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Remarks on coerciveness in Besov spaces for abstract parabolic equations

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1. Introduction

Several papers were devoted to the study of solutions in Besov spaces with values in a Banach space for the abstract linear evolution equation of parabolic type

$$(1.1)\qquad \begin{cases} D_t u(t)+Au(t)=f(t), & 0<t<T \\ u(0)=x . \end{cases}$$

Sinestrari [14], Hölder spaces; Di Blasio [5], Slobodeckii spaces and Muramatu [12], general Besov spaces. One of the main results there is that the mapping

$$L\colon u \mapsto (D_t u+Au,\, u(0))$$

is a bijection from the intersection of two Besov spaces with values in the Banach space and the domain of the operator A and with exponents $1+\theta$ and θ respectively to the space of data (f, x) satisfying suitable compatibility relations, the coerciveness in Besov spaces for the equation (1.1).

Let E and F be Banach spaces with F continuously embedded to E and a linear continuous operator A from F to E be given. Let us consider the evolution equation in E of the form (1.1). The aim in this note is to show that the parabolicity of the equation is characterized by the coerciveness in Besov spaces for the equation.

2. Notation and preliminaries

$\boldsymbol{R}$ and $\boldsymbol{C}$ denote the fields of real and complex numbers respectively. $\boldsymbol{Z}_+$ is the set of nonnegative integers.

Let E be a Banach space. $\mathcal{L}(E)$ is the space of bounded linear operators in E with uniform operator norm $\|\cdot\|_{\mathcal{L}(E)}$. For a linear operator A in E we denote the domain of A by $\mathcal{D}(A)$ and the resolvent set of A by $\rho(A)$.

Let $1\le p\le\infty$, $0<T\le\infty$ and $n\in\boldsymbol{Z}_+$ or $n=\infty$. We set several function spaces with values in E as follows.

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$\mathcal{D}'(0, T; E)$ is the space of distributions on $(0, T)$. The derivatives of $f \in \mathcal{D}'(0, T; E)$ are denoted by $D_t^n f$. $L^p(0, T; E)$ and $L_*^p(0, T; E)$ are the L^p spaces with respect to the Lebesgue measure dt and the measure $t^{-1}dt$ on $(0, T)$ respectively. Let $I=(0, T)$, $(0, T]$, $[0, T)$ or $[0, T]$. $C^n(I; E)$ is the space of n times continuously differentiable functions on I. $C_B^n(I; E)$ is the subspace of $C^n(I; E)$ which consists of functions whose derivatives of order up to n belong to $L^\infty(0, T; E)$. In the notation above we omit E when $E=\boldsymbol{R}$ or $\boldsymbol{C}$.

Analytic semigroups. The mapping $S: (0, \infty) \to \mathcal{L}(E)$ is called an analytic semigroup in E if the following conditions are satisfied.

(1) S is analytic. There exist constants $\phi \in (0, \pi/2)$ and $\omega \in \boldsymbol{R}$ such that S is extended analytically in the sector $\Sigma_\phi = \{t \in \boldsymbol{C}; |\arg t| < \phi\}$ and the extension, also denoted by S, satisfies the growth condition

$$\sup_{t \in \Sigma_\phi} \|e^{-\omega t} S(t)\|_{\mathcal{L}(E)} < \infty .$$

(2) $S(t_1+t_2)=S(t_1)S(t_2)$ holds for $t_1, t_2>0$.

(3) Let $x \in E$. If $S(t)x=0$ holds for $t>0$, then $x=0$.

The generator G of an analytic semigroup S in E is defined as follows:

$$\mathcal{D}(G)=\{x \in E; y \in E \text{ exists such that } D_t S(t)x - S(t)y=0 \text{ for } t>0.\}$$

$$Gx=y .$$

It is verified that G satisfies

$$\rho(-\omega+G) \supset \Sigma_{\phi+\pi/2}, \quad \sup_{\lambda \in \Sigma_{\phi+\pi/2}} \|\lambda(\lambda+\omega-G)^{-1}\|_{\mathcal{L}(E)} < \infty$$

for some constants $\phi \in (0, \pi/2)$ and $\omega \in \boldsymbol{R}$. The mapping $S \mapsto G$ from the set of analytic semigroups to that of closed linear operators with the properties above is bijective. The inverse of the mapping is expressed as the inverse Laplace transform.

We remark that S is not necessarily strongly continuous at $t=0$ and hence $\mathcal{D}(G)$ may not be dense in E. Analytic semigroups of such type have been already studied by Da Prato and Sinestrari [4] and Sinestrari [14], although the definitions of an analytic semigroup and its generator were not explicitly presented there.

The equation (1.1) is called of parabolic type if $-A$ is the generator of an analytic semigroup in E.

Besov spaces. Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0<\theta<\infty$ and $0<T \leq \infty$. Set $m=[\theta]+1$, where $[\theta]$ is the largest integer which does not exceed θ. For a strongly measurable function f on $(0, T)$ with values in E we put

$$[f]_{B^\theta_{p,q}(0,T;E)}=\left|\ h^{-\theta}\left|\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}f(\cdot+kh)\right|_{L^p(0,T-mh;E)}\right|_{L^q_*(0,T/m)}.$$

We define the subspaces $B^\theta_{p,q}(0,T;E)$ and $\dot{B}^\theta_{p,q}(0,T;E)$ of $L^p(0,T;E)$ as follows:

(1) $f\in L^p(0,T;E)$ belongs to $B^\theta_{p,q}(0,T;E)$ if $[f]_{B^\theta_{p,q}(0,T;E)}<\infty$.
(2) $f\in L^p(0,T;E)$ belongs to $\dot{B}^\theta_{p,q}(0,T;E)$ if

$$f\in B^\theta_{p,q}(0,T;E) \quad\text{and}\quad |h^{-\theta}|f|_{L^p(0,h;E)}|_{L^q_*(0,T)}<\infty.$$

$B^\theta_{p,q}(0,T;E)$ and $\dot{B}^\theta_{p,q}(0,T;E)$ are Banach spaces with respective norms

$$|f|_{B^\theta_{p,q}(0,T;E)}=|f|_{L^p(0,T;E)}+[f]_{B^\theta_{p,q}(0,T;E)}$$

$$|f|_{\dot{B}^\theta_{p,q}(0,T;E)}=|h^{-\theta}|f|_{L^p(0,h;E)}|_{L^q_*(0,T)}+|f|_{B^\theta_{p,q}(0,T;E)}.$$

$B^\theta_{p,q}(0,T;E)$ is called the Besov space on $(0,T)$. For Besov spaces on general regions in $\boldsymbol{R}^n$ including those with nonpositive exponents θ we refer the reader to Muramatu [9], [10]. We just collect some basic properties of $B^\theta_{p,q}(0,T;E)$ and $\dot{B}^\theta_{p,q}(0,T;E)$. In the following $B(\theta,p,q)$ stands for $B^\theta_{p,q}(0,T;E)$ or $\dot{B}^\theta_{p,q}(0,T;E)$.

Proposition 2.1. *Let $1\le p, p'\le\infty$, $1\le q, q'\le\infty$, $0<\theta, \theta'<\infty$ and $0<T\le\infty$. Let us express θ as $l+\sigma$ with $l\in\boldsymbol{Z}_+$ and $\sigma\in(0,1]$.*

(1) *$B(\theta,p,q)\subset B(\theta',p,q')$ when $\theta>\theta'$.*
(2) *$B(\theta,p,q)\subset B(\theta-p^{-1}+p'^{-1},p',q')$ when $p'\ge p$, $q'\ge q$, $\theta>p^{-1}-p'^{-1}$.*
(3) *Let $a\in C^m_B((0,T))$. We have $af\in B(\theta,p,q)$ for $f\in B(\theta,p,q)$.*
(4) *Let $T<\infty$. We have $\int_0^t f(s)ds\in B(1+\theta,p,q)$ for $f\in B(\theta,p,q)$.*
(5) *We have $D_t^kf\in B(\theta-k,p,q)$, $0\le k\le l$ for $f\in B(\theta,p,q)$.*
(6) *$\dot{B}^\theta_{p,q}(0,T;E)=\{f\in B^\theta_{p,q}(0,T;E); D_t^kf(0)=0$ for $0\le k\le l-1$, $D_t^lf\in\dot{B}^\sigma_{p,q}(0,T;E)\}$.*

When $\sigma-p^{-1}$ is not an integer, we have

$$\dot{B}^\sigma_{p,q}(0,T;E)=\begin{cases}B^\sigma_{p,q}(0,T;E), & \text{when } \sigma-p^{-1}<0\\ \{f\in B^\sigma_{p,q}(0,T;E); f(0)=0\}, & \text{when } \sigma-p^{-1}>0.\end{cases}$$

In the next section we will describe the compatibility relations between f and x for (1.1) to have a solution in some Besov spaces. To this end we prepare the following:

Proposition 2.2. *Let F be a Banach space continuously embedded to E.*

Let $1\leq p\leq\infty$, $1\leq q\leq\infty$, $p^{-1}<\theta<1+p^{-1}$ *and* $0<T\leq\infty$. *For* $u\in B_{p,q}^{1+\theta}(0, T; E)\cap B_{p,q}^{\theta}(0, T; F)$ *we have* $D_{t}u(0)\in(E, F)_{\theta-p^{-1},q}$, *the real interpolation space between* E *and* F.

Proof. Among several equivalent definitions of the real interpolation spaces we recall the K-method due to Peetre [13]:

$$(E, F)_{\eta,q}=\{x\in E;\ |t^{-\eta}K(t, x)|_{L_{*}^{q}(0,\infty)}<\infty\}\ ,\quad 0<\eta<1\ ,\quad 1\leq q\leq\infty\ ,$$

where

$$K(t, x)=\inf_{y\in F}(|x-y|_{E}+t|y|_{F})\ ,\quad \text{Peetre's } K \text{ functional}\ .$$

By the well known fact that $\min\{1, t/h\}K(h, x)\leq K(t, x)$, $t, h>0$, $x\in E$ we have only to prove that $|h^{-\theta}|K(\cdot, D_{t}u(0))|_{L^{p}(0,h)}|_{L_{*}^{q}(0,\infty)}<\infty$. This follows from the estimate

$$\begin{aligned}K(t, D_{t}u(0))&\leq|D_{t}u(0)-t^{-1}(u(t)-u(0))|_{E}+|u(t)-u(0)|_{F}\\&\leq\int_{0}^{1}|D_{t}u(tr)-D_{t}u(0)|_{E}dr+|u(t)-u(0)|_{F}\ ,\quad 0<t\leq T\end{aligned}$$

and Young's inequality. Notice that both

$$|h^{-\theta}|D_{t}u(\cdot)-D_{t}u(0)|_{L^{p}(0,h;E)}|_{L_{*}^{q}(0,T)} \text{ and } |h^{-\theta}|u(\cdot)-u(0)|_{L^{p}(0,h;F)}|_{L_{*}^{q}(0,T)}$$

are finite in view of Proposition 2.1 (6).

3. Results

Let E and F be Banach spaces with F continuously embedded to E and a linear continuous operator A from F to E be given.

Let $1\leq p\leq\infty$, $1\leq q\leq\infty$, $0<\theta<\infty$ and $0<T<\infty$.

In view of Proposition 2.1 (5) we define the linear operator $\dot{L}$ from $\mathcal{D}(\dot{L})=\dot{B}_{p,q}^{1+\theta}(0, T; E)\cap\dot{B}_{p,q}^{\theta}(0, T; F)$ to $\dot{B}_{p,q}^{\theta}(0, T; E)$ by

$$\dot{L}u=D_{t}u+Au\ ,\quad u\in\mathcal{D}(\dot{L})\ .$$

Assume that $\theta-p^{-1}$ is not an integer. Set $N=[\theta-p^{-1}]+1$. $\mathcal{D}_{0}$ is the subspace of $B_{p,q}^{\theta}(0, T; E)\times E$ which consists of elements (f, x) of $B_{p,q}^{\theta}(0, T; E)\times E$ satisfying the conditions below: We put $x_{0}=x$.

(1) If $k<N$, then $x_{k}\in F$. In this case we put $x_{k+1}=D_{t}^{k}f(0)-Ax_{k}$.

(2) $x_{N}\in(E, F)_{\theta-p^{-1}+1-N,q}$.

$\mathcal{D}_{0}$ is a Banach space with norm $|f|_{B_{p,q}^{\theta}(0,T;E)}+\sum_{k=0}^{N-1}|x_{k}|_{F}+|x_{N}|_{(E,F)_{\theta-p^{-1}+1-N,q}}$.

Remark. For $\omega\in\boldsymbol{C}$ let $\mathcal{D}_{\omega}$ be the subspace of $B_{p,q}^{\theta}(0, T; E)\times E$ defined for the operator $\omega+A$ as $\mathcal{D}_0$ for A. We see that $(f, x)\in\mathcal{D}_0$ if and only if $(e^{-\omega t}f, x)\in\mathcal{D}_\omega$.

For $u\in B_{p,q}^{1+\theta}(0, T; E)\cap B_{p,q}^{\theta}(0, T; F)$ we put

$$f=D_t u+Au\in B_{p,q}^{\theta}(0, T; E), \quad x=u(0)\in E.$$

When $N=0$, i.e., $0<\theta<p^{-1}$, we apply Proposition 2.2 to $\int_0^t u(s)ds\in B_{p,q}^{2+\theta}(0, T; E)\cap B_{p,q}^{1+\theta}(0, T; F)$ and obtain $(f, x)\in\mathcal{D}_0$. When $N\geq 1$, differentiating f succesively $(N-1)$ times and taking the traces at $t=0$ of the derivatives, we see by Proposition 2.2 that $(f, x)\in\mathcal{D}_0$. Thus $\mathcal{D}_0$ is understood as the space of data with the compatibility relations for (1.1) to have a solution in $B_{p,q}^{1+\theta}(0, T; E)\cap B_{p,q}^{\theta}(0, T; F)$.

We define the linear operator L from $\mathcal{D}(L)=B_{p,q}^{1+\theta}(0, T; E)\cap B_{p,q}^{\theta}(0, T; F)$ to $\mathcal{D}_0$ by

$$Lu=(D_t u+Au, u(0)), \quad u\in\mathcal{D}(L).$$

Our main result is described as follows:

Theorem. *The following conditions about the operator A are equivalent.*
(1) *$\dot{L}$ is bijective for any pair of (p, q, θ, T).*
(2) *$\dot{L}$ is bijective for some pair of (p, q, θ, T).*
(3) *L is bijective for any pair of (p, q, θ, T) with $\theta-p^{-1}\neq$ integer.*
(4) *L is bijective for some pair of (p, q, θ, T) with $\theta-p^{-1}\neq$ integer.*
(5) *$-A$ is the generator of an analytic semigroup in E.*

Let us sketch the proof of the theorem.

(1)$\Rightarrow$(2) and (3)$\Rightarrow$(4) are obvious. (4)$\Rightarrow$(2) follows from the facts that $(f, 0)\in\mathcal{D}_0$ for $f\in\dot{B}_{p,q}^{\theta}(0, T; E)$ and that $u\in\mathcal{D}(L)$ with $Lu=(f, 0)$ turns out to belong to $\mathcal{D}(\dot{L})$ for $f\in\dot{B}_{p,q}^{\theta}(0, T; E)$.

In the next sections we shall prove (2)$\Rightarrow$(5), (5)$\Rightarrow$(1) and (5)$\Rightarrow$(3). (2)$\Rightarrow$(5) is proved by constructing the analytic semigroup with generator $-A$. We prove (5)$\Rightarrow$(1) following the treatment of certain linear operator equations in Banach spaces due to Da Prato and Grisvard [3]. (5)$\Rightarrow$(3) is reduced to (5)$\Rightarrow$(1) and the behavior of analytic semigroups acting on the real interpolation spaces.

4. Proof of (2)$\Rightarrow$(5)

For $x\in E$ we define $U_{-m-1}x\in\dot{B}_{p,q}^{1+\theta}(0, T; E)$ by

$$(U_{-m-1}x)(t)=\frac{t^{m+1}}{(m+1)!}x, \quad 0<t<T.$$

Then the linear operators U_k: $E\rightarrow\mathcal{D}(\dot{L})$, $k\geq -m$, are defined successively by solving the equations

$$\dot{L}U_k x = D_t U_{k-1} x\,, \quad x \in E\,. \tag{4.1$_k$}$$

Putting for $k \geq -m$, $x, y \in E$

$$v_k = tU_{k-1}x + \int_0^t \{(k-2)(U_{k-1}x)(s) - (k+m)(U_k x)(s)\} ds$$

$$w_k = U_{k-1}x - U_k x - \int_0^t (U_k y)(s) ds\,,$$

we have

$$D_t v_k = (tD_t + k - 1) U_{k-1} x - (k+m) U_k x\,, \quad D_t w_k = A U_k x - U_k y\,.$$

When $k \geq -m+1$, we see that v_k, $w_k \in \mathcal{D}(\dot{L})$ and

$$\dot{L} v_k = D_t v_{k-1}\,, \quad \dot{L} w_k = D_t w_{k-1}\,.$$

Noting that $v_{-m} = 0$, we obtain the following:

Lemma 4.1. *For $k \geq -m+1$, $x \in E$ we have*

$$U_k x = \frac{1}{k+m} (tD_t + k - 1) U_{k-1} x\,. \tag{4.2}$$

We have by induction

Corollary of Lemma 4.1. *For $x \in E$ we have*

$$D_t^{1-k} U_k x = \frac{m!}{(k+m)!} t^k D_t U_0 x\,, \qquad \text{when} \quad -m \leq k \leq 1 \tag{4.3}$$

$$U_k x = \frac{m!}{(k+m)!} D_t^{k-1} \{t^k D_t U_0 x\}\,, \qquad \text{when} \quad k \geq 1\,. \tag{4.4}$$

Lemma 4.2. *Let $x, y \in E$. If $AU_k x = U_k y$ holds for some $k \geq -m$, then it holds for any $k \geq -m$.*

The relations (4.3) show that the mappings

$$x \mapsto (k+m)!\, t^{-k-m} D_t^{1-k} U_k x\,, \quad -m \leq k \leq 1$$

are the same linear operators from E to $\mathcal{D}'(0, T; F)$, which we denote by S. The derivatives of Sx are expressed as

$$t^{n+m} D_t^n Sx = (n+m)! \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} D_t U_k x\,. \tag{4.5}$$

These are derived by the identities

$$\frac{t^{n+m}}{(n+m)!}D_t^n f=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}D_t^k\left\{\frac{t^{k+m}}{(k+m)!}f\right\}, \qquad f\in\mathcal{D}'(0,\ T;F)$$

and using (4.4). (4.5) together with (4.2) shows that $Sx\in C^{\infty}((0,\ T];\ F)$ for $x\in E$.

Thus the linear operators $S(t)$: $E\to F$, $0<t\leq T$, are defined as follows:

$$S(t)x=(Sx)(t)\ , \quad x\in E\ .$$

We shall prove that $S(\cdot)$ is extended to an analytic semigroup in E and that $-A$ is its generator.

Differentiating $(4.1_{-m})(m+1)$ times, we first note the relation

$$D_tSx+ASx=0\ , \quad x\in E\ . \tag{4.6}$$

Proposition 4.1. (1) *$x\in E$ belongs to F if and only if there exists $y\in E$ such that*

$$D_tSx+Sy=0\ . \tag{4.7}$$

When this is the case, we have $y=Ax$.

(2) *If $Sx=0$, then $x=0$.*

Proof. By (4.6) and the definition of S, (4.7) is equivalent to $AU_1x=U_1y$. By Lemma 4.2 it is also equivalent to $AU_{-m}x=U_{-m}y$.

Assume that $x\in F$. Then $w_{-m}\in\mathcal{D}(\dot{L})$ and $\dot{L}w_{-m}=U_{-m-1}(Ax-y)$. Hence we have $AU_{-m}x=U_{-m}Ax$. Conversely assume that $AU_{-m}x=U_{-m}y$ holds for some $y\in E$. Then $D_tU_{-m}x+U_{-m}y=D_tU_{-m-1}x$. This implies that $x\in F$.

We proceed to the proof of (2). Suppose that $Sx=0$. Since (4.7) holds with $y=0$, we have $D_tU_{-m}x=D_tU_{-m-1}x$. Differentiating this relation m times, we obtain $x=S(t)x=0$.

Lemma 4.3. *There exist constants K_0 and K_1 such that*

$$t^{m-\theta+p^{-1}}|t^nD_t^nS(t)x|_E\leq T^{m-\theta+p^{-1}}K_1K_0^n n!|x|_E\ , \quad x\in E,\ 0<t\leq T\ . \tag{4.8}$$

Proof. Since $\dot{L}$ is bicontinuous, there exists a constant C_0 such that

$$|D_tu|_{\dot{B}^{\theta}_{p,q}(0,T;E)}\leq C_0|\dot{L}u|_{\dot{B}^{\theta}_{p,q}(0,T;E)}\ , \quad u\in\mathcal{D}(\dot{L})\ .$$

Applying this to (4.5), we obtain the estimates of the norms in $\dot{B}^{\theta}_{p,q}(0,\ T;\ E)$ of $t^{n+m}D_t^nSx$. (4.8) is a consequence of Proposition 2.1 (2) with $p'=q'=\infty$.

Lemma 4.3 implies that the mapping $(0,\ T]\to\mathcal{L}(E)$, $t\mapsto S(t)$ is analytic.

Lemma 4.4. *Let $0<t_1<T$ and $u\in\mathcal{D}'(0,\ t_1;\ F)\cap C^0((0,\ t_1);\ E)$. If $D_tu+Au=0$ in $\mathcal{D}'(0,\ t_1;\ E)$ and $\lim_{t\to 0}u(t)=0$ in E, we have $u=0$ on $(0,\ t_1)$.*

Remark. The assertion is valid if $-A$ is the generator of an analytic semigroup in E. The proof is the same as below except for the use of the analytic semigroup in place of S.

Proof. For $\tau\in[t_1, T]$ we put $v_\tau(t)=S(\tau-t)u(t)$, $0<t<t_1$. By (4.6) and Proposition 4.1 (1) it is verified that $D_t v_\tau=0$ in $\mathcal{D}'(0, t_1; E)$. Hence $v_\tau=0$ on $(0, t_1)$. In view of the analyticity of S and Proposition 4.1 (2) u vanishes on $(0, t_1)$.

Let us observe the behavior of $S(t)$ as $t\to 0$ precisely. We choose $\varphi\in C^\infty([0, \infty))$ so that φ attains the value 1 on $[0, 1]$ and vanishes on $[2, \infty)$. For $\tau\in(0, T]$ and $x\in E$ we define $u_{-m-1}\in\dot{B}^{1+\theta}_{p,q}(0, T; E)$ by

$$u_{-m-1}(t)=\varphi(t/\tau)(U_{-m-1}x)(t)\,, \quad 0<t<T$$

and $u_k\in\mathcal{D}(\dot{L})$, $k\geq -m$, by $\dot{L}u_k=D_t u_{k-1}$. Then $u_k=U_k x$, $k\geq -m$, on $(0, \tau)$ by Lemma 4.4. Repeating the argument in the proof of Lemma 4.3, we obtain

Proposition 4.2. *There exist constants K_0 and K_1 such that*

$$\|t^n D_t^n S(t)\|_{\mathcal{L}(E)}\leq K_1 K_0^n n!\,, \qquad 0<t\leq T\,.$$

Let $t_1, t_2>0$, $t_1+t_2\leq T$ and $x\in E$. Put

$$u(t)=\frac{1}{m!}\int_0^t (t-s)^m S(s+t_2)x\,ds\,, \quad 0\leq t\leq t_1\,.$$

By virtue of (4.6) $u\in C^\infty([0, t_1]; F)$ satisfies

$$\begin{cases} D_t u(t)+Au(t)=\dfrac{t^m}{m!}S(t_2)x\,, & 0\leq t\leq t_1 \\ u(0)=0\,. \end{cases}$$

Hence $u=U_{-m}S(t_2)x$ on $(0, t_1)$ by Lemma 4.4. This implies

Proposition 4.3. *Let t_1 and t_2 be as above. We have $S(t_1+t_2)=S(t_1)S(t_2)$.*

Proposition 4.1, 4.2 and 4.3 show that $S(\cdot)$ is extended to an analytic semigroup in E and that $-A$ is its generator.

5. Proofs of (5)⇒(1), (5)⇒(3)

The injectivity of $\dot{L}$ and L follows from the remark in section 4.

We shall prove that $\dot{L}$ and L are surjective. In view of the remark in section 3 we may assume that there exists a constant $\phi\in(0, \pi/2)$ such that

$$\rho(-A)\supset\Sigma_{\phi+\pi/2}\cup\{0\}\,, \qquad \sup_{\lambda\in\Sigma_{\phi+\pi/2}}\|\lambda(\lambda+A)^{-1}\|_{\mathcal{L}(E)}<\infty\,.$$

Proof of surjectivity of $\dot{L}$. Da Prato and Grisvard [3] investigated the operator equations in Banach spaces written as the sum of two closed linear operators. They developed a potential theory for what they called parabolic equations, which is described as follows:

Let $\tilde{A}$ and $\tilde{B}$ be closed linear operators in a Banach space X. We assume

(1) There exist constants ψ_1 and ψ_2 with $0<\pi-\psi_2<\psi_1<\pi$ such that $\rho(-\tilde{A})\supset\Sigma_{\psi_1}\cup\{0\}$, $\rho(-\tilde{B})\supset\Sigma_{\psi_2}\cup\{0\}$ and

$$\sup_{z\in\Sigma_{\psi_1}}\|z(z+\tilde{A})^{-1}\|_{\mathcal{L}(X)}<\infty\,,\quad \sup_{z\in\Sigma_{\psi_2}}\|z(z+\tilde{B})^{-1}\|_{\mathcal{L}(X)}<\infty\,.$$

(2) $(z_1+\tilde{A})^{-1}$ commutes with $(z_2+\tilde{B})^{-1}$ for $z_1\in\rho(-\tilde{A})$ and $z_2\in\rho(-\tilde{B})$. We set

$$D_{\tilde{B}}(\theta, q)=\{x\in X;\, |t^\theta\tilde{B}^m(t+\tilde{B})^{-m}x|_{L^q_*(0,\infty;X)}<\infty\}\,.$$

Let γ be the contour running from $e^{-i\psi}\infty$ to $e^{i\psi}\infty$ such that $\gamma=\{re^{-i\psi};\, r\geq 0\}\cup\{re^{i\psi};\, r\geq 0\}$ for some $\psi\in(\pi-\psi_2, \psi_1)$. Put

$$R=\frac{1}{2\pi i}\int_\gamma (z+\tilde{A})^{-1}(-z+\tilde{B})^{-1}dz\,.$$

Proposition 5.1. *For $x\in D_{\tilde{B}}(\theta, q)$ we have $Rx\in\mathcal{D}(\tilde{A})\cap\mathcal{D}(\tilde{B})$, $(\tilde{A}+\tilde{B})Rx=x$. Moreover, $\tilde{A}Rx$, $\tilde{B}Rx\in D_{\tilde{B}}(\theta, q)$.*

The surjectivity of $\dot{L}$ is reduced to the proposition by setting $X=L^p(0, T; E)$ and the operators $\tilde{A}$, $\tilde{B}$ as follows:

$$\mathcal{D}(\tilde{A})=L^p(0, T; \mathcal{D}(A))\,,\quad (\tilde{A}u)(t)=Au(t)$$

$$\mathcal{D}(\tilde{B})=\{u\in X;\, D_tu\in X,\, u(0)=0\}\,,\quad \tilde{B}u=D_tu\,.$$

We remark that $\dot{B}^\theta_{p,q}(0, T; E)=D_{\tilde{B}}(\theta, q)$. See [3] for the case $q=p$ and $0<\theta<1$.

Proof of Proposition. The proof goes in the same manner as in [3], where the case $0<\theta<1$ was proved. We only sketch the proof of the latter half.

As in the proof in [3], we have for $t>0$

$$(t+\tilde{B})^{-m}\tilde{B}Rx=\frac{1}{2\pi i}\int_\gamma\frac{z(z+\tilde{A})^{-1}}{(t+z)^m}(-z+\tilde{B})^{-1}xdz\,.$$

(1) The case $\psi_1\geq\psi_2$. Let T_t, $t>0$, be the holomorphic functions on some open neighborhood of γ expressed as

$$T_t(z)=\begin{cases} \dfrac{z(z+\tilde{A})^{-1}}{t+z}, & \text{when } m=1 \\ \displaystyle\int_{-z}^{z}\frac{(z-\zeta)^{m-2}\zeta(\zeta+\tilde{A})^{-1}}{(m-2)!(t+\zeta)^m}d\zeta, & \text{when } m\geq 2. \end{cases}$$

Notice that the estimate

$$\|T_t(z)\|_{\mathcal{L}(X)}\leq\frac{C_m}{|z|}\frac{(\min\{1, |z|/t\})^{m-1}}{1+t/|z|}, \qquad (t, z)\in(0, \infty)\times\gamma$$

holds for some constant C_m. Since

$$\frac{z(z+\tilde{A})^{-1}}{(t+z)^m}=D_z{}^{m-1}T_t(z)+\sum_{k=0}^{m-2}D_z{}^{m-2-k}\left\{\frac{(2z)^{m-2-k}z(-z+\tilde{A})^{-1}}{(m-2-k)!(t-z)^m}\right\},$$

we have by Cauchy's integral theorem that

$$(t+\tilde{B})^{-m}\tilde{B}Rx=\frac{1}{2\pi i}\int_\gamma D_z{}^{m-1}T_t(z)(-z+\tilde{B})^{-1}xdz.$$

Integration by parts yields the contour integral of $T_t(z)(-z+\tilde{B})^{-m}x$. Applying Young's inequality, we obtain the esimate of the norm of $t^\theta\tilde{B}^m(t+\tilde{B})^{-m}\tilde{B}Rx$ in $L^q_*(0, \infty; X)$.
(2) The case $\psi_1\leq\psi_2$. This case is proved with the use of the following expression derived by Cauchy's integral theorem:

$$(t+\tilde{B})^{-m}\tilde{B}Rx=\frac{1}{2\pi i}\int_\gamma \frac{z(z+\tilde{A})^{-1}}{(t+z)^m}(2z)^{m-1}(-z+\tilde{B})^{-1}(z+\tilde{B})^{-(m-1)}xdz.$$

Proof of surjectivity of L. Let γ be the contour running in $\Sigma_{\phi+\pi/2}$ from $e^{-i\psi}\infty$ to $e^{i\psi}\infty$ for some $\psi\in(\pi/2, \phi+\pi/2)$. The elements $S_j(t)$, $t>0$, $j\in\boldsymbol{Z}_+$, of $\mathcal{L}(E)$ given by

$$S_j(t)=\frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{-j}(\lambda+A)^{-1}d\lambda$$

satisfy the following: $S_j(\cdot)$ are analytic. The ranges of $S_j(t)$ are included in $\mathcal{D}(A)$.

$$D_tS_j(t)+AS_j(t)=D_t\left\{\frac{t^j}{j!}\right\}I, \quad t>0 \tag{5.1}$$

holds. $\sup_{t>0}\|t^{n-j}D_t{}^nS_j(t)\|_{\mathcal{L}(E)}<\infty$. If $x\in E$ belongs to the closure of $\mathcal{D}(A)$ in E, $S_0(t)x$ converges to x in E as t tends to 0. Furthermore, we have

Proposition 5.2. *Let $1\leq p\leq\infty$, $1\leq q\leq\infty$, $p^{-1}<\theta<1+p^{-1}$ and $0<T<\infty$. For $x\in(E, \mathcal{D}(A))_{\theta-p^{-1},q}$ we have $S_j(\cdot)x\in B^{j+\theta}_{p,q}(0, T; E)$. In addition, if $j+\theta>1$, we have $S_j(\cdot)x\in B^{j-1+\theta}_{p,q}(0, T; \mathcal{D}(A))$.*

Proof. The first half follows from the expression

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}S_j(t+kh)x=\frac{-1}{2\pi i}\int_{\gamma}e^{\lambda t}(e^{\lambda h}-1)^m\lambda^{-j}A(\lambda+A)^{-1}x\frac{d\lambda}{\lambda},$$

$$m\geq j+1,$$

the estimate $|e^{\lambda t}|_{L^p(0,\infty)}\leq p^{-p^{-1}}|\mathfrak{R}\lambda|^{-p^{-1}}$ for $\mathfrak{R}\lambda<0$ and Young's inequality. Recall that $(E, \mathcal{D}(A))_{\eta,q}=\{x\in E; |\lambda^\eta A(\lambda+A)^{-1}x|_{L^q_*(0,\infty;E)}<\infty\}$, $0<\eta<1$, $1\leq q\leq\infty$. See Triebel [17]. The latter from the first and (5.1).

Let $(f, x)\in\mathcal{D}_0$. We set

$$g(t)=f(t)-\sum_{k=0}^{N-1}\frac{t^k}{k!}D_t^kf(0), \quad 0<t<T.$$

Note that $g\in\dot{B}^\theta_{p,q}(0, T; E)$. The element u of $\mathcal{D}(L)$ given by

$$u(t)=\sum_{k=0}^{N-1}\frac{t^k}{k!}x_k+S_N(t)x_N+(\dot{L}^{-1}g)(t), \quad 0<t<T$$

satisfies $Lu=(f, x)$.

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