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A Kähler structure on the punctured cotangent bundle of complex and quaternion projective spaces and its application to a geometric quantization I

By

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1. Introduction

As was studied in the paper [So2], the punctured cotangent bundle $T_0^*S^n$ of the sphere S^n is identified with the phase space of the Kepler problem, leading to the correspondence of the geodesic flow of the sphere to the solution curves of the problem, and it was noticed that the phase space has a complex structure. Also in the paper [Ra1] this complex structure gives a positive complex polarization, in other words, $T_0^*S^n$ has a Kähler structure whose Kähler from coincides with the symplectic form, and this structure was used to quantize the geodesic flow of the sphere([Ra3], [MT1]).

It is well-known that the geodesic flow of the sphere is periodic. In general, if we have a free $U(1)$-action generated by a positive homogeneous Hamiltonian on the punctured cotangent bundle T_0^*M of a compact manifold M, like C_l-manifold([Be]), then we have a $\mathbf{C}^*$-free action on T_0^*M and the orbit space $\mathcal{O}_M$ becomes a compact symplectic manifold with the integral symplectic form. And then, by the symplectic embedding theorem, there exists an embedding $\mathcal{O}_M \rightarrow P^N\mathbf{C}(N \gg 1)$ and T_0^*M is identified with the pull-back of the associated $\mathbf{C}^*$-principal bundle of the tautological line bundle on $P^N\mathbf{C}$. Moreover both $\mathbf{C}^*$-actions coincide. In some cases, $\mathcal{O}_M$ is seen to have a complex structure, and expected to become a Hodge manifold. So T_0^*M will have a complex structure, and it will be interesting to study whether this complex structure defines a positive complex polarization for the symplectic manifold T_0^*M. Compact symmetric spaces of rank 1 are such manifolds (of course, the sphere is mentioned above), and in these cases $\mathcal{O}_M$ is of the form $G/Z_G(T)$. Here, G denotes $SO(n)$, $SU(n)$, $Sp(n)$, or F_4, and $Z_G(T)$ is the centralizer of a certain 1-dimensional toral subgroup T in G. The space $G/Z_G(T)$ is isomorphic to the homogeneous space $G^{\mathbf{C}}/P$, where $G^{\mathbf{C}}$ is the complexification of G and P is the parabolic subgroup correspoding to $Z_G(T)$.

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This means $\mathcal{O}_M$ becomes a complex, in fact, non-singular projective algebraic manifold([BH]). So in this paper, for complex projective spaces $P^n\mathbf{C}$ and quaternion projective spaces $P^n\mathbf{H}$ we will prove, by elementary methods, that the punctured cotangent bundle of these spaces admits a Kähler structure whose Kähler form coincides with the symplectic form, just like the case of the sphere (Theorem 2.4 and Theorem 3.8). Our arguments are based on the diagonalization of the geodesic flow (Proposition 2.5 and Proposition 3.9).

In §2 we will consider the case of the complex projective space. In §3 we will consider the case of the quaternion projective space. In §4 we will remark the relation between the case of $P^1\mathbf{H}$ and that of S^4. In §5 we will describe the automorphisms of $T_0^*P^n\mathbf{C}(n\geqq 2)$ and $T_0^*P^n\mathbf{H}(n\geqq 3$ and $n=1)$.

As an application of the result in §2, an operator from a certain Hilbert space of holomorphic functions on $T_0^*P^n\mathbf{C}$ to $L_2(P^n\mathbf{C})$ is constructed in the next paper [FY] by pairing two polarizations (real and positive complex). This operator gives a quantization of the geodesic flow on $P^n\mathbf{C}$.

2. A Kähler structure on $T_0^*P^n\mathbf{C}$

In this section we describe a Kähler structure on the punctured cotangent bundle $T_0^*P^n\mathbf{C}$ whose Kähler form coincides with the symplectic form on $T^*P^n\mathbf{C}$.

First we note the lemma 2.1 below.

Let (M, g) be a compact Riemannian manifold. We identify the cotangent bundle T^*M with the tangent bundle TM by the following map:

$$\begin{aligned} \mathcal{L}=\mathcal{L}_M &: TM \tilde{\to} T^*M \\ \mathcal{L}_M(X)(Y)&=g_x(Y, X),\ X,\ Y\in T_xM. \end{aligned} \tag{2.1}$$

Then we have,

Lemma 2.1. *Let (M, g) and (N, h) be two Riemannian manifolds, and $f: M\to N$ be a smooth map. If*

$$f^*h=cg \quad (c \text{ is a positive constant}) \tag{2.2}$$

holds, we have

$$(\mathcal{L}_N\circ df\circ\mathcal{L}_M^{-1})^*\theta_N=c\theta_M \tag{2.3}$$

where $\theta_M(\theta_N)$ denotes the canonical 1-form of $M(N)$.

Let $\mathbf{C}^n$ be the Hermitian inner product space with the inner product $(\cdot, \cdot)_\mathbf{C}$, and consider S^{2n+1} to consist of points $p\in\mathbf{C}^{n+1}$ with $(p, p)_\mathbf{C}=1$. The Fubini-Study metric g_0 on $P^n\mathbf{C}$ is defined in an obvious way through the Hopf-fibering:

(2.4)
$$S^{2n+1}\to P^n\mathbf{C},\quad p\mapsto[p],$$

where $[p]$ denotes the line in $\mathbf{C}^{n+1}$ spanned by p. Let $\mathcal{H}(n+1)$ denote the space of $(n+1)\times(n+1)$-Hermitian matrices with the Euclidean inner product $(A, B)=\operatorname{tr} AB$ where $A, B\in\mathcal{H}(n+1)$. We note that $\|A\|=(\operatorname{tr} A^2)^{\frac{1}{2}}$. There is a well-known embedding $\mathcal{P}$:

(2.5)
$$\mathcal{P} : P^n\mathbf{C}\to\mathcal{H}(n+1),\quad [p]\mapsto(p_{ij}),\ p_{ij}=p_i\bar{p}_j.$$

The matrix (P_{ij}) represents the projection onto the subspace $[p]$.

Now we have the following propositions.

Proposition 2.2.

(2.6) $\|d\mathcal{P}(X)\|^2=\operatorname{tr}(d\mathcal{P}(X)^2)=2g_0(X, X),\ X\in T(P^n\mathbf{C})$.

Proposition 2.3. *The image $d\mathcal{P}(T_0P^n\mathbf{C})$ is equal to the subset $X_{\mathbf{C}}$ of $\mathcal{H}(n+1)\times\mathcal{H}(n+1)$ where*

(2.7) $$X_{\mathbf{C}}=\{(P, Q)|P^2=P,\ \operatorname{tr} P=1,\ PQ+QP=Q,\ \operatorname{tr} Q=0,\ Q\neq 0\}.$$

By the map (2.1), we identify $T_0^*P^n\mathbf{C}$ with $T_0P^n\mathbf{C}$, and hence with $X_{\mathbf{C}}$ through the map $d\mathcal{P}$. Hereafter we will regard $X_{\mathbf{C}}$ as $T_0^*P^n\mathbf{C}$ by this identification. Then, the canonical 1-form $\theta_{P^n\mathbf{C}}$ for $T_0^*P^n\mathbf{C}$ is written on $X_{\mathbf{C}}$ as follows:

(2.8) $$\theta_{P^n\mathbf{C}}(P, Q)=\frac{1}{2}\sum_{i,j=0}^{n} Q_{ij}dP_{ji},\ P=(P_{ij}),\ Q=(Q_{ij}).$$

So it follows that the symplectic form $\omega_{P^n\mathbf{C}}$ is written as

(2.9) $$\omega_{P^n\mathbf{C}}=\frac{1}{2}\sum_{i,j=0}^{n} dQ_{ij}\wedge dP_{ji}.$$

Let $M(n, \mathbf{C})$ denotes the space of $n\times n$−complex matrices, and let $\tau_{\mathbf{C}}$: $\mathcal{H}(n+1)\times\mathcal{H}(n+1)\to M(n+1, \mathbf{C})$ be a map defined by

(2.10) $$\tau_{\mathbf{C}}(P, Q)=\frac{1}{2}\{\|PQ\|Q+\sqrt{-1}(Q^2-2\|PQ\|^2P)\},$$

where $\|PQ\|=(\operatorname{tr} PQ(PQ)^*)^{\frac{1}{2}}$.

Let $\tilde{X}_{\mathbf{C}}$ be a subspace of $M(n+1, \mathbf{C})$ such that

(2.11) $$\tilde{X}_{\mathbf{C}}=\{A\in M(n+1, \mathbf{C})|A^2=0,\ \operatorname{rank} A=1\}$$

Then $\tilde{X}_{\mathbf{C}}$ is non-singular, and we have

Theorem 2.4. *The map $\tau_{\mathbf{C}}$ defined above gives a diffeomorphism from $X_{\mathbf{C}}$*

to $\tilde{X}_{\mathbf{C}}$. *Moreover*

$$\tau_{\mathbf{C}}^{*}(2\sqrt{-1}\,\bar{\partial}\partial(\mathrm{tr}\,AA^{*})^{\frac{1}{4}})=\omega_{P^{n}\mathbf{C}}. \tag{2.12}$$

Proof. Let $X^{0}=\{(p,\ q)\in\mathbf{C}^{n+1}\times\mathbf{C}^{n+1}|(p,\ p)_{\mathbf{C}}=1,\ (p,\ q)_{\mathbf{C}}=0,\ q\neq 0\}$ and $\pi^{0}:$ $X^{0}\rightarrow X_{\mathbf{C}}$ be

$$\begin{aligned}&\pi^{0}(p,\ q)=(P,\ Q),\\ &\qquad P=(P_{ij}),\ P_{ij}=p_{i}\bar{p}_{j},\\ &\qquad Q=(Q_{ij}),\ Q_{ij}=p_{i}\bar{q}_{j}+q_{i}\bar{p}_{j},\end{aligned} \tag{2.13}$$

then $(X^{0},\ \pi^{0},\ X_{\mathbf{C}})$ is a $U(1)$-principal bundle with the $U(1)$-action $e^{\sqrt{-1}\theta}\cdot(p,\ q)$ $=(e^{\sqrt{-1}\theta}p,\ e^{\sqrt{-1}\theta}q)$. Also, let $E^{0}=\{(z,\ w)\in\mathbf{C}^{n+1}\times\mathbf{C}^{n+1}|(z,\ z)_{\mathbf{C}}=(w,\ w)_{\mathbf{C}}\neq 0,$ $\sum_{i=0}^{n}z_{i}w_{i}=0\}$ and $\Phi:\ X^{0}\rightarrow E^{0}$ be

$$\begin{aligned}&\Phi(p,\ q)=(z,\ w),\\ &\qquad z_{i}=\frac{1}{\sqrt{2}}(\|q\|p_{i}+\sqrt{-1}q_{i}),\\ &\qquad w_{i}=\frac{1}{\sqrt{2}}(-\sqrt{-1}\|q\|\bar{p}_{i}+\bar{q}_{i}),\end{aligned} \tag{2.14}$$

then we can easily see that Φ is a diffeomorphism between X^{0} and E^{0}. Let $\pi_{E}:$ $E^{0}\rightarrow\tilde{X}_{\mathbf{C}}$ be a map defined by

$$\begin{aligned}&\pi_{E}(z,\ w)=A=(a_{ij}),\\ &\qquad a_{ij}=z_{i}w_{j},\end{aligned} \tag{2.15}$$

then $(E^{0},\ \pi_{E},\ \tilde{X}_{\mathbf{C}})$ is a $U(1)$-principal bundle with the $U(1)$-action $e^{\sqrt{-1}\theta}\cdot(z,\ w)$ $=(e^{\sqrt{-1}\theta}z,\ e^{-\sqrt{-1}\theta}w)$. Now we can see

$$\pi_{E}\circ\Phi(p,\ q)=\tau_{\mathbf{C}}\circ\pi^{0}(p,\ q) \tag{2.16}$$

by a simple calculation, and so we have the following commutative diagram

$$\begin{array}{ccc} X^{0} & \xrightarrow{\ \Phi\ } & E^{0}\\ {\scriptstyle\pi^{0}}\downarrow & & \downarrow{\scriptstyle\pi_{E}}\\ X_{\mathbf{C}} & \xrightarrow[\ \tau_{\mathbf{C}}\ ]{} & \tilde{X}_{\mathbf{C}}\end{array} \tag{2.17}$$

This gives an isomorphism of $U(1)$-principal bundles.

Next we show (2.12). For this purpose we list some formulas between $(P,\ Q)\in X_{\mathbf{C}}$ obtained from the properties in (2.7). For $(P,\ Q)\in X_{\mathbf{C}}$, we have

$$PQP=0, \tag{2.18}$$

$$PQ^{2}=Q^{2}P=r^{2}P, \tag{2.19}$$

$$Q^{2}=r^{2}P+QPQ, \tag{2.20}$$

(2.21) $\quad Q^3=r^2Q,$

where we put $r=(\operatorname{tr} PQ^2)^{1/2}$.

By using these formulas we have, for $A=\tau_{\mathbf{C}}(P, Q)$

(2.22) $\quad \operatorname{tr} AA^*=r^4.$

Put $\alpha=\bar{\partial}\partial(\operatorname{tr} AA^*)^{\frac{1}{4}}$, then

$$\tau_{\mathbf{C}}^*\alpha=d(\tau_{\mathbf{C}}^*(\partial(\operatorname{tr} AA^*)^{\frac{1}{4}}))=d\left(\frac{1}{4}r^{-3}\tau_{\mathbf{C}}^*(\partial \operatorname{tr} AA^*)\right)$$

$$=d\left(\frac{1}{4}r^{-3}\tau_{\mathbf{C}}^*\left(\sum \bar{a}_{ij}da_{ij}\right)\right).$$

We calculate $\tau_{\mathbf{C}}^*\left(\sum \bar{a}_{ij}da_{ij}\right)$ by using (2.18)~(2.22):

$$\begin{aligned}
&\tau_{\mathbf{C}}^*\left(\sum \bar{a}_{ij}da_{ij}\right)\\
&=\frac{1}{4}\sum\{r\bar{Q}_{ij}-\sqrt{-1}((\bar{Q}^2)_{ij}-2r^2\bar{P}_{ij})\}\cdot d\{rQ_{ij}+\sqrt{-1}((Q^2)_{ij}-2r^2P_{ij})\}\\
&=\frac{1}{4}\sum\{rQ_{ji}d(rQ_{ij})+\sqrt{-1}rQ_{ji}d((Q^2)_{ij})-2\sqrt{-1}rQ_{ji}d(r^2P_{ij})\\
&\quad-\sqrt{-1}(Q^2)_{ji}d(rQ_{ij})+(Q^2)_{ji}d(Q^2)_{ij}-2(Q^2)_{ji}d(r^2P_{ij})\\
&\quad+2\sqrt{-1}r^2P_{ji}d(rQ_{ij})-2r^2P_{ji}d(Q^2)_{ij}+4r^2P_{ji}d(r^2P_{ij})\}\\
&=\frac{1}{4}\Big\{2r^3dr+2r^3dr-2\sqrt{-1}r^3\sum Q_{ji}dP_{ij}\\
&\quad+dr^4-2r^2dr^2\\
&\quad+2\sqrt{-1}r^3\sum P_{ji}dQ_{ij}+4r^2dr^2-2r^2dr^2\Big\}\\
&=2r^3dr-2\sqrt{-1}r^3\theta_{P^n\mathbf{C}}.
\end{aligned}$$

Here we denote the (i, j) component of Q^2 by $(Q^2)_{ij}$, and so on. Consequently,

$$\tau_{\mathbf{C}}^*\alpha=d\left\{\frac{1}{4}r^{-3}\cdot(2r^3dr-2\sqrt{-1}r^3\theta_{P^n\mathbf{C}})\right\}$$

$$=-\frac{\sqrt{-1}}{2}\omega_{P^n\mathbf{C}}.$$

Finally we have

$$\tau_{\mathbf{C}}^*(2\sqrt{-1}\,\bar{\partial}\partial(\operatorname{tr} AA^*)^{\frac{1}{4}})=\omega_{P^n\mathbf{C}}.$$

Let $X_S = \{(p, q) \in \mathbf{C}^{n+1} \times \mathbf{C}^{n+1} | \, \|p\|=1, \ \mathrm{Re}(p, q)_{\mathbf{C}}=0, \ q \neq 0\}$, then X_S is identified with T_0S^{2n+1}. We denote by $\tilde{\epsilon}_t$ $(t \in \mathbf{R})$ the geodesic flow on the sphere S^{2n+1}:

$$\begin{aligned} &\tilde{\epsilon}: \mathbf{R} \times X_S \to X_S \\ (2.23) \qquad &(t, p, q) \mapsto (\cos t \cdot p + \frac{\sin t}{\|q\|} q, \ -\|q\| \sin t \cdot p + \cos t \cdot q). \end{aligned}$$

Then $\tilde{\epsilon}$ preserves X^0 and descents to the action on $X_{\mathbf{C}} = d\mathcal{P}(T_0P^n\mathbf{C})$ through the map π^0, and the resulting action $\epsilon_t (t \in \mathbf{R})$ is the geodesic flow of $P^n\mathbf{C}$. It reduces to a periodic and free action with the period π.

Proposition 2.5. *We have the following commutative diagram:*

$$(2.24) \qquad \begin{array}{ccc} X_{\mathbf{C}} & \xrightarrow{\tau_{\mathbf{C}}} & \tilde{X}_{\mathbf{C}} \\ {\scriptstyle \epsilon_t} \downarrow & & \downarrow {\scriptstyle e^{-2\sqrt{-1}t}} \\ X_{\mathbf{C}} & \xrightarrow[\tau_{\mathbf{C}}]{} & \tilde{X}_{\mathbf{C}} \end{array}$$

where $e^{-2\sqrt{-1}t}$ denotes the scalar multiplication by $e^{-2\sqrt{-1}t}$.

When we regard $TP^n\mathbf{C}$ as the holomorphic tangent bundle, then $TP^n\mathbf{C}$ is also a complex manifold. We denote it by $\mathcal{T}P^n\mathbf{C}$.

Proposition 2.6. *Let $X_{\mathbf{C}}$ be given the complex structure by identifying $X_{\mathbf{C}}$ with $\mathcal{T}_0P^n\mathbf{C}$, then the map $\tau_{\mathbf{C}}$ is not holomorphic. The action of $SU(n+1)$ on $X_{\mathbf{C}}$ is holomorphic with respect to the both holomorphic structures.*

Proof. The last assertion is clear. The first one will be also apparent, however we see it here by representing the map $\tau_{\mathbf{C}}$ in holomorphic coordinate systems with respect to the complex structure in both senses. Let $(z_1, \cdots, z_n) \in \mathbf{C}^n$ be a holomorphic coordinate system of $P^n\mathbf{C}$ given by the correspondence:

$$(z_1, \cdots, z_n) \longleftrightarrow \frac{1}{1+\|z\|^2} \begin{pmatrix} 1 & \bar{z}_1 & \cdots & \bar{z}_j & \cdots & \bar{z}_n \\ z_1 & & & & & \\ \vdots & & & & & \\ z_i & & & z_i\bar{z}_j & & \\ \vdots & & & & & \\ z_n & & & & & \end{pmatrix} \in \mathcal{P}(P^n\mathbf{C}).$$

Then a system of holomorphic coordinates $(z, w) \in \mathbf{C}^n \times \mathbf{C}^n$ of $\mathcal{T}P^n\mathbf{C}$ is given by

$(z, w) \longleftrightarrow (P, Q) \in X_{\mathbf{C}}$,

$$P=P(z)=\frac{1}{1+\|z\|^2}\begin{pmatrix} 1 & \bar{z}_1 & \cdots & \bar{z}_n \\ z_1 & & & \\ \vdots & & z_i\bar{z}_j & \\ z_n & & & \end{pmatrix}$$

$$\begin{aligned} Q&=Q(z, w) \\ &=-\frac{(z, w)_{\mathbf{C}}+(w, z)_{\mathbf{C}}}{(1+\|z\|^2)^{\frac{3}{2}}}\begin{pmatrix} 1 & \bar{z}_1 & \cdots & \bar{z}_n \\ z_1 & & & \\ \vdots & & z_i\bar{z}_j & \\ z_n & & & \end{pmatrix} \\ &\quad+\frac{1}{\sqrt{1+\|z\|^2}}\begin{pmatrix} 1 & \bar{w}_1 & \cdots & \bar{w}_n \\ w_1 & & & \\ \vdots & & w_{ij} & \\ w_n & & & \end{pmatrix} \end{aligned}$$

where $w_{ij}=z_i\bar{w}_j+w_i\bar{z}_j$. Here (z, w) represents a holomorphic tangent vector $\sum w_i\left(\frac{\partial}{\partial z_i}\right)_{P(z)}$. Let

$$\text{(2.25)}\qquad \begin{cases} z_0=1,\ w_0=0, \\ p_i=\dfrac{z_i}{\sqrt{1+\|z\|^2}} \qquad (i=0, \cdots, n), \\ q_i=\dfrac{w_i}{\sqrt{1+\|z\|^2}}-\dfrac{(w, z)_{\mathbf{C}}}{(1+\|z\|^2)^{\frac{3}{2}}}z_i \qquad (i=0, \cdots, n), \end{cases}$$

then we have

$$\text{(2.26)}\qquad \begin{aligned} &\tau_{\mathbf{C}}(P(z), Q(z, w))=A=(a_{ij}), \\ &a_{ij}=\frac{1}{2}(\|q\|p_i+\sqrt{-1}q_i)(-\sqrt{-1}\|q\|\bar{p}_j+\bar{q}_j). \end{aligned}$$

So we have Proposition 2.6 from the expressions (2.25) and (2.26).

3. A Kähler structure on $T_0^*P^n\mathbf{H}$

In this section, we describe a Kähler structure on $T_0^*P^n\mathbf{H}$, and prove that the Kähler form coincides with the symplectic form. Our arguments are similar to that for $T_0^*P^n\mathbf{C}$.

We denote by $\mathbf{H}$ the quaternion field:

$$\text{(3.1)}\qquad \begin{aligned} &p=z+w\mathbf{j}\in\mathbf{H},\ z,\ w\in\mathbf{C}, \\ &\mathbf{j}^2=-1,\ z\mathbf{j}=\mathbf{j}\bar{z}, \\ &\bar{p}=\bar{z}-w\mathbf{j}. \end{aligned}$$

In the following we regard $\mathbf{H}^n$ as a right $\mathbf{H}$-vector space with $\mathbf{H}$-inner product $(p, q)_{\mathbf{H}}=\sum \bar{p}_iq_i$.

For $p=(p_0, \cdots, p_{n-1})\in \mathbf{H}^n$, we denote the norm of p by $\|p\|$; $\|p\|=\sqrt{\sum|p_i|^2}=\sqrt{\sum|z_i|^2+|w_i|^2}$, $p_i=z_i+w_i\mathbf{j}$.

Let $\rho: \mathbf{H}\to M(2, \mathbf{C})$ be the representation given by

$$\rho(p)=\rho(z+w\mathbf{j})=\begin{pmatrix} z & w \\ -\bar{w} & \bar{z}\end{pmatrix}, \tag{3.2}$$

then ρ is $\mathbf{R}$-linear, $\rho(pq)=\rho(p)\rho(q)$ and $\rho(\bar{p})=\rho(p)^*$. We introduce the Riemannian metric $g_{\mathbf{H}}$ on the quaternion projective space $P^n\mathbf{H}$, by the same way for $P^n\mathbf{C}$ through the Hopf-fibration:

$$\pi: S^{4n+3}\to P^n\mathbf{H}$$

$$p\mapsto[p],$$

where $p=(p_0, \cdots, p_n)\in\mathbf{H}^{n+1}$, $\|p\|=1$, and $[p]$ denotes the $\mathbf{H}$-right subspace spanned by p.

Let $\mathcal{P}_{\mathbf{H}}: P^n\mathbf{H}\to M(2n+2, \mathbf{C})$ be an embedding defined by

$$\begin{aligned}\mathcal{P}_{\mathbf{H}}([p])&=(P_{ij})\in M(2n+2, \mathbf{C})\\ P_{ij}&=\rho(p_i\bar{p}_j),\end{aligned} \tag{3.3}$$

then we have

Proposition 3.1. *$\mathcal{P}_{\mathbf{H}}(P^n\mathbf{H})=\{P\in M(2n+2, \mathbf{C})|P^2=P,\ P^*=P,\ \mathrm{tr}\,P=2,\ P\boldsymbol{J}=\boldsymbol{J}\,{}^tP\}$, where*

$$\boldsymbol{J}=\begin{pmatrix} J & & & 0\\ & J & & \\ & & \ddots & \\ 0 & & & J\end{pmatrix},\ J=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

Hence we also have

Proposition 3.2. $d\mathcal{P}_{\mathbf{H}}(T_0P^n\mathbf{H})=\{(P,\ Q)\in\mathcal{H}(2n+2)\times\mathcal{H}(2n+2)|P^2=P,\ \mathrm{tr}\,P=2,\ P\boldsymbol{J}=\boldsymbol{J}\,{}^tP,\ QP+PQ=Q,\ \mathrm{tr}\,Q=0,\ Q\boldsymbol{J}=\boldsymbol{J}\,{}^tQ,\ Q\neq0\}$.

We denote $P=(P_{ij})$, $Q=(Q_{ij})$ with P_{ij} and $Q_{ij}\in M(2, \mathbf{C})$.

Proposition 3.3

$$\|d\mathcal{P}_{\mathbf{H}}(X)\|^2=2g_{\mathbf{H}}(X,\ X),\ X\in TP^n\mathbf{H}.$$

Proposition 3.4. *The canonical 1-form $\theta_{P^n\mathbf{H}}$ of $T_0^*P^n\mathbf{H}$ is written as follows on $d\mathcal{P}_{\mathbf{H}}(T_0P^n\mathbf{H})$:*

$$\theta_{P^n\mathbf{H}}(P,\ Q)=\frac{1}{2}(d\mathcal{P}_{\mathbf{H}})^*\,\mathrm{tr}(QdP),$$

where $\mathrm{tr}(QdP)=\sum_{i,j=0}^{n}\mathrm{tr}(Q_{ij}dP_{ji})$. *Hence the symplectic form* $\omega_{P^n\mathbf{H}}$ *is written in the form :*

$$\omega_{P^n\mathbf{H}}=\frac{1}{2}(d\mathcal{P}_{\mathbf{H}})^*\ \mathrm{tr}(dQ\wedge dP).$$

Put $d\mathcal{P}_{\mathbf{H}}(T_0P^n\mathbf{H})=X_{\mathbf{H}}$ and also put $X_{\mathbf{H}}^0=\{(p,\ q)\in\mathbf{H}^{n+1}\times\mathbf{H}^{n+1}|\ \|p\|=1,\ (p,\ q)_{\mathbf{H}}=0,\ q\neq 0\}$, then we have a $Sp(1)$-principal bundle $(X_{\mathbf{H}}^0,\ \pi^0,\ X_{\mathbf{H}})$, where

$$\pi^0:\ X_{\mathbf{H}}^0\to X_{\mathbf{H}},$$

$$(p,\ q)\ \mapsto\left(\begin{pmatrix}P_{ij}\end{pmatrix},\ \begin{pmatrix}Q_{ij}\end{pmatrix}\right),$$

$$P_{ij}=\rho(p_i\bar{p}_j),$$

$$Q_{ij}=\rho(p_i\bar{q}_j+q_i\bar{p}_j).$$

Let $\mathcal{B}:\ \mathbf{C}^{2n+2}\times\mathbf{C}^{2n+2}\to\mathbf{C}$ be the bilinear form $\mathcal{B}(z,\ w)=\sum_{i=0}^{2n+1}z_iw_i$ and

$$E_{\mathbf{H}}=\{(z,\ w)\in\mathbf{C}^{2n+2}\times\mathbf{C}^{2n+2}|z\wedge w\neq 0,\ \mathcal{B}(z,\ \boldsymbol{J}w)=0\},$$

where $\boldsymbol{J}w=(w_1,\ -w_0,\ w_3,\ -w_2,\ \cdots,\ w_{2n+1},\ -w_{2n})$. Also, let $\widetilde{X}_{\mathbf{H}}$ be

$$\widetilde{X}_{\mathbf{H}}=\{A\in M(2n+2,\ \mathbf{C})|A^2=0,\ \mathrm{rank}\ A=2,\ A\boldsymbol{J}=\boldsymbol{J}\,{}^tA\},$$

and define a map $\pi_{\mathbf{H}}:\ E_{\mathbf{H}}\to\widetilde{X}_{\mathbf{H}}$ by

$$\pi_{\mathbf{H}}(z,\ w)=z\otimes\boldsymbol{J}w-w\otimes\boldsymbol{J}z.$$

Here we identify $M(2n+2,\ \mathbf{C})$ and $\mathbf{C}^{2n+2}\otimes\mathbf{C}^{2n+2}$ by the correspondence

$$z\otimes w\longleftrightarrow A=(a_{ij})\in M(2n+2,\ \mathbf{C}),$$

$$a_{ij}=z_iw_j.$$

Then we have

Proposition 3.5. $(E_{\mathbf{H}},\ \pi_{\mathbf{H}},\ \widetilde{X}_{\mathbf{H}})$ *is a holomorphic principal bundle with the structure group* $SL(2,\ \mathbf{C})$. *The action of* $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,\ \mathbf{C})$ *on* $E_{\mathbf{H}}$ *is given by* $(z,\ w)\cdot g=(\alpha z+\gamma w,\ \beta z+\delta w)$.

Proof. Let $A:\ \mathbf{C}^{2n+2}\to\mathbf{C}^{2n+2}$ be a linear map of rank two, then A is of the form $Ax=\mathcal{B}(x,\ u)z+\mathcal{B}(x,\ v)w$ with $z\wedge w\neq 0$ and $u\wedge v\neq 0$. If this linear map A satisfies the condition $A\boldsymbol{J}=\boldsymbol{J}\,{}^tA$, then $Ax=\mathcal{B}(x,\ \boldsymbol{J}w)z-\mathcal{B}(x,\ \boldsymbol{J}z)w$, that is, $A=z\otimes\boldsymbol{J}w-w\otimes\boldsymbol{J}z$. Moreover $A^2=0$ if and only if $\mathcal{B}(z,\ \boldsymbol{J}w)=0$. Hence we see that $\pi_{\mathbf{H}}$ is surjective, and the $SL(2,\ \mathbf{C})$-invariance of $\pi_{\mathbf{H}}$ can be easily seen.

Next we prove $\widetilde{X}_{\mathrm{H}}$ is non-singular. Since $(z, w) \mapsto z\otimes Jw - w\otimes Jz$ is alternative, we have a linear map $S: \overset{2}{\wedge}\mathbf{C}^{2n+2}\to\mathbf{C}^{2n+2}\otimes\mathbf{C}^{2n+2}$ satisfying $S(z\wedge w)=z\otimes Jw-w\otimes Jz$. Let $T: \mathbf{C}^{2n+2}\otimes\mathbf{C}^{2n+2}\to\overset{2}{\wedge}\mathbf{C}^{2n+2}$ be a linear map defined by $T(z\otimes w)=-z\wedge Jw$, then $T\circ S=Id$. Hence S is injective. Let $\widetilde{G}=\{X\in\overset{2}{\wedge}\mathbf{C}^{2n+2}|X\wedge X=0,\ X\neq 0\}$. Then $\widetilde{G}$ is the total space of the associated $\mathbf{C}^*$-principal bundle of the Plücker embedding of the Grassmann manifold $G_{2n+2,2}(\mathbf{C})\cong U(2n+2)/U(2)\times U(2n)$, and $X\wedge X=0$ if and only if X is of the form $X=z\wedge w(z,\ w\in\mathbf{C}^{2n+2})$([GH]). Since the bilinear form $\mathscr{B}(z, Jw)$ is $SL(2,\mathbf{C})$-invariant and its holomorphic differential is always non-zero on E_{H}, if we denote by $\bar{\mathscr{B}}$ the linear form on $\overset{2}{\wedge}\mathbf{C}^{2n+2}$ given by $\bar{\mathscr{B}}(z\wedge w)=\mathscr{B}(z, Jw)$, then $\bar{X}_{\mathrm{H}}=\{X\in\widetilde{G}|\bar{\mathscr{B}}(X)=0\}$ is non-singular. Hence $S(\bar{X}_{\mathrm{H}})=\widetilde{X}_{\mathrm{H}}$ is also non-singular.

Let $\mathcal{U}_{ij}=\{X=\sum_{\alpha<\beta}X_{\alpha\beta}e_\alpha\wedge e_\beta\in\widetilde{G}|X_{ij}\neq 0\}$, then $\bigcup_{i,j}\mathcal{U}_{ij}=\widetilde{G}$ and each $\mathcal{U}_{ij}$ is a local coordinate neighborhood of $\widetilde{G}$. A local coordinate $\phi_{ij}: \mathcal{U}_{ij}\to\mathbf{C}^{4n+1}$ is given by $\phi_{ij}(X)=(X_{i0},\cdots,\hat{X}_{ii},\cdots,\hat{X}_{ij},\cdots,X_{i2n+1},X_{j0},\cdots,\hat{X}_{jj},\cdots,X_{j2n+1})$, where we regard $X_{ij}=-X_{ji}$, and $\hat{X}_{..}$ is omitted.

Let $\widetilde{E}=\{(z,\ w)\in\mathbf{C}^{2n+2}\times\mathbf{C}^{2n+2}|z\wedge w\neq 0\}$, then $(\widetilde{E},\ \pi,\ \widetilde{G})$ is a $SL(2,\ \mathbf{C})$-principal bundle, where $\pi: \widetilde{E}\to\widetilde{G}$ is given by $\pi(z,\ w)=z\wedge w$. Because we have a local section s_{ij} on $\mathcal{U}_{ij}$ defined as follows:

$$s_{ij}: \mathcal{U}_{ij}\to\widetilde{E},$$

$$s_{ij}(X)=(z_0,\ \cdots,\ z_{2n+1},\ w_0,\ \cdots,\ w_{2n+1}),$$

$$z_\alpha=-X_{j\alpha},$$

$$w_\alpha=\frac{X_{i\alpha}}{X_{ij}},$$

for $\alpha=0,\ \cdots,\ 2n+1$. Hence $(\widetilde{E}_{\mathrm{H}},\ \pi_{\mathrm{H}},\ \widetilde{X}_{\mathrm{H}})$ is also a holomorphic principal bundle with the structure group $SL(2,\ \mathbf{C})$.

When we restrict the structure group of the principal bundle $(E_{\mathrm{H}},\ \pi_{\mathrm{H}},\ \widetilde{X}_{\mathrm{H}})$ from $SL(2,\ \mathbf{C})$ to $SU(2)$, then the total space E_{H}^0 is:

$$E_{\mathrm{H}}^0=\{(z,\ w)\in E_{\mathrm{H}}|\ \|z\|=\|w\|\neq 0,\ (z,\ w)_{\mathbf{C}}=0\}.$$

For $(p,\ q)\in X_{\mathrm{H}}^0$, we define

$$(H_0(p,\ q),\ \cdots,\ H_n(p,\ q),\ K_0(p,\ q),\ \cdots,\ K_n(p,\ q))\in\underbrace{M(2,\ \mathbf{C})\times\cdots\times M(2,\ \mathbf{C})}_{2n+2}$$

by

$$H_i = H_i(p,\ q) = \frac{1}{\sqrt{2}}\{\|q\|\rho(p_i) + \sqrt{-1}\rho(q_i)\}$$

$$K_i = K_i(p,\ q) = \frac{1}{\sqrt{2}}\{-\sqrt{-1}\|q\|\rho(p_i)^* + \rho(q_i)^*\}.$$

Then we see by an easy calculation

Proposition 3.6. $K_i = \sqrt{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^tH_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let a map Φ: $X_{\mathrm{H}}^0 \to M(2n+2,\ \mathbf{C})$ be

$$\Phi(p,\ q) = (A_{ij}),\ A_{ij} = H_i K_j \in M(2,\ \mathbf{C}),$$

then from the conditions for $(p,\ q) \in X_{\mathrm{H}}^0$ we have

Proposition 3.7. $\Phi(p,\ q) \in \tilde{X}_{\mathrm{H}}$.

Now for $(p,\ q) \in X_{\mathrm{H}}^0$ put

$$H_i = H_i(p,\ q) = \begin{pmatrix} s_{2i} & t_{2i} \\ s_{2i+1} & t_{2i+1} \end{pmatrix}$$

and let Φ^0: $X_{\mathrm{H}}^0 \to \mathbf{C}^{2n+2} \times \mathbf{C}^{2n+2}$ be

$$\Phi^0(p,\ q) = (z,\ w),$$

$$z_i = s_i,$$

$$w_i = -\sqrt{-1}t_i,$$

then we have

$$\Phi = \pi_{\mathrm{H}} \circ \Phi^0. \tag{3.4}$$

Let τ_{H}: $X_{\mathrm{H}} \to M(2n+2,\ \mathbf{C})$ be a map defined by

$$\tau_{\mathrm{H}}(P,\ Q) = \frac{1}{2}\left\{\frac{\|PQ\|}{\sqrt{2}}Q + \sqrt{-1}(Q^2 - \|PQ\|^2 P)\right\}, \tag{3.5}$$

then

Theorem 3.8.

$$\tau_{\mathrm{H}}(\rho(p_i\bar{p}_j),\ \rho(p_i\bar{q}_j + q_i\bar{p}_j)) = \pi_{\mathrm{H}} \circ \Phi^0(p,\ q) \tag{3.6}$$

and we have the isomorphism of $SU(2)$-principal bundles :

$$\begin{array}{ccc} X^0_{\mathbf{H}} & \xrightarrow{\Phi^0} & E^0_{\mathbf{H}} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_{\mathbf{H}}} \\ X_{\mathbf{H}} & \xrightarrow[\tau_{\mathbf{H}}]{} & \tilde{X}_{\mathbf{H}} \end{array} \tag{3.7}$$

Moreover

$$2^{\frac{3}{4}}\sqrt{-1}\tau^*_{\mathbf{H}}(\bar{\partial}\partial(\mathrm{tr}\, AA^*)^{\frac{1}{4}})=\omega_{P^n\mathbf{H}}. \tag{3.8}$$

Hence, through the map $\tau_{\mathbf{H}}$, we have a Kähler structure on $T_0^*P^n\mathbf{H}$ whose Kähler form coincides with the symplectic form.

Before proving Theorem 3.8, we note some formulas among $(P, Q)\in X_{\mathbf{H}}$ and $A\in\tilde{X}_{\mathbf{H}}$, which are obtained from Proposition 3.2.

$$\begin{cases} PQP=0, \\ PQ^2=Q^2P=\dfrac{1}{2}\|PQ\|^2P, \\ Q^2=\dfrac{1}{2}\|PQ\|^2P+QPQ, \\ Q^3=\dfrac{1}{2}\|PQ\|^2Q. \end{cases} \tag{3.9}$$

Put $r=\|PQ\|$, then $r=(\mathrm{tr}\, PQ^2)^{\frac{1}{2}}$, and $\mathrm{tr}\, Q^2=2r^2$, $\mathrm{tr}\, Q^3=0$, $\mathrm{tr}\, Q^4=r^4$. For $A=\frac{1}{2}\left\{\frac{r}{\sqrt{2}}Q+\sqrt{-1}(Q^2-r^2P)\right\}$ $((P, Q)\in X_{\mathbf{H}})$,

$$\|A\|=\frac{1}{\sqrt{2}}r^2, \tag{3.10}$$

and

$$\begin{cases} Q=2^{\frac{1}{4}}\dfrac{A+A^*}{\sqrt{\|A\|}}, \\ P=\dfrac{AA^*+A^*A}{\|A\|^2}-\sqrt{-1}\dfrac{A^*-A}{\sqrt{2}\|A\|}. \end{cases} \tag{3.11}$$

Also we have

$$\begin{cases} \mathrm{tr}(PdP)=\sum \mathrm{tr}(P_{ij}dP_{ji})=0, \\ \mathrm{tr}(QdQ)=\sum \mathrm{tr}(Q_{ij}dQ_{ji})=2rdr, \end{cases} \tag{3.12}$$

where $(P, Q)\in X_{\mathbf{H}}$ and $P=(P_{ij})$, $Q=(Q_{ij})$ with P_{ij}, $Q_{ij}\in M(2, \mathbf{C})$. Similarly,

$$\begin{cases} \mathrm{tr}\, Q^2dQ=\mathrm{tr}\, QdQ^2=0, \\ \mathrm{tr}\, Q^2dQ^2=2r^3dr. \end{cases} \tag{3.13}$$

Proof of Theorem 3.8. The commutativity of the diagram (3.7) is proved by a direct calculation and we have the isomorphism of $SU(2)$-principal bundles by (3.4) and the above calculations.

Now we have only to prove that the equality (3.8) holds. First we show

$$\tau_{\mathrm{H}}^{*}\partial\,\mathrm{tr}(A^{*}A)=-\frac{\sqrt{-1}}{\sqrt{2}}r^{3}\theta_{P^{n}\mathrm{H}},$$

where $A=(A_{ij})\in\tilde{X}_{\mathrm{H}}$, $A_{ij}\in M(2,\mathbf{C})$. Since $\partial(\mathrm{tr}\,A^{*}A)=\mathrm{tr}\,A^{*}dA=\sum\mathrm{tr}((A^{*})_{ij}dA_{ji})=\sum\mathrm{tr}((A_{ji})^{*}dA_{ji})$, We have

$$\tau_{\mathrm{H}}^{*}(\partial\,\mathrm{tr}\,A^{*}A)$$

$$=\sum\mathrm{tr}\,[\frac{1}{2}\{\frac{r}{\sqrt{2}}(Q_{ji})^{*}-\sqrt{-1}(((Q^{2})_{ji})^{*}-r^{2}(P_{ji})^{*})\}$$

$$\cdot d\frac{1}{2}\{\frac{r}{\sqrt{2}}Q_{ji}+\sqrt{-1}((Q^{2})_{ji}-r^{2}P_{ji})\}]$$

$$=\frac{1}{4}\sum\mathrm{tr}[\{\frac{r}{\sqrt{2}}Q_{ij}-\sqrt{-1}((Q^{2})_{ij}-r^{2}P_{ij})\}\cdot d\{\frac{r}{\sqrt{2}}Q_{ji}+\sqrt{-1}((Q^{2})_{ji}-r^{2}P_{ji})\}]$$

$$=\frac{1}{4}\sum\{\frac{r}{2}\mathrm{tr}(Q_{ij}d(rQ_{ji}))+\frac{\sqrt{-1}}{\sqrt{2}}r\,\mathrm{tr}(Q_{ij}d(Q^{2})_{ji})-\frac{\sqrt{-1}}{\sqrt{2}}r\,\mathrm{tr}(Q_{ij}d(r^{2}P_{ji}))\}$$

$$-\frac{\sqrt{-1}}{4}\sum\{\frac{1}{\sqrt{2}}\mathrm{tr}((Q^{2})_{ij}d(rQ_{ji}))+\sqrt{-1}\,\mathrm{tr}((Q^{2})_{ij}d(Q^{2})_{ji})$$

$$-\sqrt{-1}\,\mathrm{tr}((Q^{2})_{ij}d(r^{2}P_{ji}))\}$$

$$+\frac{\sqrt{-1}r^{2}}{4}\sum\{\frac{1}{\sqrt{2}}\mathrm{tr}(P_{ij}d(rQ_{ji}))+\sqrt{-1}\,\mathrm{tr}(P_{ij}d(Q^{2})_{ji})-\sqrt{-1}\,\mathrm{tr}(P_{ij}d(r^{2}P_{ji}))\}$$

$$=\frac{r}{8}\sum\{\mathrm{tr}(Q_{ij}Q_{ji})dr+r\,\mathrm{tr}(Q_{ij}dQ_{ji})\}+\frac{\sqrt{-1}}{4\sqrt{2}}r\sum\mathrm{tr}(Q_{ij}d(Q^{2})_{ji})$$

$$-\frac{\sqrt{-1}}{4\sqrt{2}}r\sum\{2r\,\mathrm{tr}(Q_{ij}P_{ji})dr+r^{2}\mathrm{tr}\,Q_{ij}dP_{ji}\}$$

$$-\frac{\sqrt{-1}}{4\sqrt{2}}\sum\{r\,\mathrm{tr}((Q^{2})_{ij}dQ_{ji})+\mathrm{tr}((Q^{2})_{ij}Q_{ji}\}dr\}+\frac{1}{4}\sum\mathrm{tr}(Q^{2})_{ij}d(Q^{2})_{ji}$$

$$-\frac{1}{4}\sum\{2r\,\mathrm{tr}((Q^{2})_{ij}P_{ji})dr+r^{2}\,\mathrm{tr}((Q^{2})_{ij}dP_{ji})\}$$

$$+\frac{\sqrt{-1}r^{2}}{4\sqrt{2}}\sum\{r\,\mathrm{tr}(P_{ij}dQ_{ji})+\mathrm{tr}(P_{ij}Q_{ji})dr\}-\frac{r^{2}}{4}\sum\mathrm{tr}\,P_{ij}d(Q^{2})_{ji}$$

$$+\frac{r^{2}}{4}\sum\{r^{2}\,\mathrm{tr}\,P_{ij}dP_{ji}+2r\,\mathrm{tr}(P_{ij}P_{ji})dr\}$$

$$=\frac{r}{8}\{2r^2dr+2r^2dr\}-\frac{\sqrt{-1}}{4\sqrt{2}}r^3\,\mathrm{tr}(QdP)+\frac{1}{4}\cdot 2r^3dr$$

$$-\frac{1}{4}\{2r\cdot r^2dr+r^2\,\mathrm{tr}(Q^2dP)\}+\frac{\sqrt{-1}}{4\sqrt{2}}r^3\,\mathrm{tr}(PdQ)-\frac{r^2}{4}\,\mathrm{tr}(PdQ^2)$$

$$=\frac{r^3}{2}dr-\frac{\sqrt{-1}}{2\sqrt{2}}r^3\,\mathrm{tr}(QdP)+\frac{r^3}{2}dr-\frac{r^3}{2}dr-\frac{r^2}{4}\,\mathrm{tr}\,(Q^2dP+PdQ^2)$$

$$=-\frac{\sqrt{-1}}{2\sqrt{2}}\|PQ\|^3\,\mathrm{tr}(QdP)+\frac{r^3}{2}dr-\frac{r^2}{4}\cdot 2rdr$$

$$=-\frac{\sqrt{-1}}{2\sqrt{2}}r^3\,\mathrm{tr}(QdP)$$

$$=-\frac{\sqrt{-1}}{\sqrt{2}}r^3(\frac{1}{2}\,\mathrm{tr}\;QdP)$$

$$=-\frac{\sqrt{-1}}{\sqrt{2}}r^3\theta_{P^n\mathbf{H}}.$$

Hence,

$$\tau_{\mathbf{H}}^*(\bar{\partial}\partial(\mathrm{tr}\,A^*A)^{\frac{1}{4}})=d\tau_{\mathbf{H}}^*(\frac{1}{4}(\mathrm{tr}\,A^*A)^{-\frac{3}{4}}\mathrm{tr}(A^*dA))$$

$$=d\left(\frac{1}{4}\cdot\left(\frac{r^4}{2}\right)^{-\frac{3}{4}}\cdot\frac{-\sqrt{-1}}{\sqrt{2}}r^3\theta_{P^n\mathbf{H}}\right)$$

$$=d\left(\frac{1}{4}\cdot 2^{\frac{3}{4}}\cdot\frac{-\sqrt{-1}}{\sqrt{2}}\theta_{P^n\mathbf{H}}\right)$$

$$=-\sqrt{-1}\frac{2^{\frac{1}{4}}}{2^2}\omega_{P^n\mathbf{H}}.$$

Therefore,

$$\tau_{\mathbf{H}}^*(2^{\frac{3}{4}}\sqrt{-1}\,\bar{\partial}\partial(\mathrm{tr}\,AA^*)^{\frac{1}{4}})=\omega_{P^n\mathbf{H}}.$$

For the geodesic flow $\epsilon_t(t\in\mathbf{R})$ of $P^n\mathbf{H}$, we have a similar representation as for the case of $P^n\mathbf{C}$.

Proposition 3.9. *The following diagram is commutative :*

$$\begin{array}{ccc} X_{\mathbf{H}} & \xrightarrow{\tau_{\mathbf{H}}} & \tilde{X}_{\mathbf{H}} \\ {\scriptstyle \epsilon_t}\downarrow & & \downarrow{\scriptstyle e^{-2\sqrt{-1}t}} \\ X_{\mathbf{H}} & \xrightarrow[\tau_{\mathbf{H}}]{} & \tilde{X}_{\mathbf{H}} \end{array} \tag{3.14}$$

where $e^{-2\sqrt{-1}t}$ *denotes the scalar multiplication by* $e^{-2\sqrt{-1}t}$.

4. $P^1\mathbf{H}$ and S^4

Let $\mathcal{S}$ be the diffeomorphism from $P^1\mathbf{H}$ to S^4 defined by the stereographic projection. $\mathcal{S}$ is realized as a map

$$\text{(4.1)}\qquad \begin{aligned}&\mathcal{S}: \mathcal{P}_{\mathbf{H}}(P^1\mathbf{H})\to S^4\\ &(P_{ij})\mapsto(x_0, x_1, x_2, x_3, x_4)\in\mathbf{R}^5\end{aligned}$$

where, for $P_{ij}\in M(2, \mathbf{C})$ $(i, j=0, 1)$ written as

$$P_{00}=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix},\ P_{11}=\begin{pmatrix}\mu & 0\\ 0 & \mu\end{pmatrix},\ \lambda,\ \mu\in\mathbf{R},$$

$$P_{01}=\begin{pmatrix}p_0 & p_1\\ -\bar{p}_1 & \bar{p}_0\end{pmatrix},\ P_{10}=(P_{01})^*,\ p_0,\ p_1\in\mathbf{C},$$

x_i's are given by;

$$\begin{cases}x_0=2\,Re\,p_0\\ x_1=2\,Im\,p_0\\ x_2=2\,Re\,p_1\\ x_3=2\,Im\,p_1\\ x_4=\mu-\lambda.\end{cases}$$

By identifying $T_0^*S^4$ with $T_0S^4=\{(x, y)\in\mathbf{R}^5\times\mathbf{R}^5 \mid \|x\|=1,\ (x, y)=0,\ y\neq0\}$, the Kähler structure on $T_0^*S^4$ is given by a map τ_S as follows:

$$\text{(4.2)}\qquad \begin{aligned}&\tau_S: T_0S^4\to\tilde{X}_S=\{z\in\mathbf{C}^5 \mid \sum_{i=0}^{4} z_i^2=0,\ z\neq0\}\\ &(x, y)\mapsto\tau_S(x, y)=z,\ z_k=\|y\|x_k+\sqrt{-1}y_k.\end{aligned}$$

Then the symplectic form ω_S on $T_0^*S^4$ is written as

$$\text{(4.3)}\qquad \omega_S=\sqrt{-2}\,\bar{\partial}\partial\|z\|,$$

where $\|z\|=\sqrt{\sum|z_i|^2}$.

Now, by a straight-forward calculation, we have the representation of the map $\Psi_0=\tau_{\mathbf{H}}\circ d\mathcal{S}^{-1}\circ\tau_S^{-1}: \tilde{X}_S\to\tilde{X}_{\mathbf{H}}$ as follows:

$$\Psi_0(z)=A,$$

$$\text{(4.4)}\qquad A=\frac{\|z\|}{8\sqrt{2}}\begin{pmatrix} iz_4 & 0 & z_1-iz_0 & z_3-iz_2\\ 0 & iz_4 & z_3+iz_2 & -(z_1+iz_0)\\ -(z_1+iz_0) & -z_3+iz_2 & -iz_4 & 0\\ -z_3-iz_2 & z_1-iz_0 & 0 & -iz_4\end{pmatrix}$$

Note that, by our definition of the Riemannian metric on $P^1\mathbf{H}$, we have

$$\|A\|=\sqrt{\operatorname{tr} AA^*}=\frac{1}{4\sqrt{2}}\|z\|^2. \tag{4.5}$$

By the expression (4.4) we have

Proposition 4.1. *Ψ_0 is not holomorphic, that, is, the differential $d\mathcal{S}$ of the diffeomorphism of $P^1\mathbf{H}$ onto S^4 defined by the stereographic projection is not holomorphic. Of course it is symplectic :*

$$\Psi_0^*(2^{\frac{3}{4}}\sqrt{-1}\,\bar{\partial}\partial\sqrt{\|A\|})=\sqrt{-2}\,\bar{\partial}\partial\|z\|. \tag{4.6}$$

Moreover the conjugation by the map Ψ_0 gives a two fold covering map of $Sp(2, \mathbf{C})$ to $SO(5, \mathbf{C})$ and this map is the extension of the covering map of $Sp(2)$ to $SO(5)$ which is defined by the conjugation of the map $\mathcal{S}$.

Remark 4.1. Of course a similar result to Proposition 4.1 holds for $P^1\mathbf{C}$ and S^2.

5. On automorphisms of $T_0^*P^n\mathbf{C}$ and $T_0^*P^n\mathbf{H}$

Let $g\in SU(n+1)$ (or $Sp(n+1)$), then the differential dg of the action g: $P^n\mathbf{C}\to P^n\mathbf{C}$ (g: $P^n\mathbf{H}\to P^n\mathbf{H}$) commutes with the adjoint action of g on $\tilde{X}_{\mathbf{C}}$ ($\tilde{X}_{\mathbf{H}}$) through the map $\tau_{\mathbf{C}}(\tau_{\mathbf{H}})$:

$$\begin{array}{ccc} X_{\mathbf{C}} & \xrightarrow{\tau_{\mathbf{C}}} & \tilde{X}_{\mathbf{C}} \\ {\scriptstyle dg}\downarrow & & \downarrow{\scriptstyle Ad_g} \\ X_{\mathbf{C}} & \xrightarrow[\tau_{\mathbf{C}}]{} & \tilde{X}_{\mathbf{C}} \end{array} \qquad \begin{array}{ccc} X_{\mathbf{H}} & \xrightarrow{\tau_{\mathbf{H}}} & \tilde{X}_{\mathbf{H}} \\ {\scriptstyle dg}\downarrow & & \downarrow{\scriptstyle Ad_g} \\ X_{\mathbf{H}} & \xrightarrow[\tau_{\mathbf{H}}]{} & \tilde{X}_{\mathbf{H}} \end{array} \tag{5.1}$$

So dg is symplectic and holomorphic.

Theorem 5.1. *Let $n\geqq 2$, and $g\in SL(n+1, \mathbf{C})$, then Ad_g preserves the symplectic form $\omega_{P^n\mathbf{C}}$ if and only if $g\in SU(n+1)$.*

Proof. Since $(Ad_g)^*\bar{\partial}\partial(\operatorname{tr} AA^*)^{\frac{1}{4}}=\bar{\partial}\partial Ad_g^*(\operatorname{tr} AA^*)^{\frac{1}{4}}$, we have

$$\partial Ad_g^*(\operatorname{tr} AA^*)^{\frac{1}{4}}-\partial(\operatorname{tr} AA^*)^{\frac{1}{4}}$$

is equal to a holomorphic 1-form ϕ. This ϕ satisfies $\partial\phi=0$, so ϕ must be exact form. Hence we have a holomorphic function f on $\tilde{X}_{\mathbf{C}}$ such that $df=\partial f=\phi$. Since $Ad_g^*(\operatorname{tr} AA^*)^{\frac{1}{4}}-(\operatorname{tr} AA^*)^{\frac{1}{4}}$ is real valued, we see

$$d(Ad_g^*(\operatorname{tr} AA^*)^{\frac{1}{4}}-(\operatorname{tr} AA^*)^{\frac{1}{4}})=d(f+\bar{f})$$

and hence

$$Ad_g^*(\mathrm{tr}\, AA^*)^{\frac{1}{4}}-(\mathrm{tr}\, AA^*)^{\frac{1}{4}}=f+\bar{f}+c_0$$

with a constant $c_0\in\mathbf{R}$. By absorbing c_0 to f we have

$$Ad_g^*(\mathrm{tr}\, AA^*)^{\frac{1}{4}}-(\mathrm{tr}\, AA^*)^{\frac{1}{4}}=f(A)+\bar{f}(A) \tag{5.2}$$

on $\tilde{X}_{\mathbf{C}}$. Now we show $f\equiv 0$. Let $t\in\mathbf{C}^*$ then $f(tA)$ is a holomorphic function on $\mathbf{C}^*$ for each fixed $A\in\tilde{X}_{\mathbf{C}}$:

$$f(tA)=\sum_{m=-\infty}^{\infty} f_m(A)t^m \tag{5.3}$$

where $f_m(A)$ is a holomorphic function on $\tilde{X}_{\mathbf{C}}$ of degree m.

Let $m<0$, and take a suitable $B\in M(n+1, \mathbf{C})$ such that

$$\begin{aligned} &\mathrm{tr}\, AB\not\equiv 0 \text{ on } \tilde{X}_{\mathbf{C}}, \\ &\mathrm{tr}\, A_0B=0 \text{ with an } A_0\in\tilde{X}_{\mathbf{C}}. \end{aligned} \tag{5.4}$$

Then, $(\mathrm{tr}\, AB)^{-m}f_m(A)$ is holomorphic and of degree zero on $\tilde{X}_{\mathbf{C}}$ and also $\tilde{X}_{\mathbf{C}}/\mathbf{C}^*$ is a compact complex manifold, so that $(\mathrm{tr}\, AB)^{-m}f_m(A)$ must be a constant on $\tilde{X}_{\mathbf{C}}$. By (5.4) this constant must be zero, which implies $f_m(A)\equiv 0$ for $m<0$. So

$$f(tA)=\sum_{m\geq 0} f_m(A)t^m,\ t\in\mathbf{C},\ A\in\tilde{X}_{\mathbf{C}}. \tag{5.5}$$

Next, from the equality

$$\begin{aligned} &(\mathrm{tr}(gtAg^{-1})^*(gtAg^{-1}))^{\frac{1}{4}}-(\mathrm{tr}(tA)^*(tA))^{\frac{1}{4}} \\ &\quad=\sqrt{|t|}((\mathrm{tr}(gAg^{-1})^*(gAg^{-1}))^{\frac{1}{4}}-(\mathrm{tr}\, A^*A)^{\frac{1}{4}}) \\ &\quad=\sum f_m(A)t^m+\sum \bar{f}_m(A)\,\bar{t}^m \end{aligned}$$

we have $f_m(A)+\bar{f}_m(A)\equiv 0$ for any $m\geqq 0$. So we have

$$Ad_g^*(\mathrm{tr}\, A^*A)=\mathrm{tr}\, A^*A \text{ on } \tilde{X}_{\mathbf{C}}. \tag{5.6}$$

Since $A\in\tilde{X}_{\mathbf{C}}$ is of the form

$$A:\ \mathbf{C}^{n+1}\to\mathbf{C}^{n+1},\ Ax=(x,\ X)_{\mathbf{C}}Y \text{ with } (X,\ Y)_{\mathbf{C}}=0, \tag{5.7}$$

we have

$$\begin{aligned} &\mathrm{tr}(A^*A)=\|X\|^2\|Y\|^2, \\ &\mathrm{tr}((gAg^{-1})^*(gAg^{-1}))=\|(g^*)^{-1}X\|^2\|gY\|^2. \end{aligned} \tag{5.8}$$

By (5.6) and (5.8) we see that, for $X(\neq 0)\in\mathbf{C}^{n+1}$, there exists a constant $c_X>0$ such that c_Xg is isometric on the subspace orthogonal to X. If $n\geqq 2$, then we can take $Y\in\mathbf{C}^{n+1}$ for any two $X_1,\ X_2\in\mathbf{C}^{n+1}$ such that $(X_1,\ Y)_{\mathbf{C}}=0$, $(X_2,\ Y)_{\mathbf{C}}=0$. So we have $c_{X_1}=c_{X_2}$, which implies that cg is unitary with a constant c

>0. Consequently $|\det cg|=|c^{n+1}\det g|=c^{n+1}=1$, and we have $g\in SU(n+1)$.

Theorem 5.2. *Let $n\geqq 3$, then the adjoint action of $g\in Sp(n+1, \mathbf{C})$ on $\tilde{X}_{\mathbf{H}}$ preserves $\omega_{P^n\mathbf{H}}$ if and only if $g\in Sp(n+1)$.*

Proof. If $g\in Sp(n+1, \mathbf{C})$ preserves the symplectic form $\omega_{P^n\mathbf{H}}$, then we have

$$Ad_g^*(\operatorname{tr} A^*A)=\operatorname{tr} A^*A \text{ on } \tilde{X}_{\mathbf{H}} \tag{5.9}$$

by the similar argument to the proof of Theorem 5.1 (from (5.2) to (5.6)).

Now $A\in\tilde{X}_{\mathbf{H}}$ is of the following form (see the proof of Proposition 3.5)

$$\begin{aligned}&A: \mathbf{C}^{2n+2}\to\mathbf{C}^{2n+2}\\ &Ax=(x, \boldsymbol{J}\bar{X})_{\mathbf{C}}Y-(x, \boldsymbol{J}\bar{Y})_{\mathbf{C}}X \text{ with } (X, \boldsymbol{J}\bar{Y})_{\mathbf{C}}=0.\end{aligned} \tag{5.10}$$

So we have

$$\operatorname{tr}(A^*A)=2\|X\|^2\|Y\|^2-2|(X, Y)_{\mathbf{C}}|^2, \tag{5.11}$$

and for $g\in Sp(n+1, \mathbf{C})$

$$\operatorname{tr}((gAg^{-1})^*(gAg^{-1}))=2\|gX\|^2\|gY\|^2-2|(gX, gY)_{\mathbf{C}}|^2. \tag{5.12}$$

Let $X_1, X_2\in\mathbf{C}^{2n+2}$. If $n\geqq 3$, then there exists a $Y\in\mathbf{C}^{2n+2}$ satisfying

$$\begin{cases}(X_1, Y)_{\mathbf{C}}=0, (X_1, \boldsymbol{J}\bar{Y})_{\mathbf{C}}=0, (gX_1, gY)_{\mathbf{C}}=0,\\ (X_2, Y)_{\mathbf{C}}=0, (X_2, \boldsymbol{J}\bar{Y})_{\mathbf{C}}=0, (gX_2, gY)_{\mathbf{C}}=0\end{cases} \tag{5.13}$$

So also by the similar arguments as in the last paragraph of the preceding proof, we have $g\in U(2n+2)$. Hence $g\in Sp(n+1, \mathbf{C})\cap U(2n+2)=Sp(n+1)$.

For $n=1$, we have

Theorem 5.3. *For $P^1\mathbf{H}$ the adjoint action of $g\in Sp(2, \mathbf{C})$ preserves the symplectic form $\omega_{P^1\mathbf{H}}$ if and only if $g\in Sp(2)$.*

Proof. By Proposition 4.1 and Theorem in the appendix of [Ra1], we have $g\in Sp(2, \mathbf{C})$ preserves $\omega_{P^1\mathbf{H}}$ if and only if $\Phi_0^{-1}\circ Ad_g\circ\Phi_0\in SO(5)$, that is, $g\in Sp(2)$.

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