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Poisson measures on the configuration space and unitary representations of the group of diffeomorphisms

By

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Introduction

Let X be a connected para-compact but not compact C^{∞}-manifold and m be a locally Euclidean measure with smooth local densities. In [6], Vershik-Gel'fand-Graev considered representations of Diff X, group of diffeomorphisms with compact supports, defined by quasi-invariant measures, especially Poisson measures P_m in the space Γ_X of infinite configurations on X. The present paper is a supplement of their works and we summarize it as follows : First in section 1 we extend the notion of configuration space Γ_X to some general topological space X and show that Γ_X is a standard space equipped with a natural measurable structure $\mathscr{C}$. Next we consider Poisson measures P_m with intensity m on the measurable space $(\Gamma_X, \mathscr{C})$ and investigate the mutual equivalence of P_m with respect to another one, say $P_{m'}$ and investigate their ergodicity with respect to action groups arising from the basic space X. These are contents in section 2. Lastly in section 3 we generalize the results obtained in [6] of the equivalence of elementary representations of Diff X generated by Poisson measures. Our main result is stated in Theorem 3.1 and its Corollary in section 3.

1. Basic properties of configuration space

1.1. Definition of configuration space. Let K be a Polish space. That is, the topology of K is derived from a metric d such that (K, d) is a complete separable metric space. And let K^n be the direct product of the n copies of K and define a metric d_K^n on K^n such that $d_K^n(x, y) = \sum_{i=1}^n d(x_i, y_i)$, for $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n) \in K^n$. Then K^n is a Polish space with the metric d_K^n. Put $\widetilde{K}^n = \{x=(x_1, \cdots, x_n) | x_i \neq x_j \text{ for all } i \neq j\}$. As $\widetilde{K}^n$ is an open set in K^n, $\widetilde{K}^n$ is again a Polish space with the induced topology. A metric δ_K^n with which $(\widetilde{K}^n, \delta_K^n)$ is a complete separable metric space is for example as follows:

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$$\delta_K^n(x, y) = \frac{d_K^n(x, y)}{d_K^n(x, y) + d_K^n(x, (\widetilde{K}^n)^c) + d_K^n(y, (\widetilde{K}^n)^c)},$$

where

$d_K^n(x, (\widetilde{K}^n)^c)$ is the distance from x to the complemented set of $\widetilde{K}^n$. Next let us consider an n-point set γ in K. The collection of all such γ's will be denoted by B_K^n. For $\gamma = \{x_1, \cdots x_n\}$, $\gamma' = \{x'_1, \cdots, x'_n\} \in B_K^n$ put

$$d_K^{(n)}(\gamma, \gamma') = \inf_{\sigma \in \mathfrak{S}_n} d_K^n((x_1, \cdots, x_n), (x'_{\sigma(1)}, \cdots, x'_{\sigma(n)}))$$

and

$\delta_K^{(n)}(\gamma, \gamma') = \inf_{\sigma \in \mathfrak{S}_n} \delta_K^n((x_1, \cdots, x_n), (x'_{\sigma(1)}, \cdots, x'_{\sigma(n)}))$, where $\mathfrak{S}_n$ is the symmetric group. It is easily checked that $d_K^{(n)}$ and $\delta_K^{(n)}$ are equivalent metrics on B_K^n and $(B_K^n, \delta_K^{(n)})$ is a complete separable metric space. Therefore B_K^n is a Polish space with this topology. The Borel σ-field on B_K^n will be denoted by $\mathscr{B}(B_K^n)$. Now for each subset A in K let us consider a number map $N_A : B_K^n \to \{0, 1, \cdots, n\}$ defined by $N_A(\gamma) = |\gamma \cap A| \equiv {}^{\#}(\gamma \cap A)$, where ${}^{\#}A$ denotes the number of elements of a set A.

Lemma 1.1. *If U is an open set in K, then $\{\gamma | N_U(\gamma) \geqq l\}$ is also open in B_K^n for each $l = 0, 1, \cdots, n$.*

Proof. There is nothing to prove for $l = 0$. So let $N_U(\gamma_0) \geqq l \geqq 1$. By the definition of N_U, some l elements $x_1, \cdots, x_l$ of γ_0 exist in U. Take $\varepsilon > 0$ such that $U_\varepsilon(x_i) \subset U$ $(i = 1, \cdots, l)$, where $U_\varepsilon(x_i) = \{x \in K | d(x, x_i) < \varepsilon\}$. Then it is easy to see that $d_K^{(n)}(\gamma, \gamma') < \varepsilon$ implies $N_U(\gamma') \geqq l$. (Q. E. D)

It is a direct consequence of the above lemma that $N_B(\cdot)$ is $\mathscr{B}(B_K^n)$-measurable for all Borel sets B in K. The converse assertion also holds. For this let us see the following lemma.

Lemma 1.2. *For any $\varepsilon > 0$ and for any $\gamma \in B_K^n$ there exists some open set $O_\varepsilon(\gamma)$ which belongs to to the smallest σ-algebra $\mathscr{B}$ with which all the functions $N_B(\cdot)$ (B is a Borel set in K) are measurable such that $\gamma \in O_\varepsilon(\gamma) \subset \{\gamma' | d_K^{(n)}(\gamma, \gamma') < \varepsilon\}$.*

Proof. For the set $\gamma = \{x_1, \cdots, x_n\}$, let us take η such that $\varepsilon > \eta > 0$ and $U_{\eta/n}(x_i) \cap U_{\eta/n}(x_j) = \phi$ $(i \neq j)$ and put $O_\varepsilon(\gamma) = \bigcap_{i=1}^n \{\gamma' | |\gamma' \cap U_{\eta/n}(x_i)| \geqq 1\}$. Then we have $\gamma \in O_\varepsilon(\gamma) \in \mathscr{B}$ and $O_\varepsilon(\gamma)$ is an open set by Lemma 1.1. And if $\gamma' = \{y_1, \cdots, y_n\} \in O_\varepsilon(\gamma)$, then by the choice of η we may conclude that $y_i \in U_{\eta/n}(x_i)$ $(i = 1, \cdots, n)$. This implies $d_K^{(n)}(\gamma, \gamma') < \varepsilon$ and the lemma is proved. (Q. E. D)

Now take any open set G in B_K^n. Then by the above lemma and the separability of B_K^n there exist some open sets $O_{\varepsilon_n}(\gamma_n)$ $(\varepsilon_n>0)$ such that $G=\bigcup_{n=1}^{\infty}O_{\varepsilon_n}(\gamma_n)$. So we have $G\in\mathscr{B}$ and therefore $\mathscr{B}(B_K^n)\subset\mathscr{B}$. Hence we have,

Theorem 1.1. *$(B_K^n, d_K^{(n)})$ is a Polish space and the Borel σ-field $\mathscr{B}(B_K^n)$ coincides with the smallest σ-algebra with which all the functions $N_B(\cdot)$ (B is a Borel set in K) are measurable.*

Next let us consider the direct sum of B_K^n $(n=0, 1, \cdots)$, $B_K=\sum_{n=0}^{\infty}B_K^n$, where $B_K^0=\{\phi\}$. It is easy to see that B_K is again a Polish space with the direct sum topology and the Borel σ-field $\mathscr{B}(B_K)$ coincides with the smallest σ-algebra with which all the functions $N_B(\cdot)$ on B_K (B: Borel sets in K) are measurable. Now consider a topological space X which satisfies following two properties.

(B.1) X is a union of increasing subsets K_n $(n=1, 2, \cdots)$, and
(B.2) K_n is a Polish space with the induced topology of X for each n.

We shall call such a sequence $\{K_n\}$ basic sequence. Since a map π_{K_n,K_m} $(n<m)$: $\gamma\in B_{K_m}\to\gamma\cap K_n\in B_{K_n}$ is measurable with rspect to $\mathscr{B}(B_{K_m})$ and $\mathscr{B}(B_{K_n})$ in virtue of Theorem 1.1, so the projective limit of (B_{K_n},π_{K_n,K_m}), $\varprojlim(B_{K_n}, \pi_{K_n,K_m})=\{(\gamma_n)\in\prod_{n=1}^{\infty}B_{K_n}|\pi_{K_n,K_m}(\gamma_m)=\gamma_n \text{ for } m>n\}$ is a Borel set in the infinite product space $\prod_{n=1}^{\infty}B_{K_n}$, and the later is a Polish space with the product topology. Thus $\varprojlim(B_{K_n}, \pi_{K_n,K_m})$ is a standard space. (See, [4].) As is easily seen, there is a one-to-one correspondence between $\varprojlim(B_{K_n}, \pi_{K_n,K_m})$ and a set $\Gamma_X=\{\gamma|\gamma\subset X \text{ such that } |\gamma\cap K_n|<\infty \text{ for all } n\}$ which is called the configuration space on X. So identifying $\varprojlim(B_{K_n}, \pi_{K_n,K_m})$ with Γ_X, we have a standard measurable structure on Γ_X. It is easy to see that its σ-algebra $\mathscr{C}$ coincides with the smallest σ-algebra with which all the functions $N_B(\cdot)$ on Γ_X (B: Borel set in X) are measurable. Thus we have,

Theorem 1.2. *The measurable space $(\Gamma_X, \mathscr{C})$, where $\mathscr{C}$ is a minimal σ-algebra with which all the functions $N_B(\cdot)$ (B: Borel set in X) are measurable is a standard space.*

For a Borel subset Y in X we put $\Gamma_Y=\{\gamma\in\Gamma_X|\gamma\subset Y\}=\{\gamma\in\Gamma_X|\ |\gamma\cap Y^c|=0\}$. Naturally Γ_Y is a measurable subspace and its σ-algebra also coincides with the minimal σ-algebra with which all the number maps $N_B(\cdot)$ (B: Borel set in Y) are measurable.

Remark 1. When X is a locally compact and σ-compact metrizable space (for example X is a para-compact manifold), there is an increasing sequence $\{X_n\}$ of open sets with compact closure such that $\bigcup_{n=1}^{\infty}X_n=X$. If we choose this sequence $\{X_n\}$ as a basic sequence, then the configuration space Γ_X consists of countable sets γ which satisfies $|\gamma\cap K|<\infty$ for all compact sets K.

As is easily seen, it is equivalent to say that γ has no accumulation points in X.

1.2. Definition of Poisson measure. Let m be a non atomic Borel measure on X such that $m(K_n)<\infty$ for all n where $\{K_n\}$ is a basic sequence. Let K be one of K_n's and put $m_K=m|K$. By the non atomic assumption the product measure m_K^n of n copies of m_K is regarded naturally as a measure on $\widetilde{K}^n$. So we can define a measure $m_{K,n}$ on $\mathscr{B}(B_K^n)$ as the image measure of m_K^n by a map $p_K^n : (x_1, \cdots, x_n)\in\widetilde{K}^n\longrightarrow\{x_1, \cdots, x_n\}\in B_K^n$.
Put $P_{K,m}=\exp(-m(K))\sum_{n=0}^{\infty}\frac{m_{K,n}}{n!}$, where $m_{K,0}$ is a probability measure on the one point set B_K^0. It is easy to see that $P_{K,m}$ is a probability measure on $\mathscr{B}(B_K)$ and the following formula holds for any non negative integers $n_1, \cdots, n_l$ and for any disjoint Borel sets $B_1, \cdots, B_l$ in K (under an agreement that $0^0=1$),

$$(1)\qquad P_{K,m}(\cap_{i=1}^{l}\{\gamma||\gamma\cap B_i|=n_i\})=\Pi_{i=1}^{l}\frac{m(B_i)^{n_i}\exp(-m(B_i))}{n_i!}$$

Especially, $|\gamma\cap B_i|$ $(i=1, \cdots, l)$ are independent random variables whose laws are 1-dimensional Poisson measures with mean $m(B_i)$. Further it is a direct consequence of the above formula that $P_{K,m}$ is consistent. That is, $\pi_{K_n,K_l}P_{K_l,m}=P_{K_n,m}$ for all $n<l$. Since B_{K_n} $(n=1, 2, \cdots,)$ are Polish spaces, so by the well-known theorem (for example, see [4]) there corresponds uniquely a probability measure P_m on the projective limit space $(\Gamma_X, \mathscr{C})$ such that $\pi_{K_n}P_m=P_{K_n,m}$ for all n, where π_{K_n} is a map : $\gamma\in\Gamma_X\longrightarrow\gamma\cap K_n\in B_{K_n}$.
The measure P_m is called the Poisson measure. The following is also a direct consequence of (1). For any non negative integers $n_1, \cdots, n_l$ and for any disjoint Borel sets $B_1, \cdots, B_l$ in X we have

$$(2)\qquad P_m(\cap_{i=1}^{l}\{\gamma||\gamma\cap B_i|=n_i\})=\Pi_{i=1}^{l}\frac{m(B_i)^{n_i}\exp(-m(B_i))}{n_i!}$$

Remark 2. Let μ_{K_l} be a probability measure on $\mathscr{B}(B_{K_l})$ defined by $\mu_{K_l}=\sum_{n=0}^{\infty}\frac{c_{l,n}}{n!}m_{K_l,n}$ where $c_{l,n}$ are non negative constants. If it happens that μ_{K_l} $(l=1, 2, \cdots,)$ is consistent by the map π_{K_n,K_l} choosing suitable constants $c_{l,n}$, then a probability measure μ arises on $(\Gamma_X, \mathscr{C})$ such that $\pi_{K_l}\mu=\mu_{K_l}$. In [3], Obata considered a characterization of such μ and obtained a result that in case $m(X)=\infty$, μ is a superposition of Poisson measures P_{cm} $(c\geqq 0)$. More exactly, μ can be represented as $\mu=\int_0^{\infty}P_{cm}\lambda(dc)$ with a suitable Borel measure λ on $[0, \infty)$.

2. Poisson measure

2.1. Basic formulas. Let X be a topological space with properties (B.1) and (B.2), $\{K_n\}$ be a basic sequence, and m be a non atomic Borel measure on X such that $m(K_n)<\infty$ for all n.

Lemma 2.1. *Let $\rho(x)$ be a non negative measurable function on X such that $\rho(x)=1$ on K_n^c and $\int_{K_n}\rho(x)\ \mathrm{m}(dx)<\infty$ for some n. Then a function $\Pi_{x\in\gamma}\ \rho(x)$ defined on Γ_X is measurable and for any non negative integers $n_1,\cdots,n_l$ and for any disjoint Borel sets $B_1,\cdots,B_l$ we have,*

$$\int_{\cap_{i=1}^{l}\{\gamma\|\gamma\cap B_i|=n_i\}}\Pi_{x\in\gamma}\ \rho(x)P_m(d\gamma)=\exp(m'(K_n)-m(K_n))\ \cdot \tag{3}$$

$P_{m'}(\cap_{i=1}^{l}\{\gamma\|\gamma\cap B_i|=n_i\})$, *where m' is a Borel measure on X defined by $m'(B)=\int_B\rho(x)m(dx)$.*

Proof. Without loss of generality we may assume that $B_i\subset K_N$ $(i=1,\cdots,l)$ for some $N(\geqq n)$. Let us approximate $\rho(x)$ with step functions $\rho_h(x)$ $(h=1,2,\cdots)$ which is increasing with respect to h : $\rho_h(x)=\Sigma_{k=1}^{s}c_k\chi_{A_k}(x)+\chi_{K_N^c}(x)$, where $\{A_1,\cdots,A_s\}$ is a Borel partition of K_N and χ_A is the indicator function of a set A. It may be assumed that $\{A_1,\cdots,A_s\}$ is a subdivision of $\{B_1,\ \cdots,\ B_l,\ K_N\cap\ (B_1\cup\cdots\cup B_l)^c\}$, so we have $B_1=\cup_{i=1}^{s_1}A_i$, $B_2=\cup_{i=s_1+1}^{s_2}A_i,\cdots,B_l=\cup_{i=s_{l-1}+1}^{s_l}A_i$ for suitable numbers $1\leqq s_1<\cdots<s_l\leqq s$.
Since $\Pi_{x\in\gamma}\rho_h(x)=\prod_{i=1}^{s}c_i^{k_i}$ on $\cap_{i=1}^{s}\{\gamma\|\gamma\cap A_i|=k_i\}$, it is a measurable function of γ for each h and so is $\Pi_{x\in\gamma}\rho(x)$.
Next as we have,

$$\int_{\cap_{i=1}^{l}\{\gamma|\gamma\cap B_i|=n_i\}}\Pi_{x\in\gamma}\rho_h(x)P_m(d\gamma)$$
$$=\Sigma'\int_{\cap_{i=1}^{s}\{\gamma|\gamma\cap A_i|=k_i\}}\Pi_{i=1}^{s}c_i^{k_i}P_m(d\gamma),$$

where Σ' is a sum for $k_1,\cdots,k_s$ such that $k_1+\cdots+k_{s_1}=n_1,\cdots,k_{s_{l-1}+1}+\cdots+k_{s_l}=n_l$ and $k_j=0,1,\cdots,(s_l+1\leqq j\leqq s)$,

$$=\Sigma'\Pi_{i=1}^{s}\frac{c_i^{k_i}m(A_i)^{k_i}\exp(-m(A_i))}{k_i!}$$
$$=\exp(-m(K_N\setminus\cup_{i=1}^{l}B_i))\exp\Big(\int_{K_N\setminus\cup_{i=1}^{l}B_i}\rho_h(x)m(dx)\Big)\ \cdot$$
$$\Pi_{i=1}^{l}\frac{\Big(\int_{B_i}\rho_h\ (x)m(dx)\Big)^{n_i}\exp(-m(B_i))}{n_i!}.$$

So (3) follows by letting $h \longrightarrow \infty$. Notice that $m'(K_N) - m(K_N) = m'(K_n) - m(K_n)$. (Q. E. D.)

The following result is derived by the same reasoning, so we omit its proof.

Lemma 2.2. *Let $\rho(x)$ be a non negative integrable function defined on K_n and put $m'(B) = \int_B \rho(x) m(dx)$ for all Borel sets B in K_n.*
Then we have

$$P_{K_n,m'}(E) = \exp(-m'(K_n) + m(K_n)) \int_E \Pi_{x\in\gamma}\rho(x) P_{K_n,m}(d\gamma) \tag{4}$$
for all $E \in \mathcal{B}(B_{K_n})$.

2.2. Mutual equivalence.

Let m and m' be non atomic Borel measures on X such that $m(K_n)$, $m'(K_n) < \infty$ for all n.

Theorem 2.1. *If $P_{m'}$ is absolutely continuous with respect to P_m ($P_m \gtrsim P_{m'}$), then $m \gtrsim m'$.*

Proof. Let $m(B) = 0$. Then $m(B \cap K_n) = 0$ for all n and $P_m(\gamma \| \gamma \cap B \cap K_n| = 1) = 0$. From the assumption, it follows that $P_{m'}(\gamma \| \gamma \cap B \cap K_n| = 1) = 0$ and therefore $m'(B \cap K_n) = 0$ for all n. Hence we have $m'(B) = 0$. (Q. E. D.)

The first part of the following theorem is already stated in [5]. However we prove it in a different even simpler manner from the original one.

Theorem 2.2. *Assume that $m \gtrsim m'$, and put $\frac{dm'}{dm}(x) = \rho(x)$. Then in order that $P_m \gtrsim P_{m'}$, it is necessary and sufficient that $\int_X |\sqrt{\rho(x)} - 1|^2 m(dx) < \infty$. Further if $\int_X |\sqrt{\rho(x)} - 1|^2 m(dx) = \infty$, then P_m and $P_{m'}$, are singular.*

Proof. As is easily seen from (4), we have $P_{K_n,m'} \lesssim P_{K_n,m}$ and $\frac{dP_{K_n,m'}}{dP_{K_n,m}}(\gamma) = \exp(-m'(K_n) + m(K_n)) \Pi_{x\in\gamma}\rho(x)$ for all n.
Hence in order that $P_{m'} \lesssim P_m$ it is necessary and sufficient that $\left\{\sqrt{\frac{dP_{K_n,m'}}{dP_{K_n,m}}(\gamma \cap K_n)}\right\}$ forms a Cauchy sequence in $L^2_{P_m}(\Gamma_X)$ which is assured by the well-known theorem. (See, [7]). So we shall calculate the values

$$\phi_{n,l} = \int_{\Gamma_X} \left| \sqrt{\frac{dP_{K_n,m'}}{dP_{K_n,m}}(\gamma \cap K_n)} - \sqrt{\frac{dP_{K_l,m'}}{dP_{K_l,m}}(\gamma \cap K_l)} \right|^2 P_m(d\gamma)$$

for $l > n$, noticing that $\Pi_{x\in\gamma\cap K_n}\rho(x)$ and $\Pi_{x\in\gamma\cap(K\backslash K_n)}\sqrt{\rho(x)}$ are independent random variables with respect to $P_{K_l,m}$. Now applying (4) to $\sqrt{\rho}$ instead of ρ we have,

$$\phi_{n,l}=2[1-\exp\{1/2(m(K_n)-m'(K_n)+m(K_l)-m'(K_l))\}\cdot$$

$$\int_{B_{K_l}}\Pi_{x\in\gamma\cap K_n}\rho(x)\ \Pi_{x\in\gamma\cap(K\setminus K_n)}\sqrt{\rho(x)}P_{K_l,m}(d\gamma)]$$

$$=2\Big[1-\exp\{1/2(-m(K_n)+m'(K_n)+m(K_l)-m'(K_l))\}\cdot$$

$$\exp\Big(\int_{K_l\setminus K_n}\sqrt{\rho(x)}\,m(dx)-m(K_l\setminus K_n)\Big)\Big]$$

$$=2\Big\{1-\exp\Big(-1/2\int_{K_l\setminus K_n}(\sqrt{\rho(x)}-1)^2m(dx)\Big)\Big\}.$$

Thus $\phi_{n,l}\to 0$ $(n, l\to\infty)$ is equivalent to $\int_X|\sqrt{\rho(x)}-1|^2m(dx)<\infty$.

If $\int_X|\sqrt{\rho(x)}-1|^2m(dx)=\infty$, then it follows from the above calculation,

$$\lim_{n\to\infty}\ \lim_{l\to\infty}\int_{\Gamma_X}\sqrt{\frac{dP_{K_n,m'}}{dP_{K_n,m}}}(\gamma\cap K_n)\ \sqrt{\frac{dP_{K_l,m'}}{dP_{K_l,m}}}(\gamma\cap K_l)P_m(d\gamma)=0. \tag{5}$$

By the way, $\frac{dP_{K_n,m'}}{dP_{K_n,m}}(\gamma\cap K_n)$ converges to a function $f_\infty(\gamma)$ for P_m — a. e. γ as $n\longrightarrow\infty$ by the martingale convergence theorem, and $f_\infty(\gamma)$ is the density function of the absolutely continuous part of $P_{m'}$ with respect to P_m. Applying Lebesgue-Fatou's lemma twice to (5), we get $\int_{\Gamma_X}f_\infty(\gamma)P_m(d\gamma)=0$ which shows P_m and $P_{m'}$ are singular. (*Q. E. D.*)

Corollary. *The Hellinger distance between P_m and $P_{m'}$ is given by*

$$\int_{\Gamma_X}\left|\sqrt{\frac{dP_{m'}}{dP_m}}(\gamma)-1\right|^2P_m(d\gamma) \tag{6}$$

$$=2\Big\{1-\exp\ (-1/2\int_X(\sqrt{\rho(x)}-1)^2m(dx))\Big\}$$

2.3. Ergodicity. Let G be a group of bimeasurable maps $\phi: X\longrightarrow X$ such that $m\simeq\phi\, m$ (image measure of m by the map ϕ) and

$\int_X\left|\sqrt{\frac{d\phi m}{dm}}(x)-1\right|^2m(dx)<\infty$. Note that $\phi m(K_n)<\infty$ for all n, because $\sqrt{\phi m(K_n)}=\Big\{\int_{K_n}\frac{d\phi m}{dm}(x)m(dx)\Big\}^{1/2}\leqq\Big\{\int_{K_n}\left|\sqrt{\frac{d\phi m}{dm}}(x)-1\right|^2m(dx)\Big\}^{1/2}+m(K_n)^{1/2}$ $<\infty$. Hence $P_{\phi m}$ is well defined and $P_{\phi m}\simeq P_m$. Next we put $\phi(\gamma)=\{\phi(x_1),\cdots,\phi(x_n),\cdots\}$ for all $\gamma=\{x_1,\cdots,x_n,\cdots\}\in\Gamma_X$. It must be noticed that $\phi(\gamma)$

does not necessarily belong to Γ_X. Nevertheless, $|\phi(\gamma)\cap K_n|=|\gamma\cap\phi^{-1}(K_n)|<\infty$ for $P_m-a.e.\ \gamma$, because $\phi m(K_n)<\infty$. So a map $T_\phi:\gamma\in\Gamma_X\longrightarrow\phi(\gamma)\in\Gamma_X$ is defined almost everywhere with respect to P_m.

Definition 1. P_m is said to be G-ergodic, if $P_m(A)=1$ or 0 provided that $P_m(A\ominus T_\phi^{-1}(A))=0$ for all $\phi\in G$.

If $m(X)<\infty$, then P_m is not ergodic, because $B_X^n\equiv\{\gamma\in\Gamma_X\,||\gamma|=n\}$ is a G-invariant set but $P_m(B_X^n)=\frac{m(X)^n}{n!}\exp(-m(X))\neq 1, 0$ for each n. Generally speaking, the ergodicity of P_m has no relation with that of m. Now we shall state sufficient conditions for the ergodicity as the following two theorems.

Theorem 2.3. *If for any $\varepsilon>0$ and for any n there exists $\phi\in G$ such that $\phi(K_n)\cap K_n=\phi$ and $\int_X\left|\sqrt{\frac{d\phi m}{dm}}(x)-1\right|^2 m(dx)<\varepsilon$, then P_m is G-ergodic.*

Proof. First of all we shall claim that

(7) $\qquad P_m(T_\phi^{-1}(E))\leqq P_m(E)+A_\phi$ for all $\phi\in G$ and for all $E\in\mathscr{C}$,

where $A_\phi=2\sqrt{2}\left\{1-\exp(-1/2\int_X\left|\sqrt{\frac{d\phi m}{dm}}(x)-1\right|^2 m(dx))\right\}^{1/2}$.

In fact we have

$$P_m(T_\phi^{-1}(E))=\int_E\frac{dP_{\phi m}}{dP_m}(\gamma)P_m(d\gamma)\leqq P_m(E)+\int_E\left|\frac{dP_{\phi m}}{dP_m}(\gamma)-1\right|P_m(d\gamma)$$

$$\leqq P_m(E)+2\left\{\int_{\Gamma_X}\left|1-\sqrt{\frac{dP_{\phi m}}{dP_m}}(\gamma)\right|^2 P_m(d\gamma)\right\}^{1/2}$$

$$=P_m(E)+2\sqrt{2}\left\{1-\exp(-1/2\int_X\left(\sqrt{\frac{d\phi m}{dm}}(x)-1\right)^2 m(dx))\right\}^{1/2},$$

where the last inequality is derived from (6).
Now let A be a measurable set such that $P_m(A\ominus T_\phi^{-1}(A))=0$ for all $\phi\in G$. We take $B_n\in\mathscr{B}(B_{K_n})$ such that $P_m(A\ominus\pi_{K_n}^{-1}(B_n))<\varepsilon$ for a given $\varepsilon>0$. Then we have $P_m(A\ominus T_\phi^{-1}\pi_{K_n}^{-1}(B_n))<\varepsilon+A_\phi$ by virtue of taking E as $A\ominus\pi_{K_n}^{-1}(B_n)$ in (7). By the assumption there exists a map $\phi\in G$ such that $\phi(K_n)\cap K_n=\phi$ and $A_\phi<\varepsilon$. It follows from the regionally independence of Poisson measure that

$(P_m(A)-2\varepsilon)\ (P_m(A^c)-\varepsilon)<P_m(T_\phi^{-1}\pi_{K_n}^{-1}(B_n))P_m(\pi_{K_n}^{-1}(B_n^c))=$
$P_m(T_\phi^{-1}\pi_{K_n}^{-1}(B_n)\cap\pi_{K_n}^{-1}(B_n^c))\leqq P_m(T_\phi^{-1}\pi_{K_n}^{-1}(B_n)\ominus A)+P_m(\pi_{K_n}^{-1}(B_n^c)\ominus A^c)$
$<\varepsilon+A_\phi+\varepsilon<3\varepsilon$.
Letting $\varepsilon\longrightarrow 0$, we have $P_m(A)P_m(A^c)=0$. (*Q. E. D.*)

Definition 2. Let $G_{K_n}=\{\phi\in G|\phi=\text{identity on } K_n^c\}$ and let f be a symmetric measurable function defined on $\widetilde{K}_n^l$ $(l=1, 2, \cdots)$.
We say that m is $G_{K_n}^l$-ergodic, if f is constant modulo null sets provided that for all $\phi\in G_{K_n}$, $f(x_1, \cdots, x_l)=f(\phi(x_1), \cdots, \phi(x_l))$ for $m_{K_n}^l-a.e.x=(x_1, \cdots, x_l)$.

Theoren 2.4. *If for any n, m is $G_{K_N}^l$-ergodic for some $N\geqq n$ and for all l, then P_m is G-ergodic provided that $m(X)=\infty$.*

Proof. If necessary taking a subsequence of the basic sequence, we may assume that m is $G_{K_n}^l$-ergodic for all n and l. Let P_n^1, P_n^2 be image measures of P_m by the maps π_{K_n}, $\pi_{K_n^c}$, $\pi_{K_n^c}(\gamma)=\gamma\cap K_n^c$, respectively. Then P_m is regarded as the product measure of P_n^1 and P_n^2. Now assume that a measurable set A satisfies $P_m(A\ominus T_\phi^{-1}(A))=0$ for all $\phi\in G$. For each n we put

$$f_n(\gamma_1)=\int_{\Gamma_{K_n^c}}\chi_A(\gamma_1\cup\gamma_2)P_n^2(d\gamma_2) \qquad \text{for } \gamma_1\in B_{K_n}.$$

Then for all $\phi\in G_{K_n}$ we have,

$$0=\int_{B_{K_n}}|f_n(\gamma_1)-f_n(\phi(\gamma_1))|P_n^1(d\gamma_1)=$$

$$\sum_{l=0}^{\infty}\frac{\exp(-m(K_n))}{l!}\int_{\widetilde{K}_n^l}|f_n(\{x_1,\cdots, x_l\})-f_n(\{\phi(x_1),\cdots, \phi(x_l)\})|m_{K_n}^l(dx).$$

Thus the symmetric function : $(x_1, \cdots, x_l)\longrightarrow f_n(\{x_1, \cdots, x_l\})$ satisfies the assumption of $G_{K_n}^l$-ergodicity, so it follows that $f_n(\{x_1, \cdots, x_l\})=\text{const}$ $(\equiv c_{n,l})$ for $m_{K_n}^l$-$a.e.x$. Define a new measure ν by $\nu(E)=P_m(A\cap E)$ for all $E\in\mathscr{C}$. Then for any $B\in\mathscr{B}(B_{K_n})$ we have,

$$\nu(\pi_{K_n}^{-1}(B))=\int_B f_n(\gamma_1)P_n^1(d\gamma_1)=\sum_{l=0}^{\infty}\frac{\exp(-m(K_n))}{l!}c_{n,l}, m_{K_n,l}(B\cap B_{K_n}^l).$$

Therefore there exists some measure λ on $[0, \infty)$ such that

$\nu=\int_0^\infty P_{cm}\lambda(dc)$ in virtue of Remark 2. As $\nu\leqq P_m$ and $\lim_{N\to\infty}\frac{1}{N}\sum_{l=1}^{N}\frac{|\gamma\cap(K_{l+1}\setminus K_l)|}{m(K_{l+1}\setminus K_l)}=c$ for $P_{cm}-a.e.\gamma$ by the law of large numbers, so we have $\lambda(\{1\}^c)=0$ and therefore $\nu=\lambda(\{1\})P_m$. This shows $P_m(A^c)=0$ if $\lambda(\{1\})>0$ and $P_m(A)=0$ if $\lambda(\{1\})=0$. (*Q.E.D.*)

The next theorem is already stated in [6] but we shall list and prove it as an application of Theorem 2.4.

Theorem 2.5. *P_m is G-ergodic under the following situation.*

(a) X is a connected para-compact but not compact C^∞-manifold,
(b) a basic sequence $\{K_n\}$ is a sequence of connected open sets with compact closure,
(c) m is a locally Euclidean infinite measure whose local densities (with respect to the Lebesgue measure) on each coordinate neighbourhood are all C^∞-functions,
(d) G is composed of all C^∞-diffeomorphisms ψ with compact supports.
That is, there exists some compact set K depending on ψ such that ψ is identity on K^c. We shall denote this group by Diff X.

Proof. Fix n and put $K_n=K$, $m|K=m_K$. Then for the proof it is sufficient to show that $m_K^l(A)m_K^l(A^c)=0$ holds for a measurable set $A\subset\widetilde{K}^l$ $(l=1, 2, \cdots)$ which satisfies $m_K^l(A\ominus T_\psi^{-1}(A))=0$ for all $\psi\in$ Diff K, where $T_\psi: x=(x_1, \cdots, x_l)\in\widetilde{K}^l\longrightarrow(\psi(x_1), \cdots, \psi(x_l))\in\widetilde{K}^l$ and Diff $K=\{\psi\in$ Diff $X|\psi=$ identity on $K^c\}$. Suppose that $m_K^l(A)>0$ and put $\mu(B)=m_K^l(B\cap A)$ for all Borel sets B in $\widetilde{K}^l$. By the assumption μ is Diff K-quasi-invariant and Diff K acts transitively on $\widetilde{K}^l$. Thus we have $\mu(U_1\times\cdots\times U_l)>0$ for all disjoint open subset $U_i\subset K$ $(i=1, \cdots, l)$. Take an arbitrary point $(x_1, \cdots, x_l)\in\widetilde{K}^l$ and take disjoint neighbourhood U_i of x_i $(i=1, \cdots, l)$ which are diffeomorphic to disks $D_i\subset\mathbf{R}^{\dim(X)}$ under maps ψ_i, and put $\psi_i(m|U_i)=\lambda_i$. $\lambda_1\times\cdots\times\lambda_l$ is equivalent to the Lebesque measure λ on $D_1\times\cdots\times D_l$. Further we put $\psi=(\psi_1, \cdots, \psi_l): U_1\times\cdots\times U_l\longrightarrow D_1\times\cdots\times D_l$ and $\widehat{A}=\psi(A\cap U_1\times\cdots\times U_l)$. Now consider a group $\widehat{\mathrm{Diff}}(D_1\times\cdots\times D_l)$ of all diffeomorphisms ϕ on $D_1\times\cdots\times D_l$ such that $\phi(t_1, \cdots, t_l)=(\phi_1(t_1), \cdots, \phi_l(t_l))$ for all $(t_1, \cdots, t_l)\in D_1\times\cdots\times D_l$, where ϕ_i is a diffeomorphism on D_i with compact support $(i=1, \cdots, l)$. It is not difficult to show that $\lambda|D_1\times\cdots\times D_l$ is $\widehat{\mathrm{Diff}}(D_1\times\cdots\times D_l)$-ergodic. (It is even $\widehat{\mathrm{Diff}}(D_1\times\cdots\times D_l, \lambda)$-ergodic in case $\dim(X)>1$, where $\widehat{\mathrm{Diff}}(D_1\times\cdots\times D_l, \lambda)=\{\phi\in\widehat{\mathrm{Diff}}(D_1\times\cdots\times D_l)|\phi\lambda=\lambda\}$.) Since $\psi^{-1}\phi\psi$ is regarded naturally as an element of Diff K, it follows that $(\lambda_1\times\cdots\times\lambda_l)$ $(\widehat{A}\ominus\phi(\widehat{A}))=m_K^l(A\cap U_1\times\cdots\times U_l)\ominus\psi^{-1}\phi\psi(A\cap U_1\times\cdots\times U_l))=m_K^l((A\ominus T_\psi^{-1}(A))\cap U_1\times\cdots\times U_l)=0$, and therefore $\lambda(\widehat{A}\ominus\phi(\widehat{A}))=0$. Hence we have $\lambda(\widehat{A})=0$ or $\lambda(\widehat{A}^c\cap D_1\times\cdots\times D_l)=0$. However $\lambda(\widehat{A})>0$ which follows from $\mu(U_1\times\cdots\times U_l)>0$. It follows that $m_K^l(A^c\cap U_1\times\cdots\times U_l)=(\lambda_1\times\cdots\times\lambda_l)$ $(\widehat{A}^c\cap D_1\times\cdots\times D_l)=0$.
By the second countable axiom we have $m_K^l(A^c)=0$. *(Q. E. D.)*

Remark 3. In a similar but rather complicated way we can show that P_m is Diff (X, m)-ergodic under the same situation with $\dim(X)>1$, where Diff (X, m) is the set of all $\psi\in$Diff X which preserve m.

3. Elementary representations of Diff *X* generated by Poisson measures

3.1. Elementary representations. *From now on we shall assume that*

(a) *X is a connected para-compact but not compact C^∞-manifold,*
(b) *the basic sequence $\{X_n\}$ is a sequence of connected open sets with compact closure,*
(c) *m is a locally Euclidean infinite measure with smooth local densities,*
(d) $G=\text{Diff } X$.

In [6], Vershik-Gel'fand-Graev defined elementary representations and discussed their several properties. Here we pick up a problem of their mutual equivalence and extend their results.

Now consider the following canonical representation of Diff X in $L^2_{P_m}(\Gamma_X)$

$$U_m(\psi): f(\gamma) \longrightarrow \sqrt{\frac{dP_{\psi m}}{dP_m}}(\gamma)\, f(\psi^{-1}(\gamma)). \tag{8}$$

U_m is an irreducible unitary representation of Diff X (See, [6]). Moreover let us consider the following representation V^ρ of another type. For this let $n \geqq 1$ be an integer and $p_n: \widetilde{X}_n \longrightarrow B^n_X$ be a map such that $(x_1, \cdots, x_n) \longrightarrow \{x_1, \cdots, x_n\}$. Then a function σ on Diff $X \times B^n_X$ with values in the symmetric group, $\mathfrak{S}_n$ is defined by the formula, $s_n(\psi^{-1}(\gamma)) = \psi^{-1}(s_n(\gamma))\,\sigma(\psi, \gamma)$, where $(x_1, \cdots, x_n)\,\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ and $s_n: B^n_X \longrightarrow \widetilde{X}_n$ is a measurable cross section of p_n. Now we associate with each pair (n, ρ), where ρ is a unitary representation of $\mathfrak{S}_n$ in a Hilbert space W, a unitary representation V^ρ of Diff X in $L^2_{m_n}(B^n_X, W)$ such that

$$V^\rho(\psi): f(\gamma) \longrightarrow \sqrt{\frac{d\psi m_n}{dm_n}}(\gamma)\,\rho(\sigma(\psi, \gamma)) f(\psi^{-1}(\gamma)), \tag{9}$$

where m_n is the image measure of the direct product of n copies of m by the map p_n and ψm_n is the image measure of m_n by a map : $\gamma \in B^n_X \longrightarrow \psi(\gamma) \in B^n_X$. If ρ is irreducible, then so is V^ρ, and two representations V^{ρ_1} and V^{ρ_2}, where ρ_1 and ρ_2 are irreducible representations of $\mathfrak{S}_{n_1}$ and $\mathfrak{S}_{n_2}$, respectively, are equivalent, if and only if $n_1 = n_2$ and ρ_1 and ρ_2 are equivalent (See, [6]). Vershik-Gel'fand-Graev called a representation of Diff X of the form

$$U^\rho_m = U_m \otimes V^\rho \tag{10}$$

elementary representation associated with the Poisson measure and obtained the following results

(a) *U^ρ_m is irreducible if ρ is so, and*
(b) *$U^{\rho_1}_{c_1 m}$ is equivalent to $U^{\rho_2}_{c_2 m}$, where c_1 and c_2 are positive constants, if and only if $c_1 = c_2$ and ρ_1 and ρ_2 are equivalent.*

In this section we shall consider the equivalence of U^ρ_m, varying m among all locally Eucidean infinite measures with smooth local densities. To see this, it is convenient to deform the representation U^ρ_m to another form. Put

$\widetilde{\mathbf{N}}^n=\{a=(i_1,\cdots,i_n)|i_j\in\mathbf{N}$ such that $i_p\neq i_q(p\neq q)\}$, $l^2(\widetilde{\mathbf{N}}^n, W)=\{\phi|\phi$ is a W-valued function defined on $\widetilde{\mathbf{N}}^n$ such that $\|\phi\|^2\equiv\sum_{a\in\widetilde{\mathbf{N}}^n}\|\phi(a)\|_W^2<\infty\}$ and $H^\rho=\{\phi\in l^2(\mathbf{N}, W)|\phi(i_{\sigma(1)},\cdots,i_{\sigma(n)})=\rho^{-1}(\sigma)\phi(i_1,\cdots,i_n)$ for all $\sigma\in\mathfrak{S}_n\}$, where ρ is a unitary representation of $\mathfrak{S}_n$ in a Hilbert space W. Further let $\mathfrak{S}^\infty$ be the set of all permutations on $\mathbf{N}$ and put $\sigma a=(\sigma(i_1),\cdots,\sigma(i_n))$ for $\sigma\in\mathfrak{S}^\infty$ and for $a\in\widetilde{\mathbf{N}}^n$. As before we define a function σ on Diff $X\times\Gamma_X$ with values in $\mathfrak{S}^\infty$ by the formula, $s(\psi^{-1}(\gamma))=\psi^{-1}(s(\gamma))\,\sigma(\psi,\gamma)$, where s is a measurable (admissible) cross section of the map $p:\widetilde{X}^\infty\ni(x_1,x_2,\cdots)\longrightarrow\{x_1,x_2,\cdots\}\in\Gamma_X$ with the following property : If we have $|\gamma\cap X_1|=k_1$, $|\gamma\cap(X_2\setminus X_1)|=k_2$, $|\gamma\cap(X_n\setminus X_{n-1})|=k_n,\cdots$, then the first k_1 element of $s(\gamma)$ are in $\gamma\cap X_1$, the next k_2 element of $s(\gamma)$ are in $\gamma\cap(X_2\setminus X_1)$ and so on. It will be useful to notice that if $|\gamma\cap X_k|=r$ and $\psi\in$ Diff $X_k=\{\psi\in$ Diff $X|\psi$ identity on $X_k^c\}$, then we have $\sigma(\psi,\gamma)\in\mathfrak{S}_r$.

Now let U_m^ρ be a unitary representation of Diff X in the space $L^2_{P_m}(\Gamma_X)\times H^\rho$ defined by

$$(11)\qquad U_m^\rho(\psi): F(\gamma,a)\longrightarrow\sqrt{\frac{dP_{\psi m}}{dP_m}}(\gamma)\ F(\psi^{-1}(\gamma),\sigma(\psi,\gamma)^{-1}a)$$

In [6] it was shown that this U_m^ρ is equivalent to that U_m^ρ defined in (10). So we shall work on $(U_m^\rho, L^2_{P_m}(\Gamma_X)\otimes H^\rho)$.

Theorem 3.1. *(Whether ρ and ρ' are irreducible or not)*
If there exists a bounded operator T: $L^2_{P_m}(\Gamma_X)\otimes H^\rho\longrightarrow L^2_{P_{m'}}(\Gamma_X)\otimes H^{\rho'}$ such that
(a) $TU_m^\rho(\psi)=U_{m'}^{\rho'}(\psi)T$ for all $\psi\in$ Diff X,
(b) $\exists\,\phi\in H^\rho$ such that $T(1\otimes\phi)\neq 0$,
then P_m and $P_{m'}$ are equivalent.

Proof. We shall divide the proof into four steps.
(I) Without loss of generality we may assume that $\|\phi\|=1$ and T is a contraction. First of all we take X_k (connected open set with compact closure) and fix it for a little while. So we put $X_k=Y$.
Further we put $P_m=\mu$, $P_{m'}=\mu'$ and put μ_1, μ_2 equal to the image measure of μ by the map : $\gamma\longrightarrow\gamma\cap Y=\gamma_1$, $\gamma\longrightarrow\gamma\cap Y^c=\gamma_2$, respectively. Now we consider a bounded operator $L^2_{\mu_1}(\Gamma_Y)\otimes H^\rho\longrightarrow L^2_{\mu'_1}(\Gamma_Y)\otimes H^{\rho'}$ defined by

$$(12)\quad T_YF(\gamma,a')=\int_{\Gamma_{Y^c}}TF(\gamma_1,\gamma_2,a')\mu'_2(d\gamma_2).$$

Here we identify an element $f\in L^2_{\mu_1}(\Gamma_Y)$ with $\hat{f}\in L^2_\mu(\Gamma_X)$ through $\hat{f}(\gamma)=f(\gamma\cap Y)$. So $L^2_{\mu_1}(\Gamma_Y)$ is regarded as a closed subspace of $L^2_\mu(\Gamma_X)$.
It is easily checked that T_YF is really a function of (γ_1, a') and that $T_YF(\gamma,a'_\sigma)=\rho'(\sigma)^{-1}T_YF(\gamma,a')$ for all $\sigma\in\mathfrak{S}_{n'}$, where $a'_\sigma=(i_{\sigma(1)},\cdots,i_{\sigma(n')})$ for an element $a'=(i_1,\cdots,i_{n'})\in\widetilde{\mathbf{N}}^{n'}$. Moreover,

$$\sum_{a' \in \tilde{N}^{n'}} \int_{\Gamma_X} \| T_Y F(\gamma, a') \|_{W'}^2 \mu'(d\gamma) \leqq$$

$$\int_{\Gamma_Y} \int_{\Gamma_{Y^c}} \sum_{a' \in \tilde{N}^{n'}} \| TF(\gamma_1, \gamma_2, a') \|_{W'}^2 \mu_1'(d\gamma_1) \mu_2'(d\gamma_2) = \|TF\|^2 \leqq \|F\|^2.$$

Thus T_Y is also a contraction. Now observe that for $\psi \in$ Diff Y, $\sigma(\psi, \gamma)$ is independent of γ_2. So we have,

(13) $\qquad T_Y U_m^\rho(\psi) = U_{m'}^{\rho'}(\psi) T_Y \qquad$ for $\psi \in$ Diff Y.

Because

$$(T_Y U_m^\rho(\psi) F)(\gamma, a') = \int_{\Gamma_{Y^c}} (U_{m'}^{\rho'}(\psi) TF)(\gamma_1, \gamma_2, a') \mu_2'(d\gamma_2) =$$

$$\int_{\Gamma_{Y^c}} \sqrt{\frac{dT_\psi \mu_1'}{d\mu_1'}}(\gamma_1) TF(\psi^{-1}(\gamma_1), \gamma_2, \sigma(\psi, \gamma)^{-1} a') \mu_2'(d\gamma_2)$$

$$= (U_{m'}^{\rho'}(\psi) T_Y F)(\gamma, a').$$

(II) Let us consider a unitary representation $Q(\sigma)$ of $\mathfrak{S}^\infty$ in the space H^ρ, $Q(\sigma)$: $\phi(a) \longrightarrow \phi(\sigma^{-1}a)$. According to section 3 in [6] We split H^ρ into the direct sum of subspaces that are primary with respect to the symmetric group $\mathfrak{S}_r \subset \mathfrak{S}^\infty$. This decomposition can be presented in the following way, $H^\rho = \sum_i^\oplus W_r^i \otimes C_r^i$, where W_r^i are the spaces in which the irreducible and pairwise inequivalent representations ρ_r^i of $\mathfrak{S}_r$ act. C_r^i is the space on which $\mathfrak{S}_r$ acts trivially. More exactly we have $Q(\sigma)\phi = \sum_i \{\rho_r^i(\sigma) \otimes id\} \phi_{r,i}$ with the decomposition $\phi = \sum_i \phi_{r,i}$, $\phi_{r,i} \in W_r^i \otimes C_r^i$. Further using a natural decomposition, $L_{\mu_1}^2(\Gamma_Y) = \sum_r^\oplus L_{\mu_1}^2(B_Y^r)$ (Note that $\Gamma_Y = \cup_{r=0}^\infty B_Y^r$: disjoint union), we have an orthogonal decomposition $L_{\mu_1}^2(\Gamma_Y) \otimes H^\rho = \sum_{r,i}^\oplus \phi_\mu(r, i)$, where $\phi_\mu(r, i) = L_{\mu_1}^2(B_Y^r) \otimes W_r^i \otimes C_r^i$ is an invariant subspace of the representation $U_m^\rho(\psi)$, $\psi \in$ Diff Y whose form on $\phi_\mu(r, i)$ are as follows.

(14) $\qquad U_m^\rho(\psi)(F \otimes w_r^i \otimes c_r^i)(\gamma, a)$

$$= \sqrt{\frac{dT_\psi \mu_1}{d\mu_1}}(\gamma_1) F(\psi^{-1}(\gamma_1)) \ (\rho_r^i(\sigma(\psi, \gamma)) \otimes id) \ (w_r^i \otimes c_r^i) \ (a).$$

Now let us put for $\psi \in$ Diff Y

(15) $\qquad U_\mu^{r,i}(\psi)(F \otimes w_r^i)(\gamma_1) = \sqrt{\dfrac{dT_\psi \mu_1}{d\mu_1}}(\gamma_1) F(\psi^{-1}(\gamma_1)) \rho_r^i(\sigma(\psi, \gamma)) w_r^i$

for $F \in L_{\mu_1}^2(B_Y^r)$ and for $w_r^i \in W_r^i$.

Then we have

(16) $\qquad U_m^\rho(\psi) = U_\mu^{r,i}(\psi) \otimes id \qquad$ on $\phi_\mu(r, i)$.

$U_{\mu}^{r,i}$ are irreducible unitary representations of Diff Y in the space $L_{\mu_1}^2(B_Y^r)\otimes W_r^i$, and $U_{\mu}^{r,i}$ and $U_{\mu}^{r',i'}$ are inequivalent unless $i=i'$ and $r=r'$. (See [6].) So it follows from (13) that there exists a unique integer J_i such $T_Y\phi_{\mu}(r, i)\subseteqq \phi_{\mu'}(r, J_i)$ unless $T_Y\phi_{\mu}(r, i)=0$, and the representations ρ_r^i and $\rho_r'^{J_i}$are equivalent. Hence we have $J_i\neq J_k$ for $i\neq k$. Let $\omega_{r,i}: W_r^i\longrightarrow W_r'^{J_i}$ be an intertwining unitary operator of the representations ρ_r^i and $\rho_r'^{J_i}$, and $J_Y: L_{\mu_1}^2(B_Y^r)\longrightarrow L_{\mu_1'}^2(B_Y^r)$ be a unitary operator defined by $J_YF(\gamma_1)=\sqrt{\frac{d\mu_1}{d\mu_1'}}(\gamma_1)F(\gamma_1)$.

Then it is easy to see that a unitary operator $T_{r,i}=J_Y\otimes\omega_{r,1}: L_{\mu_1}^2(B_Y^r)\otimes W_r^i\longrightarrow L_{\mu'_1}^2(B_Y^r)\otimes W_r'^{J_i}$ satisfies

$$U_{\mu}^{r,J_i}(\psi)T_{r,i}=T_{r,i}U_{\mu}^{r,i}(\psi)\qquad\text{for all }\psi\in\text{Diff }Y. \tag{17}$$

(III) Here we list up the following fact in the representation theory. The proof will be done at the end of this section.

Fact : Let E_i, H_i, $(i=1, 2)$ be Hilbert spaces, U_1 and U_2 be two equivalent irreducible unitary representations of a group G in the spaces H_1 and H_2, and $T: H_1\longrightarrow H_2$ be an intertwining unitary operator of the representations U_1 and U_2. Suppose that a bounded operator $\tilde{A}: H_1\otimes E_1\longrightarrow H_2\otimes E_2$ satisfies $(U_2(g)\otimes id_{E_2})\tilde{A}=\tilde{A}(U_1(g)\otimes id_{E_1})$ for all $g\in G$. Then there exists a bounded operator $A: E_1\longrightarrow E_2$ such that $\tilde{A}=T\otimes A$.

Applying this fact to the operator $T_Y|\phi_{\mu}(r, i)$, it follows from (13) (16) and (17) that there exists a bounded operator $U_{r,i}: C_r^i\longrightarrow C_r'^{J_i}$ such that $T_Y|\phi_{\mu}(r, i)=T_{r,i}\otimes U_{r,i}$ for all (r, i) unless $T_Y\phi_{\mu}(r, i)=\{0\}$. As is easily seen, $U_{r,i}$ is a contraction. Consequently for $\phi=\sum_i\phi_{r,i}$, $\phi_{r,i}\in W_{r,i}\otimes C_{r,i}$ we have

$$T_Y(1\otimes\phi)(\gamma, a')=\textstyle\sum'_{r,i}T_{r,i}\otimes U_{r,i}(\chi_{B_Y^r}\otimes\phi_{r,i})(\gamma, a')= \tag{18}$$

$$\sqrt{\frac{d\mu_1}{d\mu_1'}}(\gamma_1)\textstyle\sum'_{r,i}\chi_{B_Y^r}(\gamma_1)(\omega_{r,i}\otimes U_{r,i})(\phi_{r,i})(a'),$$

where $\sum'$ is a sum for (r, i) such that $T_Y\phi_{\mu}(r, i)\neq 0$.
Let us evaluate the norm of the right hand side of (18).

$$\begin{aligned}&\|\textstyle\sum'_{r,i}\chi_{B_Y^r}(\gamma_1)(\omega_{r,i}\otimes U_{r,i})(\phi_{r,i})(a')\|_{W'}^2\\ &=\textstyle\sum_r\chi_{B_Y^r}(\gamma_1)\|\sum'_i(\omega_{r,i}\otimes U_{r,i})(\phi_{r,i})(a')\|_{W'}^2\\ &\leqq\textstyle\sum_r\chi_{B_Y^r}(\gamma_1)\|\sum'_i(\omega_{r,i}\otimes U_{r,i})(\phi_{r,i})\|^2\\ &=\textstyle\sum_r\chi_{B_Y^r}(\gamma_1)\sum'_i\|(\omega_{r,i}\otimes U_{r,i})(\phi_{r,i})\|^2\\ &\leqq\textstyle\sum_r\chi_{B_Y^r}(\gamma_1)\sum_i\|\phi_{r,i}\|^2=1\end{aligned}$$

(IV) Therefore if it would hold that P_m and $P_{m'}$ are mutually singular, then the right hand of (18) tends to 0 for P_m -a.e.γ as $Y=X_k\uparrow X$ ($\Longleftrightarrow k\longrightarrow\infty$). On the other hand the left hand of (18) converges to $T(1\otimes\phi)(\gamma, a')$ for

$P_{m'}$-*a. e.* γ as $k \longrightarrow \infty$ by the martingale convergence theorem. Thus we have $T(1 \otimes \phi)=0$ which contradicts to the assumption.

Corollary. *(Whether ρ and ρ' are irreducible or not)*
If U_m^ρ and $U_{m'}^{\rho'}$ are equivalent as unitary representation, then P_m and $P_{m'}$ are equivalent as measure.

By the above Collorary and theorem 4 of section 4 in [6] we have,

Theorem 3.2. *If ρ and ρ' are irreducible unitary representations of $\mathfrak{S}_n$ and $\mathfrak{S}_{n'}$ and $dim(X) > 1$, then the unitary representations U_m^ρ and $U_{m'}^{\rho'}$ are equivalent if and only if the measure P_m and $P_{m'}$ are equivalent, $n=n'$ and ρ and ρ' are equivalent.*

3.2. Proof of the fact.

We shall start from the following theorem which is well-known.

Theorem 3.3. *Let H, E be complex Hilbert spaces and U be an irreducible unitary representation of a group G in the space H. And suppose that a bounded operator $\widetilde{A}$ on $H \otimes E$ satisfies $\widetilde{A}\,(U(g) \otimes id_E) = (U(g) \otimes id_E)\,\widetilde{A}$ for all $g \in G$. Then there exists a bounded operator A on E such that $\widetilde{A} = id_H \otimes A$.*

Theorem 3.4. *Let H, $E_i\,(i=1, 2)$ be complex Hilbert spaces, U be an irreducible unitary representation of a group G in the space H and put $\widetilde{U}_i(g) = U(g) \otimes id_{E_i}\,(i=1, 2)$. Suppose that a bounded operator $\widetilde{A}: H \otimes E_1 \longrightarrow H \otimes E_2$ satisfies $\widetilde{U}_2(g)\,\widetilde{A} = \widetilde{A}\,\widetilde{U}_1(g)$ for all $g \in G$. Then there exists a bounded operator $A: E_1 \longrightarrow E_2$ such that $\widetilde{A} = id_H \otimes A$.*

Proof. Case 1. First we shall assume that $\widetilde{A}$ is unitary. Without loss of generality we may assume that $\dim(E_2) \leqq \dim(E_1)$. We consider A^{-1}, if the reverse inequality holds. Take an isometric operator $V: E_2 \longrightarrow E_1$. Then we have $\widetilde{U}_1(g)\ (id_H \otimes V) = (id_H \otimes V)\ \widetilde{U}_2(g)$ for all $g \in G$, so $(id_H \otimes V)\,\widetilde{A}$ is an intertwining operator of the representation $(\widetilde{U}_1,\ H \otimes E_1)$. It follows from Theorem 3.3 that there exists a bounded operator B on E_1 such that $(id_H \otimes V)\,\widetilde{A} = id_H \otimes B$. Hence $\widetilde{A} = id_H \otimes V^* B$.

General case. Consider an orthogonal decomposition : $H \otimes E_1 = \ker \widetilde{A} \oplus (\ker \widetilde{A})^{\perp}$. Since $(\ker \widetilde{A})^{\perp}$ is an invariant subspace of the representation $(\widetilde{U}_{1'}, H \otimes E_1)$, so there exists a closed subspace F_1 of E_1 such that $(\ker \widetilde{A})^{\perp} = H \otimes F_1$. Similarly a closed subspace $F_2\,(\subseteqq E_2)$ arises such that $\overline{\widetilde{A}\,(H \otimes E_1)} = H \otimes F_2$. Put $\widetilde{A}\,|\,(\ker \widetilde{A})^{\perp} = \widetilde{T}$ and $\widetilde{U}_i(g)|H \otimes F_i = \widetilde{W}_i(g)$. Then $\widetilde{T}: H \otimes F_1 \longrightarrow H \otimes F_2$ is one-to-one and has a dense range, and $\widetilde{W}_2(g)\,\widetilde{T} = \widetilde{T}\,\widetilde{W}_1(g)$ for all $g \in G$. It follows from Theorem 3.3 that $\widetilde{T}^* \widetilde{T} = id_H \otimes T$ for some positive-definite bounded operator T on F_1. Hence $\widetilde{T}$ is decomposed as $\widetilde{T} = \widetilde{V}\,(id_H \otimes \sqrt{T})$ with an isometric operator $\widetilde{V}: \overline{\mathrm{Im}\,(id_H \otimes \sqrt{T})} \longrightarrow \overline{\mathrm{Im}\,(\widetilde{T})} = H \otimes F_2$. Since $\sqrt{T}$ is one-to-one, so $\widetilde{V}$ is unitary from $H \otimes F_1$ to $H \otimes F_2$.

Moreover it is easily checked that $\widetilde{W}_2(g)\widetilde{V}=\widetilde{V}\widetilde{W}_1(g)$ for all $g\in G$.
By virtue of case 1, we have $\widetilde{V}=id_H\otimes V$ for some bounded operator $V: F_1\longrightarrow F_2$. Thus, $\widetilde{A}=(id_H\otimes i)\widetilde{T}(id_H\otimes P_{F_1})=id_H\otimes iV\sqrt{T}P_{F_1}$, where i is the natural injection from F_2 to E_2 and P_{F_1} is a projection. (Q. E. D.)

Proof of the fact : Put $\widetilde{B}=\widetilde{A}(T\otimes id_{E_1})^{-1}=\widetilde{A}(T^{-1}\otimes id_{E_1})$. Then the bounded operator $\widetilde{B}: H_2\otimes E_1\longrightarrow H_2\otimes E_2$ satisfies $\widetilde{B}(U_2(g)\otimes id_{E_1})=(U_2(g)\otimes id_{E_2})\widetilde{B}$ for all $g\in G$. It follows from Theorem 3.4 that there exists a bounded operator $A: E_1\longrightarrow E_2$ such that $\widetilde{B}=id_{H_2}\otimes A$, and therefore $\widetilde{A}=T\otimes A$.

(Q. E. D.)

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