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Normal form of systems of partial differential and pseudo-differential operators in formal symbol classes

Dedicated to Professor Teruo IKEBE on his sixtieth birthday

By

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§ 0. Introduction

On scalar and higher order partial differential operators (or pseudo-differential operators) with characteristic roots of constant multiplicity, we know a normal form which brings a satisfactory consideration of those structures:

$$
\begin{aligned}
(0.0)\qquad p(t, x, D_t, D_x) &\equiv D_t^m + \sum_{\substack{|\alpha| \le \nu k\\ 1\le k\le m}} a_{\alpha k}(t, x) D_x^{\alpha} D_t^{m-k} \\
&= p^1(t, x, D_t, D_x)\circ\cdots\circ p^d(t, x, D_t, D_x) \mod S^{-\infty}[D_t],\\
p^j(t, x, D_t, D_x) &= (D_t - \lambda_j(t, x, D_x))^{\circ m_j}\\
&\quad + \sum_{k=1}^{m_j} b_k(t, x, D_x)\circ(D_t - \lambda_j(t, x, D_x))^{\circ(m_j-k)},\\
&\quad \operatorname{ord} b_k \le \nu k - 1, \qquad (\nu\in \boldsymbol{N}),
\end{aligned}
$$

where $\circ$ means the operator product. (See H. Kumano-go [10] and T. Nishitani [25].) In the above, the principal part of p^j is only $(D_t - \lambda_j(t, x, D_x))^{\circ m_j}$, where $\lambda_j(t, x, \xi)$ is positively homogeneous of order ν. Using this normal form, H. Kumano-go [10] and S. Mizohata [24] characterized the C^∞ well-posedness of the Cauchy problem on $p(t, x, D_t, D_x)$. We remark that the above normal form corresponds to the following system:

$$
(0.0')\qquad \begin{pmatrix} Q^1 & & & \\ & Q^2 & & \\ & & \ddots & \\ & & & Q^d \end{pmatrix},
$$

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$$Q^j = I_{m_j}(D_t - \lambda_j(t, x, D_x)) - D^j(t, x, D_x)\,; \quad m_j \times m_j\,,$$

$$D^j(t, x, \xi) = D_0{}^j(t, x, \xi) + D'^j(t, x, \xi)\,,$$

$$D_0{}^j(t, x, \xi) = J(m_j)|\xi|^\nu\,,$$

$$D'^j = \begin{pmatrix} & O & \\ * & \cdots\cdots & * \end{pmatrix}\,; \quad \textit{order} \quad \nu - 1\,,$$

where

$$J(s) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}\,; \qquad s \times s\,.$$

The principal part of the above system is $\oplus_{1 \le j \le d}\{I_{m_j}(D_t - \lambda_j(t, x, D_x)) - J(m_j)|D_x|^\nu\}$ and it is the Jordan normal form consisting of only one block for each eigenvalue. The lower order term $\oplus_{1 \le j \le d} D'^j(t, x, D_x)$ is a matrix of Sylvester type corresponding to the principal part. (See also K. Kajitani [5].)

On the other hand, on systems of partial differential operators (or ps.d.op.' s) with characteristic roots of constant multiplicity, we have not yet such simple and fructuous normal form. We know a normal form of Arnold type given by V. M. Petkov. (See V.I. Arnold [2], V.M.Petkov [26] and Theorem 2.6 in this paper.) However, it seems that we need more simple normal form in order to understand those structures.

In applications, we need 4 categories of differential operators and, corresponding to them, ω and Γ have different meanings.

1. Differential operators acting on holomorphic functions.
   In this case, ω is an open set in $\boldsymbol{C}_t{}^1 \times \boldsymbol{C}_x{}^\ell$ and Γ is an open conic set in $\boldsymbol{C}_\xi{}^\ell$.
2. Differential operators acting on real analytic functions.
   In this case, ω is an open set in $\boldsymbol{R}_t{}^1 \times \boldsymbol{R}_x{}^\ell$ and Γ is an open conic set in $\boldsymbol{R}_\xi{}^\ell$. It becomes very important to introduce an open complex neighborhood $\hat{\omega}$ of ω and an open conic complex neighborhood $\hat{\Gamma} = \{\xi \in \boldsymbol{C}^\ell;\ |\mathrm{Im}\xi| < \varepsilon|\mathrm{Re}\xi|, \mathrm{Re}\xi \in \Gamma\}$ of Γ. (ε is a positive constant.)
3. Differential operators acting on ultradifferentiable functions.
4. Differential operators acting on functions of C^∞ class.
   In cases of 3 and 4, ω is an open set in $\boldsymbol{R}_t{}^1 \times \boldsymbol{R}_x{}^\ell$ and Γ is an open conic set in $\boldsymbol{R}_\xi{}^\ell$.

Here, we say that Γ is conic when $O \notin \Gamma$ and $\xi \in \Gamma$ implies $\rho\xi \in \Gamma$ for arbitrary $\rho > 0$. Further, we often say that $\tilde{\Omega} \subset \boldsymbol{R}_t{}^1 \times \boldsymbol{R}_x{}^\ell \times \boldsymbol{R}_\xi{}^\ell$ (or $\boldsymbol{C}_t{}^1 \times \boldsymbol{C}_x{}^\ell \times \boldsymbol{C}_\xi{}^\ell$) is conic when $(t, x, O) \notin \tilde{\Omega}$ and $(t, x, \xi) \in \tilde{\Omega}$ implies $(t, x, \rho\xi) \in \tilde{\Omega}$ for arbitrary $\rho > 0$.

We consider $K\times K$ systems of differential or classical pseudo-differential operators of order ν ($\nu\in\boldsymbol{R}$, $\nu\geq 1$) on $\omega\times\Gamma$;

(0.1) $$P(t, x, D_t, D_x) = I_K D_t - A(t, x, D_x),$$

$$A(t, x, \xi)\sim\sum_{i=0}^{\infty}A_i(t, x, \xi),$$

$A_i(t, x, \xi)$; *(positively) homogeneous of order* $\nu-i$,

$(i\in\boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$,

in case of differential operator, $\nu\in\boldsymbol{N}$ and $0\leq i\leq\nu)$.

Throughout § 0 to § 3, we assume the following;

Assumption 0. The characteristic polynomial of $P(t, x, D_t, D_x)$ on λ: $\det(\lambda I_K-A_0(t, x, \xi))$ has roots $\lambda_j(t, x, \xi)$ of constant multiplicity m_j on $\omega\times\Gamma\backslash\{O\}$ ($1\leq j\leq d$, $\sum_{j=1}^{d}m_j=K$), that is, $\det(\lambda I_K-A_0(t, x, \xi))$ is decomposed on $\omega\times\Gamma\backslash\{O\}$ as $\prod_{j=1}^{d}(\lambda-\lambda_j(t, x, \xi))^{m_j}$, where $\lambda_j(t, x, \xi)\neq\lambda_{j'}(t, x, \xi)$ if $j\neq j'$ and m_j are constant natural numbers.

Remark 0.1. We shall introduce matrices with entries of "meromorphic formal symbols". For them, Assumption 0 will be applied to formal sums of type $I_K D_t-\sum_{i=0}^{\infty}A_i(t, x, \xi)$ and satisfied except on poles.

Remark 0.2. When $\omega\times\Gamma$ is not simply connected, in general, $\lambda_j(t, x, \xi)$ becomes multi-valued even if $A(t, x, D_x)$ is a differential operator. In order to give the role of true symbol of ps.d.op. to $\lambda_j(t, x, \xi)$, we need decompose the domain to some simply connected ones. However, in this paper, we treat it as a global multi-valued function, because we need only formal calculus.

Remark 0.3. Even if $\omega\times\Gamma$ is not simply connected, when every $\lambda_j(t, x, \xi)$ is real, it is single-valued and smooth under Assumption 0.

Remark 0.4. Under Assumption 0, every $\lambda_j(t, x, \xi)$ is as smooth as $A_0(t, x, \xi)$. (See Proposition 2.1.).

When we transform systems, we accept only similar transformations

(0.2) $$N^{-1}(t, x, D_x)\circ P(t, x, D_t, D_x)\circ N(t, x, D_x)$$

in order to keep the form of $I_K D_t$, where $N^{-1}(t, x, D_x)$ is the inverse of $N(t, x, D_x)$ as the matrix of operators and $A\circ B$ means the product of A and B as the (matrix of) operators. (We denote the inverse of $N_0(t, x, \xi)$ as the matrix of functions by $(N_0(t, x, \xi))^{-1}$.) We call $N(t, x, D_x)$ a transforming operator.

As a normal form of the principal part, the Jordan normal form is natural. Here, since the commutator does not take part in the principal part, the principal part is transformed by the same rule as the matrix of functions. Can we transform it to its Jordan normal form by a smooth and regular matrix $N_0(t, x, \xi)$? The answer is "No".

Example 0.1. $\ell=1$.

$$A_0(t, x, \xi)/\xi^\nu = \begin{pmatrix} tx & -t^2 \\ x^2 & -tx \end{pmatrix}. \tag{0.3}$$

$A_0(t, x, \xi)$ is nilpotent and then it satisfies Assumption 0. However, to transform the above $A_0(t, x, \xi)$ to a triangular matrix, we need a singular matrix. If we take $N_0(t, x)=\begin{pmatrix} t & 0 \\ x & 1 \end{pmatrix}$ or $N_1(t, x)=\begin{pmatrix} t & 1 \\ x & 0 \end{pmatrix}$, we have

$$(N_0)^{-1}A_0N_0 = -t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\xi^\nu, \tag{0.4}$$

$$(N_1)^{-1}A_0N_1 = x\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\xi^\nu. \tag{0.4'}$$

In the former case, N_0 is regular except on $\{t=0\}$ and in the latter case, N_1 is so except on $\{x=0\}$. In either case, we need accept an exceptional set for the regularity of matrix. If we take $\tilde{N}_0=\begin{pmatrix} 1 & 0 \\ x/t & 1 \end{pmatrix}$, we can keep the regularity of matrix. However, we lose the smoothness on $\{t=0\}$. We cannot keep both of the regularity and smoothness. (See also W. Matsumoto [12, I].) On the other hand, if we consider $A_0(t, x, \xi)$, $N_0(t, x)$ and $N_1(t, x)$ as the matrices with entries of meromorphic functions, $N_0(t, x)$ and $N_1(t, x)$ become regular and (0.4) and (0.4′) hold in such class. These are the reasons why we shall accept an exceptional set for general case and why we shall introduce meromorphic symbols for the holomorphic case and the real analytic case. Further, when we transform systems of the form by Petkov to more simple form, these ideas —the exceptional sets and the meromorphic symbols— again play essential role.

We give our main theorem, which we reformulate in § 3.

Theorem (Normal form of systems = Perfect block diagonalization). *If $P(t, x, D_t, D_x)$ in (0.1) satisfies Assumption 0, there exists an open conic dense subset $\tilde{\omega}$ of $\omega\times\Gamma$. For an arbitrary $(t_\circ, x_\circ, \xi_\circ)$ in $\tilde{\omega}$, there exist a conic neighborhood $\omega_\circ\times\Gamma_\circ$, $\{\delta_j\}_{1\le j\le d}$, $\{n_{jk}\}_{1\le j\le d, 1\le k\le\delta_j}$, ($\delta_j$, $n_{jk}\in\mathbf{N}$, $\sum_{k=1}^{\delta_j} n_{jk}=m_j$) and a matrix $N(t, x, D_x)$ invertible as the matrix of ps.d.op.'s on $\omega_\circ\times\Gamma_\circ$ such that*

(0.5) *$N^{-1}\circ P\circ N$ has the following form modulo $S^{-\infty}$.*

$$\begin{bmatrix} Q^{1,1} & & & & & & \\ & Q^{1,2} & & & & & \\ & & \ddots & & & & \\ & & & Q^{1,\delta_1} & & & \\ & & & & Q^{2,1} & & \\ & & & & & \ddots & \\ & & & & & & Q^{d,\delta_d} \end{bmatrix}$$

$$Q^{jk} = I_{n_{jk}}(D_t-\lambda_j(t,x,D_x))-D^{jk}(t,x,D_x)\,; \quad n_{jk}\times n_{jk}\,,$$

$$D^{jk}(t,x,\xi) = D_0^{jk}(t,x,\xi)+D'^{jk}(t,x,\xi)\,,$$

$$D_0^{jk}(t,x,\xi) = J(n_{jk})|\xi|^{\nu}\,,$$

$$D'^{jk} = \begin{pmatrix} O \\ *\cdots\cdots * \end{pmatrix}; \quad \text{order } \nu-1\,.$$

Further, if the symbol of $P(t,x,D_t,D_x)$ is holomorphic (*Case* 1) *or real analytic* (*Case* 2), *$\Sigma=\omega\times\Gamma\backslash\tilde{\omega}$ is an analytic set, $\{\delta_j\}_{1\le j\le d}$, $\{n_{jk}\}_{1\le j\le d,1\le k\le\delta_j}$ and $N(t,x,\xi)$ are taken globally on $\omega\times\Gamma\backslash\Sigma$ and* (0.5) *also globally holds.*

Remark 0.5. In the above theorem, if ν is an integer, if $A_i(t,x,\xi)$ is homogeneous and if $\Gamma=\Gamma_1\cup(-\Gamma_1)$ (Γ_1 is convex and $-\Gamma_1=\{\xi;\,-\xi\in\Gamma_1\}$), setting $\lambda_j(t,x,\xi)=(-1)^{|\nu|}\lambda_j(t,x,-\xi)$ in $\omega\times(-\Gamma_1)$, we can take every term in the asymptotic expansions of $N(t,x,\xi)$ and $D^{jk}(t,x,\xi)$ not only positively homogeneous but also homogeneous in $\omega\times\Gamma$. This fact often makes the considerations in applications simple and clear.

In the above theorem, (0.5) means that Q^{jk} is equivalent to the following scalar higher order operator

$$(D_t-\lambda_j(t,x,D_x))^{\circ n_{jk}}+\text{lower order terms}\,.$$

Then, $P(t,x,D_t,D_x)$ is equivalent to some of scalar higher order operators with a characteristic of constant multiplicity on $\omega_\circ\times\Gamma_\circ$. This suggests that we can obtain necessary conditions on the well-posedness by the same way as in case of single equation. For the necessity, we shall use various classes of formal symbols. Further, we shall also use the above theorem in order to show the sufficiency assuming the holomorphy or the real analyticity of the original symbols. (See § 4.)

§ 1. Formal symbols

"$K \subset\subset \Omega$" means that K is a compact subset of Ω and $\partial K \cap \partial\Omega = \phi$, where ∂A is the boundary of a set A. Let $\boldsymbol{R}$ be the set of real numbers, $\boldsymbol{C}$ be the set of complex numbers, $\boldsymbol{N}$ be $\{1, 2, \cdots\}$, $\boldsymbol{Z}_+$ be $\boldsymbol{N} \cup \{0\}$ and $\boldsymbol{R}_+$ be $\{a \in \boldsymbol{R};\ a \geq 0\}$. For $t \in \boldsymbol{R}$ and $x \in \boldsymbol{R}^\ell$ (or $t \in \boldsymbol{C}$ and $x \in \boldsymbol{C}^\ell$), we set $\tilde{x} = (t, x) = (x_0, x_1, \cdots, x_\ell)$.

For $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_\ell)$, $\alpha' = (\alpha_0', \alpha_1', \cdots, \alpha_\ell')$ in $\boldsymbol{Z}^{\ell+1}$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_\ell)$ in $\boldsymbol{Z}^\ell$, we set $|\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_\ell$, $\alpha \pm \alpha' = (\alpha_0 \pm \alpha_0', \alpha_1 \pm \alpha_1', \cdots, \alpha_\ell \pm \alpha_\ell')$, $\alpha! = \alpha_0! \alpha_1! \cdots \alpha_\ell!$, $\begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} = \dfrac{\alpha!}{\alpha'!(\alpha - \alpha')!}$, $\left(\dfrac{\partial}{\partial \tilde{x}}\right)^\alpha = \left(\dfrac{\partial}{\partial x_0}\right)^{\alpha_0} \left(\dfrac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\dfrac{\partial}{\partial x_\ell}\right)^{\alpha_\ell}$ and $D_{\tilde{x}}{}^\alpha = (-\sqrt{-1})^{|\alpha|} \left(\dfrac{\partial}{\partial \tilde{x}}\right)^\alpha$. We often identify $\alpha = (\alpha_1, \cdots, \alpha_\ell) \in \boldsymbol{Z}^\ell$ and $(0, \alpha) = (0, \alpha_1, \cdots, \alpha_\ell) \in \boldsymbol{Z}^{\ell+1}$. $\alpha' \leq \alpha$ means that $\alpha_i' \leq \alpha_i$ $(0 \leq i \leq \ell)$.

For α in $\boldsymbol{Z}^{\ell+1}$ and β in $\boldsymbol{Z}^\ell$, we set $a_{(\alpha)}^{(\beta)}(t, x, \xi) = D_{\tilde{x}}{}^\alpha \left(\dfrac{\partial}{\partial \xi}\right)^\beta a(\tilde{x}, \xi)$. For α and β in $\boldsymbol{Z}^\ell$, we denote $D_x{}^\alpha \left(\dfrac{\partial}{\partial \xi}\right)^\beta a(t, x, \xi) = a_{(0,\alpha)}^{(\beta)}(t, x, \xi)$ also by $a_{(\alpha)}^{(\beta)}(t, x, \xi)$.

We set $|\xi| = \sqrt{(\xi_1)^2 + (\xi_2)^2 + \cdots + (\xi_\ell)^2}$ not only for ξ in $\boldsymbol{R}_\xi{}^\ell$ but also for ξ in $\boldsymbol{C}_\xi{}^\ell$. Then, $|\xi|$ is holomorphic in the domain $\{|\mathrm{Im}\xi| < |\mathrm{Re}\xi|\}$. Here, we always take the branch which is positive for ξ in $\boldsymbol{R}_\xi{}^\ell \backslash \{O\}$. We denote the absolute value of $\xi \in \boldsymbol{C}_\xi{}^\ell$ by $\|\xi\|$ $(= \sqrt{|\mathrm{Im}\xi|^2 + |\mathrm{Re}\xi|^2})$.

We say that $\Omega \subset \boldsymbol{R}_t{}^1 \times \boldsymbol{R}_x{}^\ell \times \boldsymbol{R}_\xi{}^\ell$ (or $\boldsymbol{C}_t{}^1 \times \boldsymbol{C}_x{}^\ell \times \boldsymbol{C}_\xi{}^\ell$) is conic when $(t, x, O) \notin \Omega$ for arbitrary (t, x) and $(t, x, \xi) \in \Omega$ implies $(t, x, \rho\xi) \in \Omega$ for arbitrary $\rho > 0$. When a conic subset $\tilde{\Omega}$ of Ω satisfies $\tilde{\Omega} \cap \{\|\xi\| = 1\} \subset\subset \Omega \cap \{\|\xi\| = 1\}$, we say that $\tilde{\Omega}$ is a conically compact subset of Ω. We say that Σ in $\Omega \subset \boldsymbol{C}_t{}^1 \times \boldsymbol{C}_x{}^\ell \times \boldsymbol{C}_\xi{}^\ell$ is an analytic set if it is the zero set of a holomorphic function in Ω.

For an open set (or the closure of an open set) Ω in $\boldsymbol{C}^L$, we set

$$\mathcal{H}(\Omega) = \{\textit{holomorphic functions in } \Omega\},$$

$$\mathcal{M}(\Omega) = \{\textit{meromorphic functions in } \Omega\}.$$

$\mathcal{M}(\Omega)$ is the quotient field of $\mathcal{H}(\Omega)$.

Let $\{M_n\}_{n=0}^\infty$ be a positive, monotone increasing and logarithmically convex (*i.e.* $(M_n)^2 \leq M_{n-1} M_{n+1}$) sequence. For an open set (or the closure of an open set) Ω in $\boldsymbol{R}^L$, we set

$$\mathcal{E}\{M_n\}(\Omega) = \{f \in C^\infty(\Omega);\ \forall K \subset\subset \Omega,\ \exists C, R > 0,$$
$$\forall \alpha \in \boldsymbol{Z}^L,\ \sup_K |D^\alpha f| \leq C R^{|\alpha|} M_{|\alpha|}\},$$

$$\mathcal{B}\{M_n\}(\Omega) = \{f \in \mathcal{B}(\Omega);\ \exists C, R > 0,\ \forall \alpha \in \boldsymbol{Z}^L,\ \sup_\Omega |D^\alpha f| \leq C R^{|\alpha|} M_{|\alpha|}\},$$

On the topologies of the ultradifferentiable classes, see H. Komatsu [9].

Making the notation simple, we admit $M_n=\infty$ and set $\mathcal{E}\{\infty\}(\Omega)=\mathcal{E}(\Omega)$ and $\mathcal{B}\{\infty\}(\Omega)=\mathcal{B}(\Omega)$. We consider that $\{M_n\}=\{\infty\}$ satisfies all assumptions introduced hereafter. On $\{M_n\}_{n=0}^{\infty}$, we assume the following.

Assumption 1. It holds that

(1.1) $\{M_n/n!\}$ is logarithmically convex and monotone increasing and $\{M_n\}$ is differentiable (i.e. $\exists R_\circ\geq 1$ $M_{n+1}\leq R_\circ{}^n M_n$).

or

Assumption 1′. For every $k_\circ\in\boldsymbol{Z}$, replacing finite elements of $\{M_n\}_{n=0}^{\infty}$, it holds that

(1.1′) $\{M_{n+k_\circ}/n!\}$ is logarithmically convex and monotone increasing.

On the properties on $\{M_n\}_{n=0}^{\infty}$, see S. Mandelbrojt [11] and W. Matsumoto [13].

Remark 1.1. Even if we replace finite elements of $\{M_n\}_{n=0}^{\infty}$, the spaces $\mathcal{E}\{M_n\}(\Omega)$ and $\mathcal{B}\{M_n\}(\Omega)$ rest same. Then, we can relax the conditions (1.1) and (1.1′) only for $n\gg 1$. Indeed, for the condition (1.1′) we always need replace some finite elements of $\{M_n\}_{n=0}^{\infty}$ depending on $k_\circ$.

Example 1.1 (Holomorphic and analytic classes). $M_n=n!$ satisfies (1.1) but does not (1.1′).

Example 1.2 (Gevrey class). $M_n=n!^{\kappa}$ $(\kappa>1)$ satisfies both of (1.1) and (1.1′).

Example 1.3. $M_n=n!^{\kappa}\exp an^{\rho}$ $(\rho>1,\ a>0,\ \kappa\in\boldsymbol{R})$ satisfies both of (1.1) for $n\gg 1$ and (1.1′) when $1<\rho\leq 2$ and does only (1.1′) when $\rho>2$.

Remark 1.2. We can normalize $\{M_n\}_{n=0}^{\infty}$ as $M_0=1$ replacing M_n by $(M_0)^{-1}M_n$. Through this paper, we treat only normalized $\{M_n\}_{n=0}^{\infty}$. It is convenient to introduce M_n for negative n. We set $M_n=M_0=1$ for $n<0$.

Under the condition (1.1), it holds that

$$\text{(1.2)}\qquad \frac{p!}{q!r!}\leq\frac{M_p}{M_qM_r}\qquad (q+r\leq p,\quad q,r\leq p).$$

Under the condition (1.1′), replacing $\{M_n\}_{n=0}^{\infty}$ by $\{CR^nM_n\}_{n=0}^{\infty}$ $(C,R>0)$ and further M_n suitably for $0\leq n\leq 2k_\circ+1$, it holds that

$$\text{(1.2′)}\qquad \frac{p!}{q!r!}\leq\frac{M_{p+k}}{M_{q+k}M_{r+k}}\qquad (q+r\leq p,\quad q,r\leq p,\quad 0\leq k\leq k_\circ).$$

Under (1.1) or (1.1′), it holds that

$$(1.3)\qquad \left(\frac{M_n}{n!}\right)^{1/n} \le \left(\frac{M_{n+1}}{(n+1)!}\right)^{1/(n+1)} \qquad (n\ge 1)\,.$$

(In case of (1.1′), we take $k=0$ in (1.2′).)

This implies that $\mathcal{E}\{M_n\}(\Omega)$ and $\mathcal{B}\{M_n\}(\Omega)$ are closed under decomposition by $f\neq 0$ and by $|f|\ge\sigma>0$ respectively. (See W. Rudin [28].)

To consider the structure of systems of ps.d.op.'s, those asymptotic expansions are essential. (See, for example, F. Treves [30] and W. Matsumoto [14].) Then, we only consider the asymptotic expansions naming them the formal symbols and denote them by $a(t,x,\xi)=\sum_{i=0}^{\infty}a_i(t,x,\xi)$. Of course, these are formal sums. We are concerned with the formal symbols and treat neither true symbols nor true operators up to § 3. (On the construction of a true symbol from a formal symbol, see L. Boutet de Monvel and P. Krée [4], L. Boutet de Monvel [3], W. Matsumoto [14], F. Treves [30] e.t.c.)
It is essential that we only need the arithmetical operations and the differentiation in the calculation of formal symbols, both of which are the local operations.

We consider formal symbols on a conic set $\omega\times\Gamma$. Of course, we can replace $\omega\times\Gamma$ by a general conic set $\tilde{\Omega}$ in $\boldsymbol{R}_t^1\times\boldsymbol{R}_x^\ell\times\boldsymbol{R}_\xi^\ell$ (or $\boldsymbol{C}_t^1\times\boldsymbol{C}_x^\ell\times\boldsymbol{C}_\xi^\ell$). However, on the view points of applications and notation, we adopt a product set. Corresponding to the four categories of differential operators mentioned in Introduction, we introduce following categories of formal symbols.

Definition 1.1 (Meromorphic formal symbol). Let $\omega\times\Gamma$ be an open conic set in $\boldsymbol{C}_t^1\times\boldsymbol{C}_x^\ell\times\boldsymbol{C}_\xi^\ell$. We say that a formal sum $a(t,x,\xi)=\sum_{i=0}^{\infty}a_i(t,x,\xi)$ is a meromorphic formal symbol, when $\{a_i(t,x,\xi)\}$ satisfies the following;

There exist an analytic conic set Σ in $\omega\times\Gamma$ $(\omega\times\Gamma\backslash\Sigma\cap\{\xi_1=0\}=\emptyset)$ and $\kappa\in\boldsymbol{R}$, and it holds that

(1.4) $a_i(t,x,\xi)$ belongs to $\mathcal{M}(\omega\times\Gamma)\cap\mathcal{H}(\omega\times\Gamma\backslash\Sigma)$ and it is positively homogeneous of order $\kappa-i$ on ξ $(i\in\boldsymbol{Z}_+)$.

(1.5) For an arbitrary conically compact subset $\tilde{\omega}$ in $\omega\times\Gamma\backslash\Sigma$, there exist $C>0$ and $R>0$, and we have

$$|a_i(t,x,\xi)|\le CR^i i!|\xi_1|^{\kappa-i} \quad \text{on} \quad \tilde{\omega} \qquad (i\in\boldsymbol{Z}_+)\,.$$

Remark 1.3. (1.4) and (1.5) imply

(1.5′) For an arbitrary conically compact subset $\tilde{\omega}$ in $\omega\times\Gamma\backslash\Sigma$, there exist $C>0$ and $R>0$, and we have

$$|a_{i(\alpha)}^{(\beta)}(t, x, \xi)| \leq CR^{i+|\alpha|+|\beta|} i!|\alpha|!|\beta|!|\xi_1|^{\kappa-i-|\beta|} \quad \text{on} \quad \tilde{\omega}$$

$$(i \in \boldsymbol{Z}_+, \alpha \in \boldsymbol{Z}_+^{\ell+1}, \beta \in \boldsymbol{Z}_+^{\ell}).$$

On the other hand, (1.5′) implies (1.4) and (1.5) except the meromorphy and the positive homogeneity.

Remark 1.4. In the estimate (1.5) and (1.5′), ξ_1 is used as a meromorphic scale of order. We can replace it, for example, by one of ξ_i $(2 \leq i \leq \ell)$.

Remark 1.5. In Definition 1.1, Σ does not depend on i. This is very important in applications.

In Definition 1.1, if Σ is empty, we have a holomorphic formal symbol.

Definition 1.2 (Holomorphic formal symbol). We say that a formal sum $a(t, x, \xi) = \sum_{i=0}^{\infty} a_i(t, x, \xi)$ is a holomorphic formal symbol if $a(t, x, \xi)$ is a meromorphic formal symbol with $\Sigma = \emptyset$.

Corresponding to differential operators with real analytic coefficients, we use the meromorphic and holomorphic formal symbol classes on $\hat{\omega} \times \hat{\Gamma}$; $\hat{\omega}$ is a complex neighborhood of ω and $\hat{\Gamma}$ is $\{\xi \in \boldsymbol{C}_{\xi}^{\ell}; |\mathrm{Im}\xi| < \varepsilon|\mathrm{Re}\xi|, \mathrm{Re}\xi \in \Gamma\}$, $(\exists \varepsilon > 0)$. (See pp16, 2.)

Definition 1.3 (Formal symbol of class $\{M_n, N_n\}$, case of $M_n < \infty$ and $N_n < \infty$). We assume that each of $\{M_n\}_{n=0}^{\infty}$ and $\{N_n\}_{n=0}^{\infty}$ satisfies Assumption 1 or 1′, respectively. Let $\omega \times \Gamma$ be an open conic set in $\boldsymbol{R}_t^1 \times \boldsymbol{R}_x^{\ell} \times \boldsymbol{R}_{\xi}^{\ell}$. We say that a formal sum $a(t, x, \xi) = \sum_{i=0}^{\infty} a_i(t, x\ \xi)$ is a formal symbol of class $\{M_n, N_n\}$, when $\{a_i(t, x, \xi)\}$ satisfies the following;

There exists $\kappa \in \boldsymbol{R}$, and it holds that

(1.6) $a_i(t, x, \xi)$ belongs to $C^{\infty}(\omega \times \Gamma)$ and it is positively homogeneous of order $\kappa - i$ on ξ $(i \in \boldsymbol{Z}_+)$.

(1.7) there exist $C > 0$ and $R > 0$, and we have

$$|a_{i(\alpha)}^{(\beta)}(t, x, \xi)| \leq CR^{i+|\alpha|+|\beta|} M_{i+|\alpha|} N_{i+|\beta|} (i!)^{-1} |\xi|^{\kappa-i-|\beta|} \quad \text{on } \omega \times \Gamma$$

$$(i \in \boldsymbol{Z}_+, \alpha \in \boldsymbol{Z}_+^{\ell+1}, \beta \in \boldsymbol{Z}_+^{\ell}).$$

In case of $M_n = \infty$ and $N_n < \infty$, we replace (1.7) by the following;

(1.7′) there exists $R > 0$ and, for every $i \in \boldsymbol{Z}_+$ and $\alpha \in \boldsymbol{Z}_+^{\ell+1}$, there exists $C_{i\alpha} > 0$ such that

$$|a_{i(\alpha)}^{(\beta)}(t, x, \xi)| \leq C_{i\alpha} R^{|\beta|} N_{i+|\beta|} |\xi|^{\kappa-i-|\beta|} \quad \text{on} \quad \omega \times \Gamma.$$

In case of $M_n = N_n = \infty$, we replace (1.7) by the following;

(1.7″) for every $i\in \boldsymbol{Z}_+$, $\alpha\in \boldsymbol{Z}_+^{\ell+1}$, $\beta\in \boldsymbol{Z}_+^{\ell}$, there exists $C_{i\alpha\beta}>0$ and we have

$$|a_{i(\alpha)}^{(\beta)}(t,x,\xi)|\le C_{i\alpha\beta}|\xi|^{\kappa-i-|\beta|} \quad \text{on} \quad \omega\times\Gamma .$$

Remark 1.6. In case of $M_n=\infty$ and $N_n<\infty$, we set $L_n=\max\{1, \max_{i+|\alpha|\le n}\{C_{i\alpha}\}\}$ and $M_n=n!^2L_1L_2\cdots L_n$. Then, $\{M_n\}_{n=0}^{\infty}$ satisfies Assumption 1′ and $a(t,x,\xi)$ in $S\{\infty, N_n\}$ belongs to $S\{M_n, N_n\}$. In case of $M_n=N_n=\infty$, starting from $L_n=\max\{1, \max_{i+\max\{|\alpha|,|\beta|\}\le n}\{C_{i\alpha\beta}\}\}$, we set M_n by the same way. Then, $\{M_n\}_{n=0}^{\infty}$ satisfies Assumption 1′ and $a(t,x,\xi)$ in $S\{\infty,\infty\}$ belongs to $S\{M_n, M_n\}$. Therefore, on the calculus, we need consider only the case of $M_n<\infty$ and $N_n<\infty$. On the notation of $S\{M_n, N_n\}$, see the following part.

We can also treat the case of $M_n<\infty$ and $N_n=\infty$, but we shall not use this class in applications.

Remark 1.7. Through all definitions, we assume the positive homogeneity on $a_i(t,x,\xi)$. In applications, it is useful but on the view point of a closed calculus, we can drop it. By the same reason, we can also relax the meromorphy of $a_i(t,x,\xi)$ in Definition 1.1.

We call κ the order of $a(t,x,\xi)$. Let us denote the set of the meromorphic formal symbols of order κ by $S_M^{\kappa}(\omega\times\Gamma)$, that of the holomorphic formal symbols of order κ by $S_H^{\kappa}(\omega\times\Gamma)$ and that of the formal symbols of class $\{M_n, N_n\}$ of order κ by $S^{\kappa}\{M_n, N_n\}(\omega\times\Gamma)$, respectively. Further we set $S_M(\omega\times\Gamma)=\cup_{\kappa}S_M^{\kappa}(\omega\times\Gamma)$ and so on. On the other hand we set $S_{Mhom}^{\kappa}(\omega\times\Gamma)=\{a(t,x,\xi)\in S_M^{\kappa}(\omega\times\Gamma);\ a(t,x,\xi)=a_0(t,x,\xi)\}$ and so on.

Definition 1.4 (True order). When $a(t,x,\xi)=\sum_{i=0}^{\infty}a_i(t,x,\xi)$ has the order κ, $a_i(t,x,\xi)\equiv 0$ $(0\le i<i_0)$ and $a_{i_0}(t,x,\xi)\not\equiv 0$, we say that $a(t,x,\xi)$ has the true order $\kappa-i_0$ on $\omega\times\Gamma$.

The rules of calculation are common to all categories of the formal symbols. Then, we represent them by $S(\omega\times\Gamma)$ or S. We introduce a product in $S(\omega\times\Gamma)$, which corresponds to the product of ps.d.op.'s.

Definition 1.5 (Product and adjoint). For $a(t,x,\xi)=\sum_{i=0}^{\infty}a_i(t,x,\xi)$ and $b(t,x,\xi)=\sum_{i=0}^{\infty}b_i(t,x,\xi)$ in $S(\omega\times\Gamma)$, we set

$$a(t,x,\xi)\circ b(t,x,\xi)=\sum_{i=0}^{\infty}c_i(t,x,\xi), \tag{1.8}$$

$$c_i(t,x,\xi)=\sum_{j+k+|\gamma|=i,\gamma\in Z_+^{\ell}}\frac{1}{\gamma!}a_j^{(\gamma)}(t,x,\xi)b_{k(\gamma)}(t,x,\xi),$$

$$(\textit{Product of } a(t,x,\xi) \textit{ and } b(t,x,\xi) \textit{ in } S),$$

$$a^*(t,x,\xi)=\sum_{i=0}^{\infty}a_i^*(t,x,\xi), \tag{1.9}$$

$$a_i^* = \sum_{j+|\gamma|=i,\gamma\in Z_+^\ell} \frac{1}{\gamma!}\bar{a}_{j(\gamma)}^{(\gamma)}(t, x, \xi),$$

$$(Adjoint\ of\ a(t, x, \xi)\ in\ S).$$

Proposition 1.1 (L. Boutet de Monvel and P. Krée [4]). *Let S be S_M, S_H or $S\{M_n, N_n\}$. We assume Assumption 1 or 1′ on $\{M_n\}_{n=0}^\infty$ and $\{N_n\}_{n=0}^\infty$ respectively in case of $S=S\{M_n, N_n\}$. $S(\omega\times\Gamma)$ is $*$-algebra over $\mathbf{C}$. If the principal part $a_0(t, x, \xi)$ of $a(t, x, \xi)=\sum_{i=0}^\infty a_i(t, x, \xi)$ does not vanish identically in case of $S=S_M$, does not vanish in case of $S=S_H$ and $|a_0(t, x, \xi)|\geq c_\circ>0$ in case of $S\{M_n, N_n\}$, $a(t, x, \xi)$ is invertible.*

For a ring G, we set $Mat(K_1, K_2; G)=\{K_1\times K_2$ matrices with entries in $G\}$, $Mat(K; G)=Mat(K, K; G)$ and $GL(K; G)=\{$*invertible square matrices of order K with entries in G*$\}$. For $A(t, x, \xi)=(a^{jk}(t, x, \xi))_{1\leq j\leq K_1, 1\leq k\leq K_2}\in Mat(K_1, K_2; G)$, we set $A_i(t, x, \xi)=(a_i^{jk}(t, x, \xi))_{1\leq j\leq K_1, 1\leq k\leq K_2}$, where $a^{jk}(t, x, \xi)=\sum_{i=0}^\infty a_i^{jk}(t, x, \xi)$. We have $A(t, x, \xi)=\sum_{i=0}^\infty A_i(t, x, \xi)$.

Let S be $S_M(\omega\times\Gamma)$, $S_H(\omega\times\Gamma)$ or $S\{M_n, N_n\}(\omega\times\Gamma)$ $(M_n<\infty, N_n<\infty)$ and $\tilde{\omega}$ be a conically compact subset of $\omega\times\Gamma\backslash\Sigma$. For $A(t, x, \xi)=\sum_{i=0}^\infty A_i(t, x, \xi)$ in $Mat(K_1, K_2; S^\kappa(\omega\times\Gamma))$ $(A_i(t, x, \xi)=(a_i^{jk}(t, x, \xi)))$, we set

$$(1.10)\qquad \|A; T\|=\|A; T\|(\tilde{\omega})=\sum_{i\in Z_+, \alpha\in Z_+^{\ell+1}, \beta\in Z_+^\ell} \max_{j,k} \sup_{(t,x,\xi)\in\tilde{\omega}} \frac{2K\ i!}{(2\ell)^i M_{i+|\alpha|}N_{i+|\beta|}} \times |a_{i(\alpha)}^{jk(\beta)}(t, x, \xi)||\xi|^{-\kappa+i+|\beta|}T^{2i+|\alpha|+|\beta|}.$$

where $K\geq\max\{K_1, K_2\}$. $\|A; T\|(\tilde{\omega})$ converges for $0<T\ll 1$. Here, M_n, N_n and $|\xi|$ are replaced by $n!$, $n!$ and $|\xi_1|$ respectively in case of $S=S_M$ and S_H, $\Sigma=\emptyset$ in case of $S=S_H$ and $S\{M_n, N_n\}$, and $\tilde{\omega}$ can be replaced by $\omega\times\Gamma$ in case of $S=S\{M_n, N_n\}$.

Following L. Boutet de Monvel and P. Krée [4], we obtain the following propositions on matrices with entries of formal symbols.

Proposition 1.2. *Let S be S_M, S_H or $S\{M_n, N_n\}$ $(M_n<\infty$ and $N_n<\infty)$. We assume Assumption 1 or 1′ on $\{M_n\}_{n=0}^\infty$ and $\{N_n\}_{n=0}^\infty$ respectively in case of $S=S\{M_n, N_n\}$. For $A(t, x, \xi)$ in $Mat(K_1, K_2; S^\kappa(\omega\times\Gamma))$ and $B(t, x, \xi)$ in $Mat(K_2, K_3; S^{\kappa'}(\omega\times\Gamma))$, $A(t, x, \xi)\circ B(t, x, \xi)$ belongs to $Mat(K_1, K_3; S^{\kappa+\kappa'}(\omega\times\Gamma))$ and satisfies*

$$(1.11)\qquad \|A\circ B; T\|\leq\|A; T\|\|B; T\|,$$

where K in the norm (1.10) is taken as $K\geq\max_{1\leq i\leq 3}\{K_i\}$.

By virtue of Proposition 1.2, we have

Proposition 1.3. 1) *For $S=S_M$, if $A(t, x, \xi)$ belongs to $Mat(K; S^\kappa(\omega\times\Gamma))$ and satisfies $\det A_0(t, x, \xi)\not\equiv 0$, it has the inverse in $GL(K; S^{-\kappa}(\omega\times\Gamma))$.*

2) *For* $S=S_H$, *if* $A(t, x, \xi)$ *belongs to* $Mat(K; S^{\kappa}(\omega\times\Gamma))$ *and satisfies* $\det A_0(t, x, \xi)\neq 0$, *it has the inverse in* $GL(K; S^{-\kappa}(\omega\times\Gamma))$.

3) *Let* S *be* $S\{M_n, N_n\}$. *We assume Assumption* 1 *or* 1′ *on* $\{M_n\}_{n=0}^{\infty}$ *and* $\{N_n\}_{n=0}^{\infty}$ *respectively. If* $A(t, x, \xi)$ *belongs to* $Mat(K; S^{\kappa}(\omega\times\Gamma))$ *and satisfies* $|\det A_0(t, x, \xi)|\geq c_{\circ}>0$, *it has the inverse in* $GL(K; S^{-\kappa}(\omega\times\Gamma))$.

Proof. Let $I-A'$ be $A\circ(A_0)^{-1}$. A^{-1} is given by $(A_0)^{-1}\circ\sum_{h=0}^{\infty}A'^{\circ h}$, where $A'^{\circ h}$ means the operator product of A''s of h-times. In this procedure, the pole set increases only by the decomposition by det $A_0(t, x, \xi)$ in case 1).

Q.E.D.

The formal symbol $A(t, x, \xi)$ whose principal symbol satisfies the condition in Proposition 1.3 is called "non-degenerate". On the other hand, if the principal symbol of $A(t, x, \xi)$ does not satisfy the condition in Proposition 1.3, $A(t, x, \xi)$ is called "degenerate". We remark that there exist formal symbols which are degenerate but invertible.

§ 2. Separation of characteristic roots

Let G be a ring. We denote

$$B=\begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_d \end{pmatrix} \in Mat(K; G)\,, \qquad B_j\in Mat(K_j\,; G)$$

$$\Big(\sum_{j=1}^{d}K_j=K\Big)\,,$$

by $B=B_1\oplus B_2\oplus\cdots\oplus B_d=\oplus_{1\leq j\leq d}B_j$ and say that B is split to $\{B_j\}_{1\leq j\leq d}$. Further, let us set $\boldsymbol{q}^j=(q^j_{m_j}, q^j_{m_j-1}, \cdots, q^j_1)$ ($q^j_k\in \boldsymbol{Z}_+$, $\sum_{k=1}^{m_j}kq^j_k=m_j$) and $J(\lambda_j, \boldsymbol{q}^j; 1\leq j\leq d)=\oplus_{1\leq j\leq d}J(\lambda_j, \boldsymbol{q}^j)=\oplus_{1\leq j\leq d}\oplus_{m_j\geq k\geq 1}\oplus^{q^j_k}(\lambda_jI_k+J(k))$, where $\lambda_j\in G$, $\oplus_{m\geq k\geq 1}$ means the direct sum on k from m to 1, $\oplus^pA$ does $\overbrace{A\oplus\cdots\oplus A}^{p}$ and

$$J(k)=\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \in Mat(k\,; G)\,.$$

$J(\lambda_j, \boldsymbol{q}^j)$ is a Jordan normal form with respect to λ_j. q^j_k is the number of $k\times k$ blocks in $J(\lambda_j, \boldsymbol{q}^j)$.

Example 2.1. In case of $K=m_1=10$ $(d=1)$, $\lambda_1=0$ and $\boldsymbol{q}^1=(0, 0, 0, 0, 0, 0, 1, 0, 2, 2)$, $J(0, \boldsymbol{q}^1)=J(4)\oplus J(2)\oplus J(2)\oplus J(1)\oplus J(1)$ has the following form.

$$\begin{pmatrix}
0 & 1 & & & & & & & & \\
 & 0 & 1 & & & & & & & \\
 & & 0 & 1 & & & & & & \\
 & & & 0 & & & & & & \\
 & & & & 0 & 1 & & & & \\
 & & & & & 0 & & & & \\
 & & & & & & 0 & 1 & & \\
 & & & & & & & 0 & & \\
 & & & & & & & & 0 & \\
 & & & & & & & & & 0
\end{pmatrix}$$

For $A(t, x, \xi)=\sum_{i=0}^{\infty}A_i(t, x, \xi)\in Mat(K; S^{\nu}(\omega\times\Gamma))$, we give some elementary propositions.

First, we consider the case of smooth symbols. For the elements in $S_M(\omega\times\Gamma)$, we regard them as the elements in $S_H(\omega\times\Gamma\backslash\Sigma)$. We represent $\omega\times\Gamma$ ($\omega\times\Gamma\backslash\Sigma$ in case of S_M) by $\tilde{\Omega}$.

Proposition 2.1. *Let S be S_H or $S\{M_n, N_n\}$. We assume that $A_0(t, x, \xi)$ belongs to $Mat(K; S_{hom}^{\nu}(\tilde{\Omega}))$ and satisfies Assumption* 0 (*in case of $S=S\{M_n, N_n\}$, further* (1.3) *on $\{M_n\}_{n=0}^{\infty}$ and $\{N_n\}_{n=0}^{\infty}$*). *Then, the eigenvalues $\{\lambda_j(t, x, \xi)\}_{j=1}^{d}$ of $A_0(t, x, \xi)$ belong to $S_{hom}^{\nu}(\tilde{\Omega})$.*

Of course, we need only Rudin's condition —almost increasing— on $\{M_n/n!\}_{n=0}^{\infty}$ and $\{N_n/n!\}_{n=0}^{\infty}$ instead of (1.3), but we do not adhere to the best possibility on this condition. In Propositions 2.2, 2.4 and 2.5, we also assume (1.3) but also need only Rudin's condition. (See W. Rudin [28].)

Proof. By virtue of the following formula, the proposition is obvious.

$$\lambda_j(t, x, \xi)=\frac{1}{2\pi m_j\sqrt{-1}}\int_C\frac{d}{d\lambda}\log\{\det(\lambda I_K-A_0(t, x, \xi))\}d\lambda\,, \tag{2.1}$$

where C is a simple closed path around $\lambda_j(t, x, \xi)$ alone in $\boldsymbol{C}$. Q.E.D.

In general, the Jordan structure of the generalized eigenspaces changes depending on (t, x, ξ). However, as the dimension of each generalized eigenspace is constant, we have the following.

Proposition 2.2. *Let S be S_H or $S\{M_n, N_n\}$. We assume that $A_0(t, x, \xi)$ belongs to $Mat(K; S_{hom}^{\nu}(\tilde{\Omega}))$ and satisfies Assumption* 0 (*in case of $S=S\{M_n, N_n\}$, further* (1.3) *on $\{M_n\}_{n=0}^{\infty}$ and $\{N_n\}_{n=0}^{\infty}$*). *For arbitrary $(t_\circ, x_\circ, \xi_\circ)$ in $\tilde{\Omega}$, there exist a conic neighborhood $\omega_\circ\times\Gamma_\circ$ in $\tilde{\Omega}$ and a basis of the generalized* *eigenspace of* $\lambda_j(t, x, \xi)$ *in* $S^0_{hom}(\omega_\circ \times \Gamma_\circ)$.

Proof. Since the following matrix $P_j(t, x, \xi)$ gives the projection to the generalized eigenspace of $\lambda_j(t, x, \xi)$ and it is smooth, the proposition is obvious;

$$P_j(t, x, \xi) = \frac{1}{2\pi\sqrt{-1}}\int_C (\lambda I_K - A_0(t, x, \xi))^{-1} d\lambda \,, \tag{2.2}$$

where C is a simple closed path around $\lambda_j(t, x, \xi)$ alone in $\boldsymbol{C}$. Q.E.D.

By Proposition 2.2, following H. Kumano-go [10], K. Kajitani [5], T. Nishitani [25] and W. Matsumoto [15], we have the following proposition on $P(t, x, D_t, D_x)$ given in (0.1). We denote the set of differential polynomials on D_t with coefficients in S by $S[D_t]$.

Corollary 2.3 (Separation of characteristic roots). *Let* S *be* S_H *or* $S\{M_n, N_n\}$. *We assume that* $P(t, x, D_t, \xi)$ *belongs to* $Mat(K; S^\nu(\tilde{\Omega})[D_t])$ *and satisfies Assumption* 0 (*in case of* $S\{M_n, N_n\}$, *further Assumption* 1 *or* 1′ *on* $\{M_n\}_{n=0}^\infty$ *and* $\{N_n\}_{n=0}^\infty$, *respectively*). *For arbitrary* $(t_\circ, x_\circ, \xi_\circ)$ *in* $\tilde{\Omega}$, *there exist a conic neighborhood* $\omega_\circ \times \Gamma_\circ$ *in* $\tilde{\Omega}$ *and* $N(t, x, \xi)$ *in* $GL(K; S^0(\omega_\circ \times \Gamma_\circ))$, *for which it holds that*

$$N(t, x, \xi)^{-1} \circ P(t, x, D_t, \xi) \circ N(t, x, \xi) = \oplus_{1 \le j \le d} P^j(t, x, D_t, \xi) \,, \tag{2.3}$$

where

$$P^j(t, x, D_t, \xi) = I_{m_j} D_t - B^j(t, x, \xi) \,,$$

$$B^j(t, x, \xi) = \sum_{i=0}^\infty B_i^j(t, x, \xi) \,.$$

Here, $B_0^j(t, x, \xi)$ *belongs to* $Mat(m_j; S^\nu_{hom}(\omega_\circ \times \Gamma_\circ))$ *and has the unique eigenvalue* $\lambda_j(t, x, \xi)$ $(1 \le j \le d)$.

In order to obtain the above corollary in case of $S = S_H$ and $S\{M_n, N_n\}$, we can apply the proof of T. Nishitani [25] on scalar operators to systems. His proof stands on a successive approximation. On the other hand, $\{N_i(t, x, \xi)\}$ and $\{B_i^j(t, x, \xi)\}$ are obtained easily in class $S\{\infty, \infty\}$. The author gave another proof directly evaluating them. (See W. Matsumoto [15].)

On the view point of technique, Corollary 2.3 corresponds to the decomposition of scalar and higher order operators mentioned at the start of Introduction. Corollary 2.3 is the best result if we hope to keep the smoothness of symbols.

We hope to take $N(t, x, \xi)$ in Corollary 2.3 globally. However, in general, it becomes meromorphic in case of $S = S_H$. Further, if we hope to transform $B_0^j(t, x, \xi)$ to its Jordan normal form, we also need accept an exceptional set for $N(t, x, \xi)$ and $B^j(t, x, \xi)$, which is a pole set in case of S

$=S_H$. (Recall Example 0.1 near $(t, x)=O$.) By this reason, from now on, we consider S_M instead of S_H.

As we globally consider the structure of systems in case of $S=S_M$, we assume the following.

Assumption 2 (In case of $S=S_M$). The eigenvalues of $A_0(t, x, \xi)$ are holomorphic in $\omega\times\Gamma$.

We denote the number of $k\times k$ blocks in the Jordan normal form of $B_0{}^j(t, x, \xi)$ at (t, x, ξ) by $q_k^j(t, x, \xi)$ and set $\boldsymbol{q}^j(t, x, \xi)=(q_{m_j}^j(t, x, \xi), q_{m_j-1}^j(t, x, \xi), \cdots, q_1^j(t, x, \xi))$, $(\sum_{k=1}^{m_j} kq_k^j(t, x, \xi)=m_j)$. Further, we set $Q(m)=\{\boldsymbol{q}=(q_m, q_{m-1}, \cdots, q_1)\in \boldsymbol{Z}_+{}^m; \sum_{k=1}^m kq_k=m\}$ and give it the lexicographic order and the order topology. It is easily seen that $\boldsymbol{q}^j(t, x, \xi)$ is lower semi-continuous and especially, in case of $S=S_M$, $\boldsymbol{q}^j(t, x, \xi)=\max_{\omega\times\Gamma}\boldsymbol{q}^j(t, x, \xi)$ except on a conic analytic set. By these facts, we have the following proposition.

Proposion 2.4. *Let S be S_M or $S\{M_n, N_n\}$. We assume that $A_0(t, x, \xi)$ belongs to $S_{hom}^{\nu}(\omega\times\Gamma)$ and satisfies Assumption* 0 (*in case of $S=S_M$, further Assumption* 2 *and in case of $S=S\{M_n, N_n\}$, further* (1.3) *on $\{M_n\}_{n=0}^{\infty}$ and $\{N_n\}_{n=0}^{\infty}$*). *There exists an open conic dense subset $U^j=\cup_{\boldsymbol{q}\in Q(m_j)}U^j(\boldsymbol{q})$ of $\omega\times\Gamma$ such that $\boldsymbol{q}^j(t, x, \xi)=\boldsymbol{q}$ on $U^j(\boldsymbol{q})$. Especially, in case of $S=S_M$, $U^j=U^j(\boldsymbol{q}_\circ^j)$ ($\exists\, \boldsymbol{q}_\circ^j$ in $Q(m_j)$), $\omega\times\Gamma\backslash U^j$ ($=\Sigma^j$) is a conic analytic set and $\boldsymbol{q}^j(t, x, \xi)<\boldsymbol{q}_\circ^j$ on Σ^j.*

Remark 2.1. In case of $S=S\{M_n, N_n\}$, in general, U^j is composed of some of $U^j(\boldsymbol{q})$'s.

Example 2.2.

$$A_0(0, x, \xi)/\xi_1{}^{\nu}=\begin{pmatrix}0 & \mu(t)\\ 0 & 0\end{pmatrix}, \qquad \mu\begin{cases}\neq 0 & (t>0),\\ =0 & (t\leq 0).\end{cases}$$

For the above $A_0(t, x, \xi)$, $U((1, 0))=(0, \infty)$ and $U((0, 2))=(-\infty, 0)$.

The existence of the matrices of the above type causes many difficulties in the theories on the systems of differential equations with non-quasianalytic coefficients. (See W. Matsumoto [12, I] and [16] and also Remark 4.1 in § 4 of this paper.)

We set $[\boldsymbol{q}]=(\boldsymbol{q}^1, \boldsymbol{q}^2, \cdots, \boldsymbol{q}^d)$ and $\boldsymbol{Q}=Q(m_1)\times Q(m_2)\times\cdots\times Q(m_d)$. The above proposition brings the following.

Proposion 2.5. *We assume that $A_0(t, x, \xi)$ belongs to $S_{hom}^{\nu}(\omega\times\Gamma)$ and satisfies Assumption* 0 (*in case of $S=S_M$, further Assumption* 2 *and in case of $S=S\{M_n, N_n\}$, further* (1.3) *on $\{M_n\}_{n=0}^{\infty}$ and $\{N_n\}_{n=0}^{\infty}$*).

1) *Let S be S_M. There exists $N_0(t, x, \xi)$ in $Mat(K; S_{hom}^0(\omega\times\Gamma))$, invertible as a matrix of functions, such that*

$$(2.4)\qquad (N_0(t,x,\xi))^{-1}A_0(t,x,\xi)N_0(t,x,\xi) = J(\lambda_j, \boldsymbol{q}_\circ^j; 1\le j\le d)\xi_1{}^\nu\,,$$

where the pole set $\Sigma=\bigcup_{j=1}^d \Sigma^j$. ($\boldsymbol{q}_\circ^j$ *and* Σ^j *are those in Proposition* 2.4.)

2) *Let* S *be* $S\{M_n, N_n\}$. *There exists an open conic dense subset* $U=\bigcup_{[q]\in Q} U([\boldsymbol{q}])$ *of* $\omega\times\Gamma$. *For every* $(t_\circ, x_\circ, \xi_\circ)$ *in* $U([\boldsymbol{q}])$, *there exist a conic neighborhood* $\omega_\circ\times\Gamma_\circ$ *in* $U([\boldsymbol{q}])$ *and* $N_0(t,x,\xi)$ *in* $Mat(K; S^0_{hom}(\omega_\circ\times\Gamma_\circ))$, *invertible as a matrix of functions, such that*

$$(2.4')\qquad (N_0(t,x,\xi))^{-1}A_0(t,x,\xi)N_0(t,x,\xi) = J(\lambda_j, \boldsymbol{q}^j; 1\le j\le d)|\xi|^\nu \quad \text{on } \omega_\circ\times\Gamma_\circ\,.$$

In both cases 1) *and* 2), *we can take the entries of* $N_0(t,x,\xi)$ *as the polynomials of the entries of* $A_0(t,x,\xi)$, *and* $\{\lambda_j(t,x,\xi)\}$.

Remark 2.2. To see the last assertion in Proposition 2.5, we remark that we can obtain a Jordan basis through the construction of Jordan chains, that is, solving linear equations with parameters by the fundamental transformations. Here, we do not rely the formula (2.2).

By Proposition 2.5, following the proof of Proposition 2.3 and the proofs in V. I. Arnold [2] and V. M. Petkov [26], we arrive at the result on $P(t,x,D_t,D_x)$ by V. M. Petkov.

Theorem 2.6 (Normal form of V. M. Petkov [26]). *We assume that* $P(t,x,D_t,\xi)$ *belongs to* $Mat(K; S^\nu(\omega\times\Gamma)[D_t])$ *and satisfies Assumption* 0 (*in case of* $S=S_M$, *further Assumption* 2 *and in case of* $S=S\{M_n, N_n\}$, *further Assumptions* 1 *or* 1′ *on* $\{M_n\}_{n=0}^\infty$ *and* $\{N_n\}_{n=0}^\infty$).

1) *Let* S *be* S_M. *There exists* $\tilde N(t,x,\xi)$ *in* $GL(K; S^0(\omega\times\Gamma))$ *such that*

$$(2.5)\qquad \tilde N(t,x,\xi)^{-1}\circ P(t,x,D_t,\xi)\circ\tilde N(t,x,\xi) = \oplus_{1\le j\le d}P^j(t,x,\xi)\,,$$

$$P^j(t,x,\xi) = I_{m_j}(D_t-\lambda_j(t,x,\xi))-C^j(t,x,\xi)\,,$$

$$C^j(t,x,\xi) = \sum_{i=0}^{\infty} C_i^j(t,x,\xi)\in Mat(m_j\,; S^\nu)\,,$$

$$C_0^j(t,x,\xi) = J(0,\boldsymbol{q}_\circ^j)\xi_1{}^\nu\,,$$

$$C_i^j(t,x,\xi)\,;\ \textit{generalized Sylvester type}\ (i\ge 1)\,.$$

($\boldsymbol{q}_\circ^j$ *and pole set* $\Sigma=\bigcup_{j=1}^d\Sigma^j$ *are those in Propositions* 2.4 *and* 2.5.)

Here, the matrix F *of generalized Sylvester type corresponding to* $J(0,\boldsymbol{q}^j)$ *means the matrix which is decomposed to blocks* $\{F^{kh}_{k'h'}\}$ ($F^{kh}_{k'h'}$ *is the block corresponding to the* h*-th block of size* k *in the direction of row and to the* h'*-th block of size* k' *in the direction of column*, $1\le k,\ k'\le m_j,\ 1\le h,\ h'\le q^j_k$).

Further, $F^{kh}_{k'h'}$ *has the following form;*

$$(2.6)\qquad \begin{cases} \begin{pmatrix} & O & \\ * & \cdots\cdots & * \end{pmatrix} & \text{when } (k, h)\geq(k', h')\,, \\ \begin{pmatrix} * & & \\ \vdots & O & \\ * & & \end{pmatrix} & \text{when } (k, h)<(k', h')\,, \end{cases}$$

where, we give the lexicographic order to $\{(m_j-k+1, h)\,;\ 1\leq k\leq m_j,\ q^j_k\geq 1,\ 1\leq h\leq q^j_k\}$.

2) *Let* S *be* $S\{M_n, N_n\}$. *There exists an open conic dense subset* $U=\bigcup_{[q]\in Q} U([\boldsymbol{q}])$ *of* $\omega\times\Gamma$ *which is obtained in Proposition* 2.5 2). *For every* $(t_\circ, x_\circ, \xi_\circ)$ *in* $U([\boldsymbol{q}])$, *there exist a conic neighborhood* $\omega_\circ\times\Gamma_\circ$ *in* $U([\boldsymbol{q}])$ *and* $\tilde{N}(t, x, \xi)$ *in* $GL(K; S^0(\omega_\circ\times\Gamma_\circ))$ *such that* (2.6) *holds in* $\omega_\circ\times\Gamma_\circ$ *replacing* $C_0{}^j=J(0, \boldsymbol{q}^j_\circ)\xi_1{}^\nu$ *by* $C_0{}^j=J(0, \boldsymbol{q}^j)|\xi|^\nu$.

In both cases 1) *and* 2), *the entries of* $C^j_{i_\circ}(t, x, \xi)$ $(1\leq j\leq d)$ *and those of* $\tilde{N}_{i_\circ}(t, x, \xi)$ *are the polynomials of* $\{\lambda_j(t, x, \xi)\}_{1\leq j\leq d}$, *those derivatives of order up to* $i_\circ$, *the entries of* $A_i(t, x, \xi)$ *and those derivatives of order up to* $i_\circ-i$ $(0\leq i\leq i_\circ)$.

Remark 2.3. In Corollary 2.3 and Theorem 2.6, in case of $S=S\{M_n, N_n\}$, the entries of $N(t, x, \xi)$, $\tilde{N}(t, x, \xi)$, $B^j(t, x, \xi)$ and $C^j(t, x, \xi)$ belong to $S\{M_{n+3}, N_{n+3}\}$. If $\{M_n\}_{n=0}^\infty$ ($\{N_n\}_{n=0}^\infty$, respectively) satisfies Assumption 1, they also belong to $S\{M_n, N_{n+3}\}$ (to $S\{M_{n+3}, N_n\}$, respectively).

Remark 2.4. We can take $\tilde{N}_0(t, x, \xi)$ in $S^0_{Hhom}(\omega\times\Gamma)$ in case of $S=S_M$. Let us denote the pole set of $A(t, x, \xi)$ by Σ_A, that of $C^j(t, x, \xi)$ by Σ_{C^j} and the zero set of $det\tilde{N}_0(t, x, \xi)$ by $Z_{\tilde{N}_0}$. It holds that $\Sigma_{C^j}\subset\Sigma_A\cup Z_{\tilde{N}_0}$.

In order to see the form of matrix of generalized Sylvester type, we give an example corresponding to Example 2.1.

Example 2.3. In case of $K=m_1=10$ $(d=1)$, $\lambda_1=0$ and $\boldsymbol{q}^1=(0, 0, 0, 0\ 0, 0, 1, 0\ 2, 2)$, $J(0, \boldsymbol{q}^1)=J(4)\oplus J(2)\oplus J(2)\oplus J(1)\oplus J(1)$. Corresponding to $J(0, \boldsymbol{q}^1)$, the matrix of generalized Sylvester type has the following form,

$$\left(\begin{array}{cccc|cc|cc|c|c} & & & & & & & & & \\ * & * & * & * & * & * & * & * & * & * \\ \hline * & & & & & & & & & \\ * & & & & * & * & * & * & * & * \\ \hline * & & & & * & & & & & \\ * & & & & * & & * & * & * & * \\ \hline * & & & & * & & * & & * & * \\ \hline * & & & & * & & * & & * & * \end{array}\right)$$

where the entries vanish except the asterisks.

Remark 2.5. For $q^j=(q^j_{m_j}, q^j_{m_j-1}, \cdots, q^j_1)$ in $Q(m_j)$, we set $\delta_j=\sum_{k=1}^{m_j} q^j_k$ and $n_{jh}=k_\circ$ for $1+\sum_{k_\circ+1}^{m_j} q^j_k \leq h \leq \sum_{k_\circ}^{m_j} q^j_k$ ($1\leq k_\circ \leq m_j$, that is, $1\leq h\leq \delta_j$). It holds that $J(0, q^j)=\oplus_{1\leq h\leq \delta_j} J(n_{jh})$.

§ 3. Normal form of systems in formal symbol classes

If we adhere to the given principal part, Theorem 2.6 would be the best result. However, if we accept some entries of lower order terms as a part of the principal part, we can further transform $\tilde{N}^{-1}\circ P\circ\tilde{N}$ in Theorem 2.6 to more simple form. In order to carry out this, we need use degenerate and invertible formal symbols. (See the last part of § 1.)

Theorem 3.1 (Normal form of systems = Perfect block diagonalization). *We assume that $P(t, x, D_t, \xi)$ belongs to $Mat(K; S^\nu(\omega\times\Gamma)[D_t])$ and satisfies Assumption* 0 (*in case of $S=S_M$, further Assumption* 2 *and in case of $S=S\{M_n, N_n\}$, further Assumption* 1 *or* 1′ *on $\{M_n\}_{n=0}^\infty$ and $\{N_n\}_{n=0}^\infty$*).

1) *Let S be S_M. There exist $N(t, x, \xi)$ in $GL(K; S(\omega\times\Gamma))$, $\{\delta_j\}_{1\leq j\leq d}$ and $\{n_{jk}\}_{1\leq j\leq d, 1\leq k\leq \delta_j}$ (δ_j, $n_{jk}\in \mathbf{N}$, $\sum_{k=1}^{\delta_j} n_{jk}=m_j$), such that*

$$N(t, x, \xi)^{-1}\circ P(t, x, D_t, \xi)\circ N(t, x, \xi) \tag{3.1}$$

$$= \oplus_{1\leq j\leq d}\oplus_{1\leq k\leq \delta_j} Q^{jk}(t, x, D_t, \xi)$$

$$Q^{jk} = I_{n_{jk}}(D_t-\lambda_j(t, x, \xi))-D^{jk}(t, x, \xi)\,; \quad n_{jk}\times n_{jk}\,,$$

$$D^{jk}(t, x, \xi) = D_0{}^{jk}(t, x, \xi)+D'^{jk}(t, x, \xi)\,,$$

$$D_0{}^{jk}(t, x, \xi) = J(n_{jk})\xi_1{}^\nu \quad \in Mat(n_{jk}\,; S^\nu_{hom}(\omega\times\Gamma))\,,$$

$$D'^{jk}(t, x, \xi) = \begin{pmatrix} & O & \\ b^{jk}(1) & \cdots & b^{jk}(n_{jk}) \end{pmatrix} \quad \in Mat(n_{jk}\,; S^{\nu-1}(\omega\times\Gamma))\,.$$

2) *Let* S *be* $S\{M_n, N_n\}$. *There exists an open conic dense subset* $U=\bigcup_{[q]\in Q} U([\boldsymbol{q}])$ *of* $\omega\times\Gamma$ *(different from that in Theorem* 2.6). *For every* $(t_\circ, x_\circ, \xi_\circ)$ *in* $U([\boldsymbol{q}])$, *there exist a conic neighborhood* $\omega_\circ\times\Gamma_\circ$ *in* $U([\boldsymbol{q}])$ *and* $N(t, x, \xi)$ *in* $GL(K; S(\omega_\circ\times\Gamma_\circ))$ *such that* (3.1) *holds in* $\omega_\circ\times\Gamma_\circ$ *replacing* $\xi_1{}^\nu$ *by* $|\xi|^\nu$, *where* $\{\delta_j\}_{1\leq j\leq d}$ *and* $\{n_{jk}\}_{1\leq j\leq d, 1\leq k\leq\delta_j}$ *are decided by the rule given in Remark* 2.5. (*The entries of* $N(t, x, \xi)$ *and* $D^{jk}(t, x, \xi)$ *belong to* $S\{M_{n+k_\circ}, N_{n+k_\circ}\}(\omega_\circ\times\Gamma_\circ)$ *for some* $k_\circ$ *in* $\boldsymbol{Z}_+$. *If* $\{M_n\}_{n=0}^\infty$ ($\{N_n\}_{n=0}^\infty$, *respectively*) *satisfies Assumption* 1, *they also belong to* $S\{M_n, N_{n+k_\circ}\}(\omega_\circ\times\Gamma_\circ)$ (*to* $S\{M_{n+k_\circ}, N_n\}(\omega_\circ\times\Gamma_\circ)$, *respectively*).

In both cases 1) *and* 2), *we can take the entries of* $\{D_i^j(t, x, \xi)\}$ *and of* $\{N_i(t, x, \xi)\}$ *as the polynomials of the entries of* $\{A_i(t, x, \xi)\}$, $\{\lambda_j(t, x, \xi)\}_{1\leq j\leq d}$ *and those derivatives.*

Remark 3.1. The orders of the entries of $N(t, x, \xi)$ and $N(t, x, \xi)^{-1}$ are different each other and some of them may be positive.

Remark 3.2. On $\omega_\circ\times\Gamma_\circ$ (in case of $S = S_M$, we take $\omega_\circ\times\Gamma_\circ$ such as $\omega_\circ\times\Gamma_\circ\cap\Sigma=\phi$), we can construct true symbols from the fomal symbols $N(t, x, \xi)$ and $D(t, x, \xi)=\bigoplus_{1\leq j\leq d}\bigoplus_{1\leq k\leq\delta_j}D^{jk}(t, x, \xi)$. Of course, we need some additional conditions on $\{M_n\}_{n=0}^\infty$ and $\{N_n\}_{n=0}^\infty$ in case of $S=S\{M_n, N_n\}$. (See, for example, L. Boutet de Monvel and P. Krée [4], L. Boutet de Monvel [3], F. Treves [30] and W. Matsumoto [14].) Thus, Theorem 3.1 implies the theorem given in § 0.

Proof. As the proofs are parallel, we treat only the case of $S=S_M$. We start from Theorem 2.6. Since the operator is split to $P^1(t, x, D_t, \xi)$, $P^2(t, x, D_t, \xi)$, $\cdots$, and $P^d(t, x, D_t, \xi)$, we can consider each one independently. From now on, we only consider one of $\{P^j(t, x, D_t, \xi)\}$ and omit the suffix "j". We set $r=\max\{k; 1\leq k\leq m, q_k\geq 1\}$ and $r'=\min\{k; 1\leq k\leq m, q_k\geq 1\}$. Then, the size of the largest blocks in the Jordan normal form $C_0/\xi_1{}^\nu=J(0; \boldsymbol{q}_\circ)$ is $r\times r$ and the smallest one is $r'\times r'$.

We classify $C(t, x, \xi)$ to three cases.

1. There is at least an entry not vanishing identically on the first column from the $r+1$-th row to the last row.

2. Every entry on the first column from the $r+1$-th row to the last row vanishes identically but there is at least an entry not vanishing identically on the r-th row from the $r+1$-th column to the last column.

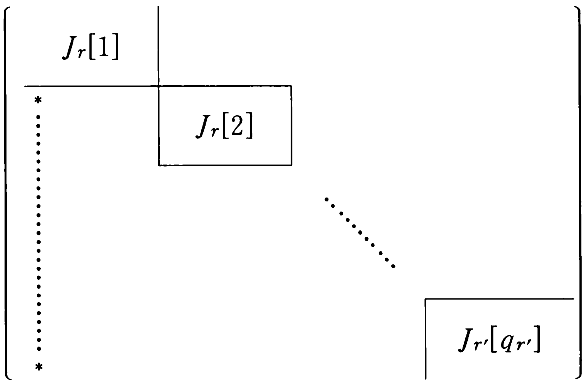
3. Every entry on the first column from the $r+1$-th row to the last row and on the r-th row from the $r+1$-th column to the last column vanishes identically.

In either case, we reduce the system as follows.

Case 1. Let μ be the maximum of the true orders of the entries on the first column from the $r+1$-th row to the last row. We take a weighting operator $W(\xi)=I_r\xi_1{}^{\nu-\mu}\oplus I_{m-r}$. Taking the similar transformation of $P(t, x, D_t, \xi)$ by $W(\xi)$, we have;

(3.2)
$$\tilde{P}(t, x, D_t, \xi) = W^{-1}(\xi)\circ P(t, x, D_t, \xi)\circ W(\xi)$$
$$= I_m(D_t-\lambda(t, x, \xi))-\tilde{C}(t, x, \xi)\,,$$
$$\tilde{C}(t, x, \xi) = \sum_{i=0}^{\infty}\tilde{C}_i(t, x, \xi) \in Mat(m\,; S^{\nu})\,.$$

$\tilde{C}_0/\xi_1{}^{\nu}$ has the following form, where $J_k[h]=J(k)$ $(r'\leq k\leq r)$.



$\tilde{C}_0$ rests nilpotent but has a Jordan chain longer than r.

Case 2. Let μ be the maximum of the true orders of the entries of the r-th row from the $r+1$-th column to the last column. We take a weighting operator $W(\xi)=I_r\xi_1{}^{\mu-\nu}\oplus I_{m-r}$ and take the similar transformation of $P(t, x, D_t, \xi)$ by $W(\xi)$;

(3.3)
$$\tilde{P}(t, x, D_t, \xi) = W^{-1}(\xi)\circ P(t, x, D_t, \xi)\circ W(\xi)$$
$$= I_m(D_t-\lambda(t, x, \xi))-\tilde{C}'(t, x, \xi)$$
$$\tilde{C}'(t, x, \xi) = \sum_{i=0}^{\infty}\tilde{C}'_i(t, x, \xi) \in Mat(m\,; S^{\nu})\,.$$

$\tilde{C}'_0/\xi_1{}^{\nu}$ has the following form.

$$\begin{pmatrix} J_r[1] & * \cdots\cdots\cdots\cdots * & & \\ & J_r[2] & & \\ & & \ddots & \\ & & & J_{r'}[q_{r'}] \end{pmatrix}$$

$\tilde{C}_0'$ rests nilpotent but has a Jordan chain longer than r.

In case of $S\{M_n, N_n\}$, the entries of $\tilde{C}(t, x, \xi)$ and $\tilde{C}'(t, x, \xi)$ belong to $S\{M_{n+\nu-\mu}, N_{n+\nu-\mu}\}$. If $\{M_n\}_{n=0}^{\infty}$ ($\{N_n\}_{n=0}^{\infty}$, resp.) satisfies Assumption 1, they also belong to $S\{M_n, N_{n+\nu-\mu}\}$ $S\{M_{n+\nu-\mu}, N_n\}$, resp.) by virtue of the differentiability of $\{M_n\}_{n=0}^{\infty}$ ($\{N_n\}_{n=0}^{\infty}$, resp.).

Case 3. $C(t, x, \xi)$ is split to the first $r\times r$ block and the rest $(m-r)\times(m-r)$ block. Here, the first block has the form of D^1 in (3.1).

In Cases 1 and 2, we apply Theorem 2.6 again, where new r is larger than the original r. In Case 3, we consider only the rest $(m-r)\times(m-r)$ block. In any case, we repeat this procedure. Since, through each procedure, new r becomes larger or else the size of system to be considered becomes smaller, we arrive at Theorem 3.1 in finite procedures. Here, $N(t, x, \xi)$ is the alternate products of non-degenerate transforming operators of type $\tilde{N}(t, x, \xi)$ in Theorem 2.6 and degenerate weighting operators of type $W(\xi)$ introduced above.

In each procedure of Cases 1 and 2, the zero set of $det\tilde{N}_0(t, x, \xi)$ is incorporated in the exceptional set.

Q.E.D.

§ 4. Applications to the Cauchy problem

§§ 4.1. Cauchy-Kowalevskaya theorem for systems. We consider the Cauchy problem in a complex domain Ω for a system of partial differential equations with holomorphic coefficients;

$$(4.1) \qquad \begin{cases} P(t, x, D_t, D_x)u = f(t, x)\,, \\ u(t_\circ, x) = \varphi(x)\,, \end{cases}$$

where

$$P(t, x, D_t, D_x) = I_K D_t - \sum_{|\alpha| \le \nu} A_\alpha(t, x) D_x{}^\alpha . \tag{4.2}$$

If $\nu=1$, (4.1) has the unique local holomorphic solution for holomorphic $\varphi(x)$ and $f(t, x)$. When $\nu>1$, we have the following theorem.

Theorem 4.1 (W. Matsumoto and H. Yamahara [17]). *In* (4.2), *we assume* $\nu>1$. *The following four conditions are equivalent.*

(a) $\forall(t_\circ, x_\circ)\in\Omega$, $\forall\omega$: *complex neighborhood of* $(t_\circ, x_\circ)$, $\forall\varphi(x)\in\mathcal{H}(\omega\cap\{t=t_\circ\})$, $\forall f(t, x)\in\mathcal{H}(\omega)$, $\exists 1 u(t, x)$: *holomorphic solution of* (4.1) *in a neighborhood of* $(t_\circ, x_\circ)$.

(b) $A_0(t, x, \xi)=\sum_{|\alpha|=\nu} A_\alpha(t, x)\xi^\alpha$ *is nilpotent and, in the normal form of* $P(t, x, D_t, \xi)$ *in Theorem* 3.1, *it holds that*

$$\text{true order of } b^{jk}(p) \le 1-(\nu-1)(n_{jk}-p), \quad (1\le p\le n_{jk}) . \tag{4.3}$$

(c) *In the meromorphic formal symbol class,* $P(t, x, D_t, \xi)$ *is transformed to a first order system.*

(d) *The determinant of* $P(t, x, D_t, D_x)$ *in sense of M. Sato and M. Kashiwara* [29] *is a Kowalevskayan polynomial, that is, its degree is* K *: the number of unknown functions and equations.*

We prove the above theorem as (a)⇒(b)⇒(c)⇒(a) and (b)⇔(d). We can prove (a)⇒(b) by the usual way as S. Mizohata [22] applying Theorem 3.1. (b)⇒(c) and (b)⇔(d) are trivial. Then, we need show (c)⇒(a). We reset $t_\circ=0$. We have the fundamental solution $E(t; x, D_x)=\sum_{k=0}^\infty t^k e(k)(x, D_x)$ for $f\equiv 0$, where $e(k)(x, D_x)$ is a differential operator of order at most νk with holomorphic coefficients. Under Condition (c), we can show the following inequality;

$$|e(k)(x, \xi)| \le \sum_{h=0}^{\infty} C R^{k-h}(k-h)!\|\xi\|^{h+\nu_\circ} , \tag{4.4}$$

on an arbitrary conically compact set in $\Omega\times \boldsymbol{C}_\xi^\ell\backslash\Sigma$, where $\nu_\circ$ is a non-negative integer, Σ is a conical analytic set and C and R are positive constants. Since $e(k)(x, \xi)$ is a holomornhic function of x and ξ, (4.4) holds on $\Omega\times \boldsymbol{C}_\xi^\ell$ by the maximum principle. This implies that $E(t; x, D_x)$ operates on the holomorphic functions of x on $\Omega\cap\{t=0\}$ for small $|t|$.

The proof of (c)⇒(a) is a modification of M. Miyake [18], where the formal solutions were considered. (See W. Matsumoto and Yamahara [17].)

A detailed proof will be given in the forthcoming paper.

§§ **4.2. Levi condition for systems.** We consider the Cauchy problem in a real domain Ω for general system of partial differential equations with real analytic coefficients;

$$(4.5)\qquad \begin{cases} P(t, x, D_t, D_x)u = f(t, x), \\ u(t_\circ, x) = \varphi(x), \end{cases}$$

where

$$(4.6)\qquad P(t, x, D_t, D_x) = I_K D_t - \sum_{|\alpha|\leq\nu} A_\alpha(t, x) D_x^\alpha .$$

By the same way as S. Mizohata [21] (or K. Kajitani [6]) and [20], we have the following theorem.

Theorem 4.2. *We assume that all coefficients of $P(t, x, D_t, D_x)$ are real analytic in Ω. If the Cauchy problem* (4.5) *is C^∞ well-posed in Ω, that is,*

(α) $\forall(t_\circ, x_\circ)\in\Omega$, $\exists\,\omega$: *neighborhood of* $(t_\circ, x_\circ)$, $\forall\varphi(x)\in C^\infty(\boldsymbol{R}^\ell)$, $\forall f(t, x)\in C^\infty(\boldsymbol{R}^{\ell+1})$, $\exists 1 u(t, x)\in C^\infty(\omega)$: *solution of* (4.5) *in* ω,

the following three equivalent conditions hold.

(β) $A_0(t, x, \xi)=\sum_{|\alpha|=\nu} A_\alpha(t, x)\xi^\alpha$ *is nilpotent and, in the normal form of* $P(t, x, D_t, \xi)$ *in Theorem* 3.1, *it holds that*

$$(4.7)\qquad \text{true order of } b^{jk}(p) \leq 1-(\nu-1)(n_{jk}-p), \quad (1\leq p\leq n_{jk}) .$$

Further, the principal part of (3.1) *in sense of Volevič has real characteristic roots.*

(γ) *In the meromorphic formal symbol class,* $P(t, x, D_t, \xi)$ *is transformed to a formally hyperbolic operator, that is, an operator of first order and with real characteristics.*

(δ) *The determinant of* $P(t, x, D_t, D_x)$ $(=\det_{SK}P)$ *in sense of M. Sato and M. Kashiwara* [29] *is a hyperbolic polynomial.* (*We say that* $p(t, x, \tau, \xi)$ *is hyperbolic if its total degree is equal to its degree on* τ *and it has only real roots on* τ *for all* (t, x, ξ) *in* $\Omega\times\boldsymbol{R}_\xi^\ell$.)

$det_{SK}P$ is a polynomial with real analytic coefficients. (See M. Sato and M. Kashiwara [29] and E. Andronikov [1].) Then, if a root on τ has a constant multiplicity, it is real analytic. Assuming that every root of $det_{SK}P$ has a constant multiplicity, we can again apply Theorem 3.1 on $N^{-1}\circ P\circ N$ in Condition (γ). We denote this new normal form by adding a tilde.

Theorem 4.3. *We assume that all coefficients of $P(t, x, D_t, D_x)$ are real analytic and every root of* $\det_{SK}P=0$ *on* τ *has a constant multiplicity. Then the following three conditions are equivalent.*

(α) *The Cauchy problem* (4.5) *is C^∞ well-posed in Ω.*

(ε) *Condition* (γ) *holds and further in the normal form of transformed operator* $N^{-1}\circ P\circ N$ *in Theorem* 3.1, *it holds that*

(4.8) $$\text{true order of } \tilde{b}^{jk}(p) \leq -(n_{jk}-p), \quad (1 \leq p \leq n_{jk}-1).$$

(ζ) *In the meromorphic formal symbol class,* $P(t, x, D_t, \xi)$ *is transformed to a formally hyperbolic system with a diagonal principal part.*

We prove the above theorem as $(\alpha)\Rightarrow(\varepsilon)\Rightarrow(\zeta)\Rightarrow(\alpha)$. We can show $(\alpha)\Rightarrow(\varepsilon)$ by the usual way as S. Mizohata [24] applying Theorem 3.1. $(\varepsilon)\Rightarrow(\zeta)$ is trivial. Then, we need to show $(\zeta)\Rightarrow(\alpha)$. We reset $t_\circ=0$. We have the fundamental solution $E(t, x, D_x)=\sum_{k=0}^{\infty} t^k e(k)(x, D_x)$ for $f\equiv 0$, where $e(k)(x, D_x)$ is a differential operator of order at most νk with holomorphic coefficients in a complex neighborhood $\hat{\Omega}$ of Ω. However, under Condition (δ), the order of $e(k)(x, D_x)$ is at most $k+\nu_\circ$. By virtue of the assumption of the constant multiplicity of characteristic roots, $E(t; x, D_x)$ is expressed as a Fourier integral operator $\sum_{j=1}^{d} E_{\phi_j}^j$: $E^j(t, x, \xi)=\sum_{k=0}^{\infty} t^k e^j(k)(x, \xi)$ modulo $\boldsymbol{S}_H^{-\infty}$, where ϕ_j is a phase function with respect to a characteristic root λ_j and $e^j(k)(x, \xi)$ belongs to $S_H^{k+\nu_\circ}(\Omega\times \boldsymbol{R}^\ell)$. Under Condition ($\zeta$), we can show the following inequality;

(4.9) $$|e^j(k)(x, \xi)| \leq CR^k k! \|\xi\|^{\nu_\circ'},$$

on an arbitrary conically compact set in $\hat{\Omega}\times\hat{\Gamma}\backslash\Sigma$, where $\nu_\circ'$ is a non-negative integer, $\hat{\Gamma}=\{\xi\in \boldsymbol{C}^\ell; |\mathrm{Im}\xi|<\varepsilon|\mathrm{Re}\xi|, \mathrm{Re}\xi\in\boldsymbol{R}^\ell\}$, Σ is a conical analytic set in $\hat{\Omega}\times\hat{\Gamma}$ and C and R are positive constants. Since $e^j(k)(x, \xi)$ is a holomorphic function of x and ξ, (4.9) holds on $\hat{\Omega}\times\hat{\Gamma}$ by the maximum principle. This implies that $E(t; x, D_x)$ operates on the functions of x in $C^\infty(\Omega\cap\{t=0\})$ for small $|t|$.

A detailed proof will be given in the forthcoming paper [16].

Remark 4.1. In Theorem 4.3, if we remove the real analyticity of coefficients, (ε) (=(ζ)) is necessary for C^∞ well-posedness but not sufficient. We can construct a counter-example:

$$I_3 D_t - \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} D_x - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu(t) \\ \nu(t) & 0 & 0 \end{pmatrix}$$

where $\mu(t)$ and $\nu(t)$ are nonnegative, they belong to $C^\infty(\boldsymbol{R})$, $\mathrm{supp}\mu=\{0\}\cup\bigcup_{n=1}^{\infty}[a_{2n}, a_{2n-1}]$ and $\mathrm{supp}\nu=\{0\}\cup\bigcup_{n=1}^{\infty}[a_{2n+1}, a_{2n}]$ $(a_n\searrow 0)$. The condition (ε) is satisfied on each $[a_{n+1}, a_n]$ $(n\in\boldsymbol{N})$, on $[a_1, \infty)$ and on $(-\infty, 0]$ but the Cauchy problem for the above operator is not C^∞ well-posed at the initial time 0. (See W. Matsumoto [16] and [12, I].)

We can also obtain the characterization of Gevrey well-posedness by the similar manner.

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