+s) - h(x-s))ds\,, \qquad y > 0,\ x \in \mathbf{R}\,. \end{aligned} \tag{7.33}$$

Let $\beta(t)$ be an orientation-preserving homeomorphism of S^1 satisfying

$$\beta(1) = 1\,. \tag{7.34}$$

We define

$$\tilde\beta(r) = \phi(\beta(\phi^{-1}(t)))\,, \qquad r \in \mathbf{R} \tag{7.35}$$

where

$$\phi(z) = -i\frac{z+1}{z-1}\,, \qquad z \in \mathbf{C}\,. \tag{7.36}$$

Then $\tilde{\beta}(r)$ is an orientation-preserving homeomorphism of $\mathbf{R}$. As far as $\phi(z)$ is conformal mapping which transforms $D^+ = \{z \in \mathbf{C} \,|\, |z| < 1\}$ onto the half-plane $\mathbf{C}^+ = \{w \in \mathbf{C} \,|\, \mathrm{Im}\, w > 0\}$ then $\beta(z)$, $z \in D^+$ is K-quasiconformal extension of $\beta(t)$ if and only if $\tilde{\beta}(w) = \phi(\beta(\phi^{-1}(w)))$, $w \in \mathbf{C}^+$ is K-quasiconformal extension of $\tilde{\beta}(r)$.

So if $\beta(t)$ satisfies (7.34) then there exists K-quasiconformal extension $\beta(z)$, $z \in D^+$ if and only if the function $\tilde{\beta}(r)$, $r \in \mathbf{R}$ defined by (7.35) satisfies (7.32) for some $M > 0$.

Lemma 7.1 ([9]). *Let $\beta(t)$ be an orientation-preserving homeomorphism of S^1 satisfying* (7.34) *and such that for some points $t_1, \ldots, t_k \in S^1$ the following conditions hold*:

(1) $\beta(t) \in \mathbf{C}(S^1) \cap \mathbf{C}^1(S^1 \backslash \{t_1, \ldots, t_k\})$;

(2) *for any $j \in \{1, \ldots, k\}$ there exist*

$$\beta_t(t_j \pm 0) = \lim_{\varphi \to \arg t_j \pm 0} \beta_t(\exp i\varphi)\,;$$

(3) $\inf\limits_{t \in S^1 \backslash \{t_1, \ldots, t_k\}} |\beta_t(t)| > 0.$

Then $\tilde{\beta}(r)$ defined by (7.35) *satisfies* (7.32) *for some constant $M > 0$.*

We denote for $\alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \backslash \{\alpha_1, \ldots, \alpha_m\}$

$$\kappa_0(t, \alpha) = \kappa(t, \alpha) + 1 - \kappa(1, \alpha)\,, \tag{7.37}$$

$$A_0(z, \alpha) = \begin{cases} A(z, \alpha)\,, & |z| < 1\,, \\ A(z + \kappa(1, \alpha) - 1, \alpha)\,, & |z| > 1\,. \end{cases} \tag{7.38}$$

Then $A(z, \alpha)$ satisfies (7.13), (7.14) if and only if $A_0(z, \alpha)$ satisfies (7.14) and

$$A_0^+(t, \alpha) - A_0^-(\kappa_0(t, \alpha), \alpha) = H(t, \alpha)\,, \qquad t \in S^1\,. \tag{7.39}$$

From (7.37) it follows that the orientation-preserving homeomorphism $\kappa_0(t, \alpha)$ satisfies (7.34).

Using (3.36), (3.48), (6.9), (7.15), (7.24), Lemma 3.1 and properties of the functions $f_k(s, \pm\alpha)$, $G_j(s, \alpha)$, $k \in \mathbf{Z}$, $j = 0, 1, 2$ we obtain that the homeomorphism $\kappa_0(t, \alpha)$ satisfies the conditions (1)–(3) of Lemma 7.1 for any $\alpha \in \left(0, \frac{\pi}{2}\right) \backslash \{\alpha_1, \ldots, \alpha_m\}$.

Let $\alpha \in \left((0, \alpha^1) \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \backslash \{\alpha_1, \ldots, \alpha_m\}$. From Theorem 7.2, Lemma 7.1 it follows that there exists $\kappa_0(z, \alpha)$, $z \in D^+$ an K-quasiconformal extension of $\kappa_0(t, \alpha)$. As far as $\hat{p}_1(r, \alpha)$, $\hat{p}_2(r, \alpha) \in \mathbf{C}^2(\mathbf{R})$ then because of (7.5), (7.11) we obtain that $H(t, \alpha) \in \mathbf{C}^1(S^1 \backslash \{\tau_1^1(\alpha), \ldots, \tau_{k_1}^1(\alpha)\})$ and for any $j \in \{1, \ldots, k_1\}$ there exist $H_t(\tau_j^1(\alpha) \pm 0, \alpha)$. Hence $H(t, \alpha)$ can be extended to a function $H(z, \alpha) \in \mathbf{W}_p^1(D^-)$ for any $p > 2$. So from Theorem 7.1 and Sokhotski's formulas it follows that there exists a constant $C(\alpha) \in \mathbf{C}$ such that

$$A_0^+(t,\alpha) = -\frac{1}{\pi}\iint_{|z|<1}\frac{\varphi(z,\alpha)}{z-\kappa_0^{-1}(t,\alpha)}dz - \frac{1}{2}H(\kappa_0^{-1}(t,\alpha),\alpha) - \frac{1}{2\pi i}\int_{|\tau|=1}\frac{H(\tau,\alpha)}{\tau-\kappa_0^{-1}(t,\alpha)}d\tau + C(\alpha)\,, \qquad t\in S^1 \tag{7.40}$$

where

$$\varphi(z,\alpha) = (I-\mu(z,\alpha)T)^{-1}\left(\frac{\mu(z,\alpha)}{2\pi i}\int_{|\tau|=1}\frac{H_\tau(\tau,\alpha)}{\tau - z}d\tau\right), \qquad |z|<1\,, \tag{7.41}$$

$$\mu(z,\alpha) = \frac{(\kappa_0(z,\alpha))_{\bar z}}{(\kappa_0(z,\alpha))_z}\,, \qquad |z|<1\,, \tag{7.42}$$

and the operator T is defined by (7.30).

Using (6.6), (7.2), (7.4), (7.7), (7.8), (7.12), (7.38), (7.40) we obtain that for any $\alpha \in \left((0,\alpha^1)\cup\left(\alpha^2,\frac{\pi}{2}\right)\right)\setminus\{\alpha_1,\dots,\alpha_m\}$ the following inversion formula holds:

$$\begin{aligned} u(x,\alpha) &= \Pi(\alpha)u(x) = \Lambda(\alpha)\hat{p}_1(r,\alpha) = \Lambda(\alpha)\,\mathrm{Im}(A_0^+(\exp(ir),\alpha)) \\ &= \Lambda(\alpha)\left(\mathrm{Im}\left[-\frac{1}{\pi}\iint_{|z|<1}\frac{\varphi(z,\alpha)}{z-\kappa_0^{-1}(\exp(ir),\alpha)}dz - \frac{1}{2}H(\kappa_0^{-1}(\exp(ir),\alpha),\alpha)\right.\right. \\ &\qquad \left.\left. -\frac{1}{2\pi i}\int_{|\tau|=1}\frac{H(\tau,\alpha)}{\tau-\kappa_0^{-1}(\exp(ir),\alpha)}d\tau\right]\right)\,. \end{aligned} \tag{7.43}$$

Consider the following Carleman boundary value problem. It is required to find a function $F(z)$ analytic in D^+ the limiting values of which on the unit circle satisfy

$$F^+(t) - F^+(\beta_0(t)) = G(t)\,, \qquad t\in S^1 \tag{7.44}$$

where $\beta_0(t)$ is an orientation-reversing homeomorphism of S^1 satisfying (7.34) and

$$\beta_0(\beta_0(t)) \equiv t\,, \qquad t\in S^1\,. \tag{7.45}$$

Necessary condition for the solvability of (7.44) is

$$G(t) = -G(\beta_0(t))\,, \qquad t\in S^1\,. \tag{7.46}$$

Let (7.46) hold. We denote

$$\Phi(z) = \begin{cases} F(z)\,, & |z|<1\,,\\ F\left(\dfrac{1}{z}\right)\,, & |z|>1\,. \end{cases} \tag{7.47}$$

Then $\Phi(z)$ satisfies (7.26) and

$$\begin{cases} \Phi(z) = \Phi\left(\dfrac{1}{z}\right)\,,\\ |\Phi(\infty)| < \infty \end{cases} \tag{7.48}$$

where the homeomorphism

$$\beta(t) = \overline{\beta_0(t)}\,, \qquad t \in S^1 \tag{7.49}$$

preserves the orientation and satisfies (7.34),

$$\overline{\beta(\overline{\beta(t)})} \equiv t\,, \qquad t \in S^1\,. \tag{7.50}$$

Let $\beta(t)$, $G(t)$ satisfy the conditions of Theorem 7.1. Then for any $C \in \mathbf{C}$ there exists a unique solution $\Phi(z, C)$ of the equations (7.26), (7.27) which can be written in the form (7.28). Consider

$$\Phi_1(z) = \Phi\left(\frac{1}{z}, 0\right). \tag{7.51}$$

From (7.26), (7.51) it follows that

$$\Phi_1^-(\bar{t}) - \Phi_1^+(\overline{\beta(t)}) = G(t)\,.$$

Hence

$$\Phi_1^-(\beta(t)) - \Phi_1^+(\overline{\beta(\overline{\beta(t)})}) = G(\overline{\beta(t)})\,.$$

Using (7.46), (7.49), (7.50) we obtain

$$\Phi_1^+(t) - \Phi_1^-(\beta(t)) = G(t)\,.$$

From Theorem 7.1 it follows that there exists a constant $C \in \mathbf{C}$ such that

$$\Phi\left(\frac{1}{z}, 0\right) = \Phi_1(z) = \Phi(z, C) = \Phi(z, 0) + C\,.$$

Therefore

$$\Phi^-(1, 0) = \Phi^+(1, 0) + C\,.$$

Because of (7.26), (7.34), (7.46) we obtain that $C = 0$. Hence

$$\Phi(z, C) = \Phi\left(\frac{1}{z}, C\right). \tag{7.51}$$

So $\Phi(z, C)$, $z \in D^+$ is the general solution of the equation (7.44).

Let $\alpha \in (\alpha^1, \alpha^2)\backslash\{\alpha_1, \ldots, \alpha_m\}$. Then the analytic in D^+ function $A_0(z, \alpha)$ satisfies (7.23). Hence there exists a constant $C(\alpha) \in \mathbf{C}$ such that

(7.52)

$$A_0(\overline{\kappa_0(z, \alpha)}, \alpha) = -\frac{1}{\pi}\iint_{|\tau|<1} \frac{\varphi(\tau, \alpha)}{\tau - z}d\tau + \frac{1}{2\pi i}\int_{|\tau|=1} \frac{H(\tau, \alpha)}{\tau - z}d\tau + C(\alpha)\,, \qquad |z| < 1$$

where $\varphi(z, \alpha)$ is defined by (7.41),

$$\mu(z, \alpha) = \frac{\overline{(\kappa_0(z, \alpha))_{\bar{z}}}}{\overline{(\kappa_0(z, \alpha))_z}} \tag{7.53}$$

$\overline{\kappa_0(z, \alpha)}$, $z \in D^+$ is an K-quasiconformal extension of $\overline{\kappa_0(t, \alpha)}$. From (6.6), (7.18)–(7.22) and Sokhotski's formulas it follows that for any $\alpha \in (\alpha^1, \alpha^2)\backslash\{\alpha_1, \dots, \alpha_m\}$ the following inversion formula holds:

$$(7.54)\quad u(x, \alpha) = \mathbf{\Pi}(\alpha)u(x) = \mathbf{\Lambda}(\alpha)\hat{p}_0(r, \alpha) = \mathbf{\Lambda}(\alpha)\,\mathrm{Im}(A_0^+(\exp(ir), \alpha))$$
$$= \mathbf{\Lambda}(\alpha)\left(\mathrm{Im}\left[-\frac{1}{\pi}\iint_{|z|<1}\frac{\varphi(z, \alpha)}{z - \overline{\kappa_0(\exp(ir), \alpha)}}dz - \frac{1}{2}H(\overline{\kappa_0(\exp(ir), \alpha)}, \alpha)\right.\right.$$
$$\left.\left. - \frac{1}{2\pi i}\int_{|\tau|=1}\frac{H(\tau, \alpha)}{\tau - \overline{\kappa_0(\exp(ir), \alpha)}}d\tau\right]\right).$$

8. Completeness of G.E.

To obtain a general representation of the solutions of the Cauchy problem (1.1) in the form (4.3) we need to prove that the system of G.E. $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ is complete in $\mathring{\mathbf{W}}_2^1(\Omega)$. It means that for any $u(x) \in \mathring{\mathbf{W}}_2^1(\Omega)$ there exists a family of G.E. $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ corresponding to $\lambda = -\cos^2\alpha$ such that

$$(8.1)\qquad u(x) = \int_0^{\pi/2} u(x, \alpha)d\alpha\,,$$

$$(8.2)\qquad \mathbf{A}u(x) = \int_0^{\pi/2} -\cos^2\alpha \cdot u(x, \alpha)d\alpha\,.$$

It should be noted that the general theory on generalized eigenfunction expansions (for example, [7]) guarantees in the case being considered by us the completeness of a system of such functions when the space is more extensive than $\mathbf{L}_2(\Omega)$.

One of the possible ways to prove the completeness is to apply the inversion formulas. Actually we constructed the operator $\mathbf{\Pi}(\alpha)$, $\alpha \in \left(0, \frac{\pi}{2}\right)\Big\backslash\{\alpha_1, \dots, \alpha_m\}$ such that

$$(8.3)\qquad u(x) = \int_0^{\pi/2} \mathbf{\Pi}(\alpha)u(x)d\alpha$$

for any

$$u(x) \in D(\mathbf{\Pi}(\alpha)) = \left\{\int_0^{\pi/2} u(x, \gamma)d\gamma \,\middle|\, u(x, \gamma) \in \mathbf{D}\right\} \subset \mathring{\mathbf{W}}_2^1(\Omega)\,.$$

So to prove (8.1), (8.2) it is sufficient to show that $\mathbf{\Pi}(\alpha)$ can be extended to an operator $\overline{\mathbf{\Pi}}(\alpha)$: $\mathring{\mathbf{W}}_2^1(\Omega) \to \mathbf{L}_2(\Omega)$, $D(\overline{\mathbf{\Pi}}(\alpha)) = \mathring{\mathbf{W}}_2^1(\Omega)$ such that (8.3) holds for any $u(x) \in \mathring{\mathbf{W}}_2^1(\Omega)$.

We shall apply another idea suggested by T. I. Zelenyak. He proposed to use the following well-known result.

Proposition 8.1. *Let* $\mathbf{A}$ *be a bounded selfadjoint operator in a Hilbert space* H. *Let* $f \in H$ *and for any* $\lambda \in [a, b] \subset \mathbf{R}$ *there exist* $u_\lambda \in H$ *satisfying*

(8.4) $$\mathbf{A}u_\lambda - \lambda u_\lambda = f\,.$$

Then $E_{[a,b)}f = (E_b - E_a)f = 0$ *where* E_λ *is the resolution of the identity corresponding to* $\mathbf{A}$.

Proof of the proposition can be found, for example, in [18].

Theorem 8.1. *For any* $u(x) \in \mathring{\mathbf{W}}_2^1(\Omega)$ *there exists a family of G.E.* $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ *corresponding to* $\lambda = -\cos^2 \alpha$, $\alpha \in \left(0, \frac{\pi}{2}\right)\Big\backslash\{\alpha_1, \dots, \alpha_m\}$ *such that* (8.1), (8.2) *hold.*

The proof of the theorem is too tiresome and needs a lot of calculations. Therefore in the present paper we only sketch it.

Sketch of the proof. To prove the theorem it is sufficient to show that for every $v(x) \in \mathring{\mathbf{C}}^\infty(\Omega) = \{v \in \mathbf{C}^\infty \,|\, \text{supp}\, v \subset \Omega\}$ there exists a family of G.E. $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ such that the following equation

(8.5) $$\mathbf{A}u - \lambda u = v(x) - \int_0^{\pi/2} u(x, \alpha) d\alpha \overset{\text{def}}{=} f(x)$$

is solvable in $\mathring{\mathbf{W}}_2^1(\Omega)$ for any $\lambda \in (-1, 0)\backslash\{\lambda_1, \dots, \lambda_m\}$, $\lambda_j = -\cos^2 \alpha_j$.

Indeed, from Proposition 8.1 and the properties of the projectors E_λ it follows that

(8.6) $$E_{[-1,\lambda_1)}f = E_{[\lambda_j,\lambda_{j+1})}f = E_{[\lambda_m,0)}f = 0\,, \qquad j = 1, \dots, m-1\,.$$

As far as $\mathbf{A}$ has no eigenfunctions then

(8.7) $$E_{-1} = 0\,, \qquad E_{\lambda_j - 0} = E_{\lambda_j + 0}\,, \qquad E_0 = I\,, \qquad j = 1, \dots, m\,.$$

Because of (8.5)–(8.7) we obtain

(8.8) $$v(x) = \int_0^{\pi/2} u(x, \alpha) d\alpha\,.$$

As far as $\mathring{\mathbf{C}}^\infty(\Omega)$ is dense in $\mathring{\mathbf{W}}_2^1(\Omega)$ then from the Parseval's equality and completeness of $\mathring{\mathbf{W}}_2^1(\Omega)$, $\mathbf{W}_2^{1/2}(0, 2\pi)$ it follows Theorem 8.1.

Let $v(x) \in \mathring{\mathbf{C}}^\infty(\Omega)$, $\lambda^0 \in (-1, 0)\backslash\{\lambda_1, \dots, \lambda_m\}$ then (8.5) holds for a function $u_{\lambda^0} \in \mathring{\mathbf{W}}_2^1(\Omega)$ if and only if

(8.9) $$(u_{\lambda^0}, \psi_{\xi^0\eta^0})_{\mathbf{L}_2(\Omega)} = -(f, \Delta\psi)_{\mathbf{L}_2(\Omega)}\,, \qquad \psi \in \mathring{\mathbf{C}}^\infty(\Omega)$$

where

(8.10) $$\begin{cases} \xi^0 = \xi^0(x) = \xi(x, \alpha) = x_1 \sin \alpha^0 + x_2 \cos \alpha^0\,, \\ \eta^0 = \eta^0(x) = \eta(x, \alpha^0) = x_1 \sin \alpha^0 - x_2 \cos \alpha^0\,, \\ \alpha^0 = \text{arcos}\sqrt{-\lambda^0}\,. \end{cases}$$

Assume that $u(x, \alpha) \in \mathbf{L}_2(\Omega)$ is a family of G.E. such that (8.5) is solvable in $\mathring{\mathbf{W}}_2^1(\Omega)$ for every $\lambda \in (-1, 0)\backslash\{\lambda_1, \ldots, \lambda_m\}$. Then (8.9) holds for any $\lambda^0 \in (-1, 0)\backslash \{\lambda_1, \ldots, \lambda_m\}$. Under some conditions about the smoothness of $u(x, \alpha)$ which will be checked later it can be shown that the following function

$$(8.11)\qquad w(x, \alpha^0) = \int_0^{\pi/2} \left(\frac{\sin(\alpha - \alpha^0)}{\sin(\alpha + \alpha^0)} + \frac{\sin(\alpha + \alpha^0)}{\sin(\alpha - \alpha^0)} - 2\cos 2\alpha^0\right) u(x, \alpha) d\alpha$$
$$- \int_0^{\xi(x,\alpha^0)} \int_0^{\eta(x,\alpha^0)} \Delta_x v\left(\frac{\xi + \eta}{2\sin\alpha^0}, \frac{\xi - \eta}{2\cos\alpha^0}\right) d\xi d\eta$$

also satisfies (8.9). Hence

$$(8.12)\qquad (u_{\lambda^0}(x) - w(x, \alpha^0), \psi_{\xi^0\eta^0})_{\mathbf{L}_2(\Omega)} = 0\,, \qquad \psi \in \mathring{\mathbf{C}}^\infty(\Omega)\,.$$

Therefore for any $\alpha^0 \in \left(0, \frac{\pi}{2}\right) \Big\backslash \{\alpha_1, \ldots, \alpha_m\}$ there exist some functions $B(\xi(x, \alpha^0), \alpha^0)$, $D(\eta(x, \alpha^0), \alpha^0)$ such that

$$(8.13)\qquad w(x, \alpha^0) + B(\xi(x, \alpha^0), \alpha^0) + D(\eta(x, \alpha^0), \alpha^0) = u_{\lambda^0}(x) \in \mathring{\mathbf{W}}_2^1(\Omega)\,.$$

So the problem of the solvability in $\mathring{\mathbf{W}}_2^1(\Omega)$ of (8.5) is reduced to the existence of the functions $B(\xi, \alpha)$, $D(\eta, \alpha)$ such that for any $\gamma \in \left(0, \frac{\pi}{2}\right) \Big\backslash \{\alpha_1, \ldots, \alpha_m\}$

$$(8.14)\qquad w(x, \gamma) + B(\xi(x, \gamma), \gamma) + D(\eta(x, \gamma), \gamma) \in \mathring{\mathbf{W}}_2^1(\Omega)\,.$$

Using the results obtained in [9] it can be shown that the solvability of (8.14) is equivalent to the solvability of

(8.15)
$$\int_{\gamma-\varepsilon}^{\gamma+\varepsilon} \frac{u(x, \alpha)}{\alpha - \gamma} d\alpha - \int_0^{\xi(x,\gamma)} \int_0^{\eta(x,\gamma)} \tilde{v}(\xi, \eta, \gamma) d\xi d\eta + B(\xi(x, \gamma), \gamma) + D(\eta(x, \gamma), \gamma) \in \mathring{\mathbf{W}}_2^1(\Omega)$$

for any $\gamma \in \left(0, \frac{\pi}{2}\right) \Big\backslash \{\alpha_1, \ldots, \alpha_m\}$ and for any small $\varepsilon > 0$, where

$$(8.16)\qquad \tilde{v}(\xi, \eta, \gamma) = \frac{\Delta_x v\left(\dfrac{\xi + \eta}{2\sin\gamma}, \dfrac{\xi - \eta}{2\cos\gamma}\right)}{\sin 2\gamma}\,.$$

The properties of the function of the following type

$$(8.17)\qquad g(x, \gamma) = \int_{\gamma-\varepsilon}^{\gamma+\varepsilon} \frac{u(x, \alpha)}{\alpha - \gamma} d\alpha$$

where $u(x, \alpha)$ is a family of G.E. are studied in [9] when the formulas (5.11), (5.12) are derived. Using the results obtained in [9] it can be shown that the solvability of (8.15) is equivalent to the solvability of the following systems:

$$(8.18)\qquad \begin{cases} \hat{p}_2(r,\gamma) - \hat{p}_1(\beta(r,\gamma),\gamma) = 0\,, & r \in [0, 2\pi]\,, \\ \dfrac{1}{2}\displaystyle\int_0^{2\pi} \hat{p}_1(\tau,\gamma)\cot\frac{\tau - r}{2}\,d\tau - \frac{1}{2}\int_0^{2\pi} \hat{p}_2(\tau,\gamma)\cot\frac{\tau - \beta(r,\gamma)}{2}\,d\tau & \\ \quad = \displaystyle\int_\Omega u(x,\gamma,r)\frac{\Delta_x v(x)}{\sin 2\gamma}\,d\Omega\,, & r \in [0, 2\pi] \end{cases}$$

if $\gamma \in \left((0,\alpha)^1 \cup \left(\alpha^2, \frac{\pi}{2}\right)\right) \backslash \{\alpha_1, \dots, \alpha_m\}$;

$$(8.19)\qquad \begin{cases} \hat{p}_0(r,\gamma) - \hat{p}_0(\mu_0(r,\gamma),\gamma) = \ , & r \in [0, 2\pi]\,, \\ \dfrac{1}{2}\displaystyle\int_0^{2\pi} \hat{p}_0(\tau,\gamma)\left[\cot\frac{\tau - \mu_0(r,\gamma)}{2} - \cot\frac{\tau - r}{2}\right]d\tau & \\ \quad = \displaystyle\int_\Omega u(x,\gamma,r)\frac{\Delta_x v(x)}{\sin 2\gamma}\,d\Omega\,, & r \in [0, 2\pi] \end{cases}$$

if $\gamma \in (\alpha^1, \alpha^2)\backslash\{\alpha_1, \dots, \alpha_m\}$. Here $\hat{p}_j(r,\alpha)$, $j = 0, 1, 2$ are 2π-periodic functions corresponding to the family of G.E. $u(x,\alpha)$. Therefore $u(x,\alpha)$ satisfies the inversion formulas (7.43), (7.54). As far as $v(x) \in \mathring{\mathbf{C}}^\infty(\Omega)$ then it can be shown that the function $u(x,\alpha)$ determined by (8.18), (8.19), (7.43), (7.54) is smooth enough and all the smoothness assumptions we used to derive (8.11)–(8.19) are valid.

Thus, using the formulas (7.43), (7.54), (8.18), (8.19) we construct for any $v \in \mathring{\mathbf{C}}^\infty(\Omega)$ a family of G.E. $u(x,\alpha) \in \mathbf{L}_2(\Omega)$ such that (8.8) holds. The proof of the theorem is completed.

9. Structure of solutions

Now we can construct the general form of the solutions of the Cauchy problem (1.1)

$$(1.1)\qquad p_{tt} = \mathbf{A}p\,, \qquad p|_{t=0} = p_0\,, \qquad p_t|_{t=0} = p_1\,.$$

Let p_0, $p_1 \in \mathring{\mathbf{W}}_2^1(\Omega)$. Then from Theorem 8.1 it follows that there exist families of G.E. $u_j(x,\alpha) \in \mathbf{L}_2(\Omega)$, $j = 0, 1$ such that

$$(9.1)\qquad p_j(x) = \int_0^{\pi/2} u_j(x,\alpha)\,d\alpha\,,$$

$$(9.2)\qquad \mathbf{A}p_j(x) = \int_0^{\pi/2} -\cos^2\alpha \cdot u_j(x,\alpha)\,d\alpha\,.$$

Then the unique solution of the problem (1.1) can be written in the following form

$$(9.3)\qquad p(x,t) = \int_0^{\pi/2}\left[\cos(t\cdot\cos\alpha)\cdot u_0(x,\alpha) + \frac{\sin(t\cdot\cos\alpha)}{\cos\alpha}\cdot u_1(x,\alpha)\right]d\alpha\,.$$

From the Parseval's equality, the inversion formulas and Theorem 8.1 it follows that the set

$$\left\{\int_0^{\pi/2} u(x, \alpha) \,\middle|\, u(x, \alpha) \in \mathbf{D}\right\}$$

is dense in $\mathring{\mathbf{W}}_2^1(\Omega)$ and the operator $\mathbf{\Pi}(\alpha)$ can be extended continuously to a bounded operator $\bar{\mathbf{\Pi}}(\alpha)$: $\mathring{\mathbf{W}}_2^1(\Omega) \to \mathbf{L}_2(\Omega)$, $D(\bar{\mathbf{\Pi}}(\alpha)) = \mathring{\mathbf{W}}_2^1(\Omega)$, $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \dots, \alpha_m\}$ such that for any $u \in \mathring{\mathbf{W}}_2^1(\Omega)$

$$u(x) = \int_0^{\pi/2} \bar{\mathbf{\Pi}}(\alpha) u(x) d\alpha \,. \tag{9.4}$$

Therefore the functions $u_j(x, \alpha)$, $j = 0, 1$ satisfying (9.1), (9.2) can be determined by the following formula

$$u_j(x, \alpha) = \bar{\mathbf{\Pi}}(\alpha) p_j(x) \,, \qquad j = 0,\ 1 \,, \qquad \alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha_1, \dots, \alpha_m\} \,. \tag{9.5}$$

Thus for any p_0, $p_1 \in \mathring{\mathbf{W}}_2^1(\Omega)$ the unique solution of the problem (1.1) can be written in the form (9.3) where the families of G.E. $u_j(x, \alpha) \in \mathbf{L}_2(\Omega)$, $j = 0, 1$ are determined by (9.5).

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