J. Math. Kyoto Univ. (JMKYAZ)
34-4 (1994) 741-754

On Gelfand pairs associated with nilpotent Lie groups

By

Katsuhiko KIKUCHI

Introduction

Let G be a locally compact group and K a compact subgroup of G. The investigation of the Banach $*$-algebra $L^1(K\backslash G/K)$ of K-biinvariant integrable functions on G is an important theme in harmonic analysis on G or G/K. When $L^1(K\backslash G/K)$ is commutative, the pair (G, K) is called a Gelfand pair, and there have been many works on Gelfand pairs until now (see for example [F] and the introduction of [BJR]). In this paper, we consider the case $G=K\ltimes N$, where N is a connected, simply connected nilpotent Lie group and K acts on N as automorphisms. We shall give a necessary and sufficient condition for the pair (G, K) to be a Gelfand pair. This is equivalent to determining a condition that the Banach $*$-algebra $L_K^1(N)$ of K-invariant integrable functions on N be commutative. We call the pair $(K ; N)$ a Gelfand pair associated with N if $L_K^1(N)$ is commutative.

Now $L_K^1(N)$ is commutative only if N is at most 2-step thanks to [BJR], and accordingly our object N is assumed to be 2-step. Our first theorem (Theorem A below) gives a way in which one reduces the matter to Heisenberg groups. Let us describe our method in detail.

Denote by $\mathfrak{n}$ the Lie algebra of N and by $\mathfrak{n}^*$ the dual vector space of $\mathfrak{n}$. For $l\in\mathfrak{n}^*$, let B_l be the alternative form corresponding to l : $B_l(X, Y)=l([X, Y])$ $(X, Y\in\mathfrak{n})$, and $\mathfrak{b}(l)$ the intersection of the radical of B_l with ker l. Then we see that $\mathfrak{n}/\mathfrak{b}(l)$ is isomorphic to a Heisenberg algebra if $l|_{[\mathfrak{n},\mathfrak{n}]}\neq 0$. Let $\pi=\pi_l$ be the irreducible unitary representation of N corresponding to l (see [Ki]) and K_π the stabilizer of π for the action of K on the unitary dual $\widehat{N}$ of N. We denote by $\Phi_\pi(K_\pi)$ the subgroup of $\mathrm{Aut}(\mathfrak{n}/\mathfrak{b}(l))$ formed by the K_π-actions induced on $\mathfrak{n}/\mathfrak{b}(l)$. Let $B(l)$ be the subgroup of N corresponding to $\mathfrak{b}(l)$. Considering the pair $(\Phi_\pi(K_\pi), B(l)\backslash N)$ with $\mathfrak{n}/\mathfrak{b}(l)$ regarded as the Lie algebra of $B(l)\backslash N$, we obtain the following theorem.

Theorem A. *Let N be a 2-step nilpotent Lie group and K a compact group acting on N as automorphisms. Then the pair $(K ; N)$ is a Gelfand pair*

Received December 13, 1993

if and only if $(\Phi_\pi(K_\pi); B(l)\backslash N)$ *is a Gelfand pair for every* $l \in \mathfrak{n}^*$.

Using Theorem A, we will show by an example a certain subtlety of 2-step nilpotent Lie groups N if the derived algebra $[\mathfrak{n}, \mathfrak{n}]$ of the Lie algebra $\mathfrak{n}$ of N is different from the center $Z(\mathfrak{n})$ of $\mathfrak{n}$. To be more precise, we decompose $\mathfrak{n}$ into K-invariant subspaces as $\mathfrak{n}=\mathfrak{n}'\oplus\mathfrak{a}\oplus[\mathfrak{n},\mathfrak{n}]$ with $Z(\mathfrak{n})=\mathfrak{a}\oplus[\mathfrak{n}, \mathfrak{n}]$. Put $\mathfrak{n}_1=\mathfrak{n}'\oplus[\mathfrak{n}, \mathfrak{n}]$. The sum is a direct sum of ideals and we have $Z(\mathfrak{n}_1)=[\mathfrak{n}_1,\mathfrak{n}_1]$. Let N_1 and A be the subgroups corresponding to $\mathfrak{n}_1$ and $\mathfrak{a}$ respectively. Consider the pair $(K; N_1)$. Since $N=N_1\times A$, we have $L^1(N)=L^1(N_1)\otimes L^1(A)$. But $L^1_K(N_1)$ and $L^1(A)$ alone do not suffice to determine the properties of $L^1_K(N)$ in general. For instance, the commutativity of $L^1_K(N_1)$ does not imply the commutativity of $L^1_K(N)$. In fact, let $\mathfrak{n}$ be the 5-dimensional Lie algebra $\boldsymbol{C}\times\boldsymbol{C}\times\boldsymbol{R}$ with the bracket product $[(z_1, z_2, t), (z_1', z_2', t')]=(0, 0, -\mathrm{Im}\, z_2\bar{z}_2')$. Let K be the one-dimensional torus $\boldsymbol{T}$ acting on $\mathfrak{n}$ by $e^{\sqrt{-1}\theta}\cdot(z_1, z_2, t)=(e^{\sqrt{-1}\theta}\cdot z_1, e^{\sqrt{-1}\theta}\cdot z_2, t)$. Then $\mathfrak{n}_1=0\times\boldsymbol{C}\times\boldsymbol{R}$, $\mathfrak{a}=\boldsymbol{C}\times 0\times 0$. For this $\mathfrak{n}$, we show

Theorem B. *Let* $N=\exp\mathfrak{n}$ *and* $N_1=\exp\mathfrak{n}_1$. *Then* $L^1_K(N_1)$ *is commutative, whereas* $L^1_K(N)$ *is not commutative.*

Finally, we treat the case where $K=\boldsymbol{T}^n$ and N is a general 2-step nilpotent Lie group, and give a necessary and sufficient condition for $(K; N)$ to be a Gelfand pair. Such a condition was given by Leptin [L] when $[\mathfrak{n}, \mathfrak{n}]=Z(\mathfrak{n})$ and the action of K is effective. In this paper, we work without these two restrictions and present a complete solution. Recall the decomposition $\mathfrak{n}=\mathfrak{n}'\oplus\mathfrak{a}\oplus[\mathfrak{n}, \mathfrak{n}]$ mentioned above. Let $\widehat{K}^r$ be the family of all equivalence classes of irreducible real K-modules. We can identify $\widehat{K}^r$ with $\boldsymbol{Z}^n/\sim$ where $\alpha\sim\beta$ if $\beta=\pm\alpha$ for $\alpha, \beta\in\boldsymbol{Z}^n$. By fixing a system of representatives, we regard $\widehat{K}^r$ as a subset of $\boldsymbol{Z}^n$ containing 0. Let $\mathfrak{n}'=\sum_{\alpha\in\widehat{K}^r} m_{\alpha,1}V_\alpha$, $\mathfrak{a}=\sum_{\alpha\in\widehat{K}^r} m_{\alpha,2}V_\alpha$ be the decompositions of $\mathfrak{n}'$, $\mathfrak{a}$ into irreducible real K-modules respectively. Put $S_i=\{\alpha\in\widehat{K}^r \mid m_{\alpha,i}\neq 0\}$ $(i=1, 2)$. Our third theorem is the following.

Theorem C. *The pair* $(K; N)$ *is a Gelfand pair if and only if the following five conditions are satisfied*:

(1) $m_{0,1}=0$,
(2) S_1 *is a linearly independent system,*
(3) $m_{\alpha,1}=1$ *for all* $\alpha\in S_1$,
(4) $\boldsymbol{R}\text{-span}(S_1)\cap\boldsymbol{R}\text{-span}(S_2)=0$,
(5) K *acts on* $[\mathfrak{n}, \mathfrak{n}]$ *trivially.*

1. Preliminaries

Let N be a connected, simply connected nilpotent Lie group. The Banach space $L^1(N)$ of integrable functions on N relative to the Haar measure has a structure of Banach $*$-algebra with convolution and involution defined respectively by

$$f*g(x)=\int_N f(y)g(y^{-1}x)dy, \quad f^*(x)=\overline{f(x^{-1})}.$$

Let K be a compact Lie group acting on N through a homomorphism $\phi: K \to \mathrm{Aut}(N)$, where $\mathrm{Aut}(N)$ denotes the automorphism group of N. Replacing K by $K/\ker\phi$ if necessary, we assume throughout this paper that K is a subgroup of $\mathrm{Aut}(N)$. Let $\mathfrak{n}$ be the Lie algebra of N. Since N is connected and simply connected, we shall identify $\mathrm{Aut}(N)$ with the automorphism group $\mathrm{Aut}(\mathfrak{n})$ of $\mathfrak{n}$. The group K acts also on $L^1(N)$ as automorphisms of $*$-algebra by $(k\cdot f)(x)=f(k^{-1}\cdot x)$. We denote by $L^1_K(N)$ the closed $*$-subalgebra of K-invariant functions in $L^1(N)$.

Definition 1.1. We call the pair $(K;N)$ a Gelfand pair if $L^1_K(N)$ is commutative.

We remark that by forming the semidirect product $K\ltimes N$, the pair $(K;N)$ is a Gelfand pair if and only if the algebra $L^1(K\backslash K\ltimes N/K)$ of K-biinvariant functions is commutative.

By Theorem 2.4 in [BJR], we assume from now on that N is a 2-step nilpotent Lie group. Let $\hat{N}$ be the unitary dual of N, that is, the space of all equivalence classes of irreducible unitary representations of N with Fell topology. For $k\in K$ and $\pi\in\hat{N}$, we define a representation π_k of N by $\pi_k(x)=\pi(k\cdot x)$ $(x\in N)$. Then, K acts continuously on $\hat{N}$ from the right. Let K_π be the stabilizer of π in K: $K_\pi=\{k\in K|\pi_k\simeq\pi\}$. For each $k\in K_\pi$, there exists a unitary operator $W_\pi(k)$ on the representation space H_π of π such that $\pi_k(x)=W_\pi(k)\pi(x)W_\pi(k)^{-1}$ for all $x\in N$. By Schur's lemma, the operator $W_\pi(k)$ is determined up to a scalar multiple of absolute value 1. On the analogy of the theory of unitary representations of compact groups, we can decompose W_π as a direct sum of irreducible projective representations of K_π:

$$W_\pi=\sum c(T, W_\pi)T,$$

where $c(T, W_\pi)$ is the multiplicity of T in W_π. We state here the following theorem due to Carcano [C, p. 1094] for later references.

Theorem 1.2 [C]. *The following three conditions are equivalent*:
(1) *$(K;N)$ is a Gelfand pair.*
(2) *One has $c(T, W_\pi)\leq 1$ for each $\pi\in\hat{N}$.*
(3) *There is a dense subset S of $\hat{N}$ such that $c(T, W_\pi)\leq 1$ for each $\pi\in S$.*

For a subset S of $\hat{N}$, we denote by $S\cdot K$ the union of all K-orbits of elements of S: $S\cdot K=\{\pi_k|\pi\in S, k\in K\}$. Then,

Corollary 1.3. *The conditions in Theorem* 1.2 *are also equivalent to*

(4) *there is a subset* S *of* $\hat{N}$ *with dense* $S\cdot K$ *in* $\hat{N}$ *such that one has* $c(T, W_\pi)\leq 1$ *for each* $\pi\in\hat{N}$.

Now, Kirillov's theory [Ki] tells us that there is a bijection between the coadjoint orbit space $\mathfrak{n}^*/N$ and the unitary dual $\hat{N}$. By [Br], this bijection is a homeomorphism when $\mathfrak{n}^*/N$ is equipped with the quotient topology. For $l\in\mathfrak{n}^*$, we denote by π_l the irreducible unitary representation of N corresponding to l. Define the right action of K on $\mathfrak{n}^*$ by

$$(l\cdot k)(X)=l(k\cdot X)\qquad (l\in\mathfrak{n}^*,\ k\in K,\ X\in\mathfrak{n}).$$

Then we see that $(\pi_l)_k\simeq\pi_{l\cdot k}$. Moreover, we have for $x\in N$

$$((\mathrm{Ad}^*(x)l)\cdot k)(X)=l(\mathrm{Ad}(x^{-1})(k\cdot X))=l(k\cdot(\mathrm{Ad}(k^{-1}\cdot x^{-1})X))$$
$$=(l\cdot k)(\mathrm{Ad}(k^{-1}\cdot x^{-1})X)=(\mathrm{Ad}^*(k^{-1}\cdot x)(l\cdot k))(X).$$

Denoting by O_l the coadjoint orbit through $l\in\mathfrak{n}^*$, we get

$$O_l\cdot k=\{(\mathrm{Ad}^*(x)l)\cdot k|x\in N\}=\{\mathrm{Ad}^*(x)(l\cdot k)|x\in N\}=O_{l\cdot k}.$$

Therefore, $(\mathrm{Ad}^*(N)l)\cdot K=\mathrm{Ad}^*(N)(l\cdot K)$. This says that in Corollary 1.3, we can take the set $\{\pi_{l_\alpha}\}_{\alpha\in A}$ as S, where $\{l_\alpha\}_{\alpha\in A}$ is a complete system of representatives of (N, K)-orbits in $\mathfrak{n}^*$ such that the union $\bigcup_{\alpha\in A}(\mathrm{Ad}^*(N)l_\alpha\cdot K)$ is dense in $\mathfrak{n}^*$.

2. Reduction to Heisenberg groups

We consider first the case of the $(2n+1)$-dimensional Heisenberg group H_n. Let $\mathfrak{h}_n$ be the Lie algebra of H_n and K a compact subgroup of $\mathrm{Aut}(\mathfrak{h}_n)$. Since the one-dimensional center $Z(\mathfrak{h}_n)$ of $\mathfrak{h}_n$ is invariant under K, and since K is compact, there is a character χ of K with image $\{1\}$ or $\{\pm 1\}$ such that

$$k\cdot X=\chi(k)X\qquad (k\in K,\ X\in Z(\mathfrak{h}_n)).$$

We suppose that K acts on $Z(\mathfrak{h}_n)$ trivially. Let T be a generator of $Z(\mathfrak{h}_n)$. Let V be a K-invariant subspace of $\mathfrak{h}_n$ complementary to $Z(\mathfrak{h}_n)$. Then we can define a symplectic form ω on V such that

$$[X, Y]=\omega(X, Y)T\qquad (X, Y\in V).$$

Since K acts on $Z(\mathfrak{h}_n)$ trivially, we have $\omega(k\cdot X, k\cdot Y)=\omega(X, Y)$ for all $k\in K$, $X, Y\in V$. Hence K can be regarded as a compact subgroup of the symplectic group $\mathrm{Sp}(\omega)$ of ω. Since $[V, V]=[\mathfrak{h}_n, \mathfrak{h}_n]=Z(\mathfrak{h}_n)$, the form ω is nondegenerate. Take a basis $\{X_1', Y_1', \cdots, X_n', Y_n'\}$ of V such that

$$\omega(X_i', Y_j')=\delta_{ij},\quad \omega(X_i', X_j')=\omega(Y_i', Y_j')=0.$$

Defining a complex structure I_0 on V by $I_0X_i'=Y_i'$, $I_0Y_i'=-X_i'$, we let

$$\omega_0(X, Y)=\omega(X, I_0 Y)-\sqrt{-1}\omega(X, Y) \qquad (X, Y\in V).$$

It is easy to see that ω_0 is a hermitian inner product on the complex vector space (V, I_0). The unitary group $\mathrm{U}(\omega_0)$ of ω_0 is a maximal compact subgroup of $\mathrm{Sp}(\omega)$. Hence there is $\psi\in\mathrm{Sp}(\omega)$ such that $K\subset\psi\mathrm{U}(\omega_0)\psi^{-1}$. Set $I=\psi I_0\psi^{-1}$ and

$$\tilde{\omega}(X, Y)=\omega(X, IY)-\sqrt{-1}\omega(X, Y) \qquad (X, Y\in V).$$

Then, since $\psi^{-1}k\psi\in\mathrm{U}(\omega_0)$ commutes with I_0, we have

$$I(k\cdot X)=\psi I_0(\psi^{-1}k\psi)\cdot(\psi^{-1}X)=\psi(\psi^{-1}k\psi)\cdot I_0(\psi^{-1}X)=k\cdot IX$$

for $k\in K$, $X\in V$. Hence we get

$$\begin{aligned}\tilde{\omega}(k\cdot X, k\cdot Y)&=\omega(k\cdot X, I(k\cdot Y))-\sqrt{-1}\omega(k\cdot X, k\cdot Y)\\&=\omega(k\cdot X, k\cdot(IX))-\sqrt{-1}\omega(k\cdot X, k\cdot Y)\\&=\omega(X, IY)-\sqrt{-1}\omega(X, Y)=\tilde{\omega}(X, Y)\end{aligned}$$

for $k\in K$, $X, Y\in V$. Consequently we see that $K\subset\mathrm{U}(\tilde{\omega})$. In what follows we regard V as an n-dimensional complex vector space (V, I) with the inner product $\tilde{\omega}$.

As is well-known, the irreducible unitary representations of H_n which are non-trivial on the center are determined by their central characters. We denote by R_λ $(\lambda\neq 0)$ the irreducible unitary representation with central character $\chi_\lambda(\exp tT)=e^{\sqrt{-1}\lambda t}$ $(t\in\boldsymbol{R})$. Moreover, $\{R_\lambda\}_{\lambda\neq 0}$ is dense in $\hat{H}_n$ with respect to Fell topology. Then, by Theorem 1.2, it is sufficient to consider the representations $\{R_\lambda\}_{\lambda\neq 0}$. We will realize R_λ by means of the Fock models (see for example [Ba]). For $\lambda>0$, let $\mathfrak{H}_\lambda$ (resp. $\mathfrak{H}_{-\lambda}$) be the Hilbert space of entire holomorphic (resp. antiholomorphic) functions f on V such that

$$\int_V |f(w)|^2 e^{-\lambda|w|^2/2}dw<+\infty.$$

The representation operators are given as follows:

$$(2.1)\qquad (R_\lambda(z, t)f)(w)=e^{\sqrt{-1}\lambda t-\lambda w\bar{z}/2-\lambda|z|^2/4}f(w+z) \qquad (\lambda>0),$$

$$(2.2)\qquad (R_{-\lambda}(z, t)f)(w)=e^{-\sqrt{-1}\lambda t-\lambda\bar{w}z/2-\lambda|z|^2/4}f(w+z) \qquad (\lambda>0).$$

If $\lambda>0$, then $\mathfrak{H}_\lambda$ contains the algebra $\boldsymbol{C}[V]$ of holomorphic polynomials densely, and $\mathfrak{H}_{-\lambda}$ contains the algebra $\overline{\boldsymbol{C}[V]}$ of antiholomorphic polynomials densely.

Recalling that K is contained in the unitary group of V, we define the unitary operator $W_\lambda(k)$ $(\lambda>0, k\in K)$ on $\mathfrak{H}_\lambda$ by

$$(W_\lambda(k)f)(w)=f(k^{-1}\cdot w).$$

Then, an easy computation shows $R_\lambda(k\cdot(z, t))W_\lambda(k)=W_\lambda(k)R_\lambda(z, t)$. Hence $W_\lambda(k)$ is an intertwining operator between R_λ and $(R_\lambda)_k$. It is obvious that $W_\lambda(k_1k_2)=W_\lambda(k_1)W_\lambda(k_2)$. Moreover, $\boldsymbol{C}[V]$ is invariant under W_λ, and the representation operators of W_λ are the same on $\boldsymbol{C}[V]$ for all $\lambda>0$. The case $\lambda<0$ being treated analogously, we have only to consider the particular case $\lambda=\lambda_0$, say.

Proposition 2.1 [**BJR**]. *The pair $(K\,;H_n)$ is a Gelfand pair if and only if W_{λ_0} decomposes into irreducibles with multiplicity one.*

We return to the case where N is a 2-step nilpotent Lie group. We will see that every infinite-dimensional irreducible unitary representation of N factors through a Heisenberg group. For $l\in\mathfrak{n}^*$, let B_l be the alternative form on $\mathfrak{n}$ defined by

$$B_l(X, Y)=l([X, Y]).$$

Define subspaces $\mathfrak{n}(l)$, $\mathfrak{b}(l)$ of $\mathfrak{n}$ as follows:

(2.3) $$\mathfrak{n}(l)=\{X\in\mathfrak{n}\,|\,B_l(X, Y)=0 \text{ for all } Y\in\mathfrak{n}\},\quad \mathfrak{b}(l)=\mathfrak{n}(l)\cap(\ker l).$$

Proposition 2.2. *Let l be a non-zero element of $\mathfrak{n}^*$. Then,*

(1) $[\mathfrak{n},\mathfrak{n}]\subset\mathfrak{n}(l)$. *In particular* $\mathfrak{n}(l)$ *is an ideal of* $\mathfrak{n}$.
(2) $[\mathfrak{n}(l),\mathfrak{n}]\subset\mathfrak{b}(l)$. *In particular* $\mathfrak{b}(l)$ *is an ideal of* $\mathfrak{n}$.
(3) $\dim(\mathfrak{n}(l)/\mathfrak{b}(l))=1$.
(4) $l|_{[\mathfrak{n},\mathfrak{n}]}\neq0$ *if and only if* $\mathfrak{n}(l)\neq\mathfrak{n}$. *In this case,* $\dim(\mathfrak{n}/\mathfrak{b}(l))>1$.

Proof. (1) Since $\mathfrak{n}$ is 2-step, $[\mathfrak{n},\mathfrak{n}]$ is included in the center $Z(\mathfrak{n})$. Therefore $B_l([\mathfrak{n},\mathfrak{n}], \mathfrak{n})\subset B_l(Z(\mathfrak{n}), \mathfrak{n})=0$. Hence $[\mathfrak{n},\mathfrak{n}]\subset\mathfrak{n}(l)$.
(2) This follows from (1) and the definition of $\mathfrak{n}(l)$.
(3) Clearly $\dim(\mathfrak{n}(l)/\mathfrak{b}(l))\leq1$. Suppose first $l|_{[\mathfrak{n},\mathfrak{n}]}\neq0$. Then we have $\mathfrak{n}(l)\neq\mathfrak{b}(l)$ by (1). Suppose next $l|_{[\mathfrak{n},\mathfrak{n}]}=0$. Then $B_l(\mathfrak{n}, \mathfrak{n})=0$, so that we get $\mathfrak{n}(l)=\mathfrak{n}$. On the other hand we have $\mathfrak{b}(l)\neq\mathfrak{n}$, because $l\neq0$.
(4) By the proof of (3), if $\mathfrak{n}(l)\neq\mathfrak{n}$, then $l|_{[\mathfrak{n},\mathfrak{n}]}\neq0$. Suppose conversely that $l|_{[\mathfrak{n},\mathfrak{n}]}\neq0$. Then there are X, $Y\in\mathfrak{n}$ such that $l([X, Y])\neq0$, so X, $Y\notin\mathfrak{n}(l)$.

Let π_l be the irreducible unitary representation of N corresponding to $l\in\mathfrak{n}^*$.

Lemma 2.3. *Let l, $l'\in\mathfrak{n}^*$. Then one has $\pi_l\simeq\pi_{l'}$ if and only if*
(1) $\mathfrak{n}(l)=\mathfrak{n}(l')$, (2) $l|_{\mathfrak{n}(l)}=l'|_{\mathfrak{n}(l)}$.

Proof. See [M], Theorem 2.3 (3).

Now, let K be a compact subgroup of $\mathrm{Aut}(\mathfrak{n})$.

Lemma 2.4. *Let* $k\in K$. *Then* $k\in K_{\pi_l}$ *if and only if*
(1) $k^{-1}\cdot\mathfrak{n}(l)=\mathfrak{n}(l)$, (2) $l\cdot k|_{\mathfrak{n}(l)}=l|_{\mathfrak{n}(l)}$.

Proof. Since $(\pi_l)_k\simeq\pi_{l\cdot k}$ for $k\in K$, we have $k\in K_{\pi_l}\Leftrightarrow\pi_{l\cdot k}\simeq\pi_l$. Now Lemma 2.4 follows from Lemma 2.3 by noting $\mathfrak{n}(l\cdot k)=k^{-1}\cdot\mathfrak{n}(l)$.

We suppose from now on that $l|_{[\mathfrak{n},\mathfrak{n}]}\neq 0$. Then, Proposition 2.2 (4) says that $\dim(\mathfrak{n}/\mathfrak{b}(l))>1$. Moreover,

$$Z(\mathfrak{n}/\mathfrak{b}(l))=\mathfrak{n}(l)/\mathfrak{b}(l),\quad (\mathfrak{n}/\mathfrak{b}(l))/(\mathfrak{n}(l)/\mathfrak{b}(l))\simeq\mathfrak{n}/\mathfrak{n}(l). \tag{2.4}$$

Since the second algebra in (2.4) is abelian by Proposition 2.2 (1), $\mathfrak{n}/\mathfrak{b}(l)$ is isomorphic to a Heisenberg algebra $\mathfrak{h}_n$ where $n=\frac{1}{2}\dim(\mathfrak{n}/\mathfrak{n}(l))$. Put $B(l)=\exp\mathfrak{b}(l)$. Denote by p_l the canonical projection of N onto $B(l)\backslash N$ and by l_0 the element of $(\mathfrak{n}/\mathfrak{b}(l))^*$ such that $l=l_0\circ p_l$.

Lemma 2.5. *Denote by* σ_{l_0} *the unitary representation of* $B(l)\backslash N$ *corresponding to* l_0. *Then, one has* $\pi_l\simeq\sigma_{l_0}\circ p_l$.

Proof. Let $\overline{\mathfrak{m}}$ be a polarization for l_0. Put $\mathfrak{m}=p_l^{-1}(\overline{\mathfrak{m}})$. Then $\mathfrak{m}$ is a polarization for l. Put $\overline{M}=\exp\overline{\mathfrak{m}}$ and $M=\exp\mathfrak{m}$ respectively. Obviously $\overline{M}=B(l)\backslash M$. Let χ_l and χ_{l_0} be the characters of M and $\overline{M}$ respectively such that

$$\chi_l(\exp X)=\exp\sqrt{-1}\,l(X)\qquad(X\in\mathfrak{m}),$$

$$\chi_{l_0}(\exp\overline{X})=\exp\sqrt{-1}\,l_0(\overline{X})\qquad(\overline{X}\in\overline{\mathfrak{m}}).$$

Then $\chi_{l_0}(p_l(x))=\chi_l(x)$ for all $x\in M$. In particular $\chi_l(b)=1$ for all $b\in B(l)$.

The representations π_l and σ_{l_0} can be considered as the induced representations Ind χ_l and Ind χ_{l_0} respectively. Let $d\mu$, $d\nu$ and $d\rho$ be Haar measures on N, M and $B(l)$ respectively. Let $d\dot{\mu}$, $d\overline{\mu}$ and $d\overline{\nu}$ be the invariant measures on $M\backslash N$, $B(l)\backslash N$ and $B(l)\backslash M$ respectively such that $d\mu=d\nu d\dot{\mu}=d\rho d\overline{\mu}$, $d\nu=d\rho d\overline{\nu}$. Identifying $M\backslash N$ with $(B(l)\backslash M)\backslash(B(l)\backslash N)$, we regard $d\dot{\mu}$ as the invariant measure on $(B(l)\backslash M)\backslash(B(l)\backslash N)$ such that $d\overline{\mu}=d\overline{\nu}d\dot{\mu}$. The representation space $\mathfrak{H}_l$ of π_l is the Hilbert space of functions f satisfying

$$f(mx)=\chi_l(m)f(x)\quad(m\in M,\ x\in N),\quad \int|f|^2d\dot{\mu}<+\infty.$$

We have a similar description for the representation space $\mathfrak{H}_{l_0}$ of σ_{l_0}. Now for $b\in B(l)$ and $x\in N$, we have $f(bx)=\chi_l(b)f(x)=f(x)$ for all $f\in\mathfrak{H}_l$. Hence the map $f\mapsto\overline{f}$, where $\overline{f}(p_l(x))=f(x)$ $(x\in N)$, gives rise to an identification of $\mathfrak{H}_l$ with $\mathfrak{H}_{l_0}$. Then, for x, $n\in N$, we get

$$\sigma_{l_0}(p_l(x))\overline{f}(p_l(n))=\overline{f}(p_l(nx))=f(nx)=\pi_l(x)f(n).$$

Therefore $\pi_l\simeq\sigma_{l_0}\circ p_l$.

Put $\pi=\pi_l$ for brevity. Consider the subgroup $\mathrm{Aut}(N)_\pi$ of $\mathrm{Aut}(N)$:

$$\mathrm{Aut}(N)_\pi=\{\varphi\in\mathrm{Aut}(N)|(\pi)_\varphi\simeq\pi\},$$

where $(\pi)_\varphi(x)=\pi(\varphi(x))$. Then K_π is a subgroup of $\mathrm{Aut}(N)_\pi$. Recalling the projection p_l: $N\to B(l)\backslash N$ we define a map Φ_π as follows:

$$\text{(2.5)} \qquad \Phi_\pi:\ \mathrm{Aut}(N)_\pi\ni\varphi\mapsto\bar{\varphi}\in\mathrm{Aut}(B(l)\backslash N),\quad \bar{\varphi}(p_l(x))=p_l(\varphi(x)).$$

Then Φ_π is well-defined thanks to Lemma 2.3, and $\Phi_\pi(K_\pi)$ stabilizes the elements in the center of $B(l)\backslash N$.

Theorem 2.6. *For $l\in\mathfrak{n}^*$, denote by $\pi=\pi_l$ the irreducible unitary representation of N corresponding to $l\in\mathfrak{n}^*$. Then, $(K;N)$ is a Gelfand pair if and only if $(\Phi_\pi(K_\pi);B(l)\backslash N)$ is a Gelfand pair for every $l\in\mathfrak{n}^*$.*

Proof. By Theorem 1.2, it suffices to treat the case $l|_{[\mathfrak{n},\mathfrak{n}]}\neq 0$. For such an l, we have $\pi_l\simeq\sigma_{l_0}\circ p_l$ for some $l_0\in(\mathfrak{n}/\mathfrak{b}(l))^*$. For $k\in K_\pi$,

$$\pi_l(k\cdot x)=\sigma_{l_0}(p_l(k\cdot x))=\sigma_{l_0}(\Phi_\pi(k)p_l(x)),$$

by (2.5). Since σ_{l_0} is unitarily equivalent to some R_{λ_0} in (2.1) or (2.2), there is an intertwining representation W of $\Phi_\pi(K_\pi)$ such that

$$\sigma_{l_0}(\Phi_\pi(k)h)=W(\Phi_\pi(k))\sigma_{l_0}(h)W(\Phi_\pi(k))^{-1},$$

for all $k\in K_\pi$, $h\in H_n$. Therefore we get

$$\pi_l(k\cdot x)=W(\Phi_\pi(k))\sigma_{l_0}(p_l(x))W(\Phi_\pi(k))^{-1}.$$

Thus we can take $W\circ\Phi_\pi$ as an intertwining representation W_π of K_π. Then it is evident that

$$c(T,W_\pi)\leq 1 \text{ for } T\in\hat{K}_\pi \iff c(T',W)\leq 1 \text{ for } T'\in\Phi_\pi(K_\pi)^\wedge.$$

This together with Theorem 1.2 and Proposition 2.1 completes the proof.

3. A counterexample

In this section we give some applications of Theorem 2.6. Let N be a 2-step nilpotent Lie group, $\mathfrak{n}$ the Lie algebra of N. We denote by $Z(\mathfrak{n})$ the center of $\mathfrak{n}$ and by $[\mathfrak{n},\mathfrak{n}]$ the derived algebra of $\mathfrak{n}$. Since $\mathfrak{n}$ is 2-step, we have $[\mathfrak{n},\mathfrak{n}]\subset Z(\mathfrak{n})$. We suppose that $[\mathfrak{n},\mathfrak{n}]\neq Z(\mathfrak{n})$. Let K be a compact group acting on $\mathfrak{n}$ as automorphisms. Then there is a K-invariant real inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$. Let $\mathfrak{n}'$(resp. $\mathfrak{a}$) be the K-invariant orthogonal complement of $Z(\mathfrak{n})$ (resp. of $[\mathfrak{n},\mathfrak{n}]$) in $\mathfrak{n}$ (resp. in $Z(\mathfrak{n})$) relative to this inner product, so that we have the following K-invariant orthogonal decompositions:

$$\text{(3.1)} \qquad \mathfrak{n}=\mathfrak{n}'\oplus\mathfrak{a}\oplus[\mathfrak{n},\mathfrak{n}],$$

(3.2) $Z(\mathfrak{n})=\mathfrak{a}\oplus[\mathfrak{n}, \mathfrak{n}]$.

Put $\mathfrak{n}_1=\mathfrak{n}'\oplus[\mathfrak{n}, \mathfrak{n}]$. Then $\mathfrak{n}_1$ is a Lie subalgebra and we have $Z(\mathfrak{n}_1)=[\mathfrak{n}_1, \mathfrak{n}_1]=[\mathfrak{n}, \mathfrak{n}]$. Moreover, $\mathfrak{a}$ and $\mathfrak{n}_1$ are ideals of $\mathfrak{n}$ and

(3.3) $\mathfrak{n}=\mathfrak{n}_1\oplus\mathfrak{a}$ (direct sum of ideals).

Let N_1, A be the normal subgroups of N corresponding to the ideals $\mathfrak{n}_1$, $\mathfrak{a}$ respectively.

Proposition 3.1. *If* $(K; N)$ *is a Gelfand pair, so is* $(K; N_1)$.

Proof. Given $l_1\in\mathfrak{n}_1^*$, we denote by l the linear form on $\mathfrak{n}$ defined by $l(X_1+Z)=l_1(X_1)$ $(X_1\in\mathfrak{n}_1, Z\in\mathfrak{a})$. By (2.3) we have easily

$$\mathfrak{n}(l)=\mathfrak{n}_1(l_1)\oplus\mathfrak{a}, \quad \mathfrak{b}(l)=\mathfrak{b}_1(l_1)\oplus\mathfrak{a}.$$

By the latter equality we can identify $\mathfrak{n}_1/\mathfrak{b}(l_1)$ with $\mathfrak{n}/\mathfrak{b}(l)$. Put $\pi_1=\pi_{l_1}\in\hat{N}_1$ for simplicity. Then π_1 is equivalent to $\pi_l|_{N_1}$. Furthermore the stabilizer K_{π_1} coincides with K_{π_l}. In fact,

$$K_{\pi_1}=\{k\in K|k^{-1}\cdot\mathfrak{n}_1(l_1)=\mathfrak{n}_1(l_1),\ l_1\cdot k|_{\mathfrak{n}_1(l_1)}=l_1|_{\mathfrak{n}_1(l_1)}\} \qquad \text{(by Lemma 2.4)}$$
$$=\{k\in K|k^{-1}\cdot\mathfrak{n}(l)=\mathfrak{n}(l),\ l\cdot k|_{\mathfrak{n}(l)}=l|_{\mathfrak{n}(l)}\}=K_{\pi_l}.$$

Let $B_1(l_1)=\exp\mathfrak{b}_1(l_1)$. Let Φ_{π_l} be the map in (2.5) and $\Phi^1_{\pi_1}$ the map $\mathrm{Aut}(N_1)_{\pi_1}\to\mathrm{Aut}(B_1(l_1)\backslash N_1)$ defined similarly through the data N_1, l_1, π_1. Then we can identify $(\Phi_{\pi_l}(K_{\pi_l}); B(l)\backslash N)$ with $(\Phi^1_{\pi_1}(K_{\pi_1}); B_1(l_1)\backslash N_1)$. Since $(K; N)$ is a Gelfand pair, so is $(\Phi_{\pi_l}(K_{\pi_l}); B(l)\backslash N)$ by Theorem 2.6. Hence $(\Phi^1_{\pi_1}(K_{\pi_1}); B_1(l_1)\backslash N_1)$ is a Gelfand pair, so that $(K; N_1)$ is also a Gelfand pair by Theorem 2.6 again.

Now, we consider the converse of Proposition 3.1. It is stated in [L, p. 59] and [BJR, p. 105] that the commutativities of $L^1_K(N)$ and $L^1_K(N_1)$ are equivalent. However it turns out that the commutativity of $L^1_K(N_1)$ does not imply the commutativity of $L^1_K(N)$ as will be shown by the following counterexample.

Let N be a nilpotent Lie group homeomorphic to $\boldsymbol{C}\times\boldsymbol{C}\times\boldsymbol{R}$, $\mathfrak{n}$ the Lie algebra of N with the bracket product given by

$$[(z_1, z_2, t), (z_1', z_2', t')]=(0, 0, -\mathrm{Im}\, z_2\bar{z}_2').$$

Let $K=\boldsymbol{T}$ act on $\mathfrak{n}$ as follows:

$$e^{\sqrt{-1}\theta}\cdot(z_1, z_2, t)=(e^{\sqrt{-1}\theta}z_1, e^{\sqrt{-1}\theta}z_2, t).$$

Using the same notations as the beginning of this section we have

$$Z(\mathfrak{n})=\boldsymbol{C}\times 0\times\boldsymbol{R}, \quad [\mathfrak{n}, \mathfrak{n}]=0\times 0\times\boldsymbol{R}, \quad \mathfrak{n}_1=0\times\boldsymbol{C}\times\boldsymbol{R}, \quad \mathfrak{a}=\boldsymbol{C}\times 0\times 0.$$

It is clear that $\mathfrak{n}_1$ is isomorphic to the 3-dimensional Heisenberg algebra. Denote by N_1 the subgroup of N corresponding to $\mathfrak{n}_1$.

Theorem 3.2. *$(K\,;N_1)$ is a Gelfand pair, whereas $(K\,;N)$ is not a Gelfand pair.*

Proof. Since N_1 is isomorphic to the 3-dimensional Heisenberg group H_1 and $K=\boldsymbol{T}$, the pair $(K\,;N_1)$ is a Gelfand pair as is well-known (see [BJR]). To show that $(K\,;N)$ fails to be a Gelfand pair we take a basis of $\mathfrak{n}$ as follows:

$$E_1^R=(1,\ 0,\ 0),\quad E_1^I=(\sqrt{-1},\ 0,\ 0),$$

$$E_2^R=(0,\ 1,\ 0),\quad E_2^I=(0,\ \sqrt{-1},\ 0),\quad T=(0,\ 0,\ 1).$$

Let l be an element of $\mathfrak{n}^*$ such that $l(z_1,\ z_2,\ t)=\mathrm{Re}\,z_1+t$. Then we have $\mathfrak{n}(l)=Z(\mathfrak{n})$, $\mathfrak{b}(l)=\boldsymbol{R}E_1^I+\boldsymbol{R}(T-E_1^R)$ and $\mathfrak{n}/\mathfrak{b}(l)$ is isomorphic to the 3-dimensional Heisenberg algebra. Let $k=e^{\sqrt{-1}\theta}\in K$. Then

$$l(e^{\sqrt{-1}\theta}\cdot(z_1,\ 0,\ t))=x_1\cos\theta-y_1\sin\theta+t,$$

where $z_1=x_1+\sqrt{-1}y_1$ $(x_1,\ y_1\in\boldsymbol{R})$. Obviously $k\cdot\mathfrak{n}(l)=\mathfrak{n}(l)$. Letting $\pi\in\hat{N}$ be corresponding to l, we have $K_\pi=\{1\}$ by Lemma 2.4. Hence $\Phi_\pi(K_\pi)=\{1\}$, so that $(\Phi_\pi(K_\pi)\,;B(l)\backslash N)$ is not a Gelfand pair. This together with Theorem 2.6 completes the proof.

By (3.3), we have $N=N_1\times A$. Hence we have $L^1(N)=L^1(N_1)\otimes L^1(A)$. But Theorem 3.2 says that $L_K^1(N)$ is not isomorphic to $L_K^1(N_1)\otimes B$ for any subalgebra B of $L^1(A)$.

4. Leptin's problem

Let N be a 2-step nilpotent Lie group and $K=\boldsymbol{T}^n$ an n-dimensional torus acting on N as automorphisms. We consider the following problem posed by Leptin [L].

Problem 4.1. *When is $(K\,;N)$ a Gelfand pair?*

For example, $(\boldsymbol{T}^n\,;H_n)$ is a Gelfand pair, [HR]. When $[\mathfrak{n},\mathfrak{n}]=Z(\mathfrak{n})$ and $\boldsymbol{T}^n$ acts on N effectively, Leptin gave an answer as follows [L]:

$(\boldsymbol{T}^n\,;N)$ is a Gelfand pair if and only if N is a quotient group of $(H_1)^n$ by a central subgroup and $\boldsymbol{T}^n$ acts on $(H_1)^n$ naturally. In this case, $\boldsymbol{T}^n$ acts on $Z(N)$ trivially.

We investigate now the case $[\mathfrak{n},\mathfrak{n}]\neq Z(\mathfrak{n})$ and give a complete solution to Problem 4.1. We have the decompositions (3.1), (3.2) and (3.3) in the previous section. We write $\hat{K}^r$ for the family of all equivalence classes of irreducible real K-modules. Then $\hat{K}^r$ is identified with $\boldsymbol{Z}^n/\sim$ where $\alpha\sim\beta$ if $\beta=\pm\alpha$ for α, $\beta\in\boldsymbol{Z}^n$. By fixing a system of representatives, we may regard $\hat{K}^r$ as a subset of $\boldsymbol{Z}^n$. We decompose further $\mathfrak{n}'$, $\mathfrak{a}$ into isotypic real K-modules:

$$\text{(4.1)} \qquad \mathfrak{n}'=\sum_{\alpha\in\hat{K}^r} V'_\alpha, \quad \mathfrak{a}=\sum_{\alpha\in\hat{K}^r} V''_\alpha.$$

For non-zero $\alpha\in\boldsymbol{Z}^n$, let V_α be the 2-dimensional real irreducible K-module $\boldsymbol{R}X_\alpha+\boldsymbol{R}Y_\alpha$ such that

$$\text{(4.2)} \qquad (\exp U\cdot X_\alpha, \exp U\cdot Y_\alpha)=(X_\alpha, Y_\alpha)\begin{pmatrix}\cos\alpha(U) & -\sin\alpha(U)\\ \sin\alpha(U) & \cos\alpha(U)\end{pmatrix},$$

where $U\in\mathfrak{k}$, the Lie algebra of K. If $\alpha=0$, V_α denotes the 1-dimensional trivial real K-module. Let

$$\text{(4.3)} \qquad V'_\alpha=m_{\alpha,1}V_\alpha, \quad V''_\alpha=m_{\alpha,2}V_\alpha,$$

where $m_{\alpha,1}$, $m_{\alpha,2}$ are the multiplicities of V_α in $\mathfrak{n}'$ and in $\mathfrak{a}$ respectively. We also write

$$\text{(4.4)} \qquad V'_\alpha=\sum_{i=1}^{m_{\alpha,1}} V'_{\alpha,i}, \quad V''_\alpha=\sum_{i=1}^{m_{\alpha,2}} V''_{\alpha,i},$$

where $V'_{\alpha,i}$, $V''_{\alpha,i}\simeq V_\alpha$ for all i, and $V'_{\alpha,i}\perp V'_{\alpha,j}$, $V''_{\alpha,i}\perp V''_{\alpha,j}$ if $i\neq j$. Let S_1, S_2 be the subsets of $\hat{K}^r$ such that

$$\text{(4.5)} \qquad S_1=\{\alpha\in\hat{K}^r \mid m_{\alpha,1}\neq 0\}, \quad S_2=\{\alpha\in\hat{K}^r \mid m_{\alpha,2}\neq 0\}.$$

Theorem 4.2. *$(K\,;N)$ is a Gelfand pair if and only if the following five conditions are satisfied*:

(1) $m_{0,1}=0$,
(2) *S_1 is a linearly independent system,*
(3) *$m_{\alpha,1}=1$ for all $\alpha\in S_1$,*
(4) $\boldsymbol{R}\text{-span}(S_1)\cap\boldsymbol{R}\text{-span}(S_2)=0$,
(5) *K acts on $[\mathfrak{n}, \mathfrak{n}]$ trivially.*

In order to prove this theorem, we need the following lemma.

Lemma 4.3. *Let K be an n-dimensional compact abelian Lie group (not necessarily connected), and suppose K acts on H_m effectively as automorphisms. Then $(K\,;H_m)$ is a Gelfand pair if and only if $n=m$.*

Proof. See [BJR, p. 103], the proof of Theorem 5.17.

Proof of Theorem 4.2. Suppose first that $(K\,;N)$ is a Gelfand pair.

Step 1: Let V_1, V_2 be mutually orthogonal K-invariant subspaces of $\mathfrak{n}'$. Then we have $[V_1, V_2]=0$ (see the proof of Leptin's theorem in [BJR, p. 107]).

In particular, $[V_0', V_\alpha']=0$ for all $\alpha \neq 0$. Clearly $[V_0', V_0']=0$ and $[V_0', Z(\mathfrak{n})]=0$. Hence $[V_0', \mathfrak{n}]=0$ by (3.1) and (3.2). This means $V_0' \subset Z(\mathfrak{n}) \cap \mathfrak{n}'=0$, whence (1). We next show (5). Indeed, for non-zero $\alpha \in \hat{K}^r$, and for $U \in \mathfrak{k}$, we have by (4.2),

$$\begin{aligned}
&[\exp U \cdot X_\alpha, \exp U \cdot Y_\alpha] \\
&\quad = [\cos \alpha(U) X_\alpha + \sin \alpha(U) Y_\alpha, \ -\sin \alpha(U) X_\alpha + \cos \alpha(U) Y_\alpha] \\
&\quad = \cos^2 \alpha(U)[X_\alpha, Y_\alpha] - \sin^2 \alpha(U)[Y_\alpha, X_\alpha] = [X_\alpha, Y_\alpha].
\end{aligned}$$

Since $[V_{\alpha,i}', V_{\alpha,j}']=0$ for $i \neq j$, we see that K acts on $[\mathfrak{n}, \mathfrak{n}]$ trivially.

Step 2: We prove (2), (3) and (4). Take bases $\{X_{\alpha,i}', Y_{\alpha,i}'\}$, $\{X_{\beta,j}'', Y_{\beta,j}''\}$ of $V_{\alpha,i}'$, $V_{\beta,j}''$ respectively similarly to (4.2). Let l be an element of $\mathfrak{n}^*$ with $l|_{\mathfrak{n}'}=0$ such that for each $\alpha \in S_1$, $l([X_{\alpha,i}', Y_{\alpha,i}']) \neq 0$ for any i with $1 \leq i \leq m_{\alpha,1}$, and for each $\beta \in S_2$, $l(X_{\beta,j}'')=1$, $l(Y_{\beta,j}'')=0$ for any j with $1 \leq j \leq m_{\beta,2}$. Then $\mathfrak{n}(l)=\mathfrak{a}+[\mathfrak{n}, \mathfrak{n}]=Z(\mathfrak{n})$ and $\mathfrak{n}/\mathfrak{b}(l) \simeq \mathfrak{h}_m$ with $m=\sum_{\alpha \in S_1} m_{\alpha,1}$. Write $\pi=\pi_l$ for simplicity. Then, we have by Lemma 2.4,

$$\begin{aligned}
K_\pi &= \{k \in K \mid l \cdot k = l \text{ on } Z(\mathfrak{n})\} \\
&= \{k \in K \mid k \cdot X = X \text{ for all } X \in Z(\mathfrak{n})\}.
\end{aligned}$$

Let $\mathfrak{k}_\pi$ be the Lie algebra of K_π. Then we have by the above,

$$\begin{aligned}
\mathfrak{k}_\pi &= \{U \in \mathfrak{k} \mid U \cdot X = 0 \text{ for all } X \in Z(\mathfrak{n})\} \\
&= \{U \in \mathfrak{k} \mid \beta(U)=0 \text{ for all } \beta \in S_2\} = \bigcap_{\beta \in S_2} \ker \beta.
\end{aligned}$$

Let Φ_π be the map $K_\pi \to \mathrm{Aut}(\mathfrak{n}/\mathfrak{b}(l))$ obtained by making the K_π-action factor through $\mathfrak{n}/\mathfrak{b}(l)$. Then for the differential $d\Phi_\pi$ which maps $\mathfrak{k}_\pi$ to the derivation algebra $\mathrm{Der}(\mathfrak{n}/\mathfrak{b}(l))$, we have

$$\ker d\Phi_\pi = \{U \in \mathfrak{k}_\pi \mid \alpha(U)=0 \text{ for all } \alpha \in S_1\} = \Big(\bigcap_{\alpha \in S_1} \ker \alpha\Big) \cap \Big(\bigcap_{\beta \in S_2} \ker \beta\Big),$$

so that $\dim \Phi_\pi(K_\pi) = \dim d\Phi_\pi(\mathfrak{k}_\pi)$. By Lemma 4.3, we obtain $\dim \Phi_\pi(K_\pi)=m$. Moreover, we have

$$\begin{aligned}
\dim \Phi_\pi(K_\pi) &= \dim \mathfrak{k}_\pi - \dim \ker d\Phi_\pi \\
&= \dim\Big(\bigcap_{\beta \in S_2} \ker \beta\Big) - \dim\Big(\Big(\bigcap_{\alpha \in S_1} \ker \alpha\Big) \cap \Big(\bigcap_{\beta \in S_2} \ker \beta\Big)\Big) \\
&\leq \dim \mathfrak{k} - \dim\Big(\bigcap_{\alpha \in S_1} \ker \alpha\Big) \leq \# S_1 \leq \sum_{\alpha \in S_1} m_{\alpha,1} = m.
\end{aligned}$$

Hence (4), (2) and (3) are proved by the first, second and the third inequality.

Suppose conversely that the conditions (1)~(5) are satisfied. We will show that $(K; N)$ is a Gelfand pair. In order to prove this, it is sufficient to deal with the elements l in the maximal dimensional coadjoint orbits in $\mathfrak{n}^*$.

We note here that (1) says every $\alpha\in S_1$ is non-zero and that (3) implies the K-module V'_α $(\alpha\in S_1)$ is irreducible. Hence for each $\alpha\in S_1$ we have $V'_\alpha=\boldsymbol{R}X'_\alpha+\boldsymbol{R}Y'_\alpha$ for some X'_α, Y'_α with the K-action (4.2). First, we show that

$$(4.6)\qquad [V'_\alpha, V'_\beta]=0 \text{ if } \alpha, \beta\in S_1,\ \alpha\neq\beta.$$

Suppose that (4.6) is not true. Then there are elements $Z_\alpha\in V'_\alpha$, $Z_\beta\in V'_\beta$ such that $[Z_\alpha, Z_\beta]\neq 0$. Transforming Z_α by an element of K if necessary, we may assume $Z_\alpha=X'_\alpha$. If $U\in\ker\beta$, then $\exp tU\cdot Z_\beta=Z_\beta$ for all $t\in\boldsymbol{R}$. We get

$$\begin{aligned}[X'_\alpha, Z_\beta]&=[\exp tU\cdot X'_\alpha, \exp tU\cdot Z_\beta] && \text{(by (5))}\\ &=[(\cos t\alpha(U))X'_\alpha+(\sin t\alpha(U))Y'_\alpha, Z_\beta]\\ &=\cos t\alpha(U)[X'_\alpha, Z_\beta]+\sin t\alpha(U)[Y'_\alpha, Z_\beta].\end{aligned}$$

If $[X'_\alpha, Z_\beta]$ and $[Y'_\alpha, Z_\beta]$ are linearly independent, then $\cos t\alpha(U)=1$, $\sin t\alpha(U)=0$ for all $t\in\boldsymbol{R}$. If $[X'_\alpha, Z_\beta]$ and $[Y'_\alpha, Z_\beta]$ are linearly dependent, then there is $c\in\boldsymbol{R}$ such that $[Y'_\alpha, Z_\beta]=c[X'_\alpha, Z_\beta]$, $\cos t\alpha(U)+c\sin t\alpha(U)=1$ for all $t\in\boldsymbol{R}$. Therefore $\alpha(U)=0$, so that $\ker\beta\subset\ker\alpha$. This contradicts (2).

By (4.6), the coadjoint orbit O_l through $l\in\mathfrak{n}^*$ is of maximal dimension if and only if $l([V'_\alpha, V'_\alpha])\neq 0$ for any $\alpha\in S_1$. Hence we may assume that $l|_{[V'_\alpha, V'_\alpha]}\neq 0$ for any $\alpha\in S_1$ and that $l|_{\mathfrak{n}'}=0$. Then $\mathfrak{n}(l)=\mathfrak{a}+[\mathfrak{n},\mathfrak{n}]=Z(\mathfrak{n})$ and $\mathfrak{n}/\mathfrak{b}(l)\simeq\mathfrak{h}_m$ with $m=\#S_1$. We denote by $\pi=\pi_l$ the irreducible unitary representation of N corresponding to l. Then we have

$$K_\pi=\{k\in K \mid l\cdot k=l \text{ on } Z(\mathfrak{n})\}.$$

Using (5), we get

$$\mathfrak{k}_\pi=\{U\in\mathfrak{k} \mid l(U\cdot X)=0 \text{ for all } X\in\mathfrak{a}\}.$$

Put $a_{\beta,j}=l(X''_{\beta,j})$, $b_{\beta,j}=l(Y''_{\beta,j})$ for simplicity. Let $S_{2,l}=\{\beta\in S_2 \mid a_{\beta,j}=b_{\beta,j}=0$ for all j with $1\leq j\leq m_{\beta,2}\}$. Then $\mathfrak{k}_\pi=\bigcap_{\beta\in S_2\setminus S_{2,l}}\ker\beta$. Consider the map Φ_π: $K_\pi\to\mathrm{Aut}(\mathfrak{n}/\mathfrak{b}(l))$ and its differential $d\Phi_\pi$ as in Step 2, then we have

$$\ker d\Phi_\pi=\Big(\bigcap_{\alpha\in S_1}\ker\alpha\Big)\cap\Big(\bigcap_{\beta\in S_2\setminus S_{2,l}}\ker\beta\Big).$$

Hence we get

$$\begin{aligned}\dim\Phi_\pi(K_\pi)&=\dim d\Phi_\pi(\mathfrak{k}_\pi)=\dim\mathfrak{k}_\pi-\dim\ker d\Phi_\pi\\ &=\dim\Big(\bigcap_{\beta\in S_2\setminus S_{2,l}}\ker\beta\Big)-\dim\Big(\Big(\bigcap_{\alpha\in S_1}\ker\alpha\Big)\cap\Big(\bigcap_{\beta\in S_2\setminus S_{2,l}}\ker\beta\Big)\Big)\\ &=\dim\mathfrak{k}-\dim\Big(\bigcap_{\alpha\in S_1}\ker\alpha\Big) && \text{(by (4))}\\ &=\#S_1=m && \text{(by (2) and (3)).}\end{aligned}$$

Then $(\Phi_\pi(K_\pi)\,;B(l)\backslash N)$ is a Gelfand pair by virtue of Lemma 4.3. Theorem 2.6 now shows that $(K\,;N)$ is a Gelfand pair.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[Ba] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform Part I, Comm. Pure Appl. Math., **14** (1961), 187-214.

[BJR] C. Benson, J. Jenkins and G. Ratcliff, On Gelfand pairs associated with solvable Lie groups, Trans. Amer. Math. Soc., **321** (1990), 85-116.

[Br] I. D. Brown, Dual topology of a nilpotent Lie group, Ann. Sci. École Norm. Sup., **6** (1973), 407-411.

[C] G. Carcano, A commutativity condition for algebras of invariant functions, Boll. Un. Mat. Italiano, **7** (1987), 1091-1105.

[F] J. Faraut, Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques, Les Cours du CIMPA, 1982, 315-446.

[HR] A. Hulanicki and F. Ricci, A tauberian theorem and tangential convergence for bounded harmonic functions on balls in $\boldsymbol{C}^n$, Invent. Math., **62** (1980), 325-331.

[Ki] A. A. Kirillov, Unitary representations of nilpotent Lie groups, Russ. Math. Surveys, **17** (1962), 53-104.

[L] H. Leptin, A new kind of eigenfunction expansions on groups, Pacific J. Math., **116** (1985), 45-67.

[M] K. G. Miller, Parametrices for hypoelliptic operators on step two nilpotent Lie groups, Comm. in Partial Diff. Eq., **5** (1980), 1153-1184.