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On the Chern character of symmetric spaces related to $SU(n)$

By

Takashi WATANABE

§ 0. Introduction

Let (G, F) be a compact symmetric pair. That is, G is a compact Lie group with an involutive automorphism $s: G\to G$ and F is a closed connected subgroup of G such that $G^S=\{x\in G;\ s(x)=x\}\supset F\supset(G^s)_e$, the identity component of G^s. Then the quotient $M=G/F$ forms a compact symmetric space. The aim of this paper is to study the Chern character homomorphism ch: $K^*(M)\to H^{**}(M; \boldsymbol{Q})$ [1, § 1] for two cases $M=SU(2n)/Sp(n)$ in section 2 and $M=SU(2n+1)/SO(2n+1)$ in section 3, where $SU(n)\subset M(n, \boldsymbol{C})$, $Sp(n)\subset M(n, \boldsymbol{H})$ and $SO(n)\subset M(n, \boldsymbol{R})$ are the n-th special unitary, symplectic and rotation groups, respectively. As a byproduct we will find a symmetry in a description of ch for $M=SU(n+1)$ at the end of section 3. Finally in section 4 we compute ch for $M=SO(2n+1)$.

Our discussion is summarized as follows. Let π: $G\to G/F$ be the projection and consider the commutative diagram

$$\begin{array}{ccc} K^*(G/F) & \xrightarrow{\ ch\ } & H^{**}(G/F;\ \boldsymbol{Q}) \\ \pi^*\downarrow & & \downarrow\pi^* \\ K^*(G) & \xrightarrow{\ ch\ } & H^{**}(G;\ \boldsymbol{Q}) \end{array} .$$

In our cases, all the rings $K^*(G/F)$, $H^*(G/F; \boldsymbol{Q})$, $K^*(G)$ and $H^*(G; \boldsymbol{Q})$ were determined and all the homomorphisms except the upper ch were described; further, the vertical homomorphisms are injective. So the upper ch can be computed.

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§ 1. Preliminaries

We first deal with $SU(2n)/Sp(n)$. Let I_n be the unit matrix of degree n, and set

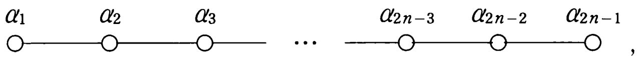
$$J_n=\begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}.$$

Define a map s_2: $SU(2n)\to SU(2n)$ by $s_2(A)=J_n\bar{A}J_n^{-1}$ for $A\in SU(2n)$, where $\bar{A}$ denotes the complex conjugate of A. Clearly s_2 is an involution. Let i_2: $Sp(n)\to SU(2n)$ be the map defined by

$$i_2(X)=\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \qquad \text{for } X=A+\boldsymbol{j}B\in Sp(n)\,,$$

where $\boldsymbol{j}$ is the element of $\boldsymbol{H}$ such that $\boldsymbol{H}=\boldsymbol{C}\{1\}\oplus \mathrm{C}\{\boldsymbol{j}\}$ and $\boldsymbol{j}^2=-1$. Clearly i_2 is a monomorphism of topological groups. It is easy to check that $(SU(2n), i_2(Sp(n)))$ is a compact symmetric pair. Thus $SU(2n)/Sp(n)$ becomes a compact symmetric space, which is denoted by AII in É. Cartan's notation.

Choose a maximal torus T of $SU(2n)$ so that $s_2(T)\subset T$. Let $L(T)$ be the Lie algebra of T. There are simple roots $\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}$: $L(T)\to \boldsymbol{R}$ such that the corresponding Dynkin diagram is



where, with respect to a certain inner product (,) on $L(T)^*=\mathrm{Hom}_{\boldsymbol{R}}(L(T), \boldsymbol{R})$, $(\alpha_i, \alpha_i)=2$ if $1\leqq i\leqq 2n-1$; $(\alpha_i, \alpha_{i+1})=-1$ if $1\leqq i<2n-1$; otherwise $(\alpha_i, \alpha_j)=0$. Hereafter we follow [4]. We may regard α_i as an element of $H^2(BT; \boldsymbol{Q})$, and then $H^*(BT; \boldsymbol{Q})$ is the polynomial algebra $\boldsymbol{Q}[\alpha_1, \cdots, \alpha_{2n-1}]$. Denote by s_2: $T\to T$ the restriction of s_2 to T. According to [9], Bs_2^*: $H^*(BT; \boldsymbol{Q})\to H^*(BT; \boldsymbol{Q})$ is given by

$$Bs_2^*(\alpha_i)=\alpha_{2n-i} \qquad (i=1, 2, \cdots, 2n-1)\,. \tag{1.1}$$

Let $\omega_1, \omega_2, \cdots, \omega_{2n-1}$ be the fundamental weights determined by $\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}$. Then we have

$$\omega_i=(1/2n)((2n-i)\sum_{j=1}^{i-1} j\alpha_j+i\sum_{j=i}^{2n-1}(2n-j)\alpha_j) \tag{1.2}$$

(see [4]). Since $H^*(BT; \boldsymbol{Z})=\boldsymbol{Z}[\omega_1, \cdots, \omega_{2n-1}]$ (see [3, § 10.1]), it follows from (1.1) and (1.2) that Bs_2^*: $H^*(BT; \boldsymbol{Z})\to H^*(BT; \boldsymbol{Z})$ is given by

(1.3) $\quad Bs_2^*(\omega_i)=\omega_{2n-i} \qquad (i=1, 2, \cdots, 2n-1)$.

Let R_i be the reflection of $L(T)^*$ relative to α_i, i.e., with respect to the hyperplane $\{x\in L(T)^*; (\alpha_i, x)=0\}$. Then $R_1, R_2, \cdots, R_{2n-1}$ generate the Weyl group $W(SU(2n))$ and act on $H^2(BT; \boldsymbol{Z})$ by the formulas

$$R_i(\omega_i)=-\omega_i-\sum_{j\neq i}(2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j))\omega_j \qquad \text{and}$$

$$R_i(\omega_j)=0 \quad \text{if} \quad i\neq j.$$

In this case, we have

$$R_1(\omega_1)=-\omega_1+\omega_2,$$

$$R_i(\omega_i)=\omega_{i-1}-\omega_i+\omega_{i+1} \qquad (i=2, 3, \cdots, 2n-2),$$

$$R_{2n-1}(\omega_{2n-1})=\omega_{2n-2}-\omega_{2n-1}.$$

Put

$$t_1=\omega_1,$$

$$t_i=R_{i-1}(t_{i-1})=-\omega_{i-1}+\omega_i \qquad (i=2, 3, \cdots, 2n-1),$$

$$t_{2n}=R_{2n-1}(t_{2n-1})=-\omega_{2n-1}.$$

Then $H^*(BT; \boldsymbol{Z})=\boldsymbol{Z}[t_1, \cdots, t_{2n}]/(c_1)$, where $c_1=t_1+\cdots+t_{2n}$, and it follows from (1.3) that

(1.4) $\quad Bs_2^*(t_i)=-t_{2n+1-i} \qquad (i=1, 2, \cdots, 2n)$.

For a commutative ring R with a unit $1\in R$, let $\sigma_i(x_1, x_2, \cdots, x_n)$ denote the i-th elementary symmetric polynomial in $R[x_1, x_2, \cdots, x_n]$. For a compact connected Lie group G with a maximal torus T, we denote by i: $T\rightarrow G$ the inclusion, by σ^*: $H^*(BG; R)\rightarrow H^{*-1}(G; R)$ the cohomology suspension and by $H^*(BT; R)^{W(G)}$ the subalgebra of $H^*(BT; R)$ invariant under the action of the Weyl group $W(G)$.

Let $c_{i+1}=\sigma_{i+1}(t_1, \cdots, t_{2n})\in H^{2i+2}(BT; \boldsymbol{Z})$. Since $W(SU(2n))$ acts on $H^2(BT; \boldsymbol{Z})$ as the group of permutations on $\{t_1, \cdots, t_{2n}\}$, we have $H^*(BT; \boldsymbol{Z})^{W(SU(2n))}=\boldsymbol{Z}[c_2, c_3, \cdots, c_{2n}]$ and it follows from (1.4) that

(1.5) $\quad Bs_2^*(c_{i+1})=(-1)^{i+1}c_{i+1} \qquad (i=1, 2, \cdots, 2n-1)$.

Since $H^*(SU(2n); \boldsymbol{Z})$ has no torsion, by [2, § 29] Bi: $BT\rightarrow BSU(2n)$ induces an isomorphism $H^*(BSU(2n); \boldsymbol{Z})\cong H^*(BT; \boldsymbol{Z})^{W(SU(2n))}$ (so we shall identify them). Therefore $H^*(BSU(2n); \boldsymbol{Z})=\boldsymbol{Z}[c_2, c_3, \cdots, c_{2n}]$. Let $x_{2i+1}=\sigma^*(c_{i+1})\in H^{2i+1}(SU(2n); \boldsymbol{Z})$. Then

(1.6) $\quad H^*(SU(2n); \boldsymbol{Z})$ is the exterior algebra $\Lambda_Z(x_3, x_3, \cdots, x_{4n-1})$

(see [2, § 19]).

Proposition 1.1. *s_2^*: $H^*(SU(2n); \boldsymbol{Z}) \to H^*(SU(2n); \boldsymbol{Z})$ is given by*

$$s_2^*(x_{2i+1}) = (-1)^{i+1} x_{2i+1} \qquad (i=1, 2, \cdots, 2n-1).$$

Proof. We have

$$\begin{aligned} s_2^*(x_{2i+1}) &= s_2^*(\sigma^*(c_{i+1})) \\ &= \sigma^*(Bs_2^*(c_{i+1})) \quad \text{by the naturality of } \sigma^* \\ &= \sigma^*((-1)^{i+1} c_{i+1}) \quad \text{by (1.5)} \\ &= (-1)^{i+1} x_{2i+1}. \end{aligned}$$

Choose a maximal torus T' of $Sp(n)$ so that $i_2(T') \subset T$. Let $L(T')$ be the Lie algebra of T'. There are simple roots $\alpha_1', \alpha_2', \cdots, \alpha_n'$: $L(T') \to \boldsymbol{R}$ such that the corresponding Dynkin diagram is

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where $(\alpha_i', \alpha_i') = 2$ if $1 \leqq i < n$; $(\alpha_n', \alpha_n') = 4$; $(\alpha_i', \alpha_{i+1}') = -1$ if $1 \leqq i < n-1$; $(\alpha_{n-1}', \alpha_n') = 2$; otherwise $(\alpha_i', \alpha_j') = 0$. Then $H^*(BT'; \boldsymbol{Q}) = \boldsymbol{Q}[\alpha_1', \cdots, \alpha_n']$. Denote by i_2: $T' \to T$ the restriction of i_2 to T'. According to [9], Bi_2^*: $H^*(BT; \boldsymbol{Q}) \to H^*(BT'; \boldsymbol{Q})$ is given by

$$(1.7) \qquad Bi_2^*(\alpha_i) = \alpha_i' = Bi_2^*(\alpha_{2n-i}) \qquad (i=1, 2, \cdots, n).$$

Let $\omega_1', \omega_2', \cdots, \omega_n'$ be the fundamental weights determined by $\alpha_1', \alpha_2', \cdots, \alpha_n'$. Then we have

$$(1.8) \qquad \omega_i' = \sum_{j=1}^{i-1} j\alpha_j' + i\left(\sum_{j=i}^{n-1} \alpha_j' + (1/2)\alpha_n'\right).$$

Since $H^*(BT'; \boldsymbol{Z}) = \boldsymbol{Z}[\omega_1', \cdots, \omega_n']$, it follows from (1.2), (1.7) and (1.8) that Bi_2^*: $H^*(BT; \boldsymbol{Z}) \to H^*(BT'; \boldsymbol{Z})$ is given by

$$(1.9) \qquad Bi_2^*(\omega_i) = \omega_i' = Bi_2^*(\omega_{2n-i}) \qquad (i=1, 2, \cdots, n).$$

Let R_i' be the reflection of $L(T')^*$ relative to α_i'. Then the action of $W(Sp(n))$ on $H^2(BT'; \boldsymbol{Z})$ is given by

$$\begin{aligned} R_i'(\omega_1') &= -\omega_1' + \omega_2', \\ R_i'(\omega_i') &= \omega_{i-1}' - \omega_i' + \omega_{i+1}' \qquad (i=2, 3, \cdots, n-1), \\ R_n'(\omega_n') &= 2\omega_{n-1}' - \omega_n'. \end{aligned}$$

Put

$$t'_1 = \omega'_1 \,,$$

$$t'_i = R'_{i-1}(t'_{i-1}) = -\omega'_{i-1} + \omega'_i \qquad (i=2, 3, \cdots, n)\,.$$

Then $H^*(BT'; \boldsymbol{Z}) = \boldsymbol{Z}[t'_1, \cdots, t'_n]$ and it follows from (1.9) that

$$(1.10) \qquad Bi_2^*(t_i) = t'_i \qquad (i=1, 2, \cdots, n)\,,$$

$$Bi_2^*(t_{2n+1-i}) = -t'_i \qquad (i=1, 2, \cdots, n)\,.$$

Let $q_i = \sigma_i(t_1'^2, t_2'^2, \cdots, t_n'^2) \in H^{4i}(BT'; \boldsymbol{Z})$. Since $W(Sp(n))$ acts on $H^2(BT'; \boldsymbol{Z})$ as the group of permutations on $\{t'_1, \cdots, t'_n\}$ together with substitutions $t'_i \to -t'_i$, we have $H^*(BT'; \boldsymbol{Z})^{W(Sp(n))} = \boldsymbol{Z}[q_1, q_2, \cdots, q_n]$ and it follows from (1.10) that

$$(1.11) \qquad Bi_2^*(c_{2i}) = (-1)^i q_i \qquad (i=1, 2, \cdots, n)\,,$$

$$Bi_2^*(c_{2i+1}) = 0 \qquad (i=1, 2, \cdots, n-1)\,.$$

By [2], $H^*(BSp(n); \boldsymbol{Z}) \cong H^*(BT'; \boldsymbol{Z})^{W(Sp(n))}$. Therefore

$$(1.12) \qquad H^*(BSp(n); \boldsymbol{Z}) = \boldsymbol{Z}[q_1, q_2, \cdots, q_n]\,.$$

Let $x'_{4i-1} = \sigma^*(q_i) \in H^{4i-1}(Sp(n); \boldsymbol{Z})$. Again by [2],

$$(1.13) \qquad H^*(Sp(n); \boldsymbol{Z}) = \Lambda_{\boldsymbol{Z}}(x'_3, x'_7, \cdots, x'_{4n-1})\,.$$

Proposition 1.2. *i_2^*: $H^*(SU(2n); \boldsymbol{Z}) \to H^*(Sp(n); \boldsymbol{Z})$ is given by*

$$i_2^*(x_{4i-1}) = (-1)^i x'_{4i-1} \qquad (i=1, 2, \cdots, n)\,,$$

$$i_2^*(x_{4i+1}) = 0 \qquad (i=1, 2, \cdots, n-1)\,.$$

Proof. This follows from (1.11).

We next deal with $SU(2n+1)/SO(2n+1)$. Define a map s_1: $SU(2n+1) \to SU(2n+1)$ by $s_1(A) = \bar{A}$ for $A \in SU(2n+1)$. Clearly s_1 is an involution. Let i_1: $SO(2n+1) \to SU(2n+1)$ be the map derived from the inclusion $\boldsymbol{R} \subset \boldsymbol{C}$. Clearly i_1 is a monomorphism of topological groups. It is easy to check that $(SU(2n+1), i_1(SO(2n+1)))$ is a compact symmetric pair. Thus $SU(2n+1)/SO(2n+1)$ becomes a compact symmetric space, which is denoted by AI in É. Cartan's notation.

Choose a maximal torus T of $SU(2n+1)$ so that $s_1(T) \subset T$. There are simple roots $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$: $L(T) \to \boldsymbol{R}$ such that the corresponding Dynkin diagram is

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where $(\alpha_i, \alpha_i)=2$ if $1\leqq i\leqq 2n$; $(\alpha_i, \alpha_{i+1})=-1$ if $1\leqq i<2n$; otherwise $(\alpha_i, \alpha_j)=0$. Then $H^*(BT;\,\boldsymbol{Q})=\boldsymbol{Q}[\alpha_1, \cdots, \alpha_{2n}]$. Denote by $s_1\colon T\to T$ the restriction of s_1 to T. According to [9], $Bs_1^*\colon H^*(BT;\,\boldsymbol{Q})\to H^*(BT;\,\boldsymbol{Q})$ is given by

$$Bs_1^*(\alpha_i)=\alpha_{2n+1-i} \qquad (i=1, 2, \cdots, 2n)\,. \tag{1.14}$$

Let $\omega_1, \omega_2, \cdots, \omega_{2n}$ be the fundamental weights determined by $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$. Then we have

$$\omega_i=(1/(2n+1))((2n+1-i)\sum_{j=1}^{i-1} j\alpha_j+i\sum_{j=i}^{2n}(2n+1-j)\alpha_j)\,. \tag{1.15}$$

Since $H^*(BT;\,\boldsymbol{Z})=\boldsymbol{Z}[\omega_1, \cdots, \omega_{2n}]$, it follows from (1.14) and (1.15) that Bs_1^*: $H^*(BT;\,\boldsymbol{Z})\to H^*(BT;\,\boldsymbol{Z})$ is given by

$$Bs_1^*(\omega_i)=\omega_{2n+1-i} \qquad (i=1, 2, \cdots, 2n)\,. \tag{1.16}$$

Let R_i be the reflection relative to α_i. Then the action of $W(SU(2n+1))$ on $H^2(BT;\,\boldsymbol{Z})$ is given by

$$R_1(\omega_1)=-\omega_1+\omega_2\,,$$

$$R_i(\omega_i)=\omega_{i-1}-\omega_i+\omega_{i+1} \qquad (i=2, 3, \cdots, 2n-1)\,,$$

$$R_{2n}(\omega_{2n})=\omega_{2n-1}-\omega_{2n}\,.$$

Put

$$t_1=\omega_1\,,$$

$$t_i=R_{i-1}(t_{i-1})=-\omega_{i-1}+\omega_i \qquad (i=2, 3, \cdots, 2n)\,,$$

$$t_{2n+1}=R_{2n}(t_{2n})=-\omega_{2n}\,.$$

Then $H^*(BT;\,\boldsymbol{Z})=\boldsymbol{Z}[t_1, \cdots, t_{2n+1}]/(c_1)$, where $c_1=\sigma_1(t_1, \cdots, t_{2n+1})$, and it follows from (1.16) that

$$Bs_1^*(t_i)=-t_{2n+2-i} \qquad (i=1, 2, \cdots, 2n+1)\,. \tag{1.17}$$

Let $c_{i+1}=\sigma_{i+1}(t_1, \cdots, t_{2n+1})\in H^{2i+2}(BT;\,\boldsymbol{Z})$. Then $H^*(BT;\,\boldsymbol{Z})^{W(SU(2n+1))}=\boldsymbol{Z}[c_2, c_3, \cdots, c_{2n+1}]$ and it follows from (1.17) that

$$Bs_1^*(c_{i+1})=(-1)^{i+1}c_{i+1} \qquad (i=1, 2, \cdots, 2n)\,. \tag{1.18}$$

By [2], $H^*(BSU(2n+1);\,\boldsymbol{Z})\cong H^*(BT;\,\boldsymbol{Z})^{W(SU(2n+1))}=\boldsymbol{Z}[c_2, c_3, \cdots, c_{2n+1}]$. Let $x_{2i+1}=\sigma^*(c_{i+1})\in H^{2i+1}(SU(2n+1);\,\boldsymbol{Z})$. Then

$$H^*(SU(2n+1);\,\boldsymbol{Z})=\Lambda_{\boldsymbol{Z}}(x_3, x_5, \cdots, x_{4n+1})\,. \tag{1.19}$$

Proposition 1.3. *$s_1^*\colon H^*(SU(2n+1);\,\boldsymbol{Z})\to H^*(SU(2n+1);\,\boldsymbol{Z})$ is given by*

$$s_1^*(x_{2i+1})=(-1)^{i+1}x_{2i+1} \qquad (i=1, 2, \cdots, 2n) .$$

Proof. This follows from (1.18).

Choose a maximal torus T' of $SO(2n+1)$ so that $i_1(T')\subset T$. There are simple roots $\alpha_1', \alpha_2', \cdots, \alpha_n'$: $L(T')\to \boldsymbol{R}$ such that the corresponding Dynkin diagram is

where $(\alpha_i', \alpha_i')=2$ if $1\leqq i<n$; $(\alpha_n', \alpha_n')=1$; $(\alpha_i', \alpha_{i+1}')=-1$ if $1\leqq i<n$; otherwise $(\alpha_i', \alpha_j')=0$. Then $H^*(BT'; \boldsymbol{Q})=\boldsymbol{Q}[\alpha_1', \cdots, \alpha_n']$. Denote by i_1: $T'\to T$ the restriction of i_1 to T'. According to [9], Bi_1^*: $H^*(BT; \boldsymbol{Q})\to H^*(BT'; \boldsymbol{Q})$ is given by

$$\text{(1.20)} \qquad Bi_1^*(\alpha_i)=\alpha_i'=Bi_1^*(\alpha_{2n+1-i}) \qquad (i=1, 2, \cdots, n) .$$

Let $\omega_1', \omega_2', \cdots, \omega_n'$ be the fundamental weights determined by $\alpha_1', \alpha_2', \cdots, \alpha_n'$. Then we have

$$\text{(1.21)} \qquad \omega_i'=\sum_{i=1}^{i-1} j\alpha_j'+i\sum_{j=i}^{n}\alpha_j' \qquad (i=1, 2, \cdots, n-1) ,$$

$$\omega_n'=(1/2)\sum_{j=1}^{n} j\alpha_j' .$$

Since $H^*(BT'; \boldsymbol{Z})=\boldsymbol{Z}[\omega_1', \cdots, \omega_n']$, it follows from (1.15), (1.20) and (1.21) that Bi_1^*: $H^*(BT; \boldsymbol{Z})\to H^*(BT'; \boldsymbol{Z})$ is given by

$$\text{(1.22)} \qquad Bi_1^*(\omega_i)=\omega_i'=Bi_1^*(\omega_{2n+1-i}) \qquad (i=1, 2, \cdots, n-1) ,$$

$$Bi_1^*(\omega_n)=2\omega_n'=Bi_1^*(\omega_{n+1}) .$$

Let R_i' be the reflection relative to α_i'. Then the action of $W(SO(2n+1))$ on $H^2(BT'; \boldsymbol{Z})$ is given by

$$R_1'(\omega_1')=-\omega_1'+\omega_2' ,$$

$$R_i'(\omega_i')=\omega_{i-1}'-\omega_i'+\omega_{i+1}' \qquad (i=2, 3, \cdots, n-2) ,$$

$$R_{n-1}'(\omega_{n-1}')=\omega_{n-2}'-\omega_{n-1}'+2\omega_n' ,$$

$$R_n'(\omega_n')=\omega_{n-1}'-\omega_n' .$$

Put

$$t_1'=\omega_1' ,$$

$$t_i'=R_{i-1}'(t_{i-1}')=-\omega_{i-1}'+\omega_i' \qquad (i=2, 3, \cdots, n-1) ,$$

$$t'_n = R'_{n-1}(t'_{n-1}) = -\omega'_{n-1} + 2\omega'_n .$$

Then $H^*(BT'; \boldsymbol{Z}) = \boldsymbol{Z}[t'_1, \cdots, t'_n]$ and it follows from (1.22) that

(1.23) $$Bi_1^*(t_i) = t'_i \qquad (i=1, 2, \cdots, n),$$

$$Bi_1^*(t_{n+1}) = 0 ,$$

$$Bi_1^*(t_{2n+2-i}) = -t'_i \qquad (i=1, 2, \cdots, n).$$

Let $p_i = \sigma_i(t_1'^2, t_2'^2, \cdots, t_n'^2) \in H^{4i}(BT'; \boldsymbol{Z})$. Since $W(SO(2n+1))$ acts on $H^2(BT'; \boldsymbol{Z})$ as the group of permutations on $\{t'_1, \cdots, t'_n\}$ together with substitutions $t'_i \to -t'_i$, we have $H^*(BT'; \boldsymbol{Z})^{W(SO(2n+1))} = \boldsymbol{Z}[p_1, p_2, \cdots, p_n]$ and it follows from (1.23) that

(1.24) $$Bi_1^*(c_{2i}) = (-1)^i p_i \qquad (i=1, 2, \cdots, n),$$

$$Bi_1^*(c_{2i+1}) = 0 \qquad (i=1, 2, \cdots, n).$$

Suppose given a field $\boldsymbol{k}$ of characteristic $p \neq 2$. Since $H^*(SO(2n+1); \boldsymbol{Z})$ has no odd torsion, by [2, § 29] $H^*(BSO(2n+1); \boldsymbol{k}) \cong H^*(BT'; \boldsymbol{k})^{W(SO(2n+1))}$. Therefore

(1.25) $$H^*(BSO(2n+1); \boldsymbol{k}) = \boldsymbol{k}[p_1, p_2, \cdots, p_n] .$$

Let $x'_{4i-1} = \sigma^*(p_i) \in H^{4i-1}(SO(2n+1); \boldsymbol{k})$. By [2],

(1.26) $$H^*(SO(2n+1); \boldsymbol{k}) = \Lambda_{\boldsymbol{k}}(x'_3, x'_7, \cdots, x'_{4n-1}) .$$

In this way, from (1.24) we obtain a result on the behavior of i_1^*: $H^*(SU(2n+1); \boldsymbol{k}) \to H^*(SO(2n+1); \boldsymbol{k})$, which is quite similar to Proposition 1.2. However, in section 4 we will prove its integral version (Proposition 4.1).

§ 2. The Chern character of $SU(2n)/Sp(n)$

The cohomology of $SU(2n)/Sp(n)$ is known (e.g., see [10, Vol. I, Chap. 3, Theorem 6.7, (1)]).

Proposition 2.1. *$H^*(SU(2n)/Sp(n); \boldsymbol{Z})$ has no torsion and there exist elements $e_{4i+1} \in H^{4i+1}(SU(2n)/Sp(n); \boldsymbol{Z})$ $(i=1, 2, \cdots, n-1)$ such that*

$$H^*(SU(2n)/Sp(n); \boldsymbol{Z}) = \Lambda_{\boldsymbol{Z}}(e_5, e_9, \cdots, e_{4n-3}) .$$

If π_2: $SU(2n) \to SU(2n)/Sp(n)$ is the projection, π_2^*: $H^*(SU(2n)/Sp(n); \boldsymbol{Z}) \to H^*(SU(2n); \boldsymbol{Z})$ satisfies $\pi_2^*(e_{4i+1}) = x_{4i+1}$.

Proof. Consider the Serre spectral sequence $\{E_r, d_r\}$ for the integral cohomology of the fibration

$$SU(2n) \xrightarrow{\pi_2} SU(2n)/Sp(n) \longrightarrow BSp(n)$$

induced by Bi_2: $BSp(n) \to BSU(2n)$. By (1.12) and (1.6),

$$E_2 = H^*(BSp(n); \boldsymbol{Z}) \otimes H^*(SU(2n); \boldsymbol{Z})$$
$$= \boldsymbol{Z}[q_1, q_2, \cdots, q_n] \otimes \Lambda_{\boldsymbol{Z}}(x_3, x_5, \cdots, x_{4n-1}) .$$

Since each $x_{2i+1} \in H^{2i+1}(SU(2n); \boldsymbol{Z})$ transgresses to $c_{i+1} \in H^{2i+2}(BSU(2n); \boldsymbol{Z})$ in the Serre spectral sequence of the universal $SU(2n)$-bundle, it follows from (1.11) that

$$d_{4i}(1 \otimes x_{4i-1}) = (-1)^i q_i \otimes 1 \qquad (i=1, 2, \cdots, n) ,$$

$$d_r(1 \otimes x_{4i+1}) = 0 \qquad (i=1, 2, \cdots, n-1;\ r \geqq 2) .$$

Then a routine spectral sequence argument yields the result.

Let $U(n)$ and U be the n-th and infinite unitary groups, respectively. A representation of a compact Lie group G is a homomorphism $G \to U(n)$ of topological groups, where n is its dimension. The representation ring $R(G)$ of G has the structure of a λ-ring (see [8, 12(1.1)]) given by the exterior power operations λ^k: $R(G) \to R(G)$ for $k \geqq 0$. Let β: $R(G) \to \tilde{K}^{-1}(G)$ be the homomorphism of abelian groups defined by assigning to a representation ρ: $G \to U(n)$ the homotopy class $\beta(\rho) = [\iota_n \rho] \in [G, U] = \tilde{K}^{-1}(G)$, where ι_n: $U(n) \to U$ is the canonical injection. Then β has the following properties ([7, *p.* 8]):

(2.1) *if ρ_1, ρ_2 are representations of G of dimensions n_1, n_2 respectively, then $\beta(\rho_1\rho_2) = n_2\beta(\rho_1) + n_1\beta(\rho_2)$;*

(2.2) *if n denotes the trivial representation of G of dimension n, then $\beta(n) = 0$ for all $n \in \boldsymbol{Z}$.*

Consider the inclusion λ_1: $SU(2n) \to U(2n)$. It gives rise to an element $\lambda_1 \in R(SU(2n))$. Since λ_1 admits a highest weight $\omega_1 = t_1$, we see that $\{t_i;\ i=1, 2, \cdots, 2n\}$ is the set of weights of λ_1. If we write $\lambda_k = \lambda^k(\lambda_1)$, then

(2.3) $$R(SU(2n)) = \boldsymbol{Z}[\lambda_1, \lambda_2, \cdots, \lambda_{2n-1}]$$

(see [8, 13(3.1)]) and s_2^*: $R(SU(2n)) \to R(SU(2n))$ is given by

(2.4) $$s_2^*(\lambda_k) = \lambda_{2n-k} \qquad (k=1, 2, \cdots, 2n-1) .$$

This is equivalent to (1.1), because $\lambda_1, \lambda_2, \cdots, \lambda_{2n-1}$ are the irreducible representations determined by $\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}$ through the fact that each λ_k admits a highest weight ω_k.

Consider the composite $\lambda_1' = \lambda_1 i_2$: $Sp(n) \to U(2n)$. It gives rise to an element $\lambda_1' \in R(Sp(n))$. Since λ_1' admits a highest weight $\omega_1' = t_1'$, we see that $\{\pm t_i';$

$i=1, 2, \cdots, n\}$ is the set of weights of λ_1'. If we write $\lambda_k'=\lambda^k(\lambda_1')$, then

$$R(Sp(n))=\boldsymbol{Z}[\lambda_1', \lambda_2', \cdots, \lambda_n']$$

(see [8, 13(6.1)]) and i_2^*: $R(SU(2n))\rightarrow R(Sp(n))$ is given by

$$i_2^*(\lambda_k)=\lambda_k'=i_2^*(\lambda_{2n-k}) \qquad (k=1, 2, \cdots, n). \tag{2.5}$$

This is equivalent to (1.7) because $\lambda_1', \lambda_2', \cdots, \lambda_n'$ are the irreducible representations determined by $\alpha_1', \alpha_2', \cdots, \alpha_n'$.

For any compact connected Lie group G with torsion-free fundamental group, the $\boldsymbol{Z}/(2)$-graded K-ring of G was determined by Hodgkin [7, Theorem A]. His result is stated as follows: $K^*(G)$ has no torsion and therefore it has the structure of a $\boldsymbol{Z}/(2)$-graded Hopf algebra; if G is semi-simple and $R(G)=\boldsymbol{Z}[\rho_1, \cdots, \rho_l]$ for some representations ρ_i, then $K^*(G)$ is the exterior algebra $\Lambda_{\boldsymbol{Z}}(\beta(\rho_1), \cdots, \beta(\rho_l))$, where the $\beta(\rho_i)$ are primitive. In particular, for $G=SU(n+1)$ we have

$$K^*(SU(n+1))=\Lambda_{\boldsymbol{Z}}(\beta(\lambda_1), \beta(\lambda_2), \cdots, \beta(\lambda_n))$$

(see (2.3) and (3.2)).

Moreover, the Chern character of $SU(n+1)$ was computed in [12] for all $n\geqq 1$. We recall the result. Define a function ϕ: $\boldsymbol{N}\times\boldsymbol{N}\times\boldsymbol{N}\rightarrow\boldsymbol{Z}$ by

$$\phi(n, k, q)=\sum_{i=1}^{k}(-1)^{i-1}\binom{n}{k-i}i^{q-1}. \tag{2.6}$$

Then by [12, (2.2) and Lemma 1] we have

Proposition 2.2. *ch*: $K^*(SU(n+1))\rightarrow H^{**}(SU(n+1); \boldsymbol{Q})$ *is given by*

$$ch(\beta(\lambda_k))=\sum_{i=1}^{n}((-1)^i/i!)\phi(n+1, k, i+1)x_{2i+1} \qquad (k\geqq 1).$$

The K-theory of $SU(2n)/Sp(n)$ was determined by Minami [11]. To state his result, we need some notation. Let G and F be as in section 0. If two representations ρ_1, ρ_2: $G\rightarrow U(n)$ satisfy $\rho_1|F=\rho_2|F$, we have a map f: $G/F\rightarrow U(n)$ defined by $f(xF)=\rho_1(x)\rho_2(x)^{-1}$ for $xF\in G/F$. We denote by $\beta(\rho_1-\rho_2)$ the homotopy class $[\iota_n f]\in[G/F, U]=\tilde{K}^{-1}(G/F)$. If π: $G\rightarrow G/F$ is the projection, as noted in [5, p. 325], π^*: $\tilde{K}^{-1}(G/F)\rightarrow\tilde{K}^{-1}(G)$ satisfies

$$\pi^*(\beta(\rho_1-\rho_2))=\beta(\rho_1)-\beta(\rho_2). \tag{2.7}$$

Applying this construction to the case $G/F=SU(2n)/Sp(n)$, by (2.5) we get elements $\beta(\lambda_k-\lambda_{2n-k})\in\tilde{K}^{-1}(SU(2n)/Sp(n))$ $(k=1, 2, \cdots, n-1)$.

Proposition 2.3. ([11, Proposition 6.1]). *With notation as above,*

$$K^*(SU(2n)/Sp(n))=\Lambda_Z(\beta(\lambda_1-\lambda_{2n-1}), \cdots, \beta(\lambda_{n-1}-\lambda_{n+1})) .$$

Now we are ready to state our main result.

Theorem 2.4. *With notation as in Propositions* 2.3 *and* 2.1, *ch*: $K^*(SU(2n)/Sp(n))\to H^{**}(SU(2n)/Sp(n); \boldsymbol{Q})$ *is given by*

$$ch(\beta(\lambda_k-\lambda_{2n-k}))=\sum_{i=1}^{n-1}(2/(2i)!)\phi(2n, k, 2i+1)e_{4i+1} \qquad (k=1, 2, \cdots, n-1) .$$

Proof. We have

$$\begin{aligned} \pi_2^*(ch(\beta(\lambda_k-\lambda_{2n-k})))&=ch(\pi_2^*(\beta(\lambda_k-\lambda_{2n-k}))) \\ &=ch(\beta(\lambda_k)-\beta(\lambda_{2n-k})) \quad \text{by} \quad (2.7) \\ &=ch(\beta(\lambda_k))-ch(\beta(\lambda_{2n-k})) . \end{aligned}$$

By Proposition 2.2,

$$(2.8) \qquad ch(\beta(\lambda_k))=\sum_{i=1}^{2n-1}((-1)^i/i!)\phi(2n, k, i+1)x_{2i+1}$$

and

$$(2.9) \qquad ch(\beta(\lambda_{2n-k}))=\sum_{i=1}^{2n-1}((-1)^i/i!)\phi(2n, 2n-k, i+1)x_{2i+1} .$$

But

$$\begin{aligned} ch(\beta(\lambda_{2n-k}))&=ch(\beta(s_2^*(\lambda_k))) \qquad \text{by (2.4)} \\ &=s_2^*(ch(\beta(\lambda_k))) \\ &=s_2^*(\sum_{i=1}^{2n-1}((-1)^i/i!)\phi(2n, k, i+1)x_{2i+1}) \qquad \text{by (2.8)} \\ &=\sum_{i=1}^{2n-1}((-1)^i/i!)\phi(2n, k, i+1)(-1)^{i+1}x_{2i+1} \quad \text{by Proposition 1.1} \end{aligned}$$

and so

$$(2.10) \qquad ch(\beta(\lambda_{2n-k}))=-\sum_{i=1}^{2n-1}(1/i!)\phi(2n, k, i+1)x_{2i+1} .$$

Therefore

$$\begin{aligned} &ch(\beta(\lambda_k))-ch(\beta(\lambda_{2n-k})) \\ &=\sum_{i=1}^{2n-1}(((-1)^i+1)/i!)\phi(2n, k, i+1)x_{2i+1} \quad \text{by (2.8) and (2.10)} \end{aligned}$$

$$=\sum_{i=1}^{n-1}(2/(2i)!)\phi(2n, k, 2i+1)x_{4i+1}$$

$$=\pi_2^*(\sum_{i=1}^{n-1}(2/(2i)!)\phi(2n, k, 2i+1)e_{4i+1})$$

since $\pi_2^*(e_{4i+1})=x_{4i+1}$ by Proposition 2.1.

Since π_2^*: $H^*(SU(2n)/Sp(n); \boldsymbol{Q})\to H^*(SU(2n); \boldsymbol{Q})$ is injective by Proposition 2.1 and (1.6), the result follows.

For example, if $n=2$, 3 or 4, the equalities of this theorem are seen to be: if $n=2$,

$$ch(\beta(\lambda_1-\lambda_3))=e_5;$$

if $n=3$,

$$ch(\beta(\lambda_1-\lambda_5))=e_5+(1/12)e_9\,,$$

$$ch(\beta(\lambda_2-\lambda_4))=2e_5-(5/6)e_9\,,$$

if $n=4$,

$$ch(\beta(\lambda_1-\lambda_7))=e_5+(1/12)e_9+(1/360)e_{13}\,,$$

$$ch(\beta(\lambda_2-\lambda_6))=4e_5-(2/3)e_9-(7/45)e_{13}\,,$$

$$ch(\beta(\lambda_3-\lambda_5))=5e_5-(19/12)e_9+(49/72)e_{13}\,.$$

§ 3. The Chern character of $SU(2n+1)/SO(2n+1)$

The cohomology of $SU(2n+1)/SO(2n+1)$ is known (e.g., see [10, Vol. I, Chap. 3, Theorem 6.7, (2) and (3)]).

Proposition 3.1. *Let* $\boldsymbol{k}$ *be a field. Then*

(i) *if the characteristic of* $\boldsymbol{k}$ *is* $p\neq 2$,

$$H^*(SU(2n+1)/SO(2n+1); \boldsymbol{k})=\Lambda_{\boldsymbol{k}}(e_5, e_9, \cdots, e_{4n+1})\,,$$

where $e_{4i+1}\in H^{4i+1}(SU(2n+1)/SO(2n+1); \boldsymbol{k})$;

(ii) *if the characteristic of* $\boldsymbol{k}$ *is* 2,

$$H^*(SU(2n+1)/SO(2n+1); \boldsymbol{k})=\Lambda_{\boldsymbol{k}}(e_2, e_3, \cdots, e_{2n+1})\,,$$

where $e_{i+1}\in H^{i+1}(SU(2n+1)/SO(2n+1); \boldsymbol{k})$ *and*

$$Sq^1(e_{2i})=e_{2i+1},\ Sq^1(e_{2i+1})=0 \quad (i=1, 2, \cdots, n)\,.$$

Thus $H^*(SU(2n+1)/SO(2n+1); \boldsymbol{Z})$ *has* 2*-torsion and there exist elements*

$e_{4i+1}\in H^{4i+1}(SU(2n+1)/SO(2n+1);\ \boldsymbol{Z})$ $(i=1, 2, \cdots, n)$ *such that*

$$H^*(SU(2n+1)/SO(2n+1);\ \boldsymbol{Z})/\mathrm{Tor}=\Lambda_{\boldsymbol{Z}}(e_5, e_9, \cdots, e_{4n+1}).$$

If π_1: $SU(2n+1)\rightarrow SU(2n+1)/SO(2n+1)$ *is the projection,* π_1^*: $H^*(SU(2n+1)/SO(2n+1);\ \boldsymbol{Z})\rightarrow H^*(SU(2n+1);\ \boldsymbol{Z})$ *satisfies* $\pi_1^*(e_{4i+1})=2x_{4i+1}$.

Proof. Consider the Serre spectral sequence $\{E_r, d_r\}$ for the cohomology with R-coefficients of the fibration

$$SU(2n+1)\xrightarrow{\pi_1}SU(2n+1)/SO(2n+1)\xrightarrow{j_1}BSO(2n+1)$$

induced by Bi_1: $BSO(2n+1)\rightarrow BSU(2n+1)$. If $R=\boldsymbol{k}$ is a field of characteristic $p\neq 2$, by (1.25) and (1.19),

$$E_2=H^*(BSO(2n+1);\ \boldsymbol{k})\otimes H^*(SU(2n+1);\ \boldsymbol{k})$$
$$=\boldsymbol{k}[p_1, p_2, \cdots, p_n]\otimes\Lambda_{\boldsymbol{k}}(x_3, x_5, \cdots, x_{4n+1}).$$

Since each $x_{2i+1}\in H^{2i+1}(SU(2n+1);\ \boldsymbol{k})$ transgresses to $c_{i+1}\in H^{2i+2}(BSU(2n+1);\ \boldsymbol{k})$ in the Serre spectral sequence of the universal $SU(2n+1)$-bundle, it follows from (1.24) that

$$d_{4i}(1\otimes x_{4i-1})=(-1)^i p_i\otimes 1\quad (i=1, 2, \cdots, n),$$
$$d_r(1\otimes x_{4i+1})=0\quad (i=1, 2, \cdots, n;\ r\geqq 2).$$

Then a routine spectral sequence argument yields that $E_\infty=\Lambda_{\boldsymbol{k}}(x_5, x_9, \cdots, x_{4n+1})$. So there exist elements $e_{4i+1}\in H^{4i+1}(SU(2n+1)/SO(2n+1);\ \boldsymbol{k})$ such that $\pi_1^*(e_{4i+1})=x_{4i+1}$. Hence (i) follows.

If $R=\boldsymbol{k}$ is a field of characteristic 2, by [3, § 30] and (1.19),

$$E_2=H^*(BSO(2n+1);\ \boldsymbol{k})\otimes H^*(SU(2n+1);\ \boldsymbol{k})$$
$$=\boldsymbol{k}[w_2, w_3, \cdots, w_{2n}, w_{2n+1}]\otimes\Lambda_{\boldsymbol{k}}(x_3, x_5, \cdots, x_{4n+1}),$$

where $w_{i+1}\in H^{i+1}(BSO(2n+1);\ \boldsymbol{k})$ and

(3.1) $$Sq^1(w_{2i})=w_{2i+1},\quad Sq^1(w_{2i+1})=0\quad (i=1, 2, \cdots, n).$$

Since Bi_1^*: $H^*(BSU(2n+1);\ \boldsymbol{k})\rightarrow H^*(BSO(2n+1);\ \boldsymbol{k})$ satisfies $Bi_1^*(c_{i+1})=w_{i+1}^2$ (see [10, Vol. I, Chap. 3]), it follows that

$$d_{2i+2}(1\otimes x_{2i+1})=w_{i+1}^2\otimes 1\quad (i=1, 2, \cdots, 2n).$$

Then a routine spectral sequence argument yields that $E_\infty=\Delta_{\boldsymbol{k}}(w_2, w_3, \cdots, w_{2n+1})$, where $\Delta_{\boldsymbol{k}}$ denotes the $\boldsymbol{k}$-algebra having a simple system of generators. So there exist elements $e_{i+1}\in H^{i+1}(SU(2n+1)/SO(2n+1);\ \boldsymbol{k})$ such that $j_1^*(w_{i+1})=e_{i+1}$. From this and (3.1) we deduce the last two equalities of (ii).

Since the composite $(Bi_1)j_1$ is null homotopic, we have

$$0=j_1^*Bi_1^*(c_{i+1})=j_1^*(w_{i+1}^2)=e_{i+1}^2 .$$

Hence the remaining part of (ii) follows.

The above facts imply that if $R=\boldsymbol{Z}$, then each $x_{4i-1}\in H^{4i-1}(SU(2n+1);\boldsymbol{Z})$ transgresses to a generator of a summand $\boldsymbol{Z}$ in $H^{4i}(BSO(2n+1);\boldsymbol{Z})$; each $x_{4i+1}\in H^{4i+1}(SU(2n+1);\boldsymbol{Z})$ transgresses to a generator of a summand $\boldsymbol{Z}/(2)$ in $H^{4i+2}(BSO(2n+1);\boldsymbol{Z})$; and $2x_{4i+1}\in H^{4i+1}(SU(2n+1);\boldsymbol{Z})$ survives to E_∞. This proves the existence of an element $e_{4i+1}\in H^{4i+1}(SU(2n+1)/SO(2n+1);\boldsymbol{Z})$ such that $\pi_1^*(e_{4i+1})=2x_{4i+1}$, and the rest follows from (i) and (ii).

Consider the inclusion λ_1: $SU(2n+1)\to U(2n+1)$. It gives rise to an element $\lambda_1\in R(SU(2n+1))$. Since λ_1 admits a highest weight $\omega_1=t_1$, we see that $\{t_i;\ i=1,2,\cdots,2n+1\}$ is the set of weights of λ_1. If we write $\lambda_k=\lambda^k(\lambda_1)$, then

(3.2) $$R(SU(2n+1))=\boldsymbol{Z}[\lambda_1,\lambda_2,\cdots,\lambda_{2n}]$$

(see [8, 13(3.1)]) and s_1^*: $R(SU(2n+1))\to R(SU(2n+1))$ is given by

(3.3) $$s_1^*(\lambda_k)=\lambda_{2n+1-k} \qquad (k=1,2,\cdots,2n) .$$

This is equivalent to (1.14).

Consider the composite $\lambda_1'=\lambda_1 i_1$: $SO(2n+1)\to U(2n+1)$. It gives rise to an element $\lambda_1'\in R(SO(2n+1))$. Since λ_1' admits a highest weight $\omega_1'=t_1'$, we see that $\{\pm t_i', 0;\ i=1,2,\cdots,n\}$ is the set of weights of λ_1'. If we write $\lambda_k'=\lambda^k(\lambda_1')$, then

(3.4) $$R(SO(2n+1))=\boldsymbol{Z}[\lambda_1',\lambda_2',\cdots,\lambda_n']$$

(see [8, 13(10.3)]) and i_1^*: $R(SU(2n+1))\to R(SO(2n+1))$ is given by

(3.5) $$i_1^*(\lambda_k)=\lambda_k'=i_1^*(\lambda_{2n+1-k}) \qquad (k=1,2,\cdots,n) .$$

This is equivalent to (1.20).

The K-theory of $SU(2n+1)/SO(2n+1)$ was also determined by Minami [11]. Applying the previous construction to the case $G/F=SU(2n+1)/SO(2n+1)$, by (3.5) we get elements $\beta(\lambda_k-\lambda_{2n+1-k})\in\tilde{K}^{-1}(SU(2n+1)/SO(2n+1))$ $(k=1,2,\cdots,n)$.

Proposition 3.2 ([11, Proposition 8.1]). *With notation as above,*

$$K^*(SU(2n+1)/SO(2n+1))=\Lambda_Z(\beta(\lambda_1-\lambda_{2n}),\cdots,\beta(\lambda_n-\lambda_{n+1})) .$$

Now we are ready to state our main result.

Theorem 3.3. *With notation as in Propositions* 3.2 *and* 3.1, *ch*: $K^*(SU(2n+1)/SO(2n+1))\to H^{**}(SU(2n+1)/SO(2n+1);\boldsymbol{Q})$ *is given by*

$$ch(\beta(\lambda_k-\lambda_{2n+1-k}))=\sum_{i=1}^{n}(1/(2i)!)\phi(2n+1,\ k,\ 2i+1)e_{4i+1}$$

$$(k=1,\ 2,\ \cdots,\ n)\ .$$

Proof. We have

$$\begin{aligned}\pi_1^*(ch(\beta(\lambda_k-\lambda_{2n+1-k})))&=ch(\pi_1^*(\beta(\lambda_k-\lambda_{2n+1-k})))\\&=ch(\beta(\lambda_k)-\beta(\lambda_{2n+1-k}))\quad\text{by (2.7)}\\&=ch(\beta(\lambda_k))-ch(\beta(\lambda_{2n+1-k}))\ .\end{aligned}$$

By Proposition 2.2,

(3.6) $$ch(\beta(\lambda_k))=\sum_{i=1}^{2n}((-1)^i/i!)\phi(2n+1,\ k,\ i+1)x_{2i+1}$$

and

(3.7) $$ch(\beta(\lambda_{2n+1-k}))=\sum_{i=1}^{2n}((-1)^i/i!)\phi(2n+1,\ 2n+1-k,\ i+1)x_{2i+1}\ .$$

But

$$\begin{aligned}&ch(\beta(\lambda_{2n+1-k}))=ch(\beta(s_1^*(\lambda_k)))\qquad\text{by (3.3)}\\&=s_1^*(ch(\beta(\lambda_k)))\\&=s_1^*(\sum_{i=1}^{2n}((-1)^i/i!)\phi(2n+1,\ k,\ i+1)x_{2i+1})\qquad\text{by (3.6)}\\&=\sum_{i=1}^{2n}((-1)^i/i!)\phi(2n+1,\ k,\ i+1)(-1)^{i+1}x_{2i+1}\qquad\text{by Proposition 1.3}\end{aligned}$$

and so

(3.8) $$ch(\beta(\lambda_{2n+1-k}))=-\sum_{i=1}^{2n}(1/i!)\phi(2n+1,\ k,\ i+1)x_{2i+1}\ .$$

Therefore

$$\begin{aligned}&ch(\beta(\lambda_k))-ch(\beta(\lambda_{2n+1-k}))\\&=\sum_{i=1}^{2n}(((-1)^i+1)/i!)\phi(2n+1,\ k,\ i+1)x_{2i+1}\qquad\text{by (3.6) and (3.8)}\\&=\sum_{i=1}^{n}(2/(2i)!)\phi(2n+1,\ k,\ 2i+1)x_{4i+1}\\&=\pi_1^*(\sum_{i=1}^{n}(1/(2i)!)\phi(2n+1,\ k,\ 2i+1)e_{4i+1})\end{aligned}$$

since $\pi_1^*(e_{4i+1})=2x_{4i+1}$ by Proposition 3.1.

Since π_1^*: $H^*(SU(2n+1)/SO(2n+1); \boldsymbol{Q})\to H^*(SU(2n+1); \boldsymbol{Q})$ is injective by Proposition 3.1 and (1.19), the result follows.

For example, if $n=1$, 2 or 3, the equalities of this theorem are seen to be: if $n=1$,

$$ch(\beta(\lambda_1-\lambda_2))=(1/2)e_5\,;$$

if $n=2$,

$$ch(\beta(\lambda_1-\lambda_4))=(1/2)e_5+(1/24)e_9\,,$$

$$ch(\beta(\lambda_2-\lambda_3))=(1/2)e_5-(11/24)e_9\,,$$

if $n=3$,

$$ch(\beta(\lambda_1-\lambda_6))=(1/2)e_5+(1/24)e_9+(1/720)e_{13}\,,$$

$$ch(\beta(\lambda_2-\lambda_5))=(3/2)e_5-(3/8)e_9-(19/240)e_{13}\,,$$

$$ch(\beta(\lambda_3-\lambda_4))=e_5-(5/12)e_9+(151/360)e_{13}\,,$$

By comparing (2.9) with (2.10), we find that the relation $\phi(2n, 2n-k, i+1)=(-1)^{i+1}\phi(2n, k, i+1)$ holds for $i, k=1, 2, \cdots, 2n-1$. By comparing (3.7) with (3.8), we also find that the relation $\phi(2n+1, 2n+1-k, i+1)=(-1)^{i+1}\phi(2n+1, k, i+1)$ holds for i, $k=1, 2, \cdots, 2n$. Summing up, by means of topology we have shown that

(3.9) *the relation* $\phi(n+1, n+1-k, i+1)=(-1)^{i+1}\phi(n+1, k, i+1)$ *holds for* $i, k=1, 2, \cdots, n$.

In view of Proposition 2.2, this relation expresses a symmetry in a description of *ch* of $SU(n+1)$ (see [12, Theorem 2]).

There is another curious relation concerning the function ϕ of (2.6). In $R(SU(n+1))=\boldsymbol{Z}[\lambda_1, \lambda_2, \cdots, \lambda_n]$ we have $\lambda_{n+1}=1$ and $\lambda_k=0$ for $k>n+1$. From this and (2.2), $\beta(\lambda_k)=0$ for $k\geqq n+1$. Combining this with Proposition 2.2, we find that

(3.10) *the relation* $\phi(n+1, k, i+1)=0$ *holds for* $k\geqq n+1$ *and* $i=1, 2, \cdots, n$.

Of course, it is not immediate to deduce (3.9) and (3.10) directly from (2.6). The author would like to thank Drs. Shin-ichiro Hara, Susumu Kono and Jun Murakami who taught various such proofs independently.

§ 4. The Chern character of $SO(2n+1)$

Since $H^*(SO(2n+1); \boldsymbol{Q})$ is an exterior algebra generated by primitive elements $x'_{4i-1} \in H^{4i-1}(SO(2n+1); \boldsymbol{Q})$ $(i=1, 2, \cdots, n)$, using the Poincaré duality, we can take elements $x'_{4i-1} \in H^{4i-1}(SO(2n+1); \boldsymbol{Z})$ such that

(4.1) $$H^*(SO(2n+1); \boldsymbol{Z})/\mathrm{Tor} = \Lambda_Z(x'_3, x'_7, \cdots, x'_{4n-1})$$

and the image of each x'_{4i-1} under the coefficient group homomorphism $H^{4i-1}(SO(2n+1); \boldsymbol{Z}) \to H^{4i-1}(SO(2n+1); \boldsymbol{Q})$ is primitive.

Proposition 4.1. *With notation as in* (1.19) *and* (4.1), i_1^*: $H^*(SU(2n+1); \boldsymbol{Z}) \to H^*(SO(2n+1); \boldsymbol{Z})$ *is given by*

$$i_1^*(x_{4i-1}) = (-1)^i 2x'_{4i-1} \qquad (i=1, 2, \cdots, n),$$

$$i_1^*(x_{4i+1}) = 0 \qquad (i=1, 2, \cdots, n).$$

Proof. Consider the Serre spectral sequence $\{E_r, d_r\}$ for the cohomology with R-coefficients of the fibration

$$SO(2n+1) \xrightarrow{i_1} SU(2n+1) \xrightarrow{\pi_1} SU(2n+1)/SO(2n+1)$$

induced by j_1: $SU(2n+1)/SO(2n+1) \to BSO(2n+1)$. If $R=\boldsymbol{k}$ is a field of characteristic $p \neq 2$, by Proposition 3.1(i) and (1.26),

$$E_2 = H^*(SU(2n+1)/SO(2n+1); \boldsymbol{k}) \otimes H^*(SO(2n+1); \boldsymbol{k})$$

$$= \Lambda_{\boldsymbol{k}}(e_5, e_9, \cdots, e_{4n+1}) \otimes \Lambda_{\boldsymbol{k}}(x'_3, x'_7, \cdots, x'_{4n-1}).$$

Since each $x'_{4i-1} \in H^{4i-1}(SO(2n+1); \boldsymbol{k})$ transgresses to $p_i \in H^{4i}(BSO(2n+1); \boldsymbol{k})$ in the Serre spectral sequence of the universal $SO(2n+1)$-bundle and $j_1^*(p_i) = 0$ (see the proof of Proposition 3.1(i)), it follows that

$$d_r(1 \otimes x'_{4i-1}) = 0 \qquad (i=1, 2, \cdots, n;\ r \geqq 2)$$

and hence $E_2 = E_\infty$.

If $R=\boldsymbol{k}$ is a field of characteristic 2, by Proposition 3.1 (ii) and [2],

$$E_2 = H^*(SU(2n+1)/SO(2n+1); \boldsymbol{k}) \otimes H^*(SO(2n+1); \boldsymbol{k})$$

$$= \Lambda_{\boldsymbol{k}}(e_2, e_3, \cdots, e_{2n+1}) \otimes \Delta_{\boldsymbol{k}}(x'_1, x'_2, \cdots, x'_{2n}).$$

where $x'_i \in H^i(SO(2n+1); \boldsymbol{k})$. Since each $x'_i \in H^i(SO(2n+1); \boldsymbol{k})$ transgresses to $w_{i+1} \in H^{i+1}(BSO(2n+1); \boldsymbol{k})$ in the Serre spectral sequence of the universal $SO(2n+1)$-bundle and $j_1^*(w_{i+1}) = e_{i+1}$ (see the proof of Proposition 3.1(ii)), it follows that

(4.2) $$d_{i+1}(1\otimes x'_i)=e_{i+1}\otimes 1 \qquad (i=1, 2, \cdots, 2n).$$

From (3.1) we also have

(4.3) $$Sq^1(x'_{2i-1})=x'_{2i}, \quad Sq^1(x'_{2i})=0 \qquad (i=1, 2, \cdots, n).$$

Let ρ: $H^*(SO(2n+1); \boldsymbol{Z})\to H^*(SO(2n+1); \boldsymbol{Z}/(2))$ be the coefficient group homomorphism induced by reduction mod 2. Using (4.2) and (4.3) we observe that for $i=1, 2, \cdots, n$

$$\rho(x'_{4i-1})=\begin{cases} x'_{2i-1}x'_{2i}-x'_{4i-1} & \text{if } 4i-1\leqq 2n \\ x'_{2i-1}x'_{2i} & \text{if } 4i-1>2n \end{cases}$$

and

$$d_{2i}(1\otimes x'_{2i-1}x'_{2i})=e_{2i}\otimes x'_{2i}$$
$$=(1\otimes Sq^1)(e_{2i}\otimes x'_{2i-1}).$$

Then a routine spectral sequence argument yields that

$$E_\infty=\Lambda_k(e_2\otimes x'_1, e_3\otimes x'_2, \cdots, e_{2n+1}\otimes x'_{2n}).$$

The above facts imply that if $R=\boldsymbol{Z}$, then for $i=1, \cdots, n$ d_{2i} sends $x'_{4i-1}\in E_2^{0,4i-1}$ to a generator, which is represented by $e_{2i}\otimes x'_{2i-1}$, of a summand $\boldsymbol{Z}/(2)$ in $E_2^{2i,2i}$; and $2x'_{4i-1}$ survives to E_∞. This proves that $i_1^*(x_{4i-1})=2x'_{4i-1}$ up to sign. (Here we put the sign $(-1)^i$ on the right side of this equality for fitting it to suit the first equality of (1.24).)

The second equality is obvious for dimensional reasons.

The K-theory of $SO(2n+1)$ was determined by Held and Suter [6, Satz (5.15)]. We recall their result. The spinor group Spin$(2n+1)$ appears as the universal covering group of $SO(2n+1)$. Let p: Spin$(2n+1)\to SO(2n+1)$ be the two-fold covering projection. Consider the composite $\lambda_1=\lambda'_1 p$: Spin$(2n+1)\to U(2n+1)$. It gives rise to an element $\lambda_1\in R(\text{Spin}(2n+1))$. Write $\lambda_k=\lambda^k(\lambda_1)$ and let Δ_{2n+1}: Spin$(2n+1)\to U(2^n)$ be the spin representation. Then

$$R(\text{Spin}(2n+1))=\boldsymbol{Z}[\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \Delta_{2n+1}],$$

where the relation

(4.4) $$\Delta_{2n+1}{}^2=\lambda_n+\lambda_{n-1}+\cdots\lambda_1+1$$

holds (see [8, 13(10.3)]). From this, by the theorem of Hodgkin [7] we have

$$K^*(\text{Spin}(2n+1))=\Lambda_{\boldsymbol{Z}}(\beta(\lambda_1), \cdots, \beta(\lambda_{n-1}), \beta(\Delta_{2n+1})).$$

With this notation (and (3.4)), they showed that there are two extra elements $\varepsilon_{2n+1}\in K^{-1}(SO(2n+1))$ and $\xi_{2n+1}\in K^0(SO(2n+1))$ such that

$$K^*(SO(2n+1))=[\Lambda_Z(\beta(\lambda'_1), \cdots, \beta(\lambda'_{n-1}), \varepsilon_{2n+1})\otimes T_{2n+1}]/(\varepsilon_{2n+1}\otimes\xi_{2n+1}),$$

where $T_{2n+1}=\boldsymbol{Z}\{1\}\oplus\boldsymbol{Z}/(2^n)\{\xi_{2n+1}\}$, and that p^*: $K^*(SO(2n+1))\to K^*$ (Spin$(2n+1)$) satisfies

(4.5) $\quad p^*(\beta(\lambda'_k))=\beta(\lambda_k) \qquad (k\geqq 1)$

and

(4.6) $\quad p^*(\varepsilon_{2n+1})=2\beta(\Delta_{2n+1})$.

Thus we have

Proposition 4.2 ([6, Korollar (2.10)]). *With notation as above,*

$$K^*(SO(2n+1))/\mathrm{Tor}=\Lambda_Z(\beta(\lambda'_1), \cdots, \beta(\lambda'_{n-1}), \varepsilon_{2n+1}).$$

Lemma 4.3. *In* $K^*(SO(2n+1))/\mathrm{Tor}$ *the following relation holds*:

$$2^n\varepsilon_{2n+1}=\sum_{k=1}^{n}\beta(\lambda'_k).$$

Proof. We have

$$\begin{aligned} p^*(2^n\varepsilon_{2n+1}) &= 2^{n+1}\beta(\Delta_{2n+1}) && \text{by (4.6)} \\ &= \beta(\Delta_{2n+1}{}^2) && \text{by using (2.1)} \\ &= \beta(\sum_{k=0}^{n}\lambda_k) && \text{by (4.4)} \\ &= \sum_{k=0}^{n}\beta(\lambda_k) && \text{since } \beta \text{ is additive} \\ &= \sum_{k=1}^{n}\beta(\lambda_k) && \text{by (2.2)} \\ &= \sum_{k=1}^{n}p^*(\beta(\lambda'_k)) && \text{by (4.5)} \\ &= p^*(\sum_{k=1}^{n}\beta(\lambda'_k)). \end{aligned}$$

Since p^* clearly gives an injection $K^*(SO(2n+1))/\mathrm{Tor}\to K^*(\mathrm{Spin}(2n+1))$, the result follows.

Since the range of ch: $K^*(SO(2n+1))\to H^{**}(SO(2n+1);\ \boldsymbol{Q})$ is a vector space over $\boldsymbol{Q}$, it factors to give

$$ch\colon K^*(SO(2n+1))/\mathrm{Tor}\to H^{**}(SO(2n+1);\ \boldsymbol{Q}).$$

Theorem 4.4. *With notation as in Proposition* 4.2 *and* (4.1), *ch:*

$K^*(SO(2n+1))/\mathrm{Tor} \to H^{**}(SO(2n+1); \boldsymbol{Q})$ *is given by*

$$ch(\beta(\lambda'_k)) = \sum_{i=1}^{n}((-1)^{i-1}2/(2i-1)!)\phi(2n+1, k, 2i)x'_{4i-1}$$

$$(k=1, 2, \cdots, n-1),$$

$$ch(\varepsilon_{2n+1}) = \sum_{i=1}^{n}((-1)^{i-1}2/(2i-1)!)((1/2^n)\sum_{k=1}^{n}\phi(2n+1, k, 2i))x'_{4i-1}.$$

Proof. Apply i_1^*: $H^*(SU(2n+1); \boldsymbol{Q}) \to H^*(SO(2n+1); \boldsymbol{Q})$ to the equality (3.6). Then the left hand side is

$$i_1^*(ch(\beta(\lambda_k))) = ch(\beta(i_1^*(\lambda_k))) = ch(\beta(\lambda'_k)) \qquad \text{by (3.5)}$$

and the right hand side is

$$i_1^*(\sum_{i=1}^{2n}((-1)^i/i!)\phi(2n+1, k, i+1)x_{2i+1})$$

$$= \sum_{i=1}^{n}((-1)^{2i-1}/(2i-1)!)\phi(2n+1, k, 2i)(-1)^i 2x'_{4i-1} \text{ by Proposition 4.1}$$

$$= \sum_{i=1}^{n}((-1)^{i-1}2/(2i-1)!)\phi(2n+1, k, 2i)x'_{4i-1}.$$

This proves the first equality.

By using Lemma 4.3, the second equality is obtained from the first.

For example, if $n=1$, 2 or 3, the equalities of this theorem are seen to be: if $n=1$,

$$ch(\varepsilon_3) = x'_3;$$

if $n=2$,

$$ch(\beta(\lambda'_1)) = 2x'_3 - (1/3)x'_7,$$

$$ch(\varepsilon_5) = 2x'_3 + (1/6)x'_7;$$

if $n=3$,

$$ch(\beta(\lambda'_1)) = 2x'_3 - (1/3)x'_7 + (1/60)x'_{11},$$

$$ch(\beta(\lambda'_2)) = 10x'_3 + (1/3)x'_7 - (5/12)x'_{11},$$

$$ch(\varepsilon_7) = 4x'_3 + (1/3)x'_7 + (1/30)x'_{11}.$$

Department of Applied Mathematics
Osaka Women's University

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