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Morse function and attaching map

By

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1. Introduction

Let M be a closed manifold and f be a Morse function, a differentiable function on M with isolated, non-degenerate, critical points. Associated to a Morse function there is a cell complex L which is homotopy equivalent to M, see Milnor [M2]. In the following we identify M with such a cell complex L though there is not a canonical way to construct L from M. Therefore we assume that M has a cell structure. By $M^{(n)}$ we denote the n-skeleton of M, whereas we set $M^c = f^{-1}((-\infty, c])$ and $M_c = f^{-1}(c)$ for a real number c. Since there is a j-cell for each critical point of index j, we identify a cell with the corresponding critical point. Thus the integral homology group of M is computed as the homology group of the cell complex. For a j-cell α its bounday $\partial\alpha$ is described as follows: When we write $\partial\alpha = \Sigma n_i \cdot \beta_i$, then n_i is the intersection number of the unstable manifold of $-grad\ f$ at α with the stable manifold of $-grad\ f$ at β_i. Here we fix a Riemannian metric of M. That is, the composition of the attaching map of the j-cell α $M^{(j-1)}$ and the collapsing map

$$M^{(j-1)} \to M^{(j-1)}/M^{(j-2)} \approx \bigvee_i S^{j-1} \to S^{j-1}$$

(This composition shall be called the attaching map of the j-cell α to the $(j-1)$-cell β_i.) is described as the intersection number of the unstable and stable manifolds of critical points.

Now we assume that the Morse function f on the manifold M has no critical points of indices $j+1, \ldots, j+\ell-1$, where ℓ is a positive integer.

The unstable framed cobordism group consists of the equivalence clsses of n-dimensional manifolds embedded in S^{n+k} with a framing of the trivial normanl bundle and the group is isomorphic to $\pi_{n+k}(S^k)$, the homotopy group of maps from S^{n+k} to S^k, see Milnor [M1]. The purpose of this paper is to prove the following thorem and to give its application to the projective spaces.

Therrem. *The attaching map of the $j+\ell$-cell α to the j-cell β is described as the (transversal) intersection manifold of the unstable manifold of $-$ grad f at α with the stable manifold of $-$grad f at β considered as a framed manifold embedded in the unstable manifold.*

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We make clear the framing of the intersection manifold. Let c be the critical value of f at β. By the Morse lemma there is a coordinate system $x_1, \ldots, x_n$ in a neighborhood U of β so that the idenity

$$f(x_1, \ldots, x_n) = c - x_1^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_n^2$$

holds throughout U. We choose a sufficiently small positive real number ε. In $M_{c+\varepsilon}$ we consider the stable manifold

$$S_L = \{(0, \ldots, 0, x_{j+1}, \ldots, x_n) \in U \mid x_{j+1}^2 + \cdots + x_n^2 = \varepsilon\}.$$

S_L has the natural framing in $M_{c+\varepsilon}$ given by $\left\{\frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_j}\right\}$. This gives the framing of the intersection manifold of the unstable manifold with the stable manifold embedded in the unstable manifold.

Since a 0-dimensional framed manifold is interpretted as the intersection number of the unstable and the stable manifolds, our theorem is a generalization of the classical fact.

2. Proof of Main Theorem

Proof of Theorem. For simplicity we consider the case where there is only one critical point α with index $j+\ell$ and only one critical point β with index j. Let $f(\alpha) = c$ and $f(\beta) = d$, $d < c$. Furthermore we assume that there is no critical value of f in the range (d, c).

For all sufficiently small ε, the set $M^{c+\varepsilon}$ is homotopy equivalent to $M^{c-\varepsilon}$ with a $(j+\ell)$-cell attached. The embedding of the unstable manifold to the level surface represents the attaching map. Since $M^{c-\varepsilon}$ deformed to $M^{d+\delta}$ along the gradient flow, where δ is again choosed sufficiently small, $M^{c+\varepsilon}$ is homotopy equivalent to $M^{d+\delta} \cup e^{j+l}$, where the attaching map is given by the unstable manifold at α. Again $M^{d+\delta}$ is deformed to $M^{d-\delta} \cup e^j$ and the deformation is given as follows: Choose a coordinate system $x_1, \ldots x_n$ in a neighborhood U of β so that the identity

$$f = d - x_1^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_{\mathrm{n}}^2$$

holds throughout U. Thus the critical point β has coordinates

$$x_1(\beta) = \cdots = x_n(\beta) = 0.$$

Choose $\delta > 0$ sufficiently small so that the image of U by the coordinatj system contains the closed ball

$$\{(x_1, \ldots, x_n) \mid \Sigma x_i^2 \leq 2\delta\}.$$

Then e^j is defined by

$$e^j=\{(x_1,\ldots,x_n)\in U\,|x_1^2+\ldots+x_j^2\le\delta,\ x_{j+1}=\cdots=x_n=0\}\ .$$

Then the deformation is given by

$$(x_1,\ldots,x_n)\longmapsto(x_1\ldots,x_j,tx_{j+1},\ldots tx_n).$$

Thus the restriction of this deformation to the stable manifold gives the attaching map g of the $(j+\ell)$-cell to the j-cell. β is a regular value of the map g and g^{-1} (β) is the intersection manifold of the unstable and stable manifolds. The framing is given by the way mentioned as above.

3. Attching Map of projective spaces

In this section we consider the attaching map of the top cell of the complex projective space $\boldsymbol{C}P^n$ and the quaternion projective space $\boldsymbol{H}P^n$.

At first we consider the complex projective space. The function f defined by

$$f([z_0:\cdots:z_n])=(\sum_{i=0}^{n}i\,|z_i|^2)(\sum_{i=0}^{n}|z_i|^2)^{-1}$$

is a Morse function. Its critical points are $[0:\cdots:0:1:0:\cdots:0]$, where 1 is in i-th coordinate, with index $2i$. In the neighborhood

$$U=\{[z_0:\cdots:z_n]\in\boldsymbol{C}P^n\,|z_{n-1}\ne0\}$$

of $[0:\cdots:0:1:0]$ we choose the standard coordinate system

$$[z_0:\cdots:z_n]\longmapsto\left(\frac{z_0}{z_{n-1}},\ldots,\frac{z_{n-2}}{z_{n-1}},\frac{z_n}{z_{n-1}}\right)$$
$$=(x_1,y_1,\ldots,x_n,y_n).$$

In U we have

$$f=n-1+\frac{-\sum_{j=1}^{n-1}(n-j)(x_j^2+y_j^2)+x_n^2+y_n^2}{1+\|z\|^2}$$

where $\|z\|^2=\sum x_i^2+y_i^2$. We set

$$u_i=\sqrt{\frac{n-i}{1+\|z\|^2}}\cdot x_i,\quad v_i=\sqrt{\frac{n-i}{1+\|z\|^2}}\cdot y_i\quad\text{for}\quad 1\le i\le n-1$$

and

$$u_n=\frac{x_n}{\sqrt{1+\|z\|^2}},\quad v_n=\frac{y_n}{\sqrt{1+\|z\|^2}}.$$

Then $u_1,v_1,\cdots,u_n,v_n$ is a new coordinate system of U and the identity

$$f=n-1-u_1^2-v_1^2-\ldots-u_{n-1}^2-v_{n-1}^2+u_n^2+v_n^2$$

holds throughout U. We choose a sufficiently small real number ε. To examine the attaching map of the $2n$-cell to the $(2n-2)$-cell, it is sufficient to consider the framed manifold

$$S^1=\{0,\ldots,0,u_n,v_n)\,|u_n^2+v_n^2=\varepsilon\}$$

embedded in $(\boldsymbol{CP}^n)_{n-1+\varepsilon}$ which is the unstable manifold at $[0:\cdots:0:1]$. The framing of S^1 is given by $\left\{\frac{\partial}{\partial u_1},\frac{\partial}{\partial v_1},\ldots,\frac{\partial}{\partial u_{n-1}},\frac{\partial}{\partial v_{n-1}}\right\}$. In terms of standard coordinate system

$$S^1=\left\{0,\ldots,0,x_n,y_n)\,|x_n^2+y_n^2=\frac{\varepsilon}{1-\varepsilon}\right\}$$

embedded in

$$S^{2n-1}=\{[z_0:\cdots:z_n]\in \boldsymbol{C}P^n|\sum_{i=0}^{n-1}(n-i-1+\varepsilon)\ |z_i|^2=(1-\varepsilon)\,|z_n|^2\}$$

with the framing given by $\left\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial x_{n-1}},\frac{\partial}{\partial y_{n-1}}\right\}$. By the diffeomorphism of $\boldsymbol{C}P^n$ defined by $[z_0:\cdots:z_n]\longmapsto[w_0:\ldots:w_n]$, where $w_i=\sqrt{n-i-1+\varepsilon}\cdot z_i$ for $0\leq i\leq n-1$ and $w_n=\sqrt{1-\varepsilon}\cdot z_n$, the framed manifold S^1 is mapped to

$$S^1=\{[0:\cdots:0:1:z_n]\in \boldsymbol{C}P^n|\ |z_n|^2=1\}$$

embedded in

$$S^{2n-1}=\{[z_0:\cdots:z_n]\in \boldsymbol{C}P^n|\sum_{i=0}^{n-1}|z_i|^2=|z_n|^2\}$$

with the framing given by $\left\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial x_{n-1}},\frac{\partial}{\partial y_{n-1}}\right\}$. This framed manifold is equivalent to

$$S^1=\{(0,\ldots,0,z)\in\boldsymbol{C}^n||z|=1\}$$

embedded in

$$S^{2n-1}=\{(z_1,\ldots,z_n)\in\boldsymbol{C}^n|\sum_{i=1}^{n}|z_i|^2=1\}$$

with the framing given by $\left\{\left(\frac{\partial}{\partial x_1},\frac{\partial}{y_1}\right)z,\ldots,\left(\frac{\partial}{\partial x_{n-1}},\frac{\partial}{\partial y_{n-1}}\right)z\right\}$, where

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)z=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

for $z=\cos\theta+i\sin\theta$. Thus this framing differs from the trivial one by the map

$$S^1\xrightarrow{\triangle}\underbrace{S^1\times\cdots\times S^1}_{n-1\text{ times}}\subset SO(2n-2),$$

where $\triangle$ is the diagonal map. By $\eta:S^3\to S^2$ we denote the Hopf map and by the same letter we denote its suspension. Since η is the attaching map of the top cell of $\mathbf{C}P^2$, the above framing of S^1 for $n=2$ corresponds to the map $\eta \in \pi_3(S^2)$. $\triangle \in \pi_1(SO(2n-2)) \cong \mathbf{Z}$ represents $n-1$ times the generator. Thus the attaching map of the top cell of $\boldsymbol{C}P^n$ is $(n-1)\eta \in \pi_{2n-1}(S^{2n-2})$ which is isomorphic to $\mathbf{Z}\{\eta\}$ for $n=2$ and to $\mathbf{Z}/2\{\eta\}$ for $n \geq 3$.

Similarly by $\nu : S^7 \to S^4$ we denote the Hopf map and by the same letter $(n-1)\nu \in \pi_{4n-1}(S^{4n-4})$ which is isomorphic to $Z\{\nu\} \oplus \mathbf{Z}/12$ for $n=2$ and to $\mathbf{Z}/24\{\nu\}$ for $n \geq 3$. What we have to do to prove this fact is only replacing the complex number with the quaternion in the above explanation.

Of course the result itself is well known, but our proof is much simpler than the usual one whcih involves the calculation of the e-invariant.

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