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Complex manifolds modeled on a complex Minkowski space

By

Tadashi AIKOU

§0. Introduction

In the present paper, we investigate the differential geometry of complex Finsler manifolds. The main purpose is to introduce a connection on a complex Finsler manifold as the transversal connection constructed by the same method as used in [3], and to discuss some properties of *complex manifolds modeled on a complex Minkowski space*, which is a complex version of the notion due to Ichijyō [8].

We denote by $\boldsymbol{C}^n$ the complex vector space of n-tuples of complex numbers. A function $f(\xi)$ defined on $\boldsymbol{C}^n$ is said to be a *Finsler metric* if it satisfies the following properties:

(i) $f(\xi)\geq 0$, the equality holds if and only if $\xi=(\xi^1, \dots, \xi^n)=\boldsymbol{0}$,

(ii) $f(\xi)$ is C^∞ on $\boldsymbol{C}^n$- $\{O\}$, and continuous on $\boldsymbol{C}^n$,

(iii) $f(\lambda\xi)=|\lambda|^2 f(\xi)$ for $\forall\lambda\in\boldsymbol{C}$,

(iv) $f(\xi)$ is strictly plurisubharmonic outside of the origin O, that is, the Hermitian matrix $(\partial^2 f/\partial\xi^\alpha\partial\overline{\xi}^\beta)$ is positive-definite.

The condition (iv) is equivalent to the strict pseudoconvexity of the indicatrix $I=\{\xi\in\boldsymbol{C}^n; f(\xi)<1\}$. Conversely, if a *complete proper circular domain* I in $\boldsymbol{C}^n$ with smooth boundary is strictly pseudoconvex, the Minkowski functional of I defines a Finsler metric on $\boldsymbol{C}^n$ whose indicatrix becomes the given I ([13]). Any Hermitian metric on $\boldsymbol{C}^n$ belongs to the class of Finsler metrics, and is characterized by one of the following three equivalent conditions (see Corollary 3.2 in [13]):

(1) The indicatrix I is biholomorphic to the unit ball in $\boldsymbol{C}^n$.

(2) The function $f(\xi)$ is C^∞ at the origin O.

(3) The function $f(\xi)$ is expressed as $f(\xi)=\sum_{i=1}^{n}\left|\sum_{m=1}^{n}A^i_m\xi^m\right|^2$ for $\exists\,(A^i_j)\in GL(n,\boldsymbol{C})$.

In the present paper, following to Ichijyō [9], we call a Finsler metric f on $\boldsymbol{C}^n$ a *complex Minkowski metric* on $\boldsymbol{C}^n$, and the pair $(\boldsymbol{C}^n, f)$ a *complex Minkowski space*.

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In the similar way a *Finsler metric* F of a complex vector bundle $\pi: \boldsymbol{E} \to M$ is generally defined as a function on its total space $\boldsymbol{E}$ (§1). We find many papers on the complex differential geometry of complex manifolds with a Finsler metric ([2], [6], [14], [16], [24], etc.). In the real case, Bao-Chern [4], Chern [5] and Shen [18] have recently developed the theory of connections in Finsler geometry by using the projective bundle, and obtained some results. On the other hand, if a real Finsler metric F is given on a C^∞ manifold M, then its tangent bundle TM admits a natural Sasaki-type metric, and has the structure of foliated Riemannian manifold. Suggested by these facts, in the previous paper [3] the author has introduced a connection on a real Finsler manifold (M, F), and given some characterization of special Finsler manifolds. The connection in [3] was defined as the transversal Levi-Civita connection which plays an important role in differential geometry of foliated Riemannian manifolds ([20]).

On a complex Finsler manifold (M, F), that is, a complex manifold M with a complex Finsler metric F, a connection is introduced in the same way (§2). Based on this connection, we treat a complex manifold modeled on a complex Minkowski space (§3, §4, §5).

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§1. Finsler metrics on complex vector bundles

Let M be a connected C^∞ manifold, and $\pi: \boldsymbol{E} \to M$ a C^∞ complex vector bundle of rank $\boldsymbol{E} = r$. If we fix a local frame field $s = \{s_1, \cdots, s_r\}$ of $\boldsymbol{E}$ over a neighborhood U of M, we have the identification $\pi^{-1}(U) \cong U \times \boldsymbol{C}^r$. If we put $\xi = \sum_\alpha \xi^\alpha s_\alpha$, the component $(\xi^1, \cdots, \xi^r)$ defines the complex fibre coordinate of $\pi^{-1}(U)$. We denote a point of $\pi^{-1}(U)$ by (x, ξ), where $x \in U$ and $\xi \in \boldsymbol{C}^r$.

Definition 1.1. A function $F(x, \xi)$ on $\boldsymbol{E}$ is said to be a *complex Finsler metric* if satisfies the following conditions:

(1) $F(x, \xi) \geq 0$, the equality holds if and only if $\xi = \mathbf{0}$,

(2) $F(x, \xi)$ is C^∞ on $\boldsymbol{E}$- {zero sections}, and continuous on $\boldsymbol{E}$,

(3) $F(x, \lambda\xi) = |\lambda|^2 F(x, \xi)$ for $\forall \lambda \in \boldsymbol{C}$,

(4) the following Hermitian matrix $(F_{\alpha\bar{\beta}})$ is positive-definite:

$$F_{\alpha\bar{\beta}}(x, \xi) = \frac{\partial^2 F}{\partial \xi^\alpha \partial \bar{\xi}^\beta}$$

If a complex Finsler metric $F(x, \xi)$ is given on $\boldsymbol{E}$, each fibre $\boldsymbol{E}_p$ is considered as a complex Minkowski space $(\boldsymbol{C}^r, \|\cdot\|_p)$ with the norm function $\|\xi\|_p^2 := F(p, \xi)$. Given $\xi \in C^\infty(\boldsymbol{E})$, the norm of $\xi(x)$ is defined by $\|\xi(x)\|_x^2 = F(x, \xi(x))$, where $C^\infty(\boldsymbol{E})$ denotes the linear space of all C^∞ sections of $\boldsymbol{E}$.

For later discussions, we fix any point $p \in M$ and denote by G the isometric group of the norm on $\boldsymbol{E}_p$:

$$G = \{g \in GL(r, \boldsymbol{C});\ F(p, g\xi) = F(p, \xi) \text{ for } \forall \xi \in \boldsymbol{E}_p\} .$$

By using the condition (3) in Definition 1.1 and the continuity of the norm, we can prove the following lemma by the same method as Wang [22] or Yano [26].

Lemma 1.1. *The isometric group G is a compact Lie group.*

We denote by J the given complex structure on $\boldsymbol{E}$, that is, J is an automorphism of $\boldsymbol{E}$ satisfying $J^2 = -1_{\boldsymbol{E}}$. A connection $\nabla: C^\infty(\boldsymbol{E}) \to C^\infty(\boldsymbol{E} \otimes TM^*)$ of $\boldsymbol{E}$ is said to be *complex* if it satisfies $\nabla J = 0$. Generalizing the method in [19] to our case, we have

Theorem 1.1. *Let $\boldsymbol{E}$ be a C^∞ complex vector bundle over M with a complex Finsler metric $F(x, \xi)$. We suppose that $\boldsymbol{E}$ admits a complex connection ∇ on $\boldsymbol{E}$ which preserves the norm invariant under the parallel displacement. Then there exists a Hermitian metric h on $\boldsymbol{E}$ such that ∇ is a metrical connection of $(\boldsymbol{E}, h)$.*

Proof. Since M is connected, we denote by H the holonomy group of ∇ with reference point $p \in M$. By hypothesis, H is a subgroup of G. Then we define an inner product $\langle , \rangle_p$ on $\boldsymbol{E}_p$ by

$$\langle \xi, \psi \rangle_p = \int_G (g\xi, g\psi)\, dg ,$$

where $(,)$ is an arbitrary Hermitian inner product on $\boldsymbol{E}_p$, and dg is the bi-invariant Haar measure on G. Then we have

$$\begin{aligned} \langle J\xi, J\psi \rangle_p &= \int_G (g(J\xi), g(J\psi))\, dg = \int_G (J(g\xi), J(g\psi))\, dg \\ &= \int_G (g\xi, g\psi)\, dg = \langle \xi, \psi \rangle_p , \end{aligned}$$

that is, $\langle , \rangle_p$ is a Hermitian inner product on $\boldsymbol{E}_p$. By the construction, $\langle , \rangle_p$ is G-invariant, and furthermore, it is also H-invariant.

Using the parallel displacement with respect to ∇, we can extend $\langle , \rangle_p$ to a Hermitian metric h of $\boldsymbol{E}$. Let x be an arbitrary point of M, and $c(t)$ $(0 \leq t \leq 1)$ a C^∞ curve such that $c(0) = p$ and $c(1) = x$. For $\forall \xi, \psi \in \boldsymbol{E}_x$, we define

$$h(\xi, \psi) := \langle P_c^{-1}\xi, P_c^{-1}\psi \rangle_p ,$$

where P_c: $\boldsymbol{E}_p \to \boldsymbol{E}_x$ is the parallel displacement with respect to ∇ along $c(t)$. Since $\langle , \rangle_p$ is H-invariant, this definition is independent on the choice of $c(t)$ on M, and by $\nabla J = 0$ the metric h is a Hermitian metric. In this way, we can define a Hermitian metric h on $\boldsymbol{E}$. By the construction of h, we have easily

$$dh(\xi, \psi) = h(\nabla \xi, \psi) + h(\xi, \nabla \psi)$$

for $\forall\xi, \psi\in C^{\infty}(\boldsymbol{E})$. Hence, ∇ is metrical with respect to h. Q.E.D.

Remark 1.1. In Theorem 1.1, if M is a complex manifold and ∇ is of $(1, 0)$-type, then ∇ is the Hermitian connection of $(\boldsymbol{E}, h)$.

From the discussions in [26], it follows that if a suitable basis is chosen, all elements of G are orthogonal. Hence, all elements of G are contained in $U(r)=O(2r)\cap GL(r, \boldsymbol{C})$. In the proof above, we have constructed an Hermitian metric on $\boldsymbol{E}_p$ which is invariant under the action of G. We shall use this fact in §4.

§2. Complex Finsler manifolds and Finsler connections

Let M be a connected complex manifold of $\dim_{\boldsymbol{C}}M=n$, and π: $TM\rightarrow M$ its holomorphic tangent bundle. The total space TM is a complex manifold of $\dim_{\boldsymbol{C}}TM=2n$. We denote by $\{\pi^{-1}(U), (z^i, \eta^i)\}$ $(1\leq i\leq n)$ the canonical covering of TM induced from a covering by the system of complex coordinate neighborhoods $\{U, (z^i)\}$ on M. Suppose that a complex Finsler metric $F(z, \eta)$ is given on TM. Then we call the pair (M, F) a *complex Finsler manifold*. By the condition (4) in Definition 1.1, the following Hermitian matrix $(F_{i\bar{j}})$ is positive-definite:

$$(2.1)\qquad F_{i\bar{j}}(z, \eta):=\frac{\partial^2 F}{\partial\eta^i\partial\overline{\eta}^j}\ .$$

In the following, we put $(F^{\bar{j}i})=(F_{i\bar{j}})^{-1}$.

Complex Finsler metrics include the following important classes which will be characterized in terms of a connection in the later:

(1) *Hermitian metries*: $F(z, \eta)=\sum_{i,j}h_{i\bar{j}}(z)\eta^i\overline{\eta}^j$,

(2) *locally Minkowski metrics*: $F=F(\eta^1, \cdots, \eta^n)$ by taking a suitable system of complex coordinate neighborhoods $\{U, (z^i)\}$ on M.

Let (M, F) be a complex Finsler manifold. For studying Finsler geometry, we introduce a connection which is a natural generalization of real case ([3]). We denote by VTM the holomorphic tangent bundle of the fibres of TM. Since VTM is a holomorphic sub-bundle of TTM, we have the following exact sequence of holomorphic vector bundles:

$$\boldsymbol{0}\longrightarrow VTM\stackrel{i}{\longrightarrow}TTM\stackrel{<>}{\longrightarrow}\boldsymbol{Q}\longrightarrow\boldsymbol{0}\ ,$$

where $\boldsymbol{Q}$ is the quotient bundle TTM/VTM. Since $\boldsymbol{Q}$ is naturally identified with $\pi^{-1}TM$, the natural frame $\{\partial/\partial z^i\}$ $(1\leq i\leq n)$ of TM over U may be considered as a local holomorphic frame field of $\boldsymbol{Q}$ over $\pi^{-1}(U)$. Then we introduce a Hermitian metric $h_{\boldsymbol{Q}}$ on $\boldsymbol{Q}$ by

$$h_Q\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) := F_{i\bar{j}}(z, \eta) \quad . \tag{2.2}$$

As a connection of $(\boldsymbol{Q}, h_Q)$, it is natural to use the Hermitian connection, but we use a transversal connection of $(\boldsymbol{Q}, h_Q)$ which is defined as follows.

First we introduce a C^∞ splitting σ: $\boldsymbol{Q} \to TTM$ of the exact sequence above by

$$\sigma\left(\frac{\partial}{\partial z^i}\right) = \frac{\partial}{\partial z^i} - \sum_m N_i^m \frac{\partial}{\partial \eta^m} \ ,$$

where N_j^i $(1 \leq i, j \leq n)$ are C^∞ functions on $\pi^{-1}(U)$ defined by

$$N_j^i(z, \eta) := \sum_{m,r} F^{\bar{r}i} \frac{\partial F_{m\bar{r}}}{\partial z^j} \eta^m \quad . \tag{2.3}$$

Then the tangent bundle TTM has a C^∞ decomposition $TTM = VTM \oplus HTM$, where we put $HTM = \sigma(\boldsymbol{Q})$. Putting $X_i := \sigma(\partial/\partial z^i)$ and $Y_i := \partial/\partial \eta^i$, then $\{X_i\}$ and $\{Y_i\}$ $(1 \leq i \leq n)$ define a local frame field on $\pi^{-1}(U)$ of HTM and VTM respectively. In the dual frame field $\{dz^i, \theta^i\}$ $(1 \leq i \leq n)$ of $\{X_i, Y_i\}$, we introduce a Hermitian metric h_{TM} on TM by

$$h_{TM} = \sum_{i,j} F_{i\bar{j}} dz^i \otimes d\bar{z}^j + \sum_{i,j} F_{i\bar{j}} \theta^i \otimes \bar{\theta}^j \ , \tag{2.4}$$

where we put $\theta^i := d\eta^i + \sum_m N_m^i dz^m$. This is a natural metric from the standpoint of the geometry of tangent bundles ([17]). Then (TM, h_{TM}) has the structure of foliated Hermitian manifold, and $\boldsymbol{Q}$ is the transversal distribution in TM.

We denote by ∇^{TM} the Hermitian connection of (TM, h_{TM}). For $\forall \xi \in C^\infty(\boldsymbol{Q})$, there exists a unique $X_\xi \in C^\infty(HTM)$ such that $\langle X_\xi \rangle$ is the natural projection of X_ξ to the quotient bundle $\boldsymbol{Q}$. Now we introduce a connection on $(\boldsymbol{Q}, h_Q)$ as follows:

Definition 2.1. The (1, 0)-type connection ∇ on $(\boldsymbol{Q}, h_Q)$ defined by

$$\nabla_Z \xi := \begin{cases} \langle [Z, X_\xi] \rangle & \text{if } Z \in C^\infty(VTM) \\ \langle \nabla_Z^{TM} X_\xi \rangle & \text{if } Z \in C^\infty(HTM) \end{cases} \tag{2.5}$$

is called the *Finsler connection* of (M, F).

Since the complex structure of $\boldsymbol{Q}$ is given by $\pi^{-1} J$ for J of TM, it is obvious that ∇ satisfies $\nabla J = 0$. Corresponding to the decomposition $TTM = VTM \oplus HTM$, the differential operator d on functions and the Finsler connection ∇ are decomposed as $d = d_H + d_V$ and $\nabla = \nabla^H + \nabla^V$ respectively. We also decompose d_H and d_V into (1, 0)-part and (0, 1)-part as $d_H = \partial_H + \bar{\partial}_H$,

$d_V=\partial_V+\bar{\partial}_V$ respectively, where we put

$$\partial_H f=\sum_m (X_m f)\,dz^m,\ \partial_V f=\sum_m (Y_m f)\,\theta^m$$

for a C^∞ function $f(z, \eta)$ on TM.

By the definition, it is obvious that ∇ is not always metrical with respect to h_Q, but we have

Proposition 2.1. *The Finsler connection* ∇ *of* (M, F) *satisfies*

$$d_H h_Q(\xi, \phi)=h_Q(\nabla^H\xi, \phi)+h_Q(\xi, \nabla^H\phi) \tag{2.6}$$

for $\xi, \phi\in C^\infty(\boldsymbol{Q})$.

Proof. From (2.2) and (2.4) we have $h_Q(\xi, \phi)=h_{TM}(X_\xi, X_\phi)$. Since ∇^{TM} is the Hermitian connection of (TM, h_{TM}), we have

$$dh_Q(\xi, \phi)=h_{TM}(\nabla^{TM}X_\xi, X_\phi)+h_{TM}(X_\xi, \nabla^{TM}X_\phi)\ ,$$

whose restriction to the transversal part implies (2.6). Q.E.D.

For $\forall\xi\in C^\infty(\boldsymbol{Q})$, we have $\nabla_{Y_i}\xi=\langle[Y_i, X_\xi]\rangle=\partial_V\xi(Y_i)$, and since the connection is (1, 0)-type, we have $\nabla^V=d_V$. Furthermore, the connection form of ∇ is written as $\omega^i_j=\sum_m F^i_{jm}dz^m$. So, by (2.6) we get

$$d_H F_{i\bar{j}}=h_Q\Big(\sum_m \omega^m_i\frac{\partial}{\partial z^m}, \frac{\partial}{\partial z^j}\Big)+h_Q\Big(\frac{\partial}{\partial z^i}, \sum_m \omega^m_j\frac{\partial}{\partial z^m}\Big)\ ,$$

which is written as $\partial_H F_{i\bar{j}}+\bar{\partial}_H F_{i\bar{j}}=\sum_m F_{m\bar{j}}\omega^m_i+\sum_m F_{i\bar{m}}\bar{\omega}^m_j$. Hence we have

$$\omega^i_j=\sum_m F^{\bar{m}i}\partial_H F_{j\bar{m}}\ .$$

The coefficients in ω^i_j are given by $F^i_{jk}(z, \eta)=\sum_m F^{\bar{m}i}X_k F_{j\bar{m}}$, from which we get N^i_j of (2.3) as

$$N^i_j=\sum_m \eta^m F^i_{mj}\ . \tag{2.7}$$

Defining a section $\varepsilon\in C^\infty(\boldsymbol{Q})$ by

$$\varepsilon(z, \eta)=\sum_m \eta^m\frac{\partial}{\partial z^m}\ ,$$

we have $\nabla^H\varepsilon=0$ from (2.7). By using the property $h_Q(\varepsilon, \varepsilon)=F(z, \eta)$, we get from (2.6) the identity:

(2.8) $\qquad d_H F=0$.

Then we have

Theorem 2.1. (1) *A Finsler metric F on M is Hermitian if and only if its Finsler connection ∇ is metrical.*
(2) *A Finsler metric F on M is locally Minkowski if and only if its Finsler connection ∇ is flat.*

Proof. Since F is Hermitian if and only if $\partial_V F_{i\bar{j}}=0$, the first statement is obvious from (2.6). It is shown from (2.7) that the metric F is locally Minkowski if and only if $\partial_H F_{i\bar{j}}=0$ on a suitable coordinate system $\{U, (z^i)\}$ on M. Hence the connection form of ∇ vanishes identically on such a coordinate system, and so ∇ is flat. Thus the second statement has been proved. Q.E.D.

Remark 2.1. Finsler geometry is sometimes studied by using the projective bundle PM instead of TM ([4], [5], [10], [18]). The following Hermitian form Φ_U on $\pi^{-1}(U)$ is invariant by replacing η by $\lambda\eta$ for $\forall\lambda\in \boldsymbol{C}-\{0\}$:

$$\Phi_U=\sum_{i,j}\left(\frac{F_{i\bar{j}}}{F}-\sum_{l,m}\frac{F_{i\bar{l}}F_{m\bar{j}}\bar{\eta}^l\eta^m}{F^2}\right)\theta^i\otimes\bar{\theta}^j .$$

Furthermore, it is easy to show that $\Phi_U=\Phi_V$ on $\pi^{-1}(U)\cap\pi^{-1}(V)$. Hence $\{\Phi_U\}$ defines a global form Φ on PM. Then we define a Hermitian metric h_{PM} by

$$h_{PM}=\sum_{i,j}F_{i\bar{j}}dz^i\otimes d\bar{z}^j+\Phi$$

instead of (2.4) ([21]). Since the bundle PM has also a natural foliation VPM and the exact sequence of holomorphic vector bundles $\mathbf{0}\to VPM\to TPM\to \boldsymbol{Q}\to\mathbf{0}$, we can define a Hermitian metric $h_{\boldsymbol{Q}}$ on the quotient bundle $\boldsymbol{Q}$ and a connection ∇ on $(\boldsymbol{Q}, h_{\boldsymbol{Q}})$ by (2.5). In this case, we can do the discussions similar to the above.

The section $\varepsilon\in C^\infty(\boldsymbol{Q})$ defines a holomorphic line bundle $\boldsymbol{L}$ over PM. Kobayashi [10] showed that a Hermitian metric on $\boldsymbol{L}$ defines Finsler metric on M, and vice-versa. Furthermore he showed that the negativity of $\boldsymbol{L}$, which is equivalent to the negativity of the tangent bundle TM, implies the positive-definiteness of $(F_{i\bar{j}})$. As to the existence of Finsler manifolds with negative tangent bundle, see [24].

§3. Kähler condition of a complex Finsler manifold

In this section, we shall state some remarks on the Kähler condition of a complex Finsler manifold (M, F). First we note that around $\forall P\in TM$ we can always take a coordinate neighborhood $\{\pi^{-1}(U), (z^i, \eta^i)\}$ which is *semi-normal at P*, that is, a neighborhood satisfying

$$(3.1) \qquad F_j{}^i{}_k = -F_k{}^i{}_j$$

at P. In fact, for a given complex coordinate system $\{U, (z^i)\}$ on M, we define a new coordinate system $\{U, (z'^i)\}$ on M by

$$z'^i = (z^i - z_0^i) - \frac{1}{2}\sum_{j,k} F_j{}^i{}_k(P)(z^j - z_0^j)(z^k - z_0^k) ,$$

where we put $P = (z_0^i, \eta_0^i)$. Then it is easily seen that the coordinate system on TM induced from $\{U, (z'^i)\}$ satisfies the condition (3.1) at P. Furthermore, if a semi-normal coordinate system $\{\pi^{-1}(U), (z^i, \eta^i)\}$ at P is said to be *normal* if the following condition is satisfied:

$$F_{i\bar{j}}(P) = \delta_{ij} \quad \text{and} \quad F_{jk}^i(P) = 0 .$$

If the Finsler connection ∇ of (M, F) is the transversal Levi-Civita connection of $(\boldsymbol{Q}, h_{\boldsymbol{Q}})$ in the sense of Tondeur [20], we say (M, F) satisfies the *Kähler condition*. A Finsler manifold (M, F) satisfies the Kähler condition if and only if the coefficients $F_j{}^i{}_k$ satisfies the symmetry

$$(3.2) \qquad F_{jk}^i = F_{kj}^i .$$

If we put $\Theta = \sqrt{-1}\sum_{i,j} F_{i\bar{j}} dz^i \wedge d\bar{z}^j$, it is directly shown that this condition is equivalent to $d_H \Theta = 0$. Then, from (3.1) and (3.2) we have

Theorem 3.1 ([2]). *A complex Finsler manifold (M, F) satisfies the Kähler condition if and only if around any point P of TM there exists a complex coordinate system which is normal at P.*

The functions $N_j^i(z, \eta)$ in (2.3) are also found in [14]. From (2.7) we get easily $F_{jk}^i = \partial N_j^i / \partial z^k$. Hence the Finsler connection ∇ coincides with the one introduced in [14]. The function $N_j^i(z, \eta)$ are derived from the variational problem as follows.

Let $c(t)$ be a C^∞ curve on a complex Finsler manifold (M, F). The Euler-Lagrange equation with respect to F is given by

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \eta^i}\right) = \frac{\partial F}{\partial z^i} .$$

For an arbitrary point (z, ξ) in TM, there exists a holomorphic map ψ: $\Delta(r) \to M$ satisfying $\psi(0) = z$, $\psi_*(0) := \psi_*((\partial/\partial\zeta)_0) = \xi$, where $\Delta(r)$ is the disk in $\boldsymbol{C}$ of radius r centered the origin. We give on $\Delta(r)$ the Poincaré metric g_r:

$$g_r = \frac{r^2}{(r^2 - |\zeta|^2)^2} d\zeta d\bar{\zeta} \ .$$

Now we assume Royden's condition in [14]:

"for any $(z, \xi) \in TM$, there exists a holomorphic map ψ: $\Delta(r) \to M$ such
(3.3) that $\psi(0) = z$, $\psi_*(0) = \xi$ and the curve $\gamma(t) = \psi(e^{\sqrt{-1}\theta}t)$ in (M, F) is a geodesic tangent to the common complex line $\boldsymbol{C} \cdot \xi$ at z for each $\theta \in \boldsymbol{R}$",

that is, the disk $\psi(\Delta(r))$ is the union of such geodesics. Then, corresponding to (16) and (K_3) of [14], we have from the Euler-Lagrange equation

$$\begin{cases} \dot{\xi}^i + \sum_j N^i_j(z, \xi)\xi^j = 0 \ , \\ \sum_{i,j,m} F_{i\bar{m}}(F^i_{jk} - F^i_{kj})\xi^j\bar{\xi}^m = 0 \ , \end{cases}$$

where we put $\psi = (\psi^1, \cdots, \psi^n)$, and $\xi^i = \partial\varphi^i/\partial\zeta$, $\dot{\xi}^i = \partial\xi^i/\partial\zeta$ for the coordinate ζ of $\Delta(r)$. The first equation is a differential equation for geodesics, and the second is an algebraic condition. The second equation is satisfied if (M, F) satisfies the Kähler condition.

The discussions for geodesics in complex Finsler manifolds are also found in Abate-Patrizio [1]. It is noted that the second condition above is equivalent to the Kähler condition in the case where the metric is a Hermitian metric: $F(z, \eta) = \sum_{i,j} h_{i\bar{j}}(z)\eta^i\bar{\eta}^j$.

§4. Complex manifolds modeled on a complex Minkowski space

Another important class of Finsler metrics is the one whose Finsler connection ∇ is *basic*: $F_j{}^i{}_k = F_j{}^i{}_k(z)$. This property is, of course, independent on the choice of complex coordinate system on M. In this case, ∇ of $(\boldsymbol{Q}, h_{\boldsymbol{Q}})$ is considered as the pull-back of an (1, 0)-type connection of TM. It is obvious that any Hermitian metric and locally Minkowski metric belong to this class.

Now, assume that ∇ is basic, and we consider ∇ as a connection on TM. Let $c(t)$ be a C^∞ curve on M. We denote by ξ_t the parallel displacement of $\xi \in T_pM$ along $c(t)$ with respect to ∇. The norm $\|\xi_t\|_{c(t)}$ of ξ_t is given by $\|\xi_t\|^2_{c(t)} = F(c(t), \xi_t)$. Then we have

Proposition 4.1. *If ∇ is basic, $\|\xi_t\|_{c(t)}$ is invariant under the parallel displacement with respect to ∇, that is, $d\|\xi_t\|_{c(t)}/dt = 0$.*

Proof. If we put $\xi_t = \sum_m \xi^m(\partial/\partial z^m)$, ξ_t satisfies

$$\frac{d\xi^i(t)}{dt}+\sum_m \omega^i_m\left(\frac{dz}{dt}\right)\xi^m(t)=0 \ .$$

Then we have

$$\begin{aligned}\frac{d\|\xi_t\|^2_{c(t)}}{dt}&=\sum_m \frac{\partial F(c(t),\xi_t)}{\partial z^m}\frac{dz^m}{dt}+\sum_m \frac{\partial F(c(t),\xi_t)}{\partial \eta^m}\frac{d\xi^m}{dt}+(\text{conj.})\\ &=\sum_k\left(\frac{\partial F}{\partial z^k}-\sum_{j,m}\frac{\partial F}{\partial \eta^m}F^m_{jk}\xi^j\right)\frac{dz^k}{dt}+(\text{conj.})\\ &=\sum_k\left(\frac{\partial F}{\partial z^k}-\sum_{j,m}N^m_k(c(t),\xi_t)\frac{\partial F}{\partial \eta^m}\right)\frac{dz^k}{dt}+(\text{conj.})\\ &=\partial_H F+\bar{\partial}_H F=d_H F(c(t),\xi_t) \ .\end{aligned}$$

Hence, our assertion is derived from (2.8). Q.E.D.

Proposition 4.1 means that if the Finsler connection of $(\boldsymbol{Q}, h_{\boldsymbol{Q}})$ is basic, there exists a (1, 0)-type connection ∇ on TM which preserves the norm $\|\xi_t\|_{c(t)}$ invariant under the parallel displacement. Hence, by Theorem 1.1, we have

Theorem 4.1. *Let (M, F) be a complex Finsler manifold whose Finsler connection ∇ is basic. Then there exists a Hermitian metric h_M on M such that ∇ is the pull-back of the Hermitian connection of h_M.*

Since the parallel displacement gives a complex linear isomorphism between tangent spaces, Proposition 4.1 says that if the Finsler connection ∇ is basic, each tangent space is isometric to a fixed complex Minkowski space $(\boldsymbol{C}^n, f)$ with $f(\xi)=F(p, \xi)$. In the real case, such a manifold belongs to the class of *manifolds modeled on a Minkowski space* due to Ichijyō [8]. In the following, we shall consider the notion in the case of complex manifolds.

We state some terminology. Let G be a Lie group. We say that a C^∞ manifold M admits a *G-structure* if there exists a covering $\{U\}$ with local frame fields $\{\boldsymbol{e}_U\}$ such that the transition functions $\{g_{UV}\}$ are all G-valued function. Such a frame $\{\boldsymbol{e}_U\}$ is said to be *adapted*. A linear connection D is called a *G-connection* of the G-structure if the connection form with respect to an adapted frame $\{\boldsymbol{e}_U\}$ takes its values in the Lie algebra of G.

Let $f(\xi)$ be a complex Minkowski metric on $\boldsymbol{C}^n$, and G the isometric group of $f(\xi)$ (cf. Lemma 1.1). Let $\{\boldsymbol{e}_i\}$ $(1\leq i\leq n)$ be a frame of TM, which we express as $\boldsymbol{e}_i=\sum_m A^m_i(\partial/\partial z^m)$, where A^i_j: $U\to GL(n, \boldsymbol{C})$ $(1\leq i, j\leq n)$ are C^∞ functions. For $\forall\, \eta=\sum_m \xi^m \boldsymbol{e}_m=\sum_m \eta^m(\partial/\partial z^m)$, we define a function F: $\pi^{-1}(U)\to\boldsymbol{R}$ by

$$(4.1) \qquad F(z, \eta) := f(\xi^i) = f(\sum_m B^i_m(z)\eta^m) \ ,$$

where $B=(B^i_j)$ is the inverse of $A=(A^i_j)$. The function $F(z, \eta)$ is defined globally on TM and becomes a Finsler metric, if and only if M has a G-structure and $\{e_i\}$ is an adapted frame of the G-structure.

Definition 4.1. A complex Finsler manifold (M, F) is said to be a *complex manifold modeled on a complex Minkowski space* $(\boldsymbol{C}^n, f)$ if M admits a G-structure and the metric F is written in the form of (4.1).

Let (M, F) be a complex manifold modeled on a complex Minkowski space $(\boldsymbol{C}^n, f)$. So, in the following we always assume that M admits a G-structure, and $\{e_i\}$ is an adapted frame of this G-structure. With respect to $\{e_i\}$, each tangent space of (M, F) may be considered as the given Minkowski space $(\boldsymbol{C}^n, f)$.

Let D be a G-connection of the G-structure. We put $De_i = \sum_m \Phi^m_i e_m$. Since the matrix $\Phi=(\Phi^i_j)$ is a 1-form which values in the Lie algebra of G, we get

Proposition 4.2. *Let (M, F) be a complex manifold modeled on a complex Minkowski space $(\boldsymbol{C}^n, f)$. The Finsler connection ∇ of (M, F) is given by the $(1, 0)$-part of $\Phi=(\Phi^i_j)$, and so ∇ is basic.*

Proof. If we denote by $\theta=(\theta^i_j)$ the $(1, 0)$-part of $\Phi=(\Phi^i_j)$, the 1-form θ takes the value in the Lie algebra of G. Thus the equality $f((\exp t\theta)\xi)=f(\xi)$ holds for $\forall t \in \boldsymbol{R}$. Differentiating this equation at $t=0$, we get

$$(4.2) \qquad \sum_{l,m} \frac{\partial f}{\partial \xi^l}\theta^l_m \xi^m + \sum_{l,m} \frac{\partial f}{\partial \bar{\xi}^l}\overline{\theta^l_m \xi^m} = 0 \ .$$

We express θ by $\omega=(\omega^i_j)$ with respect to the natural frame field $\{\partial/\partial z^i\}$, where we put $\omega^i_j = \sum_m \Gamma^i_{jm}(z)dz^m$. Then we have $\theta = BdA + B\omega A = B\partial A + B\omega A + B\bar{\partial}A$, where we used the matrix notation. Substituting this into (4.2), we get

$$(4.3) \qquad \sum_{j,l,m} \frac{\partial f}{\partial \xi^l} B^l_m \Big(\frac{\partial A^m_j}{\partial z^i} + \sum_r \Gamma^m_{ri} A^r_j \Big) \xi^j + \sum_{j,l,m} \frac{\partial f}{\partial \bar{\xi}^l} \bar{B}^l_m \frac{\partial \bar{A}^m_j}{\partial z^i} \bar{\xi}^j = 0 \ .$$

We define the function $N^i_j(z, \eta)$ in (2.3) by $N^i_j(z, \eta) = \sum_m \Gamma_m{}^i{}_j(z)\eta^m$. Using $\eta^i = \sum_m A^i_m(z)\xi^m$ and $\xi^i = \sum_m B^i_m(z)\eta^m$, we can show $\partial_H F = 0$. In fact, we have

$$X_iF=\frac{\partial F}{\partial z^i}-\sum_m N_i^m\frac{\partial F}{\partial \eta^m}$$

$$=\sum_l\Big(\frac{\partial f}{\partial \xi^l}\frac{\partial \xi^l}{\partial z^i}+\frac{\partial f}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial z^i}\Big)-\sum_{l,m}N_i^m\frac{\partial f}{\partial \xi^l}\frac{\partial \xi^l}{\partial \eta^m}-\sum_{l,m}N_i^m\frac{\partial f}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial \eta^m}$$

$$=\sum_{k,l}\Big\{\frac{\partial f}{\partial \xi^l}\frac{\partial B_k^l}{\partial z^i}\eta^k+\frac{\partial f}{\partial \bar{\xi}^l}\frac{\partial \bar{B}_k^l}{\partial z^i}\bar{\eta}^k-\Big(\sum_m \Gamma_k{}^m{}_i B_m^l\Big)\eta^k\frac{\partial f}{\partial \xi^l}\Big\}$$

$$=-\sum_{k,l,r}\Big\{\frac{\partial f}{\partial \xi^l}B_k^l\frac{\partial A_r^k}{\partial z^i}\xi^r+\Big(\sum_m \Gamma_k{}^m{}_i B_m^l\Big)A_r^k\xi^r\frac{\partial f}{\partial \xi^l}+\frac{\partial f}{\partial \bar{\xi}^l}\bar{B}_k^l\frac{\partial \bar{A}_r^k}{\partial z^i}\bar{\xi}^r\Big\}$$

$$=-\sum_{j,l,m}\frac{\partial f}{\partial \xi^l}B_m^l\Big(\frac{\partial A_j^m}{\partial z^i}+\sum_r \Gamma_{ri}^m A_j^r\Big)\xi^j-\sum_{j,l,m}\frac{\partial f}{\partial \bar{\xi}^l}\bar{B}_m^l\frac{\partial \bar{A}_j^m}{\partial z^i}\bar{\xi}^j ,$$

which is equal to zero from (4.3). Furthermore, from $\partial^2(X_kF)/\partial\eta^i\partial\bar{\eta}^j=0$ we get $\omega_j^i=\sum_m F^{\bar{m}i}\partial_H F_{j\bar{m}}$, which shows that the connection ∇ defined by $\nabla e_i=\sum_m \theta_i^m e_m$ is the Finsler connection of (M, F). Q.E.D.

Therefore we have proved

Theorem 4.2. *A complex Finsler manifold (M, F) is modeled on a complex Minskowski space if and only if the Finsler connection (M, F) is basic.*

Let (M, F) be a complex manifold modeled on a complex Minkowski space $(\boldsymbol{C}^n, f)$, and $\{e_i\}$ an adapted frame of the G-structure. Since any element of G is given by a unitary matrix with respect to $\{e_i\}$, we can define a Hermitian metric h_M on M by

$$h_M(e_i, e_j)=\delta_{ij} , \quad \text{or} \quad h_M\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\Big):=h_{i\bar{j}}(z)=\sum_m B_i^m(z)\bar{B}_j^m(z) . \tag{4.4}$$

We call h_M the *associated Hermitian metric* to (M, F). Since G is a subgroup of $U(n)$, the connection form $\theta=(\theta_j^i)$ of ∇ in Proposition 4.2 satisfies $\theta+{}^t\bar{\theta}=0$, and so ∇ is the Hermitian connection of h_M. Therefore the pull-back of the Hermitian connection of h_M defines the Finsler connection of (M, F). The associated Hermitian metric h_M is a metric whose existence is asserted in Theorem 4.1.

Proposition 4.3. *Let (M, F) be a complex manifold modeled on a complex Minkouski space. The Finsler connection of (M, F) is given by the Hermitian connection of the associated Hermitian metric h_M.*

Furthermore, from this proposition and the Kähler condition (3.2), we get

Proposition 4.4. *Let (M, F) be a complex manifold modeled on a complex Minkowski space. (M, F) satisfies the Kähler condition if and only if the associated Hermitian metric h_M is a Kähler metric on M.*

Let (M, F) be a complex manifold modeled on a complex Minkowski space $(\boldsymbol{C}^n, f)$. In $(\boldsymbol{C}^n, f)$, the indicatrix $I=\{\xi\in\boldsymbol{C}^n; f(\xi)<1\}$ is bounded and strictly pseudoconvex, and there exists a unique Euclidian sphere S centered at the origin inscribed about the indicatrix I. We may assume S is the unit sphere

$$\langle\xi, \xi\rangle_P=\sum_i|\xi^i|^2=1 ,$$

which is the boundary of the indicatrix in T_pM of the associated Hermitian metric h_M. The associated Hermitian metric h_M defines a function f_M on TM by

$$f_M(z, \eta)=\sum_{i,j}h_{i\bar{j}}(z)\eta^i\bar{\eta}^j . \tag{4.5}$$

The definition (4.4) yields the following inequality:

$$f_M(z, \eta)\geq F(z, \eta) . \tag{4.6}$$

Then we can show

Theorem 4.3. *Let (M, F) be a complex manifold modeled on a complex Minkowski space $(\boldsymbol{C}^n, f)$. F is a Hermitian metric on M if and only if the isometric group G acts transitively on the boundary ∂I at each point.*

Proof. Let ξ be an element such that $f(\xi)=1=\langle\xi, \xi\rangle_p$, and ψ another element with unit norm: $f(\psi)=1$. If the isometric group G of f acts on ∂I transitively, there exists $g\in G$ with $\psi=g(\xi)$. Since the imner product $\langle,\rangle_p$ is also invariant under $g\in G$, we get

$$f(\psi)=1=\langle\xi, \xi\rangle_p=\langle g(\xi), g(\xi)\rangle_p=\langle\psi, \psi\rangle_p .$$

Thus the point ψ lies on S, and the boundary ∂I coincides with S. The converse is also true. Q.E.D.

Example 4.1 (*Complex parallelisable manifolds*). Let M be a complex parallelisable manifold, e.g., M is a complex multi-torus. Then, its holomorphic tangent bundle admits a globally defined holomorphic frame field $\{\boldsymbol{e}_i\}$ $(1\leq i\leq n)$, that is, M admits a $\{1\}$-structure ([7]). In this case, the function B^i_j: $U\to GL(n, \boldsymbol{C})$ are holomorphic. Since the curvature is given by $\Omega^i_j=\sum_m\bar{\partial}(h^{\bar{m}i}\partial h_{j\bar{m}})$, the Hermitian connection of the associated Hermitian metric h_M of (4.4) is of zero-curvature, and so the Finsler connection of (M, F) constructed in Proposition 4.3 is flat. Such a complex Finsler manifold (M, F) is locally Minkowski. Therefore it is possible to introduce a locally Minkowski metric on any complex parallelisable manifold.

§5. Holomorphic sectional curvature

In this section, we shall treat a complex Finsler manifold (M, F) and its holomorphic sectional curvature. We denote by Ω the curvature form of the Finsler connection ∇ of (M, F). Ω is a $C^\infty(\boldsymbol{Q}\otimes\boldsymbol{Q}^*)$-valued 2-form on TM, and by direct calculations we have

$$\Omega=\bar{\partial}\omega+\partial_V\omega \ .$$

Therefore the holomorphic sectional curvature $H(z, \xi)$ at $(z, \xi)\in TM$ defined in [10] is written as

$$H(z, \xi)=2\frac{h_{\boldsymbol{Q}}(\Omega(X_\xi, \bar{X}_\xi)\xi, \xi)}{F(z, \xi)^2} \ .$$

Then, by direct calculations, we get a local expression of $H(z, \xi)$ as follows:

$$(5.1) \qquad H(z, \xi)=\frac{2}{F(z, \xi)^2}\sum_{k,l,s,t}\Big(\sum_{i,m}F^{m\bar{i}}\frac{\partial F_{s\bar{m}}}{\partial z^k}\frac{\partial F_{i\bar{t}}}{\partial \bar{z}^l}-\frac{\partial^2 F_{s\bar{t}}}{\partial z^k\partial \bar{z}^l}\Big)\xi^s\bar{\xi}^t\xi^k\bar{\xi}^l$$

$$=\frac{2}{F(z, \xi)^2}\sum_{k,l}\Big(\sum_{i,j}F_{i\bar{j}}N^i_k\bar{N}^j_l-\frac{\partial^2 F}{\partial z^k\partial \bar{z}^l}\Big)\xi^k\bar{\xi}^l \ ,$$

where we used (2.3). It is noted that if the given metric F is a Hermitian metric on M, $H(z, \xi)$ is just the holomorphic sectional curvature in the usual sence ([25]).

On the other hand, for an arbitrary point $(z, \xi)\in TM$ there exists a holomorphic map φ: $\Delta(r)\rightarrow M$ satisfying

$$(5.2) \qquad \varphi(0)=z, \ \varphi_*(0)=\xi \ .$$

Then, for the given Finsler metric $F(z, \eta)$, a Hermitian metric $\varphi^* F$ on $\Delta(r)$ is introduced by

$$\varphi^*F=E(\zeta)d\zeta\otimes d\bar{\zeta} \ ,$$

where we put $E(\zeta)=F(\varphi(\zeta), \varphi_*(\zeta))$. The Gauss curvature of φ^*F is defined by

$$K(\varphi^*F)=-2\Big(\frac{1}{E}\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\log E\Big)_{\zeta=0} \ ,$$

and, according to Wong [23], the *holomorphic sectional curvature* $K(z, \xi)$ of (M, F) at (z, ξ) is defined by

$$K(z, \xi):=\sup\{K(\varphi^*F)\} \ ,$$

where φ ranges over all holomorphic maps satisfying (5.2).

In the following, we assume that $\|\xi\|_z^2=F(z, \xi)=1$, for simplicity. Then we may always choose a coordinate system on $\Delta(r)$ satisfying $(\partial E/\partial\zeta)_{\zeta=0}=0$ and $(\partial E/\partial\bar{\zeta})_{\zeta=0}=0$. Hence, in such a coordinate system on $\Delta(r)$, $K(z, \xi)$ can be written as

$$K(z, \xi)=2\sup\left\{\left(-\frac{\partial^2 E}{\partial\zeta\partial\bar{\zeta}}\right)_{\zeta=0}\right\} .$$

By direct calculations, using (2.3) and (5.1) we get

$$\begin{aligned}\left(\frac{\partial^2 E}{\partial\zeta\partial\bar{\zeta}}\right)_{\zeta=0}&=\sum_{i,j}\left\{\frac{\partial^2 F}{\partial z^i\partial\bar{z}^j}\xi^i\bar{\xi}^j+F_{i\bar{j}}\ddot{\varphi}_0^i\sum_m\overline{(N_m^j\xi^m)}+F_{i\bar{j}}\left(\sum_m N_m^i\xi^m\right)\overline{\ddot{\varphi}_0^j}+F_{i\bar{j}}\ddot{\varphi}_0^i\overline{\ddot{\varphi}_0^j}\right\}\\&=\sum_{i,j}F_{i\bar{j}}\left(\ddot{\varphi}_0^i+\sum_m N_m^i\xi^m\right)\overline{\left(\ddot{\varphi}_0^j+\sum_m N_m^j\xi^m\right)}-H(z, \xi) ,\end{aligned}$$

from which we have

$$H(z, \xi)=K(\varphi^*F)+2\left\|\sum_i\left(\ddot{\varphi}_0^i+\sum_m N_m^i\xi^m\right)\frac{\partial}{\partial z^i}\right\|_z^2 ,$$

where we put $\ddot{\varphi}_0^i=(\partial^2\varphi^i/\partial\zeta^2)_{\zeta=0}$. This yields the inequality $H(z, \xi)\geq\sup\{K(\varphi^*F)\}$. Royden [14] showed that $\sup\{K(\varphi^*F)\}$ attains to the maximum when φ is a complex line in the semi-normal coordinate system at $P=(z, \xi)$. The equality above shows that the maximum is given by the previous $H(z, \xi)$. In fact, in a semi-normal coordinate system at $P=(z, \xi)$, we get from (2.3) and (3.1)

$$\sum_m N_m^i(z, \xi)\xi^m=\sum_{l,m}F_{lm}^i(z, \xi)\xi^l\xi^m=0 .$$

Therefore we have proved

Proposition 5.1. *The holomorphic sectional curvature $K(z, \xi)$ coincides with the one $H(z, \xi)$ constructed from the curvature Ω of the Finsler connection ∇ of (M, F).*

In the case of Hermitian metric on M, this fact is well-known ([25]).

Now, we suppose that $H(z, \xi)$ is bounded above by a negative constant $-k$ $(k>0)$. Proposition 5.1 implies that $K(\varphi^*F)\leq -k$ for an arbitrary φ. Then we have the following Schwartz-type lemma ([15]).

Proposition 5.2. *Let φ: $\Delta(r)\rightarrow M$ be a holomorphic map of a small disk into a complex Finsler manifold with holomorphic sectional curvature at most $-k$.* Then we have

$$4\left(\frac{r}{r^2-|\zeta|^2}\right)^2|v|^2\geq kF(\varphi(\zeta), \varphi_*(\boldsymbol{v})) \quad \text{for} \quad \boldsymbol{v}=v\left(\frac{\partial}{\partial\zeta}\right)_\zeta\in T_\zeta\Delta(r) \tag{5.3}$$

For an arbitrary $(z, \xi) \in TM$, we take a holomorphic map φ: $\Delta(r) \rightarrow M$ satisfying (5.2). Then, by Proposition 5.2 we have

$$4\frac{r^2}{(r^2-|0|^2)^2}=\frac{4}{r^2}\geq kF(\varphi(0),\ \varphi_*(0))=kF(z,\ \xi)\ ,$$

from which we get the following inequality:

$$4F_M(z,\ \xi)^2:=4(\inf\{\frac{1}{r}\})^2\geq kF(z,\ \xi)\ ,$$

where F_M is the *Kobayashi metric* on M:

$$F_M(z,\ \xi):=\inf\{\frac{1}{r};\ \varphi\colon \Delta(r)\rightarrow M \text{ is a holomorphic map satisfying } \varphi(0)=z,\ \varphi_*(0)=\xi\}\ .$$

Hence we have

Theorem 5.1. *Let* (M, F) *be a complex manifold whose holomorphic sectional curvature* $H(z, \xi)$ *is bounded above by a negative constant* $-k$. *Then we have*

$$4F_M^2\geq kF\ .$$

Let (M, F) be a complex manifold modeled on a complex Minkowski space $(\boldsymbol{C}^n, f)$. By Proposition 4.3, the holomorphic sectional curvature $H(z, \xi)$ of (M, F) is given by the one of the associated Hermitian metric h_M. If $H(z, \xi)$ is bounded above by a negative constant $-k$, the following inequality is well-known ([25]):

$$4F_M^2\geq kf_M\ . \tag{5.4}$$

Thus Theorem 5.1 is a generalization of this estimate. Then, from (4.6) and (5.4) we get

Theorem 5.2. *Let* (M, F) *be a complex manifold modled on a Minkowski space* $(\boldsymbol{C}^n, f)$. *If its holomorphic sectional curvature* $H(z, \xi)$ *is bounded above by a negative constant* $-k$, *we have*

$$4F_M^2\geq kf_M\geq kF\ .$$

Next, we are interested in the class of complex Finsler manifolds whose holomorphic sectional curvature $H(z, \xi)$ is constant. In Hermitian geometry, the following result is well-known (Chapter IX of [11]):

A simply connected and complete Kähler manifold of constant holomorphic sectional curvature c is holomorphically isometric to the following three classes according to (i) $c<0$, (ii) $c=0$ or (iii) $c>0$:

(i) the open unit ball D_n in $\boldsymbol{C}^n$ with the metric

$$(5.5)\qquad ds^2=-\frac{4}{c}\frac{(1-\sum_k z^k\bar{z}^k)(\sum_k dz^k d\bar{z}^k)+(\sum_k \bar{z}^k dz^k)(\sum_k z^k d\bar{z}^k)}{(1-\sum_k z^k\bar{z}^k)^2},$$

(ii) the space $\boldsymbol{C}^n$ with the metric

$$(5.6)\qquad ds^2=\sum_k dz^k d\bar{z}^k ,$$

(iii) the complex projective space $P^n(C)$ with the metric

$$(5.7)\qquad ds^2=-\frac{4}{c}\frac{(1+\sum_k z^k\bar{z}^k)(\sum_k dz^k d\bar{z}^k)-(\sum_k \bar{z}^k dz^k)(\sum_k z^k d\bar{z}^k)}{(1+\sum_k z^k\bar{z}^k)^2},$$

On the other hand, Pang [12] has shown the following proposition.

Proposition 5.3. *If a complete complex Finsler manifold (M, F) of constant holomorphic sectional curvature $H(z, \xi)=-4$ satisfies the property* (3.3), *then the Finsler metric F coincides with the Kobayashi metric F_M.*

Now, we shall consider an application of these results to a simply connected and complete complex manifold (M, F) modeled on a complex Minkowski space $(\boldsymbol{C}^m, f)$. Suppose that (M, F) satisfies (3.2) and (3.3). Then, by Proposition 4.4, the associated Hermitian manifold (M, h_M) is a simply connected and complete Kähler manifold. Moreover, if (M, F) is of constant holomorphic sectional curvature c, (M, h_M) is also of constant holomorphic sectional curvature c.

First we consider the case of $c<0$. Then Proposition 5.3 and Theorem 5.2 show that the given Finsler metric F, the function f_M defined by (4.5) from h_M and Kobayashi metric F_M on M coincide with each other, that is, (M, F) is a simply connected and complete Kähler manifold of negative constant holomorphic sectional curvature. Hence (M, F) is holomorphically isometric to the unit open ball D_n in $\boldsymbol{C}^n$ with the metric (5.5).

In the case of $c=0$, (M, h_M) is holomorphically isometric to $\boldsymbol{C}^n$ with the metric (5.6). So the curvature of the Finsler connection ∇ of (M, F) vanishes identically. Thus (M, F) is locally Minkowski, and holomorphically isometric to the complex Minkowski space $(\boldsymbol{C}^n, f)$.

As to the case of $c>0$, the associated Hermitian manifold (M, h_M) is holomorphically isometric to the complex projective space $P^n(\boldsymbol{C})$ with the metric (5.7). Consequently we get

Theorem 5.3. *Let (M, F) be a simply connected and complete complex manifold modeled on a complex Mikowski space $(\boldsymbol{C}^n, f)$. Suppose that (M, F) satisfies the Kähler condition* (3.2) *and the property* (3.3), *and furthermore, (M, F) is of* *constant holomorphic sectional curvature c.*

(i) *If* $c<0$, (M, F) *is a Kähler manifold which is holomorphically isometric to the open unit ball* D_n *in* $\boldsymbol{C}^n$ *with the metric* (5.5):

$$F(z, \xi)=-\frac{4}{c}\frac{(1-\sum_k z^k\bar{z}^k)(\sum_k \xi^k\bar{\xi}^k)+(\sum_k \bar{z}^k\xi^k)(\sum_k z^k\bar{\xi}^k)}{(1-\sum_k z^k\bar{z}^k)^2},$$

(ii) *if* $c=0$, (M, F) *is a locally Minkowski space which is holomorphically isometric to the complex Minkowski space* $(\boldsymbol{C}^n, f)$ *with the metric*

$$F(z, \xi)=f(\xi)\leq\sum_k|\xi^k|^2,$$

(iii) *if* $c>0$, *the following inequality holds*:

$$F(z, \xi)\leq-\frac{4}{c}\frac{(1+\sum_k z^k\bar{z}^k)(\sum_k \xi^k\bar{\xi}^k)-(\sum_k \bar{z}^k\xi^k)(\sum_k z^k\bar{\xi}^k)}{(1+\sum_k z^k\bar{z}^k)^2}.$$

DEPARTMENT OF MATHEMATICS
KAGOSHIMA UNIVERSITY

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Added in proof: Let (M, F) be a complex manifold modeled on a complex Minkowski space. If (M, F) satisfies the property (3.3), the Kähler condition (3.2) is derived from the Euler-Lagrange equation in §3. Hence the assumption of Kähler condition (3.2) in Theorem 5.3 can be omitted.