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Index for factors generated by extended Jones' projections

By

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0. Introduction

The index theory for a pair of type II_1-factors was introduced by V. Jones in [2]. In his paper, he constructed a sequence of projections $\{e_i; i=1,2,\cdots\}$ satisfying the following conditions:

(a) $e_i e_{i\pm1} e_i = \lambda e_i$ for $i \geq 1$ with a fixed constant λ $(0<\lambda<1)$,

(b) $e_i e_j = e_j e_i$ for $|i-j| \geq 2$,

(c) $tr(e_i \omega) = \lambda tr(\omega)$ for any word ω on $e_1, \cdots, e_{i-1}$,

where tr is the canonical trace on $\{e_i; i=1,2,\cdots\}''$.

In this paper, generalizing the above conditions, we consider a family of projections $\{e_i, f_j; i=1,2,\cdots, 1\leq j\leq l\}$ such that

(R-1) $e_i e_{i+1} e_i = \lambda e_i$ for $i\geq 1$,

(R-2) $e_i e_{i-1} e_i = \lambda e_i$ for $i \geq 2$,

$e_1 f_j e_1 = \alpha_j e_1$ for $1 \leq j \leq l$,

(R-3) $e_i e_j = e_j e_i$ for $|i-j| \geq 2$,

$e_i f_j = f_j e_i$ for $i \geq 2$, $1 \leq j \leq l$,

(R-4) $tr(e_i \omega) = \lambda tr(\omega)$ for any word ω on $f_1, \cdots, f_l, e_1, \cdots, e_{i-1}$,

where tr is the canonical trace on $\{e_i, f_j; i=1,2,\cdots, 1\leq j\leq l\}''$,

(R-5) $\sum_j f_j = 1$,

where $\lambda^{-1} = 4\cos^2(\pi/(n+2))$, $\alpha_j \in \boldsymbol{R}$ and $0<\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_l$.

These projections are called *extended Jones' projections* with a data $(n; \alpha_1, \cdots,$

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α_l). And extended Jones' projections corresponding to the data $(n; \lambda, 1-\lambda)$ are nothing but the original Jones' projections. An existence condition for such a family is given by Theorems 2.1, 2.2 and 2.3, which can be put together into the following theorem.

Theorem (Theorems 2.1-2.3). *There exists a family of extended Jones' projections corresponding to the data* $(n; \alpha_1, \cdots, \alpha_l)$, *if and only if*

$$(n; \alpha_1, \alpha_2)=(n; \lambda_k, \lambda_{n-k-2}) \quad \textit{for} \quad 0\leq k\leq[(n-2)/2], \quad n\geq 2 \quad \textit{or}$$

$$(n; \alpha_1, \alpha_2, \alpha_3)=(2k; \lambda_0, \lambda_0, \lambda_{k-2}) \quad \textit{for} \quad k\geq 2,$$

$$(10; \lambda_0, \lambda_1, \lambda_1), \quad (16; \lambda_0, \lambda_1, \lambda_2) \textit{ or } (28; \lambda_0, \lambda_1, \lambda_3),$$

where $\lambda_k=\sin(k+1)\theta_n/(2\cos\theta_n\sin(k+2)\theta_n)$ *and* $\theta_n=\pi/(n+2)$.

This is the first important result of this paper.

For a family of extended Jones' projections satisfying the condition in this theorem, we put $A=\{e_i, f_j; i=1, 2, \cdots, 1\leq j\leq l\}''$ and $B=\{e_i; i=1, 2, \cdots\}''$. The next main purpose of this paper is to calculate the index $[A: B]$ and to show that the relative commutant $B'\cap A$ is trivial. The indices are given in the next Theorem 4.1.

Theorem 4.1. *Let* $\{e_i, f_j; i\geq 1, 1\leq j\leq l\}$ *be a family of extended Jones' projections corresponding to* $(n; \alpha_1, \cdots, \alpha_l)$ *and* $A=\{e_i, f_j; i\geq 1, 1\leq j\leq l\}''$, $B=\{e_i; i\geq 1\}''$. *Then* A *and* B *are hyperfinite type* II_1*-factors and index* $[A: B]$ *is given as follows:*

1) *Case of* $(n; \alpha_1, \alpha_2)=(n; \lambda_k, \lambda_{n-k-2})$ $(0\leq k\leq[(n-2)/2])$

$$[A: B]=\frac{\sin^2(k+2)\theta_n}{\sin^2\theta_n}, \qquad \textit{with} \quad \theta_n=\frac{\pi}{n+2}.$$

2) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(2k; \lambda_0, \lambda_0, \lambda_{k-2})$ $(k\geq 2)$

$$[A: B]=2\cot^2\theta_n.$$

3) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(10; \lambda_0, \lambda_1, \lambda_1)$

$$[A: B]=18+10\sqrt{3}.$$

4) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(16; \lambda_0, \lambda_1, \lambda_2)$

$$[A: B]=9\left\{2\sin^2\theta_n\left(\frac{\sin^2 2\theta_n}{\sin^2 4\theta_n}+\frac{\sin^2\theta_n}{\sin^2 3\theta_n}+1\right)\right\}^{-1}.$$

5) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(28; \lambda_0, \lambda_1, \lambda_3)$

$$[A: B]=15\left\{2\sin^2\theta_n\left(\frac{\sin^2\theta_n+\sin^2 3\theta_n}{\sin^2 5\theta_n}+\frac{\sin^2\theta_n}{\sin^2 3\theta_n}+1\right)\right\}^{-1}.$$

Furthermore we specify the fixed point subalgebras $A^{\sigma}\subset A$ of automorphisms σ: $A\rightarrow A$, defined by permutations of $\{f_i; 1\leq i\leq l\}$, and calculate the indices $[A: A^{\sigma}]$.

The contents of this paper are as follows.

In section 1, we add one projection to a sequence of Jones' projections and impose a relaxed Jones' relation on them. Then the existence condition for such a family of projections is given by Proposition 1.1. Moreover we introduce a graph to show the relation between projections.

In section 2, the definition of extended Jones' projections is introduced and the necessary and sufficient condition for the existence of them is given. We shall make use of string algebras of Dynkin diagrams to construct extended Jones' projections.

Section 3 is mainly devoted to preparing for the calculation of the index $[A:B]$. We study the structure of $A_n=\{e_i, f_j: 1\leq i\leq n, 1\leq j\leq l\}''$ and $B_n=\{e_i; 1\leq i\leq n\}''$, and prove that the inclusion matrices $[A_n\rightarrow A_{n+1}]$, $[B_n\rightarrow B_{n+1}]$ and $[B_n\rightarrow A_n]$ are periodic for sufficiently large n.

In section 4, we calculate the index $[A: B]$ by using Wenzl's index formula and results in section 3 and show that the relative commutant $B'\cap A$ is trivial (Theorem 4.3). Put $A(j)=\{e_i, f_j; i=1, 2, \cdots\}''$ and let A^{σ} by a fixed point algebra, where σ is an automorphism of A defined by a permutation of $\{f_j; 1\leq j\leq l\}$. The indices $[A: A(j)]$ and $[A^{\sigma}: B]$ are also computed (Theorems 4.2 and 4.4).

1. Family of extended Jones' projections

1.1. Jones' projections. In 1983, Jones defined the index for a pair of II_1-factors $M\supset N$. In that paper, he constructs the sequence $\{e_i; i=1, 2, \cdots\}$ of projections which satisfies the following relations:

(a) $e_ie_{i\pm 1}e_i=\lambda e_i$ for $i\geq 1$ with a fixed constant λ $(0<\lambda<1)$,

(b) $e_ie_j=e_je_i$ for $|i-j|\geq 2$,

(c) $tr(e_i\omega)=\lambda tr(\omega)$ for any word ω on $e_1, \cdots, e_{i-1}$,
where tr is the canonical trace on $\{e_i; i=1, 2, \cdots\}''$.

We define $A=\{e_i; i\geq 1\}''$ and $B=\{e_i; i\geq 2\}''$, then A and B are type II_1-factors and index $[A: B]$ is λ^{-1}. Moreover relative commutant $B'\cap A$ is trivial if and only if $\lambda^{-1}<4$. And if $\lambda^{-1}<4$, we have $\lambda^{-1}\in\{4\cos^2(\pi/(n+2)); n\in\boldsymbol{N}\}$.

1.2. Extended Jones' projections. At first, we add a projection e_0 to $\{e_i; i=1, 2, \cdots\}$, and consider the sequence $\{e_i; i=0, 1, \cdots\}$ of projections of M satisfying the following relations:

(a$'_1$) $e_ie_{i+1}e_i=\lambda e_i$ for $i\geq 1$, (a$_2'$) $e_ie_{i-1}e_i=\begin{cases}\lambda e_i & \text{if } i\geq 2\\ \alpha e_1 & \text{if } i=1\,,\end{cases}$

(b′) $e_ie_j=e_je_i$ for $|i-j|\geq 2$,

(c′) $tr(e_i\omega)=\lambda tr(\omega)$ for any word ω on $e_0, \cdots, e_{i-1}$,
where tr is the canonical trace on M.

In this paper, we treat the case when $\lambda>1/4$. The next proposition gives a necessary and sufficient condition for the existence of the above sequence of projections.

Definition. We define the polynomials $P_k(\lambda)$ by $P_{-1}(\lambda)=P_0(\lambda)=1$ and $P_k(\lambda)=P_{k-1}(\lambda)-\lambda P_{k-2}(\lambda)$ for $k\geq 1$.

The polinomials P_k are called Jones' polynomials.

Proposition 1.1. *Let M be a type II_1-factor and $\lambda>1/4$. Then, there exists a sequence $\{e_i; i=0, 1, \cdots\}$ of projections of M satisfying the above relations* (a$_1'$), (a$_2'$), (b′), (c′), *if and only if*

$$\lambda\in\left\{\left(4\cos^2\frac{\pi}{n+2}\right)^{-1};\ n\in \boldsymbol{N}\right\} \text{ and } \alpha\in\left\{\frac{\lambda P_{k-1}(\lambda)}{P_k(\lambda)};\ 0\leq k\leq n-1\right\},$$

where $P_k(\lambda)$ is Jones' polynomial.

Proof. Suppose the existence of a such sequence of projections. Then $\{e_i; i=1, 2, \cdots\}$ satisfies the relations (a), (b), (c), and so λ belongs to $\{(4\cos^2(\pi/(n+2)))^{-1}; n\in \boldsymbol{N}\}$ by the result of Jones ([2] Theorem 4.11). By relations (a$_2'$), (b′), (c′) and Popa's result ([4] Theorem 2.11), it follows that α is in $\{\lambda P_{k-1}(\lambda)/P_k(\lambda); 0\leq k\leq n-1\}$.

Conversely take λ and α satisfying the above conditions. Let $M_0\subset M_1$ be a pair of II_1-factors with $[M_1: M_0]=\lambda^{-1}$. Iterating basic construction, we obtain a tower of II_1-factors $M_0\subset M_1\subset M_2\subset\cdots\subset M_i\subset M_{i+1}\subset\cdots$ and a sequence $\{e_i; i=1, 2, \cdots\}$ with $M_i=\langle M_{i-1}, e_{i-1}\rangle$, $e_i=e_{M_{i-1}}$. By section 3. of [2], a sequence $\{e_i; i=1, 2, \cdots\}$ satisfies the relations (a$_1'$), (a$_2'$), (b′). Let M_∞ be a von Neumann algebra generated by $\cup_{i\in N}M_i$, then M_∞ is a II_1-factor, and the sequence $\{e_i; i=1, 2, \cdots\}$ satisfies the relation (c′) for the canonical trace tr of M_∞.

Now, we define $\Lambda(M_1, M_0)=\{\alpha\in \boldsymbol{C};\ \exists f\in M_1$ projection s.t. $E_{M_0}(f)=\alpha 1_{M_0}\}$, then

$$\Lambda(M_1, M_0)=\left\{\frac{\lambda P_{k-1}(\lambda)}{P_k(\lambda)};\ 0\leq k\leq n-1\right\}\cup\{0\}$$

by Theorem 5.1 of [4]. So, for any $\alpha\in\{\lambda P_{k-1}(\lambda)/P_k(\lambda); 0\leq k\leq n-1\}$, we get a projection $e_0\in M_1$ such that $E_{M_0}(e_0)=\alpha 1_{M_0}$. Hence $e_1e_0e_1=E_{M_0}(e_0)e_1=\alpha e_1$ and for $i\geq 2$, $e_ie_0=e_0e_i$ because $e_i\in M_1'$. Since tr is a (λ, M_i) trace, we have $tr(e_i\omega)=\lambda tr(\omega)$ for $\omega\in alg\{1, e_0, \cdots, e_{i-1}\}$, $i\in\boldsymbol{N}$. Here $alg\{\cdots\}$ denotes the algebra generated by $\{\cdots\}$. From the above argument, the sequence $\{e_i; i=0, 1, \cdots\}$ of non-zero projections of M_∞ satisfies the relations (a_1'), (a_2'), (b') and (c'). As it is shown later, $\{e_i; i=0, 1, \cdots\}''$ is a hyperfinite II_1-factor, so the existence follows in case that M is hyperfinite. In the general case, M has a hyperfinite II_1-factor as a subfactor, so the existence follows from the above special case.

Now we consider the case where the exceptional projection e_0 is replaced by $f_1, \cdots, f_l$, that is a family of projections satisfying the following relations. Let l, $n\in\boldsymbol{N}$ and $\{e_i, f_j; i\geq 1, 1\leq j\leq l\}$ be a family of non-zero projections of M, such that

(R-1) $e_ie_{i+1}e_i=\lambda e_i$ for $i\geq 1$,

(R-2) $e_ie_{i-1}e_i=\lambda e_i$ for $i\geq 2$,

$e_1f_je_1=\alpha_je_1$ for $1\leq j\leq l$,

(R-3) $e_ie_j=e_je_i$ for $|i-j|\geq 2$,

$e_if_j=f_je_i$ for $i\geq 2$, $1\leq j\leq l$,

(R-4) $\mathrm{tr}(e_i\omega)=\lambda tr(\omega)$ for any word ω on $f_1, \cdots, f_l, e_1, \cdots, e_{i-1}$, where tr is the canonical trace on M,

(R-5) $\sum_j f_j=1$

where $\lambda^{-1}=4\cos^2(\pi/(n+2))$, $\alpha_j\in\boldsymbol{R}$, $0<\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_l$.

We call the above relations (R-1)~(R-5) the extended Jones' relations or EJ relation, and projections $\{e_i, f_j; i\geq 1, 1\leq j\leq l\}$ extended Jones' projections.

Remark that we do not add the condition $f_je_1f_j\in\boldsymbol{C}f_j$ to (R-1). However it can be satisfied in some cases. The following proposition gives the necessary and sufficient condition for $f_je_1f_j\in\boldsymbol{C}f_j$.

Proposition 1.2. *Let $\{e_i, f_j; i\geq 1, 1\leq j\leq l\}$ be a family of extended Jones' projections, $\{\lambda, \alpha_j; 1\leq j\leq l\}$ be scalars that appear in the extended Jones' relations. Then*

$$f_je_1f_j\in\boldsymbol{C}f_j\Longleftrightarrow\alpha_j=\lambda\,.$$

Further $f_je_1f_j\in\boldsymbol{C}f_j$ implies $f_je_1f_j=\lambda f_j$.

Proof. If $f_je_1f_j=\beta f_j$ for some $\beta\in\boldsymbol{C}$, we have $\beta f_je_1=f_je_1f_je_1=\alpha_jf_je_1$. Since $e_1f_je_1=\alpha_je_1\neq 0$, we obtain $\alpha_j=\beta$. On the other hand, we have $\beta tr(f_j)=tr(f_je_1f_j)=tr(f_je_1)=tr(f_j)tr(e_1)=\lambda tr(f_j)$, so $\beta=\lambda=\alpha_j$.

Conversely, if $\alpha_j=\lambda$, then $tr((f_je_1f_j-\lambda f_j)^2)=0$. Since $(f_je_1f_j-\lambda f_j)^2\geq 0$ and

tr is faithful, we obtain $f_j e_1 f_j = \lambda f_j$.

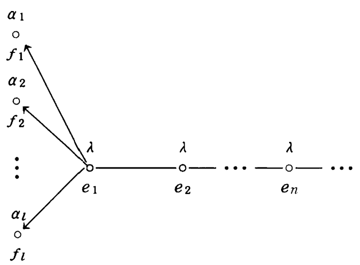
Next we consider the expression of the extended Jones' relations by a certain graph. We express a projection by a vertex $\circ$, under which we denote its name, and above which its trace. We introduce the following symbols for expressing of relations between two projections.

(1)  $\Longleftrightarrow$ $fef=\alpha f$, $efe=\beta e$

(2)  $\Longleftrightarrow$ $efe=\alpha e$

(3)  $\Longleftrightarrow$ $ef=fe$

In these notations, we can express the relations (R-1)~(R-3) by the following graph:



2. Existence of the family of extended Jones' projections

2.1. Condition for existence of extended Jones' projections. In this section, we give a necessary and sufficient condition for the existence of a family of extended Jones' projections.

Let $\{e_i, f_j;\ i\geq 1, 1\leq j\leq l\}$ be a family of extended Jones' projections in a type II_1-factor M and $\{\lambda, \alpha_j;\ 1\leq j\leq l\}$ be scalar corresponding to the family. Recall that $\lambda^{-1}=4\cos^2(\pi/(n+2))$ for some $n\in \boldsymbol{N}$, so we denote the data for the family by $(n;\ \alpha_1, \cdots, \alpha_l)$. Taking f_j as e_0, the sequence of projections $\{e_i;\ i\geq 0\}$ satisfies the relations (a$_1'$), (a$_2'$), (b$'$), (c$'$). So by Theorem 1.1 we obtain

$$\alpha_j\in\left\{\frac{\lambda P_{k-1}(\lambda)}{P_k(\lambda)};\quad 0\leq k\leq n-1\right\}=\left\{\frac{\sin(k+1)\theta_n}{2\cos\theta_n\sin(k+2)\theta_n};\ 0\leq k\leq n-1\right\},$$

where $\theta_n=\pi/(n+2)$.

We put $\lambda_k=\sin(k+1)\theta_n/(2\cos\theta_n\sin(k+2)\theta_n)$ for $0\leq k\leq n-1$, then $\lambda=\lambda_0<\lambda_1<\cdots<\lambda_{n-1}$ and $\alpha_j\in\{\lambda_k;\ 0\leq k\leq n-1\}$. Using this, we seek the condition for existence of a family of extended Jones' projections by means of data $(n;\ \alpha_1, \cdots, \alpha_l)$. At first, we consider the case $l=2$. Assume that there exists a family of extended Jones' projections corresponding to the data $(n;\ \alpha_1, \alpha_2)$.

Since $\alpha_1 \leq \alpha_2$, $\alpha_1 + \alpha_2 = 1$ and $\alpha_j \in \{\lambda_k; 0 \leq k \leq n-1\}$, we obtain $(n; \alpha_1, \alpha_2) = (n; \lambda_k, \lambda_{n-k-2})$ for $0 \leq k \leq [(n-2)/2]$, $n \geq 2$. On the other hand, by Theorem 1.1 for any $\alpha \in \{\lambda_k; 0 \leq k \leq [(n-2)/2]\}$, we get a sequence of projections $\{e_i; i \geq 0\}$ satisfying the relations (a_1'), (a_2'), (b') and (c'). We set $f_1 = e_0$, $f_2 = 1 - e_0$, then $\{e_i, f_j; i \geq 1, j = 1, 2\}$ satisfies the EJ relation corresponding to the data $(n; \lambda_k, \lambda_{n-k-2})$. From the above arguments, we have obtained the next theorem about the necessary and sufficient condition for the existence of a family of extended Jones' projectons in case $l = 2$.

Theorem 2.1. *Let M be a type II_1-factor. Then there exists a family of extended Jones' projections corresponding to the data $(n; \alpha_1, \alpha_2)$ if and only if $(n; \alpha_1, \alpha_2) = (n; \lambda_k, \lambda_{n-k-2})$ for some k, $0 \leq k \leq [(n-2)/2]$.*

Secondly we consider the case $l \geq 3$. In this case, everything can be done by simple but lengthy calculations, and we get following Theorem 2.2.

Theorem 2.2. *Let M be a type II_1-factor and $l \geq 3$. If there exists a family of extended Jones' projections corresponding to the data $(n; \alpha_1, \cdots, \alpha_l)$, then $l = 3$, $n \geq 4$ and $(n; \alpha_1, \alpha_2, \alpha_3)$ is one of the following:*

$$(2k; \lambda_0, \lambda_0, \lambda_{k-2}), \quad (10; \lambda_0, \lambda_1, \lambda_1), \quad (16; \lambda_0, \lambda_1, \lambda_2), \quad (28; \lambda_0, \lambda_1, \lambda_3).$$

Proof. By $\alpha_j \geq \lambda_0 = \lambda$, we get $1 = \sum_{j=1}^{l} \alpha_j \geq l\lambda$, hence $l \leq \lambda^{-1} < 4$. So we obtain $l = 3$ and $\lambda^{-1} \geq 3$. Since $\lambda^{-1} = 4\cos^2(\pi/(n+2))$, we have $n \geq 4$. And $\alpha_1 \leq \alpha_j$ implies that $\alpha_1 \leq 1/3$. On the other hand $\lambda_0 < \lambda_1 < \cdots < \lambda_{n-1}$ and $\lambda_1 > 1/3$, so $\alpha_1 = \lambda_0$. By $\lambda_0 + \alpha_1 + \alpha_2 = 1$ and $\alpha_2 \leq \alpha_3$, we get $\alpha_2 \leq (1-\lambda_0)/2$. Moreover $\lambda_2 > (1-\lambda_0)/2$, $\alpha_2 = \lambda_0$ or λ_1.

a) Case of $\alpha_2 = \lambda_0$:

$\alpha_3 = 1 - 2\lambda_0$. Since $\alpha_3 \in \{\lambda_i; 0 \leq i \leq n-1\}$, we can denote α_3 by λ_k for some k, $0 \leq k \leq n-1$. Then $\lambda_k = 1 - 2\lambda_0$. By simple calculation, we get $n = 2k+4$. So n is even and $k = (n-4)/2$.

b) Case of $\alpha_2 = \lambda_1$:

$\alpha_3 = 1 - \lambda_0 - \lambda_1 = \sin\theta_n \sin 5\theta_n / (2\cos\theta_n \sin 2\theta_n \sin 3\theta_n)$. We obtain $\alpha_3 = \lambda_1$, λ_2 or λ_3 by $\lambda_4 > \sin\theta_n \sin 5\theta_n / (2\cos\theta_n \sin 2\theta_n \sin 3\theta_n)$. Assume that $\alpha_3 = \lambda_1$, then we get trigonometric equation

$$\frac{\sin\theta_n \sin 5\theta_n}{2\cos\theta_n \sin 2\theta_n \sin 3\theta_n} = \frac{\sin 2\theta_n}{2\sin\theta_n \sin 3\theta_n}.$$

Solving this equation, we obtain $n = 10$. Similarly $\alpha_3 = \lambda_2$ (resp. $\alpha_3 = \lambda_3$) implies $n = 16$ (resp. $n = 28$).

As we prove in 2.2., for any of the above data $(n; \alpha_1, \alpha_2, \alpha_3)$, there exists a family of extended Jones' projections, or we have the following existence theorem.

Theorem 2.3. *Let M be a type II_1-factor. Then for every data $(2k; \lambda_0, \lambda_0, \lambda_{k-2})$ $(k\geq 2)$, $(10; \lambda_0, \lambda_1, \lambda_1)$, $(16; \lambda_0, \lambda_1, \lambda_2)$ or $(28; \lambda_0, \lambda_1, \lambda_3)$, there exists a family of extended Jones' projections corresponding to them.*

2.2. Construction of a family of extended Jones' projections. In this subsection, we construct a family of extended Jones' projections by use of string algebra.

Let G be an unoriented pointed graph. Moreover we require that G be bipartite, locally finite and accessible. We denote a distinguished point by $*$.

Definition. For $x, y\in G^{(0)}$ (=vertex set of G), $n\in \boldsymbol{N}$, we put

$Path_x^{(n)}$=the set of paths of length n with source x,

$Path_{x,y}^{(n)}=\{\xi\in Path_x^{(n)};\ r(\xi)=y\}$,

$String_x^{(n)}$=the set of strings of length n with source x,

H_n=*Hilbert space with orthonormal basis* $Path_*^{(n)}$,

(cf. [3]).

For a string $\rho=(\rho_+, \rho_-)\in String_*^{(n)}$, we represent an operator ρ on H_n by $\rho\xi=\delta(\rho_-, \xi)\rho_+$ $\xi\in H_n$ and denote by A_n a finite dimensional C^*-algebra generated by $String_*^{(n)}$. Moreover for $k\leq n$, we define linear maps i_k^n: $A_k\to A_n$ by

$$i_k^n(\rho)=\sum_{\xi\in Path_{r(\rho)}^{(n-k)}}\rho\circ(\xi, \xi)\ , \quad \rho\in String_*^{(k)}\ .$$

Let μ be a weight which is a map $G^{(0)}\to \boldsymbol{R}^+$ with $\mu(*)=1$, and Λ be Laplacian of G. We require that μ is harmonic i.e. $\Lambda\mu=\beta\mu$ with $\beta\in\boldsymbol{R}^+$. We define a trace tr on A_n by $tr(\rho)=\beta^{-n}\mu(r(\rho))\delta(\rho_+, \rho_-)$ for $\rho=(\rho_+, \rho_-)\in String_*^{(n)}$. For $n\in\boldsymbol{N}$, we define a projection $e_n\in A_{n+1}$ by

$$e_n=\beta^{-1}\sum_{\alpha\in Path_*^{(n-1)}}\ \sum_{\xi,\eta\in Path_{r(\alpha)}^{(1)}}\frac{\sqrt{\mu(r(\xi))\mu(r(\eta))}}{\mu(r(\alpha))}(\alpha\circ\xi\circ\xi^{\sim}, \alpha\circ\eta\circ\eta^{\sim})\in A_{n+1}\ .$$

Then we can prove that the sequence $\{e_n; n=1, 2, \cdots\}$ satisfies the following relations by calculations (cf. [3]):

(a) $e_ne_{n\pm1}e_n=\beta^{-2}e_n$ for $n\in\boldsymbol{N}$,

(b) $e_ne_m=e_me_n$ for $|m-n|\geq 2$,

(c) $tr(e_m\omega)=\lambda tr(\omega)$ for any word ω on $e_1, \cdots, e_{m-1}$.

Moreover for $x\in G^{(0)}$ such that $Path_{*,x}^{(1)}\neq\emptyset$, we define a projection $f_x\in A_1$ by $f_x=\sum_{\xi\in Path_{*,x}^{(1)}}(\xi, \xi)$. Then the next proposition gives the relations between f_x and e_n.

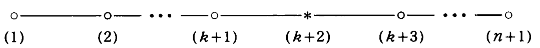
Proposition 2.4. (1) $e_1 f_x e_1 = \#(Path^{(1)}_{*,x})\mu(x)\beta^{-1}e_1$,

(2) $f_x e_n = e_n f_x$ *for* $n \geq 2$.

Proof. It follows from simple calculation.

Construction of a family of extended Jones' projections.

1) Case of $(n; \alpha_k, \alpha_{n-k-2})$: Let G be a Dynkin diagram of type A_{n+1} and the distinguished point $*$ be a vertex with distance $k+1$ from the end vertex.

A_{n+1} 

Then $\beta = 2\cos(\pi/(n+2))$, $\mu((i)) = \sin i\theta_n / \sin(k+2)\theta_n$. We take e_n, f_x with $x = (k+1)$, $(k+3)$, and denote $f_{(k+1)}$, $f_{(k+3)}$ by f_1, f_2. From [3] and Proposition 2.4

$$e_m e_{m\pm1} e_m = \beta^{-2} e_m = (4\cos^2(\pi/(n+2)))^{-1} e_m \,,$$

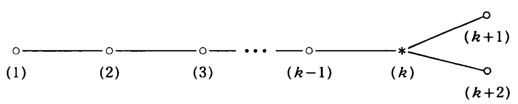
$$e_1 f_1 e_1 = \#(Path^{(1)}_{*,(k+1)})\mu((k+1))\beta^{-1}e_1 = \frac{\sin(k+1)\theta_n}{2\cos\theta_n \sin(k+2)\theta_n} e_1 \,,$$

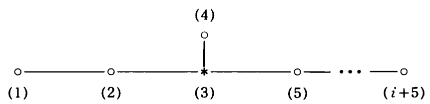
$$e_1 f_2 e_1 = \#(Path^{(1)}_{*,(k+3)})\mu((k+3))\beta^{-1}e_1 = \frac{\sin(n-k-1)\theta_n}{2\cos\theta_n \sin(k+2)\theta_n} e_1 \,.$$

So $\{e_n, f_1, f_2; n \geq 1\}$ is a family of extended Jones' projections corresponding to $(n; \alpha_k, \alpha_{n-k-2})$.

2) Case of $(2k; \lambda_0, \lambda_0, \lambda_{k-2})$ or $(n; \lambda_0, \lambda_1, \lambda_i)$ $(1 \leq i \leq 3)$ $(n_1 = 10,\ n_2 = 16,\ n_3 = 28)$:

Let G be a Dynkin diagram of type D_{k+2} or E_{i+5} respectively and the distinguished point $*$ be a vertex which is a source point of three edges.

D_{k+2} 

E_{i+5} 

Similarly we can construct a family of extended Jones' projections.

Remark that construction in case 1 is another proof of Theorem 2.1. and one in case 2 gives proof of Theorem 2.3.

3. Structure of A_m

In this section, for a family of extended Jones' projections $\{e_i, f_j; i\geq 1, 1\leq j\leq l\}$, we define von Neumann algebras $A=\{e_i, f_j; i\geq 1, 1\leq j\leq l\}''$ and $B=\{e_i; i\geq 1\}''$. To calculate the index $[A: B]$, we use subalgebras of A, $A_m=\{e_i, f_j; 1\leq i\leq m, 1\leq j\leq l\}''$, $B_m=\{e_i; 1\leq i\leq m\}''$ and $A_0=\{f_j; 1\leq j\leq l\}''$, $A_{-1}=B_0=B_{-1}=\boldsymbol{C}$. We search the structure of A_m and inclusion matrix for $B_m\subset A_m$.

3.1. Structure of A_m. Let ω be a word on $\{e_i, f_j; i\geq 1, 1\leq j\leq l\}$. We call ω *reduced* if it is of minimal length for the following gramatical rules of replacements:

(a) $e_ie_{i\pm 1}e_i \longleftrightarrow e_i$ for $i\geq 1$,

$e_1f_je_1 \longleftrightarrow e_1$ for $1\leq j\leq l$,

(b) $e_ie_j \longleftrightarrow e_je_i$ for $|i-j|\geq 2$,

$e_if_j \longleftrightarrow f_je_i$ for $i\geq 2, 1\leq j\leq l$,

(c) $e_i^2 \longleftrightarrow e_i$ for $i\geq 1$,

$f_j^2 \longleftrightarrow f_j$ for $1\leq j\leq l$,

(d) $f_if_j \longleftrightarrow 0$ for $i\neq j$.

Lemma 3.1. *Let ω be a reduced word and $m(\omega)$ a maximal index i of e_i which appears in ω. If any e_i does not appear in ω, we put $m(\omega)=0$. Then if $m(\omega)\geq 1$, $e_{m(\omega)}$ appears only once in ω, and if $m(\omega)=0$, $\omega\in\{f_j; 1\leq j\leq l\}$.*

Proof. We denote length of ω by $l(\omega)$ and show it by induction of $l(\omega)$. It is trivial in case $l(\omega)=1$. Suppose true for words ω of $l(\omega)\leq k$. Let ω be a reduced word of length $k+1$. Then $l(\omega)\geq 2$, and by (d) we obtain $m(\omega)\geq 1$. Suppose $\omega=\omega_1e_{m(\omega)}\omega_2e_{m(\omega)}\omega_3$ where ω_i is a reduced word and $m(\omega_2)\leq m(\omega)-1$.

1) Case of $m(\omega_2)\leq m(\omega)-2$: Since $e_{m(\omega)}$ commutes with $e_i(1\leq i\leq m(\omega)-2)$ and f_j, $\omega=\omega_1e_{m(\omega)}\omega_2e_{m(\omega)}\omega_3 \longleftrightarrow \omega_1e^2_{m(\omega)}\omega_2\omega_3 \longleftrightarrow \omega_1e_{m(\omega)}\omega_2\omega_3$. So length of ω is shortened using (c):

2) Case of $m(\omega_2)=m(\omega)-1$:

2a) Case of $m(\omega)-1=0$: Since $l(\omega_2)\leq k-1$, by induction hypothesis $\omega_2=f_1$ $(1\leq j\leq l)$. Then $\omega=\omega_1e_1f_je_1\omega_3 \longleftrightarrow \omega_1e_1\omega_3$. So length of ω is redued using (a).

2b) Case of $m(\omega)-1\geq 1$: Since $l(\omega_2)\leq k-1$, by induction hypothesis $\omega_2=v_1e_{m(\omega)-1}v_2$, here v_i is a reduced word of length $\leq k-2$. Then $\omega \longleftrightarrow \omega_1v_1e_{m(\omega)}e_{m(\omega)-1}e_{m(\omega)}v_2\omega_3 \longleftrightarrow \omega_1v_1e_{m(\omega)}v_2\omega_3$. So length of ω is shortened using (a).

From the above arguments, the assertion of the lemma is true for any words of length $k+1$.

Next using Lemma 3.1, we show A_m is finite dimensional.

Proposition 3.2. *For any $m\geq 0$, A_m and B_m are finite dimensional.*

Proof. We prove this by induction on m. It is trivial in case $m=0$. Suppose it is true for m. Let ω be a reduce word on $\{e_1, \cdots, e_{m+1}, f_1, \cdots, f_l\}$. If $m(\omega)=m+1(\geq 1)$, by previous lemma $\omega=\omega_1 e_{m(\omega)}\omega_2$, where ω_i is in A_m. And if $m(\omega)\leq m$, ω belongs to A_m. By induction A_m is finite dimensional, so the number of reduced word in A_{m+1} is finite. Since A_{m+1} is generated by reduced words on $\{e_1, \cdots, e_{m+1}, f_1, \cdots, f_l\}$, A_{m+1} is finite dimensional.

Now we show the relation of e_m's and the expectations E_{A_m}'s.

Proposition 3.3. *For any $m\geq 0$ and $x\in A_m$, $e_{m+1}xe_{m+1}=E_{A_{m-1}}(x)e_{m+1}$.*

Proof. In case of $x\in A_{m-1}$, since x commutes with e_{m+1}, we have $e_{m+1}xe_{m+1}=xe_{m+1}=E_{A_{m-1}}(x)e_{m+1}$. Let ω be a reduced word in A_m and $m(\omega)=m$.

a) Case of $m=0$: By Lemma 3.1 $\omega=f_j$ for some j $(1\leq j\leq l)$. Then $e_1\omega e_1=e_1 f_j e_1=\alpha_j e_1$. On the other hand, since $E_{A_{-1}}(f_j)\in \boldsymbol{C}$, $E_{A_{-1}}(f_j)=tr(E_{A_{-1}}(f_j))=tr(f_j)=\alpha_j$. Therefore $e_1\omega e_1=E_{A_{-1}}(\omega)e_1$.

b) Case of $m\geq 1$: By Lemma 3.1 $\omega=\omega_1 e_m\omega_2$ where ω_i belongs to A_{m-1}. Then $e_{m+1}\omega e_{m+1}=e_{m+1}\omega_1 e_m\omega_2 e_{m+1}=\omega_1 e_{m+1}e_m e_{m+1}\omega_2=\lambda\omega_1 e_{m+1}\omega_2=\lambda\omega_1\omega_2 e_{m+1}$. On the other hand, for any $x\in A_{m-1}$, $tr(e_m x)=tr(e_m)tr(x)=\lambda tr(x)$, so $E_{A_{m-1}}(e_m)=\lambda$ and $E_{A_{m-1}}(\omega)=E_{A_{m-1}}(\omega_1 e_m\omega_2)=\omega_1 E_{A_{m-1}}(e_m)\omega_2=\lambda\omega_1\omega_2$. Hence $e_{m+1}\omega e_{m+1}=E_{A_{m-1}}(\omega)e_{m+1}$.

Let k_i be a non-negative integer such that $\alpha_i=\lambda_{k_i}$ and we set $C_m=alg(A_{m-1}e_m A_{m-1})$, $p_{i,m}=1-(1-f_i)\vee e_1\vee\cdots\vee e_m$ for $m\geq 1$ and $p_{i,0}=f_i$.

Lemma 3.4.

1) $p_{i,m}=0$ *for* $m\geq k_i+1$,

2) $f_i-p_{i,m}$, $1-\sum_{i=1}^{l}p_{i,m}\in C_m$ *for* $m\geq 1$.

Proof. 1) By Corollary 2.8 and Lemma 2.10 of [4].

$$tr(p_{i,k_i+1})=P_{k_i+1}(\lambda)-(1-\alpha_i)P_{k_i}(\lambda)=P_{k_i+1}(\lambda)-\left(1-\frac{\lambda P_{k_i-1}(\lambda)}{P_{k_i}(\lambda)}\right)P_{k_i}(\lambda)=0\,.$$

Hence $p_{i,k_i+1}=0$. For $m\geq k_i+1$, since $0\leq p_{i,m}\leq p_{i,k_i+1}$, we have $p_{i,m}=0$.

2) We prove the assertion by induction. In case of $m=1$, by Theorem 2.7 of [4], $p_{i,1}=f_i-c_i p_{i,0}e_1 p_{i,0}$, $(c_i=P_0(1-\alpha_i, \lambda)/P_0(\alpha_i, \lambda)$ where $P_{i+1}(\alpha, \lambda)=P_i(\lambda)-\alpha P_{i-1}(\lambda))$. Therefore $f_i-p_{i,1}=c_i p_{i,0}e_1 p_{i,0}=c_i f_i e_1 f_i\in C_1$. Suppose it is true

for $m(\leq k_i)$. By Theorem 2.7 of [4], $p_{i,m+1}=p_{i,m}-c_{i,m}p_{i,m}e_{m+1}p_{i,m}$ ($c_{i,m}=P_m(1-\alpha_i, \lambda)/P_{m+1}(1-\alpha_i, \lambda)$), so $f_i-p_{i,m+1}=f_i-p_{i,m}+c_{i,m}p_{i,m}e_{m+1}p_{i,m}\in C_m+C_{m+1}$. Since $C_m=alg(A_{m-1}e_mA_{m-1})=alg(A_{m-1}e_me_{m+1}e_mA_{m-1})\subset C_{m+1}$, we get $f_i-p_{i,m+1}\in C_{m+1}$. Therefore for any $m\leq k_i+1$; $f_i-p_{i,m}\in C_m$. Now we set $m=k_i+1$, then since $p_{i,k_i+1}=0$ we obtain $f_i\in C_{k_i+1}$. So for any $m\geq k_i+1$, $f_i-p_{i,m}=f_i\in C_{k_i+1}\subset C_m$. Since $1-\sum_{i=1}^{l}P_{i,m}=\sum_{i=1}^{l}(f_i-p_{i,m})$, we have $1-\sum_{i=1}^{l}p_{i,m}\in C_m$ for $m\geq 1$.

Now we show theorems which give the structure of subalgebras $\{A_m\}_{m\geq -1}$.

Theorem 3.5. *Let M be a type II_1-factor and $\{e_i, f_j; i\geq 1, j=1, 2\}$ be a family of extended Jones' projections in M corresponding to $(n; \lambda_k, \lambda_{n-k-2})$ $(0\leq k\leq[(n-2)/2])$. And A_m is as above.*

(*a*) *The factorization of the algebra A_m is*

(1) *when $m\leq k$,*

$$A_m=\bigoplus_{j=0}^{m+1}A_{m,j}\,, \quad A_{m,j}\cong M_{a_{m,j}}(\boldsymbol{C}) \quad \textit{with} \quad a_{m,j}=\binom{m+1}{j},$$

(2) *when $k+1\leq m\leq n-k-2$,*

$$A_m=\bigoplus_{j=0}^{[(m+k)/2]+1}A_{m,j}\,, \quad A_{m,j}\cong M_{a_{m,j}}(\boldsymbol{C})$$

$$\textit{with} \quad a_{m,j}=\binom{m+1}{j}-\binom{m+1}{j-k-2},$$

(3) *when $m\geq n-k-1$,*

$$A_m=\bigoplus_{j=[(m-n+k+1)/2]+1}^{[(m+k)/2]+1}A_{m,j}\,, \quad A_{m,j}\cong M_{a_{m,j}}(\boldsymbol{C})$$

$$\textit{with} \quad a_{m,j}=\binom{m+1}{j}-\binom{m+1}{j-k-2}-\binom{m+1}{j+n-k}.$$

(*b*) *Let $\Lambda_m=[A_{m-1}\rightarrow A_m]$ be the inclusion matrix of A_{m-1} in A_m. Then*

(1) *when $m\leq k$,*

$$\Lambda_m=(d_{i,j}),\ d_{i,j}=\begin{cases}1 & j=i,\ i+1\,,\\ 0 & \textit{otherwise}\,,\end{cases}$$

for $i=0, 1, \cdots, m; j=0, 1, \cdots, m+1$,

(2) *when $k+1\leq m\leq n-k-2$,*

$$\Lambda_m=(d_{i,j}),\ d_{i,j}=\begin{cases}1 & j=i,\ i+1\ ,\\ 0 & otherwise\ ,\end{cases}$$

for $i=0, 1, \cdots, [(m+k+1)/2]+1; j=0, 1, \cdots, [(m+k)/2]+1$,

(3) *when* $m \geq n-k-1$,

$$\Lambda_m=\Lambda_{m-1}^t\ .$$

(*c*) *Trace of a minimal projection in* $A_{m,j}$ *is*

$$t_{m,j}=\frac{\sin(m-2j+k+3)\theta_n}{2^{m+1}\cos^{m+1}\theta_n\sin(k+2)\theta_n}$$

$$=\begin{cases}\lambda^j P_{m+1-2j}(\lambda_k, \lambda) & j \leq (m+1)/2\ ,\\ \lambda^{m+1-j} P_{2j-m-1}(\lambda_{n-k-2}, \lambda) & j>(m+1)/2\ .\end{cases}$$

Theorem 3.6. *Let* M *be a type* II_1*-factor and* $\{e_i, f_j; i \geq 1, j=1, 2, 3\}$ *be a family of extended Jones' projections in* M *corresponding to* $(2k+4; \lambda_0, \lambda_0, \lambda_k)$ $(k \geq 0)$. *And* A_m *is as above.*

(*a*) *The factorization of the algebra* A_m *is*

(1) *when* $m \leq k$,

$$A_m=\bigoplus_{j=1}^{2[m/2]-[(m+1)/2]+3} A_{m,j}\ ,$$

(2) *when* $m \geq k+1$,

$$A_m=\bigoplus_{j=[(m-k+3)/2]}^{2[m/2]-[(m+1)/2]+3} A_{m,j}\ .$$

Here $A_{m,j}$ *is a full matrix algebra of certain order.*

(b) Let $\Lambda_m=[A_{m-1} \rightarrow A_m]$ be the inclusion matrix of A_{m-1} in A_m. Then

(1) when $m \leq k$,

$$\Lambda_m=(d_{i,j}),\ d_{i,j}=\begin{cases}1 & j=i,\ i+1\ ,\\ 1 & (i, j)=(2[(m-1)/2]-[m/2]+3,\ 2[m/2]\\ & \qquad -[(m+1)/2]+3)\ ,\\ 0 & \text{otherwise}\ ,\end{cases}$$

for $i=1, 2, \cdots, 2[(m-1)/2]-[m/2]+3;$
$j=1, 2, \cdots, 2[m/2]-[(m+1)/2]+3$,

(2) when $m \geq k+1$,

$$\Lambda_m=\Lambda_{m-1}^t\ .$$

(c) Trace of a minimal projection in $A_{m,j}$ is

$$t_{m,j}=\begin{cases}\dfrac{\sin(k-m+2j-1)\theta_n}{2^{m+1}\cos^{m+1}\theta_n\sin(k+2)\theta_n} & j\leq[m/2]+1\,,\\ \lambda^{[m/2]+1} & j\geq[m/2]+2\,.\end{cases}$$

Theorem 3.7. *Let M be a type II_1-factor and $\{e_i, f_j;\, i\geq 1, j=1, 2, 3\}$ be a family of extended Jones' projections in M corresponding to $(n; \lambda_0, \lambda_1, \lambda_k)$ $(1\leq k\leq 3)$. And A_m is as above.*

(*a*) *The factorization of the algebra A_m is*

(1) *when $0\leq m\leq k$,*

$$A_m=\bigoplus_{j=1}^{2[m/2]+3}A_{m,j}\,,$$

(2) *when $m\geq k+1$,*

$$A_m=\bigoplus_{j=[(m-k+3)/2]}^{[m/2]+3}A_{m,j}\,.$$

Here $A_{m,j}$ is a full matrix algebra of certain order.

(b) Let $\Lambda_m=[A_{m-1}\rightarrow A_m]$ be the inclusion matrix of A_{m-1} in A_m. Then

(1) when $1\leq m\leq k$,

(i) in case m is odd

$$\Lambda_m=(d_{i,j}),\ d_{i,j}=\begin{cases}1 & j=i,\ i+1\ (i, j\leq(m+3)/2)\,,\\ 1 & j=i-1,\ i\ (i=(m+5)/2)\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

for $i,j=1, 2, \cdots, (m+5)/2$,

(ii) in case m is even

$$\Lambda_m=(d_{j,j}),\ d_{i,j}=\begin{cases}1 & j=i,\ i+1\ (i\leq m/2+1, j\leq m/2+2)\,,\\ 1 & j=i+1,\ i+2\ (j=m/2+3)\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

for $i=1, 2, \cdots, m/2+2;\ j=1, 2, \cdots, m/2+3$,

(2) when $m\geq k+1$,

$$\Lambda_m=\Lambda_{m-1}^t\,.$$

(c) Trace of a minimal projetion in $A_{m,j}$ is

$$t_{m,j}=\begin{cases}\dfrac{\sin(k-m+2j-1)\theta_n}{2^{m+1}\cos^{m+1}\theta_n\sin(k+2)\theta_n} & j\leq[m/2]+1\,,\\ \lambda^{[m/2]+1} & j=[m/2]+2\,.\\ \dfrac{\sin a_m\theta_n}{2^{m+1}\cos^{m+1}\theta_n\sin 3\theta_n} & j=[m/2]+3\,,\end{cases}$$

where $a_m=\begin{cases}2 & \text{when } m \text{ is even}\,,\\ 1 & \text{when } m \text{ is odd}\,.\end{cases}$

Proof of Theorems 3.5 ~ 3.7. We prove Theorem 3.5 by induction on m. It is trivial in case of $m=-1, 0$. Suppose the assertions are true all $l\leq m$. By basic construction for $A_{m-1}\subset A_m$, we get a projection $f(=e_{A_{m-1}})$ and von Neumann algebra $\langle A_m, f\rangle$ generated by $\{A_m, f\}$. We define $\tilde{A}_{m+1}=alg\{A_mfA_m\}$, then since $\tilde{A}_{m+1}$ is uw-closed two sided ideal of $\langle A_m, f\rangle$, there exists a central projection p in $\langle A_m, f\rangle$ such that $\tilde{A}_{m+1}=\langle A_m, f\rangle p$. By $f\in\tilde{A}_{m+1}$, $fp=f$, so we have $z(f)\leq p$ where $z(f)$ is a central support of f. On the other hand $z(f)=1$, hence $p=1$ and $\tilde{A}_{m+1}=\langle A_m, f\rangle$.

Now let $\langle A_m, f\rangle=\oplus_{j=1}^{m+1}\tilde{A}_{m+1,j}$, $(\tilde{A}_{m+1,j})_f\cong A_{m-1,j-1}$, and p_j be a minimal projection in $A_{m-1,j}$. Since $A_{m-1,j}\ni x\mapsto xf\in(\tilde{A}_{m+1,j+1})_f$ is $*$-isomorphism and onto, p_jf is a minimal projection in $\tilde{A}_{m+1,j+1}$. We define a trace Tr on $\langle A_m, f\rangle=\oplus_{j=1}^{m+1}\tilde{A}_{m+1,j}$ by $Tr(p_jf)=\lambda tr(p_j)$, then for any j, $Tr(f_jf)>0$, so Tr is faithful. And by the definition of Tr, for any $x\in A_{m-1}$ $Tr(fx)=\lambda tr(x)$. Therefore for any $x\in A_m$, $Tr(fx)=Tr(fxf)=Tr(E_{A_{m-1}}(x)f)=\lambda tr(x)$.

Now we define a homomorphism Φ: $\langle A_m, f\rangle=alg\{A_mfA_m\}\to C_{m+1}=alg\{A_me_{m+1}A_m\}$ by $\Phi(\sum_i a_ifb_i)=\sum_i a_ie_{m+1}b_i$, where $a_i, b_i\in A_m$. Then $Tr(\sum_i a_ifb_i)=\sum_i Tr(fb_ia_i)=\sum_i\lambda tr(b_ia_i)=tr(\sum_i a_ie_{m+1}b_i)$. Since Tr and tr are faithful, Φ is well-defined and injective. Moreover by a simple calculation we obtain that Φ is $*$-isomorphism and onto. Because $C_{m+1}=alg\{A_me_{m+1}A_m\}$ is a uw-closed two sided ideal of A_{m+1}, there exists a central projection q in A_{m+1} such that $C_{m+1}=(A_{m+1})q$. Then, since Φ: $\langle A_{m+1}, f\rangle\to A_{m+1}q$ is $*$-isomorphism and onto, we have $\Phi(1)=q$. And by $\Phi(f)=e_{m+1}$, $z(e_{m+1})=\Phi(z(f))=\Phi(1)=q$. $p_{i,m}$ is central projection in A_{m+1} and $e_{m+1}(1-p_{1,m+1}-p_{2,m+1})=e_{m+1}$, so $q=z(e_{m+1})\leq 1-p_{1,m+1}-p_{2,m+1}$. On the other hand by Lemma 3.4, $1-p_{1,m+1}-p_{2,m+1}\in C_{m+1}=(A_{m+1})q$. Therefore we obtain $1-p_{1,m+1}-p_{2,m+1}=q$. Since $1\in\langle A_m, f\rangle=alg\{A_mfA_m\}$, there exist a_i, $b_i\in A_m$ $(1\leq i\leq l)$ such that $\sum_{i=1}^{l}a_ifb_i=1$. Then $q=\Phi(1)=\Phi(\sum a_ifb_i)=\sum a_ie_{m+1}b_i$ and for any $x\in A_m$, $\Phi(x)=\Phi(\sum xa_ifb_i)=\sum xa_ie_{m+1}b_{id}=xq$. Hence we have

$$\Phi(A_m)=(A_m)q\ ,\quad \Phi(A_{m,j})=(A_{m,j})q$$

and

$$[(A_m)_q\to(A_{m+1})_q]=[A_m\to\langle A_m, f\rangle]=[A_{m-1}\to A_m]^t=\Lambda_m{}^t\ .$$

Next we consider the inclusion matrix $[(A_m)_{p_{1,m+1}+p_{2,m+1}}\to(A_{m+1})_{p_{1,m+1}+p_{2,m+1}}]$.

1) Case of $m\leq k-1$: Since $(A_m)_{p_{i,m}p_{i,m+1}}=(A_{m+1})_{p_{i,m}p_{i,m+1}}=\mathbf{C}p_{i,m+1}$ and $dim(A_m)_{p_{i,m+1}}=1$, we obtain

$$[A_m \to A_{m+1}]=\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & & & & \vdots \\ \vdots & & \Lambda_m{}^t & & \vdots \\ \vdots & & & & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

2) Case of $k\leq m\leq n-k-3$: Since $(A_m)_{p_{2,m}p_{2,m+1}}=(A_{m+1})_{p_{2,m}p_{2,m+1}}=\mathbf{C}p_{2,m+1}$ and $dim(A_m)_{p_{2,m+1}}=1$ and $p_{1,m+1}=0$, we have

$$[A_m \to A_{m+1}]=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \Lambda_m{}^t & \\ 0 & & & \end{pmatrix}.$$

3) Case of $m\geq n-k-2$: Since $p_{i,m+1}=0$ by Lemma 3.4, we have

$$[A_m \to A_{m+1}]=\Lambda_m{}^t .$$

Next we consider a trace of minimal projection in $A_{m+1,j}$. Because Φ is onto, *-isomorphic and fp_j is a minimal projection in $\tilde{A}_{m+1,j+1}$, $\Phi(fp_j)=e_{m+1}p_j$ is a minimal projection in $A_{m+1,j+1}$ for $0\leq j\leq m$. So, for $0\leq j\leq m$, $t_{m+1,j+1}=tr(e_{m+1}f_j)=Tr(ff_j)=\lambda tr(f_j)=\lambda t_{m-1,j}$.

1) Case of $j+1\leq\{(m+1)+1\}/2$:

$$t_{m+1,j+1}=\lambda\cdot\lambda^j P_{(m-1)+1-2j}(\lambda_k, \lambda)=\lambda^{j+1}P_{m-2j}(\lambda_k, \lambda) .$$

2) Case of $j+1>\{(m+1)+1\}/2$:

$$t_{m+1,j+1}=\lambda\cdot\lambda^{m-1-j}P_{2j-(m-1)-1}(\lambda_{n-k-2}, \lambda)=\lambda^{m-j}P_{2j-m}(\lambda_k, \lambda).$$

And by Corollary 2.8 of [4] we have $t_{m+1,0}=tr(p_{2,m+1})=P_{m+2}(\lambda_k, \lambda)$, $t_{m+1,m+2}=tr(p_{1,m+1})=P_{m+2}(\lambda_{n-k-2}, \lambda)$, so the assertion (c) has proved.

Let $\overrightarrow{a_m}=(a_{m,0}, \cdots, a_{m,m+1})$, then since $\overrightarrow{a_{m+1}}=\overrightarrow{a_m}[A_m \to A_{m+1}]$, the assertion (a) is obvious. Therefore the induction step for $m+1$ has been completed and Theorem 3.5 has been proved. Similarly we can prove Theorems 3.6 and 3.7.

3.2. Relation between A_m and B_m. In this subsection, we show a diagram

$$\begin{array}{ccc} B_{m+1} & \subset & A_{m+1} \\ \cup & & \cup \\ B_m & \subset & A_m \end{array}$$

is a commuting square for all $m\geq 0$ and the periodicity of inclusion matrix $[B_m \to A_m]$.

Proposition 3.8. *For any* $m\geq 0$,

$$\begin{array}{ccc} B_{m+1} & \subset & A_{m+1} \\ \cup & & \cup \\ B_m & \subset & A_m \end{array}$$

is a commuting square.

Proof. Let ω be a reduced word in B_{m+1}.

1) When $\omega\in B_m\subset A_m$, $E_{A_m}(\omega)=\omega\in B_m$.

2) When $\omega\notin B_m$ (i.e. $m(\omega)=m+1$), by Lemma 3.2 there exist reduced words $\omega_1, \omega_2\in B_m$ such that $\omega=\omega_1 e_{m+1}\omega_2$. Then $E_{A_m}(\omega)=E_{A_m}(\omega_1 e_{m+1}\omega_2)=\omega_1 E_{A_m}(e_{m+1})\omega_2$. Since $tr(xe_{m+1})=tr(x)tr(e_{m+1})=\lambda tr(x)$ for any $x\in A_{m+1}$, $E_{A_m}(e_{m+1})=\lambda$. So we have $E_{A_m}(\omega)=\lambda\omega_1\omega_2\in B_m$. Hence $E_{A_m}(B_{m+1})\subset B_m$, namely the diagram is a commuting square.

Proposition 3.9. *For any* $m\geq n-2$

$$[B_m \to A_m]=[B_{m+2} \to A_{m+2}]\,.$$

Proof. Let p (resp. q) be a minimal central projection in A_m (resp. B_m).

1) Case of $l=[(A_m)_{pq}\colon (B_m)_{pq}]^{1/2}\neq 0$: Let f be a minimal projection in B such that $f\leq q$. Since $(B_m)_q\ni x\mapsto xp\in(B_m)_{pq}$ is $*$-isomorphic and onto, fp is a minimal projection in $(B_m)_{pq}$. Then by $[(A_m)_{pq}\colon (B_m)_{pq}]^{1/2}=l$, there exists a family of mutually orthogonal non-zero projections $\{p_i;\ i=1, 2, \cdots, l\}$ in $(A_m)_{pq}$ and $fp=\sum_i p_i$. Because $B_m\ni x\mapsto xe_{B_m}\in e_{B_m}\langle B_{m+1}, e_{B_m}\rangle e_{B_m}$ is $*$-isomorphic and onto, fe_{B_m} is a minimal projection in $e_{B_m}\langle B_{m+1}, e_{B_m}\rangle e_{B_m}$. And by $e_{B_m}\langle B_{m+1}, e_{B_m}\rangle e_{B_m}\subset\langle B_{m+1}, e_{B_m}\rangle$, fe_{B_m} is a minimal projection in $\langle B_{m+1}, e_{B_m}\rangle$. Now let J_m (resp. J'_m) be a canonical conjugation of $L^2(A_m, tr)$ (resp. $L^2(B_m, tr)$), then $J'_{m+1}qJ'_{m+1}$ is a minimal central projection in $\langle B_{m+1}, e_{B_m}\rangle$ and $fe_{B_m}=(fe_{B_m})(J'_{m+1}qJ'_{m+1})$ is a minimal projection in $\langle B_{m+1}, e_{B_m}\rangle(J'_{m+1}qJ'_{m+1})$. Since

$$\Phi'\colon\ \langle B_{m+1}, e_{B_m}\rangle\ni\sum a_i e_{B_m} b_i\mapsto\sum a_i e_{m+2} b_i\in B_{m+2},\ a_i,\ b_i\in B_{m+1}\,,$$

is $*$-isomorphic and onto, we see that $q\equiv\Phi'(J'_{m+1}qJ'_{m+1})$ is a minimal central projection in B_{m+2} and $fe_{m+2}=\Phi'(fe_{B_m})$ is a minimal projection in $(B_{m+2})_{q'}$. On the other hand

$$\Phi\colon\ \langle A_{m+1}, e_{A_m}\rangle\ni\sum a_i e_{A_m} b_i\mapsto\sum a_i e_{m+2} b_i\in A_{m+2},\ a_i,\ b_i\in A_{m+1}\,,$$

is $*$-isomorphic and onto, so $p\equiv\Phi(J_{m+1}qJ_{m+1})$ is a minimal central projection in A_{m+2} and $(fe_{m+2})p'=\sum p_j e_{m+2}$. Since $A_m\ni x\mapsto xe_{A_m}\in e_{A_m}\langle A_{m+1}, e_{A_m}\rangle e_{A_m}$ is $*$-isomorphic and onto, we obtain that $p_i e_{A_m}$ is a minimal projection. And by $\langle A_{m+1}, e_{A_m}\rangle\cong A_{m+2}$, $p_i e_{m+2}=\Phi(p_i e_{A_m})$ is a minimal projection in A_{m+2}.

Moreover $p_i e_{m+2} \leq p'q'$ implies that $p_i e_{m+2}$ is a minimal projection in $(A_{m+2})_{p'q'}$. Hence a minimal projection in $(B_{m+2})_{p'q'}$ is a sum of l mutually orthogonal minimal projections in $(A_{m+2})_{p'q'}$. So $[(A_{m+2})_{p'q'}: (B_{m+2})_{p'q'}]^{1/2} = l = [(A_m)_{pq}: (B_m)_{pq}]^{1/2}$.

2) Case of $[(A_m)_{pq}: (B_m)_{pq}] = 0$: For any minimal projection f in $(B_m)_q$, $fp = 0$. Also fe_{m+2} is a minimal projection in $(B_{m+2})_{q'}$, and $(fe_{m+2})p' = fpe_{m+2} = 0$. Because for any minimal projection f' in $(B_{m+2})_{q'}$, there exists a partial isometry v in $(B_{m+2})_{q'}$ such that $f' = v(fe_{m+2})v^*$, so $f'p' = v(fe_{m+2})v^*p' = v(fe_{m+2}p')v^* = 0$. Hence $p'q' = 0$ and $[(A_{m+2})_{p'q'}: (B_{m+2})_{p'q'}] = 0 = [(A_m)_{pq}: (B_m)_{pq}]$.

From 1) and 2) we obtain $[B_m \to A_m] = [B_{m+2} \to A_{m+2}]$ for any $m \geq n-2$.

4. The indices of the pairs of II_1-factors

4.1. Calculation of the index $[A: B]$. In this subsection, for a pair of type II_1-factors $A \supset B$ generated by a family of extended Jones' projections, we calculate index $[A: B]$ by using Wenzl's index formula. Moreover we define some II_1-subfactors of A.

Theorem 4.1. *Let M be a type II_1-factor, $\{e_i, f_j; i \geq 1, 1 \leq j \leq l\}$ be a family of extended Jones' projections in M corresponding to $(n; \alpha_1, \cdots, \alpha_l)$ and $A = \{e_i, f_j; i \geq 1, 1 \leq j \leq l\}''$, $B = \{e_i: i \geq 1\}''$. Then A and B are hyperfinite type II_1-factors and index $[A: B]$ is given as follows:*

1) *Case of* $(n; \alpha_1, \alpha_2) = (n; \lambda_k, \lambda_{n-k-2})$ $(0 \leq k \leq [(n-2)/2])$

$$[A: B] = \frac{\sin^2(k+2)\theta_n}{\sin^2\theta_n}, \quad \textit{with} \quad \theta_n = \frac{\pi}{n+2}.$$

2) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3) = (2k; \lambda_0, k_0, \lambda_{k-2})$ $(k \geq 2)$

$$[A: B] = 2\cot^2\theta_n.$$

3) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3) = (10; \lambda_0, \lambda_1, \lambda_1)$

$$[A: B] = 18 + 10\sqrt{3}.$$

4) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3) = (16; \lambda_0, \lambda_1, \lambda_2)$

$$[A: B] = 9\left\{2\sin^2\theta_n\left(\frac{\sin^2 2\theta_n}{\sin^2 4\theta_n} + \frac{\sin^2\theta_n}{\sin^2 3\theta_n} + 1\right)\right\}^{-1}.$$

5) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3) = (28; \lambda_0, \lambda_1, \lambda_3)$

$$[A: B] = 15\left\{2\sin^2\theta_n\left(\frac{\sin^2\theta_n + \sin^2 3\theta_n}{\sin^2 5\theta_n} + \frac{\sin^2\theta_n}{\sin^2 3\theta_n} + 1\right)\right\}^{-1}.$$

Proof. From Proposition 3.8, $\{A_m; m \geq -1\}$, $\{B_m; m \geq -1\}$ satisfy the first hypothesis of Wenzl's index formula. And by Theorems 3.5~3.7 and 3.9, for $m\geq n-2$, the inclusion matrices $[A_m\to A_{m+1}]$, $[B_m\to B_{m+1}]$ and $[A_m\to B_m]$ are periodic and those $[A_m\to A_{m+2}]$, $[B_m\to B_{m+2}]$ are primitive. Therefore $\{A_m; m\geq -1\}$, $\{B_m; m\geq -1\}$ satisfy the second hypothesis of Wenzl's index formula. Hence A and B are hyperfinite II_1-factors, and for any $m\geq n-2$,

$$[A: B]=\|\overrightarrow{s_m}\|^2/\|\overrightarrow{t_m}\|^2$$

where $\overrightarrow{t_m}$ (resp. $\overrightarrow{s_m}$) is trace vector for A_m (resp. B_m).

At first, we consider trace vector $\overrightarrow{s_{n-1}}$. Since B_{n-1} in any case is isomorphic to A_{n-1} in case of $(n; \alpha_1, \alpha_2)=(n; \lambda_0, \lambda_{n-2})$, by Theorem 3.5 trace vector $\overrightarrow{s_{n-1}}=(s_{n-1,j})_{j=0,\cdots,[n/2]}$ is given by

$$s_{n-1,j}=\frac{\sin(2j+1)\theta_n}{2^n\cos^n\theta_n\sin\theta_n}$$

and

$$\|\overrightarrow{s_{n-1}}\|^2=(2^{2n}\cos^{2n}\theta_n\sin^2\theta_n)^{-1}\sum_{j=0}^{[n/2]}\sin^2(2j+1)\theta_n\,.$$

Next we calculate the square of norm $\|\overrightarrow{t_{n-1}}\|^2$.

1) Case of $(n; \alpha_1, \alpha_2)=(n; \lambda_k, \lambda_{n-k-2})$ $(0\leq k\leq[(n-2)/2])$: By Theorem 3.5, trace vector $\overrightarrow{t_{n-1}}=(t_{n-1,j})_{j=[k/2]+1,\cdots,[(n+k+1)/2]}$ is given by

$$t_{n-1,j}=\frac{\sin(2j-k)\theta_n}{2^n\cos^n\theta_n\sin(k+2)\theta_n}$$

and

$$\|\overrightarrow{t_{n-1}}\|^2=(2^{2n}\cos^{2n}\theta_n\sin^2(k+2)\theta_n)^{-1}\sum_{j=[k/2]+1}^{[(n+k+1)/2]}\sin^2(2j-k)\theta_n\,.$$

Then

$$\begin{aligned}[A: B]&=\|\overrightarrow{s_{n-1}}\|^2/\|\overrightarrow{t_{n-1}}\|^2\\ &=\frac{\sin^2(k+2)\theta_n\sum_{j=0}^{[n/2]}\sin^2(2j+1)\theta_n}{\sin^2\theta_n\sum_{j=[k/2]+1}^{[(n+k+1)/2]}\sin^2(2j-k)\theta_n}\\ &=\frac{\sin^2(k+2)\theta_n}{\sin^2\theta_n}\,.\end{aligned}$$

2) Case of $(n; \alpha_1, \alpha_2, \alpha_3)=(2k; \lambda_0, \lambda_0, \lambda_{k-2})$ $(k\geq 2)$: By Theorem 3.6, trace vector $\overrightarrow{t_{n-1}}=(t_{n-1})_{j=[k/2]+2,\cdots,k+1}$ is given by

$$t_{n-1,j}=\begin{cases}\dfrac{\sin(2j-k-2)\theta_n}{2^n\cos^{n+1}\theta_n} & \text{for } i\leq k\\ (2^n\cos^n\theta_n)^{-1} & \text{for } j=k+1\end{cases}$$

$$=\frac{\sin(2j-k-2)\theta_n}{2^n\cos^{n+1}\theta_n}$$

and

$$\|\overrightarrow{t_{n-1}}\|^2=(2^{2n}\cos^{2n+2}\theta_n)^{-1}\sum_{j=[k/2]+2}^{k+1}\sin^2(2j-k-2)\theta_n .$$

Then

$$\begin{aligned}[A: B]&=\|\overrightarrow{s_{n-1}}\|^2/\|\overrightarrow{t_{n-1}}\|^2\\ &=\frac{2\cos^2\theta_n\sum_{j=0}^{[n/2]}\sin^2(2j+1)\theta_n}{\sin^2\theta_n\sum_{j=[k/2]+2}^{k+1}\sin^2(2j-k-2)\theta_n}\\ &=2\cot^2\theta_n .\end{aligned}$$

Similarly we can obtain index $[A: B]$ in case of $(n; \alpha_1, \alpha_2, \alpha_3)=(10, \lambda_0, \lambda_1, \lambda_1)$, $(16, \lambda_0, \lambda_1, \lambda_2)$, $(28, \lambda_0, \lambda_1, \lambda_3)$ by using Theorem 3.7.

Now for a family of extended Jones' projections $\{e_i, f_j;\ i\geq 1, 1\leq j\leq 3\}$, we define von Neumann subalgebras $A(j)$ of A $(j=1, 2, 3)$ by $A(j)=\{e_i, f_j;\ i\geq 1\}''$. Since $\{e_i, f_j, 1-f_j;\ i\geq 1\}$ is a family of extended Jones' projections corresponding to $(n; \alpha_j, 1-\alpha_j)$, by Theorem 3.5 $A(j)$ is a hyperfinite II_1-factor and

$$[A(j): B]=\frac{\sin^2(k_j+2)\theta_n}{\sin^2\theta_n},$$

where k_j is an integer such that $\lambda_{k_j}=\alpha_j$.

Theorem 4.2. *Let A and $A(j)$ are as above. Then index for a pair $A\supset A(j)$ is given as follows.*

1) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(2k; \lambda_0, \lambda_0, \lambda_{k-2})$ $(k\geq 2)$

$$[A: A(1)]=[A: A(2)]=(2\sin^2\theta_n)^{-1} ,$$

$$[A: A(3)]=2 .$$

2) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(10; \lambda_0, \lambda_1, \lambda_1)$

$$[A: A(1)]=6+2\sqrt{3} ,$$

$$[A: A(2)]=[A: A(3)]=3+\sqrt{3} .$$

3) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(16; \lambda_0, \lambda_1, \lambda_2)$

$$[A: A(j)]=9\beta\{2\sin^2(j+1)\theta_n\}^{-1}\quad (j=1, 2, 3)$$

where $\beta^{-1}=\sin^2 2\theta_n/(\sin^2 4\theta_n)+\sin^2\theta_n/(\sin^2 3\theta_n)+1$.

4) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(28; \lambda_0, \lambda_1, \lambda_3)$

$$[A: A(j)]=15\gamma\{2\sin^2(k_j+2)\theta_n\}^{-1} \quad (j=1, 2, 3)$$

where $\gamma^{-1}=(\sin^2\theta_n+\sin^2\theta_n)/(\sin^2 5\theta_n)+\sin^2\theta_n/(\sin^2 3\theta_n)+1$ *and* $(k_1, k_2, k_3)=(0, 1, 3)$.

Proof. Since $[A: B]=[A: A(j)][A(j): B]$, this follows by Theorem 3.10 and simple calculation.

4.2. Relative commutant $B'\cap A$.

Theorem 4.3. *Let M be a type II_1-factor, $\{e_i, f_j; i\geq 1, 1\leq j\leq l\}$ be a family of extended Jones' projections in M corresponding to $(n; \alpha_1, \cdots, \alpha_l)$ and $A=\{e_i, f_j; i\geq 1, 1\leq j\leq l\}''$, $B=\{e_i; i\geq 1\}''$. Then relative commutant $B'\cap A$ is trivial.*

Proof. Here we give the proof in case of $(n; \alpha_1, \alpha_2)=(n; \lambda_k, \lambda_{n-k-2})$ $(0\leq k\leq[(n-2)/2])$. Other cases can be treated similarly.

Let G be a Dynkin diagram of type A_{n+1}, the distinguished point $*$ be a vertex with distance $k+1$ from the end vertex and $A(G)$ be a hyperfinite II_1-factor generated by string algebras of G. From 2.2 we can construct a family of extended Jones' projections $\{e_i, f_j; i\geq 1, 1\leq j\leq 2\}$ corresponding to $(n; \alpha_1, \alpha_2)=(n; \lambda_k, \lambda_{n-k-2})$ and put $A=\{e_i, f_j; i=1, 2, \cdots, 1\leq j\leq 2\}''$ and $B=\{e_i; i=1, 2, \cdots\}''$. From Theorem 4.1, we have $[A: B]=\sin^2(k+2)\theta_n/(\sin^2\theta_n)$. On the other hand, $[A(G): B]=\sin^2(k+2)\theta_n/(\sin^2\theta_n)$ by Prop. 4.5.2. of [1]. Since $A(G)\supset A\supset B$, we obtain $A(G)=A$. So by $B'\cap A(G)=\boldsymbol{C}$, it follow that $B'\cap A=\boldsymbol{C}$.

4.3. Fixed point subalgebras for permutations of f_j's. Now we consider automorphisms of A by permutation of $\{f_i; 1\leq i\leq l\}$. If $\sigma\in\mathrm{Aut}(A)$ and $\sigma(f_i)=f_j$, then $tr(f_j)=tr(\sigma(f_i))=tr(f_i)$ i.e. $\alpha_i=\alpha_j$. So there exists such an automorphism, if and only if

$(n; \alpha_1, \alpha_2)=(2k; \lambda_{k-1}, \lambda_{k-1})$ for $k\geq 1$, or

$(n; \alpha_1, \alpha_2, \alpha_3)=(2k; \lambda_0, \lambda_0, \lambda_{k-2})$ $(k\geq 2)$, $(10; \lambda_0, \lambda_1, \lambda_1)$.

In each case, we consider fixed point algebras.

1) Case of $(n; \alpha_1, \alpha_2)=(2k; \lambda_{k-1}\ \lambda_{k-1})$ for $k\geq 2$: We define $\sigma\in\mathrm{Aut}(A)$ by $\sigma(f_1)=f_2$, $\sigma(f_2)=f_1$ and $\sigma(e_i)=e_i$ for $i\geq 1$. Since $A\supset A^\sigma\supset B$ and $B'\cap A=\boldsymbol{C}$, σ is an outer automorphism of A. Hence $[A: A^\sigma]=|\langle\sigma\rangle|=2$. On the other hand, $[A: B]=(\sin^2\theta_n)^{-1}$ from Theorem 3.10. Further we devide in two cases.

1a) Case of $k=1$: In this case $[A: B]=[A: A^\sigma]=2$, so $A^\sigma=B$.

1b) Case of $k\geq 2$: Since $[A: B]=(\sin^2\theta_n)^{-1}\neq 2=[A: A^\sigma]$, we have $A^\sigma\supsetneqq B$ and $[A^\sigma: B]=(2\sin^2\theta_n)^{-1}$. And from $B'\cap A=\boldsymbol{C}$ it follows that $(A^\sigma)'\cap A=\boldsymbol{C}$.

2) Case of $(n; \alpha_1, \alpha_2, \alpha_3)=(4; \lambda_0, \lambda_0, \lambda_0)$: We define ρ: $S_3 \in \sigma \mapsto \rho_\sigma \in \mathrm{Aut}(A)$ a homomorphism by $\rho_\sigma(f_j)=f_{\sigma(j)}$, $\rho(e_i)=e_i$ $(1\leq j\leq 3, i\geq 1)$. Since $B\subset A^{\rho_\sigma}$ and $B'\cap A=\boldsymbol{C}$, for any $\sigma\in S_3$, $\sigma\neq e$, ρ_σ is outer. Hence $[A: A^{S_3}]=|S_3|=6$. On the other hand, $[A: B]=2\cot^2(\pi/6)=6$ from Theorem 3.10, so $A^{S_3}=B$.

For $\{i, j, k\}=\{1, 2, 3\}$, $[A: A^{(ij)}]=|\langle(i\ j)\rangle|=2$, $[A: A(k)]=2$ and $A(k)\subset A^{(ij)}$, therefore $A^{(ij)}=A(k)$, where $(i\ j)$=the transposition i and j.

3) Case of $(n; \alpha_1, \alpha_2, \alpha_3)=(2k; \lambda_0, \lambda_0, \lambda_{k-2})$ for $k\geq 3$: We define $\sigma\in\mathrm{Aut}(A)$ by $\sigma(f_1)=f_2$, $\sigma(f_2)=f_1$, $\sigma(f_3)=f_3$ and $\sigma(e_i)=e_i$ for $i\geq 1$. Since $A(3)\subset A^\sigma$ and $A(3)'\cap A=\boldsymbol{C}$, σ is outer. Hence $[A: A^\sigma]=|\langle\sigma\rangle|=2$. On the other hand, $[A: A(3)]=2$ from Theorem 3.10. Therefore we obtain $A(3)=A^\sigma$.

4) Case of $(n; \alpha_1, \alpha_2, \alpha_3)=(10; \lambda_0, \lambda_1, \lambda_1)$: We define $\sigma\in\mathrm{Aut}(A)$ by $\sigma(f_1)=f_1$, $\sigma(f_2)=f_3$, $\sigma(f_3)=f_2$ and $\sigma(e_i)=e_i$ for $i\geq 1$. Since $A(1)\subset A^\sigma$ and $A(1)'\cap A=\boldsymbol{C}$, σ is outer and $[A: A^\sigma]=|\langle\sigma\rangle|=2$. On the other hand, $[A: A(1)]=6+2\sqrt{3}$ from Theorem 3.10. Therefore $A^\sigma \supsetneqq A(1)$ and $[A^\sigma: A(1)]=3+\sqrt{3}$.

From above argument, we obtain next theorem.

Theorem 4.4. *Notation is as above.*

1) *Case of* $(n; \alpha_1, \alpha_2)=(2; \lambda_0, \lambda_0)$

$$A^{S_3}=B\,.$$

2) *Case of* $(n; \alpha_1, \alpha_2)=(2k; \lambda_{k-1}, \lambda_{k-1})$ $(k\geq 2)$

$$A^{S_2}\supsetneqq B\,,\quad [A^{S_2}: B]=(2\sin^2\theta_n)^{-1}\,,\quad B'\cap A^{S_2}=\boldsymbol{C}\,.$$

3) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(4; \lambda_0, \lambda_0, \lambda_0)$

$$A^{S_3}=B\,,\quad A^{(ij)}=A(k)\,,\quad \textit{where } \{i, j, k\}=\{1, 2, 3\}\,.$$

4) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(2\lambda; \lambda_0, \lambda_0, \lambda_{k-2})$ *for* $k\geq 3$

$$A^{S_2}=A(3)\,.$$

5) *Case of* $(n; \alpha_1, \alpha_2, \alpha_3)=(10; \lambda_0, \lambda_1, \lambda_1)$

$$A^{S_2}\supsetneqq A(1)\,,\quad [A^{S_2}: A(1)]=3+\sqrt{3}\,,\quad A(1)'\cap A^{S_2}=\boldsymbol{C}\,.$$

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References

[1] F. M. Goodman, P. de la Harpe and V. F. R. Jones, Coxeter Graphs and Towers of Algebras, MSRI publications 14, Springer-Verlag, New York, 1989.

[2] V. F. R. Jones, Index for subfactors, Invent. Math., **72** (1983), 1-25.

[3] A. Ocneanu, Graph geometry, quantized groups and nonamenable subfactors, Lake Tahoe

Lectures, June-July (1989).
[4] S. Popa, Relative dimension, tower of projections and commuting squares of subfactors, Pacific J. Math., **137** (1989), 181-207.
[5] H. Wenzl, Hecke algebras of type A_n and subfactors, Invent. Math., **92** (1988), 349-383.