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On the index of reducibility of parameter ideals and Cohen-Macaulayness in a local ring

By

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1. Introduction

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of A. Then Northcott proved that the number of irreducible components of $\mathfrak{a}$, which is called the index of reducibility of $\mathfrak{a}$, is equal to the length of $\mathrm{Hom}_A(A/\mathfrak{m}, A/\mathfrak{a})$ [10]. In general, let M be a finitely generated A-module and N be a submodule of M such that M/N has finite length. Then the number of irreducible components of N in M, which is also called the index of reducibility of N in M, is equal to the length of $\mathrm{Hom}_A(A/\mathfrak{m}, M/N)$. It is well-known that if M is a Cohen-Macaulay A-module of dimension d, then the index of reducibility of a submodule $\boldsymbol{x}M$ of M, where $\boldsymbol{x}=x_1, \cdots, x_d$ is a system of parameters for M, depends only on M, and not on the choice of the system of parameters.

On the other hand, it is known that in a local ring A, if the index of reducibility of any parameter ideal is equal to one, then A is Cohen-Macaulay, and hence Gorenstein [11]. But there are examples of a non-Cohen-Macaulay ring such that the index of reducibility of any parameter ideal is equal to a constant not depending on the choice of the system of parameters [3].

Concerning these results, the following question may be raised:

Let A be a Noetherian local ring such that the index of reducibility of any parameter ideal for A is equal to some constant. What makes A Cohen-Macaulay?

The aim of this paper is to answer this question and to generalize it for modules.

2. Preliminaries

In this section, we state some definitions and recall some facts on a dualizing complex and on a module with finite local cohomologies. Throughout this paper, A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let

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M be a finitely generated A-module. We denote by $\mathrm{Ass}_A M$ the set of associated prime ideals of M, and we put $\mathrm{Assh}_A M=\{\mathfrak{p}\in \mathrm{Ass}_A M \mid \dim A/\mathfrak{p}=\dim M\}$. We say that M is unmixed if $\mathrm{Ass}_{\hat{A}}\hat{M}=\mathrm{Assh}_{\hat{A}}\hat{M}$, where $\hat{A}$ (resp. $\hat{M}$) denotes the $\mathfrak{m}$-adic completion of A (resp. M) [9]. If A is a homomorphic image of a Cohen-Macaulay ring, then M is unmixed if and only if $\mathrm{Ass}_A M=\mathrm{Assh}_A M$ [8].

Definition (2.1). We set

$$r_A{}^c(M)=\sup\{l_A(\mathrm{Hom}_A(A/\mathfrak{m}, M/\mathfrak{q}M)) \mid \mathfrak{q} \text{ is a parameter ideal for } M\},$$

where $l_A(N)$ denotes the length of an A-module N. And we call $r_A{}^c(M)$ the *classical type* of M.

It is not difficult to derive $r_A{}^c(M)=r_{\hat{A}}{}^c(\hat{M})$.

Remark (2.2). Vasconcelos used the term "type" to name the d-th Bass number of M, where $d=\dim M$ [19]. If M is Cohen-Macaulay, then the type of M is equal to the classical type of M. But in general, the two types are not equal, and so, we use the word "*classical*" to distinguish between them.

Let T be an A-module. $H_{\mathfrak{m}}{}^i(T)$ denotes the i-th local cohomology module of T with respect to $\mathfrak{m}$, and $E_A(T)$ denotes the injective envelope of T. The following proposition is known as Matlis duality. We refer the reader to [7] or [17] for details.

Proposition (2.3). (1) *If an A-module M has finite length, then* $\mathrm{Hom}_A(M, E_A(A/\mathfrak{m}))$ *has finite length and its length is equal to the length of M.* (2) *Let M be a finitely generated A-module. Then* $\mathrm{Hom}_A(M, E_A(A/\mathfrak{m}))$ *is an Artinian A-module. Furthermore* $\mathrm{Hom}_A(A/\mathfrak{m}, \mathrm{Hom}_A(M, E_A(A/\mathfrak{m})))$ *has finite length and its length is equal to $\mu_A(M)$, where $\mu_A(M)$ denotes the minimal number of generators for M.*

Now we state the definition of a dualizing complex after Roberts. Refer to [12], [16] or [18] for details.

Definition (2.4). A complex $D^\bullet$ over A is called a dualizing complex of A if it satisfies the following conditions:

(1) $D^i=\bigoplus_{\substack{\mathfrak{p}\in \mathrm{Spec}A \\ \dim A/\mathfrak{p}=-i}} E_A(A/\mathfrak{p})$;

(2) $H^i(D^\bullet)$ is a finitely generated A-module for every i.

The dualizing complex of A is usually denoted by $D_A^\bullet$, if it exists.

For example, let A be a Gorenstein local ring and $D^\bullet$ be the minimal injective resolution of A. Then $D^\bullet[\dim A]$ is a dualizing complex of A.

Suppose that A has a dualizing complex $D_A^{\bullet}$. Then for an ideal $\mathfrak{a}$ (resp. a prime ideal $\mathfrak{p}$) of A, $\mathrm{Hom}_A(A/\mathfrak{a}, D_A^{\bullet})$ (resp. $(D_A^{\bullet})_{\mathfrak{p}}[-\dim A/\mathfrak{p}]$) is a dualizing complex of $A/\mathfrak{a}$ (resp. $A_{\mathfrak{p}}$). In particular, any complete local ring has a dualizing complex.

Theorem (2.5). (local duality theorem). *Let M be a finitely generated A-module. If A has a dualizing complex $D_A^{\bullet}$, then*

$$H_{\mathfrak{m}}{}^i(M) \cong \mathrm{Hom}_A(H^{-i}(\mathrm{Hom}_A(M, D_A^{\bullet})), E_A(A/\mathfrak{m}))\,.$$

Finally we state the definition of a module with finite local cohomologies or a generalized Cohen-Macaulay module, and we give a characterization of it.

Definition (2.6). A finitely generated A-module M is referred to as a module with finite local cohomologies if $H_{\mathfrak{m}}{}^i(M)$ has finite length for all $i \neq \dim M$.

For example, any finitely generated module of dimension one is a module with finite local cohomologies. It is obvious that M is a module with finite local cohomologies if and only if $\hat{M}$ is also.

Lemma (2.7). ([15]). *Let M be a finitely generated A-module.*

(1) *If M is a module with finite local cohomologies, then* $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$ *and, $M_{\mathfrak{p}}$ is a Cohen-Macaulay $A_{\mathfrak{p}}$-module for all $\mathfrak{p}$ in* $\mathrm{Supp} M \setminus \{\mathfrak{m}\}$.
(2) *If A has a dualizing complex, then the converse to* (1) *holds.*

3. Main theorem and its proof

This section is devoted to a proof of the following theorem and corollaries.

Theorem (3.1). *Let A be a homomorphic image of a Cohen-Macaulay ring and M be a finitely generated A-module. Let n be an integer. If M satisfies the following conditions:*

i) $r_A{}^c(M) = n$;

ii) $M_{\mathfrak{p}}$ *is a Cohen-Macaulay $A_{\mathfrak{p}}$-module for all $\mathfrak{p}$ in SuppM such that* $\dim M_{\mathfrak{p}} < n$;

iii) $\mathrm{Assh}_A M = \mathrm{Ass}_A M$,

then M is Cohen-Macaulay.

Corollary (3.2). *Let A be an unmixed local ring such that the index of reducibility of any parameter ideal is equal to two. Then A is Cohen-Macaulay.*

Corollary (3.3). *Let $n\geq 3$ be an integer and A be a complete local ring such that the index of reducibility of any parameter ideal is equal to n. If A satisfies Serre's condition (S_{n-1}), (i.e. $\operatorname{depth} M_{\mathfrak{p}}\geq\min\{n-1, \dim M_{\mathfrak{p}}\}$ for any $\mathfrak{p}$ in $\operatorname{Supp} M$), then A is Cohen-Macaulay.*

We need the following proposition to prove this theorem. It is difficult to compute the classical type in general, but for a module with finite local cohomologies we have the following result.

Proposition (3.4). *Let M be a module of dimension d with finite local cohomologies. Then we have*

(1)

$$r_A{}^c(M)\leq\sum_{i=0}^{d-1}\binom{d}{i}l_A(H_{\mathfrak{m}}{}^i(M))+l_A(\operatorname{Hom}_A(A/\mathfrak{m}, H_{\mathfrak{m}}{}^d(M))) .$$

(2) *For any system of parameters $x_1, \cdots, x_d$ for M, there are some positive integers $\alpha_1, \cdots, \alpha_d$ such that*

$$l_A(\operatorname{Hom}_A(A/\mathfrak{m}, M/(x_1{}^{\alpha_1}, \cdots, x_d{}^{\alpha_d})M))=\sum_{i=0}^{d}\binom{d}{i}l_A(\operatorname{Hom}_A(A/\mathfrak{m}, H_{\mathfrak{m}}{}^i(M))) .$$

Therefore $r_A{}^c(M)\geq\sum_{i=0}^{d}\binom{d}{i}l_A(\operatorname{Hom}_A(A/\mathfrak{m}, H_{\mathfrak{m}}{}^i(M)))$.

Proof. See [4].

We note that, if A has a dualizing complex $D_A^{\bullet}$ and M is a module of dimension d with finite local cohomologies, then

$$l_A(\operatorname{Hom}_A(A/\mathfrak{m}, M/(x_1{}^{\alpha_1}, \cdots, x_d{}^{\alpha_d})M))=\sum_{i=0}^{d}\binom{d}{i}\mu_A(H^{-i}(\operatorname{Hom}_A(M, D_A^{\bullet}))) ,$$

holds for any system of parameters $x_1, \cdots, x_d$ for M, and for some positive integers $\alpha_1, \cdots, \alpha_d$ depending on the choice of the system of parameters.

We prove the following theorem which is slightly stronger than Theorem (3.1).

Theorem (3.5). *Let A be a local ring which has a dualizing complex $D_A^{\bullet}$ and M be a finitely generated A-module of dimension d. Let n be an integer. If M satisfies the following conditions:*

i$'$) *For any chain of prime ideals $\mathfrak{p}_0\subset\mathfrak{p}_1\subset\cdots\subset\mathfrak{p}_{d-1}\subset\mathfrak{p}_d=\mathfrak{m}$ (strict inclusions) in $\operatorname{Supp}_A M$, there exists a system of parameters $x_1, \cdots, x_d$ for M, which satisfies*

$$(\#)\quad \begin{cases} x_1, \cdots, x_i \in \mathfrak{p}_i\,, & \textit{for all } i\,, \\ l_A(\mathrm{Hom}_A(A/\mathfrak{m}, M/(x_1^{\alpha_1}, \cdots, x_d^{\alpha_d})M)) \leq n\,, & \textit{for arbitrary positive} \\ & \textit{integers } \alpha_1, \cdots, \alpha_d\,; \end{cases}$$

ii) $M_{\mathfrak{p}}$ *is a Cohen-Macaulay* $A_{\mathfrak{p}}$*-module for all* $\mathfrak{p}$ *in* $\mathrm{Supp}M$ *such that* $\dim M_{\mathfrak{p}} < n$*;*

iii) $\mathrm{Assh}_A M = \mathrm{Ass}_A M$,

then M *is Cohen-Macaulay.*

Proof. We work by induction on d. There is nothing to prove if $d<n$. Assume that $d \geq n$. Let $\mathfrak{p}$ be a prime ideal in $\mathrm{Supp}_A M$ such that $\dim A/\mathfrak{p}=1$. We show that $M_{\mathfrak{p}}$ satisfies the assumption of the theorem. For any chain of prime ideals $\mathfrak{p}_0 A_{\mathfrak{p}} \subset \mathfrak{p}_1 A_{\mathfrak{p}} \subset \cdots \subset \mathfrak{p}_{d-1} A_{\mathfrak{p}} = \mathfrak{p} A_{\mathfrak{p}}$ in $\mathrm{Supp}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$, $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{d-1} \subset \mathfrak{m}$ is a chain of prime ideals in $\mathrm{Supp}_A M$. Then there is a system of parameters $x_1, \cdots, x_d$ for M which satisfies (#). A set of elements $x_1, \cdots, x_{d-1}$ is a system of parameters for $M_{\mathfrak{p}}$. Let $\alpha_1, \cdots, \alpha_{d-1}$ be arbitrary positive integers and $M' = M/(x_1^{\alpha_1}, \cdots, x_{d-1}^{\alpha_{d-1}})M$. Then M' is a module with finite local cohomologies, and x_d is a system of parameters for M', and so there is some integer α_d, such that

$$\begin{aligned} n \geq l_A(\mathrm{Hom}_A(A/\mathfrak{m}, M'/x_d^{\alpha_d}M')) &= \mu_A(H^{-1}(\mathrm{Hom}_A(M', D_A^{\bullet}))) \\ &\quad + \mu_A(H^0(\mathrm{Hom}_A(M', D_A^{\bullet}))) \\ &\geq \mu_{A_{\mathfrak{p}}}(H^0(\mathrm{Hom}_{A_{\mathfrak{p}}}(M'_{\mathfrak{p}}, D_{A_{\mathfrak{p}}}^{\bullet})))\,. \end{aligned}$$

Since $M'_{\mathfrak{p}} = M_{\mathfrak{p}}/(x_1^{\alpha_1}, \cdots, x_{d-1}^{\alpha_{d-1}})M_{\mathfrak{p}}$ has finite length,

$$\begin{aligned} l_{A_{\mathfrak{p}}}(\mathrm{Hom}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}, M'_{\mathfrak{p}})) &= \mu_{A_{\mathfrak{p}}}(\mathrm{Hom}_{A_{\mathfrak{p}}}(M'_{\mathfrak{p}}, E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}))) \\ &= \mu_{A_{\mathfrak{p}}}(H^0(\mathrm{Hom}_{A_{\mathfrak{p}}}(M'_{\mathfrak{p}}, D_{A_{\mathfrak{p}}}^{\bullet}))) \leq n\,. \end{aligned}$$

Hence $M_{\mathfrak{p}}$ satisfies condition i′). Clearly $M_{\mathfrak{p}}$ satisfies condition ii) and iii), hence $M_{\mathfrak{p}}$ is a Cohen-Macaulay module, by induction hypothesis.

Therefore M is a module with finite local cohomologies. If M is not Cohen-Macaulay, then

$$\sum_{i=0}^{d} \binom{d}{i} l_A(\mathrm{Hom}_A(A/\mathfrak{m}, H_{\mathfrak{m}}^{i}(M))) > d\,,$$

because $\mathrm{Hom}_A(A/\mathfrak{m}, H_{\mathfrak{m}}^{d}(M)) \neq 0$ and $H_{\mathfrak{m}}^{0}(M)=0$, which contradicts the assumption that $d \geq n$.

Theorem (3.1) and Corollary (3.2) are immediately derived from Theorem (3.5). In addition, complete local rings satisfying (S_2) are unmixed [2], hence Theorem (3.1) leads to Corollary (3.3).

4. Examples

We consider whether we can weaken the assumption of Theorem (3.5), giving several examples.

Example (4.1). Let k be a field and $A=k[\![x, y, z]\!]/(xy, xz)=k[\![x, y, z]\!]/(x)\cap(y, z)$ where x, y, z are indeterminates over k. Then A is a complete local ring of dimension two, and $A_{\mathfrak{p}}$ is a Cohen-Macaulay ring for all $\mathfrak{p}$ in Spec $A\backslash\{\mathfrak{m}\}$. Furthermore $r_A{}^c(A)=2$, because, $x-z$, y^2 is a system of parameters for A, and $l_A(\mathrm{Hom}_A(A/\mathfrak{m}, A/(x-z, y^2)))=2$. On the other hand, there exists an exact sequence:

$$0\to A/(y, z)\to A\to A/(x)\to 0\,.$$
$$1 \mapsto x$$

Since $A/(y, z)$ and $A/(x)$ are regular local rings, for any parameter ideal $\mathfrak{q}$ for A,

$$0\to A/\{\mathfrak{q}\colon x\}\to A/\mathfrak{q}\to A/\{\mathfrak{q}+(x)\}\to 0\,,$$

is exact and $\{\mathfrak{q}\colon x\}/(y, z)$ (resp. $\{\mathfrak{q}+(x)\}/(x)$) is a parameter ideal for $A/(x, y)$ (resp. $A/(x)$). Therefore $l_A(\mathrm{Hom}(A/\mathfrak{m}, A/\mathfrak{q}))\leq 2$. But A is not Cohen-Macaulay, because A is not unmixed.

Local rings of classical type one are automatically unmixed, because they are Cohen-Macaulay. But local rings of classical type two are not necessarily unmixed. Therefore we can not omit the condition "$\mathrm{Ass}_A A=\mathrm{Assh}_A A$" from Theorem (3.5).

Example (4.2). Let A be a Gorenstein ring with maximal ideal $\mathfrak{m}$ of dimension $d>1$. Then $\mathfrak{m}$ is a non-Cohen-Macaulay unmixed module of dimension d. By considering an exact sequence $0\to\mathfrak{m}\to A\to A/\mathfrak{m}\to 0$, we have

$$H_{\mathfrak{m}}{}^i(\mathfrak{m})=\begin{cases} H_{\mathfrak{m}}{}^d(A)\,, & i=d\,;\\ A/\mathfrak{m}\,, & i=1\,;\\ 0\,, & i\neq 1, d\,.\end{cases}$$

Hence A is a module with finite local cohomologies of classical type $d+1$.

This example shows that we can not weaken condition ii) of Theorem (3.5) for modules. For rings, can we weaken condition ii) of Theorem (3.5)? The author has no example of a non-Cohen-Macaulay unmixed local ring A with finite local cohomologies of classical type $\dim A+1$. But we have the following example.

Example (4.3). Let k be a field and $n\geq 1$. We consider semigroup rings $A=k[\![x^2, x^3, xy_1, y_1, \cdots, xy_n, y_n]\!]$ and $S=k[\![x, y_1, \cdots, y_n]\!]$. Then, $\dim A=\dim S$

$=n+1$, and there exists an exact sequence $0\to A\to S\to k\to 0$. And so,

$$H_{\mathfrak{m}}^{i}(A)=\begin{cases} H_{\mathfrak{m}}^{n+1}(S), & i=n+1; \\ k, & i=1; \\ 0, & \text{otherwise}. \end{cases}$$

Hence, A is a ring with finite local cohomologies of classical type $\dim A+2$.

Question (4.4). *Does there exist an unmixed local ring A with finite local cohomologies of classical type* $\dim A+1$, *but that is not Cohen-Macauly?*

If there exists such a ring, then its dimension must be equal to three. Because if $l_A(\mathrm{Hom}_A(A/\mathfrak{m}, H_{\mathfrak{m}}^{d}(A)))=1$ then A must be a quasi-Gorenstein ring and $l_A(H_{\mathfrak{m}}^{i}(A))=l_A(H_{\mathfrak{m}}^{d-i+1}(A))$, where $d=\dim A$(see [1] and [14]). If there is no such ring, then the following conjecture holds.

Conjecture (4.5). *Let A be a homomorphic image of a Cohen-Macaulay ring. If A satisfies the following conditions:*

(1) $r_A{}^{c}(A)=n\geq 3$;

(2) $A_{\mathfrak{p}}$ *is a Cohen-Macaulay ring for all* $\mathfrak{p}$ *in* $\mathrm{Spec}A$ *such that* $\mathrm{ht}\mathfrak{p}<n-1$;

(3) $\mathrm{Ass}_A A=\mathrm{Assh}_A A$,

than A is Cohen-Macaulay.

Finally we remark the "type" in Vasconcelos' sense. Roberts showed that local rings of type one are Cohen-Macaulay [12] and Marley showed that unmixed local rings of type two are Cohen-Macaulay [6]. Furthermore the author obtains a theorem like Corollary (3.3) on "type" [5].

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