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Affine lines on Q-homology planes

By

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1. Introduction

An algebraic surface X defined over $\mathbf{C}$ is called a $\mathbf{Q}$ (respectively $\mathbf{Z}$) *-homology Plane* if $H_i(X, \mathbf{Q})=0$ (resp. $H_i(X, \mathbf{Z})=0$) for all $i>0$. By a result of T. Fujita, a $\mathbf{Q}$-homology plane is an affine surface. $\mathbf{Q}$-homology planes occur naturally and "abundantly" as follows. Let Z be a smooth rational surface and D a simply connected curve on Z whose irreducible components generate $H_2(Z; \mathbf{Q})$ freely. Then $X:=Z-D$ is a $\mathbf{Q}$-homology plane (cf. Lemma 5).

Following results about the existence of contractible algebraic curves on $\mathbf{Q}$-homology planes are known.

(i) If $\bar{\kappa}(X)=-\infty$, then there is a morphism $\phi: X \to B$ where B is a nonsingular curve, such that a general fibre of ϕ is isomorphic to $\mathbf{C}$, and hence there are infinitely many contractible curves on X (cf. [M], Chapter I, Theorem 3.13).

(ii) If $\bar{\kappa}(X)=1$, then X contains at least one and at most two contractible curves (cf. [M-S], Lemma 2.15). If X is a $\mathbf{Z}$-homology plane with $\bar{\kappa}(X)=1$, then X contains a unique contractible curve and it is smooth (cf. [G-M]).

(iii) If $\bar{\kappa}(X)=2$, then X contains no contractible algebraic curve (cf. [M-T2]).

In this paper we complete the picture by proving the following (somewhat unexpected) result. For the terminology used in the statement of the theorem, see §1.

Theorem. *Let X be a $\mathbf{Q}$-homology plane with $\bar{\kappa}(X)=0$. Then the following assertions are true.*

(*i*) *If X is not NC-minimal, then X contains a unique contractible curve C. Moreover C is smooth with $\bar{\kappa}(X-C)=0$.*

(*ii*) *If X is NC-minimal and not the surface $H[k, -k]$ in Fujita's classification, then X has no contractible curves.*

(*iii*) *If X is NC-minimal and is isomorphic to $H[k, -k]$ with $k\geq 2$, then there is a unique contractible curve C on X and it is smooth. Further, $\bar{\kappa}(X-C)=0$.*

(*iv*) *The surface $X=H[1, -1]$ has exactly two contractible curves, say C*

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and L. Further, both the curves are smooth, $\bar{\kappa}(X-C)=0$ and $\bar{\kappa}(X-L)=1$. The curves C and L intersect each other transversally in exactly two points.

It should be remarked that by a beautiful result of Fujita, there does not exist a $\mathbf{Z}$-homology plane X with $\bar{\kappa}(X)=0$. This follows from the complete classification of NC-minimal $\mathbf{Q}$-homology planes with $\bar{\kappa}(X)=0$ due to Fujita (cf. [F, §8.64]). A direct and short proof of this was recently found by the first author and M. Miyanishi. In this paper we use this classification of Fujita in a crucial way.

Combining the results in this paper with the earlier known results, we get the following.

Corollary. *A $\mathbf{Q}$-homology plane with three contractible curves is of logarithmic Kodaira dimension $-\infty$.*

2. Notations and preliminaries

All algebraic varieties considered in this paper are defined over the field of complex numbers $\mathbf{C}$.

For any topological space X, $e(X)$ denotes its topological Euler characteristic.

Given a connected, smooth, quasiprojective variety V, $\bar{\kappa}(V)$ denotes the logarithmic Kodaira dimension of V as defined by S. Iitaka (cf. [I]).

By a $(-n)$-curve on a smooth algebraic surface we mean a smooth rational curve with self-intersection $-n$. By a *normal crossing divisor* on a smooth algebraic surface we mean a reduced algebraic curve C such that every irreducible component of C is smooth, no three irreducible components pass through a common point and all intersections of the irreducible components of C are transverse. For brevity, we will call a normal crossing divisor an n.c. divisor. Let D be an n.c. divisor on a smooth surface. We say that D is a *minimal normal crossing divisor* if any (-1)-curve in D intersects at least three other irreducible components of D. A minimal normal crossing divisor will be called an m.n.c. divisor for brevity.

Following Fujita, we call a divisor D on a smooth projective surface Y *pseudo-effective* if $H\cdot D\geq 0$ for every ample divisor H on Y.

For the convenience of the reader, we now recall some basic definitions which are used in the results about Zariski-Fujita decomposition of a pseudo-effective divisor (cf. [F], §6; [M-T], Chapter 1).

Let (Y, D) be a pair of a nonsingular surface Y and a normal crossing divisor D. A connected curve T consisting of irreducible curves in D (a connected curve in D, for short) is a *twig* if the dual graph of T is a linear chain and T meets $D-T$ in a single point at one of the end points of T; the other end of T is called a *tip* of T. A connected curve R (resp. F) in D is a *club* (resp. an *abnormal club*) if R (resp. F) is a connected component of D and the dual graph of R (resp. F) is a linear chain (resp. the dual graph of the exceptional curves of a minimal resolution of singularities of a non-cyclic quotient singularity). A connected curve B in D is *rational* (resp. *admissible*) if each irreducible component of B is rational (resp. if none of the irreducible components of B is a (-1)-curve and the intersection matrix of B is negative definite). An admissible rational twig T is *maximal* if T is not contained in an admissible rational twig with more irreducible components.

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$ and $\{F_\nu\}$) be the set of all admissible rational maximal twigs (resp. admissible rational maximal clubs and admissible rational maximal abnormal clubs). Then there exists a decomposition of D into a sum of effective $\mathbf{Q}$-divisors, $D=D^{\#}+Bk(D)$, such that $\mathrm{Supp}(Bk(D))=(\cup_\lambda T_\lambda)\cup(\cup_\mu R_\mu)\cup(\cup_\nu F_\nu)$ and $((K_Y+D^{\#})\cdot Z)=0$ for every irreducible component Z of Supp $(Bk(D))$. The divisor $Bk(D)$ is called the *bark* of D, and we say that $K_Y+D^{\#}$ is produced by the *peeling* of D. For details of how $Bk(D)$ is obtained from D, see [M-T].

The Zariski-Fujita decomposition of K_Y+D, in case K_Y+D is pseudo -effective, is as follows:

There exist $\mathbf{Q}$-divisors P, N such that $K_Y+D\approx P+N$ where, $\approx$ denotes numerical equivalence, and

(a) P is numerically effective (nef, for short). If $\bar{\kappa}(Y-D)=0$, then $P\approx 0$ by a fundamental result of Kawamata (cf. [Ka2]).

(b) N is effective and the intersection form on the irreducible components of N is negative definite

(c) $P\cdot D_i=0$ for every irreducible component D_i of N.

N is unique and P is unique upto numerical equivalence. If some multiple of K_Y+D is effective, then P is also effective.

The following result from [F, Lemma 6.20] is very useful.

Lemma 1. *Let (Y, D) be as above. Assume that all the maximal rational twigs, maximal rational clubs and maximal abnormal rational clubs of D are admissible. Let $\bar{\kappa}(Y-D)\geq 0$. As above, let $P+N$ be the Zariski decomposition of K_Y+D. If $N\neq Bk(D)$, then there exists a (-1)-curve L, not contained in D, such that one of the following holds:*

(i) L is disjoint from D

(ii) $L\cdot D=1$ and L meets an irreducible component of $Bk(D)$

(iii) $L\cdot D=2$ and L meets two different connected components of D such that one of the connected components is a maximal rational club R_ν of D and L meets a tip of R_ν

Further, $\bar{\kappa}(V-D-L)=\bar{\kappa}(Y-D)$.

Following Fujita, we will say that a smooth affine surface V with $\bar{\kappa}(V)\geq 0$ is *NC-minimal* if it has a smooth projective completion $\bar{V}$ such that $D:=\bar{V}-V$ is an m.n.c. divisor and $N=Bk(D)$, where $P+N$ is the Zariski-Fujita decomposition of $K_{\bar{V}}+D$.

The following results proved by Kawamata will be used frequently.

Lemma 2. (cf. [Ka1]). *Let Y be a smooth quasi-projective algebraic surface and f: $Y \to B$ be a surjective morphism to a smooth algebraic curve such that a general fibre F of f is irreducible. Then* $\bar{\kappa}(Y) \geq \bar{\kappa}(B) + \bar{\kappa}(F)$.

Lemma 3, (cf. [Ka2]). *Let Y be a smooth quasi-projective algebraic surface with* $\bar{\kappa}(Y) = 1$. *Then there is a Zariski-open subset U of Y which admits a morphism f: $U \to B$ onto a smooth algebraic curve B such that a general fibre of f is isomorphic to either* $\mathbf{C}^*$ *or an elliptic curve.*

We call such a fibration a $\mathbf{C}^*$-fibration or an elliptic fibration respectively.

Similarly, we can define a $\mathbf{C}$-fibration and a $\mathbf{P}^1$-fibration on a smooth projective surface.

As mentioned in the introduction, the next result follows from R. Kobayashi's inequality and plays an important role in the proof of the theorem.

Lemma 4. (cf. [M-T2]). *Let V be a smooth affine surface with* $e(V) \leq 0$. *Then* $\bar{\kappa}(V) \leq 1$.

We begin with some properties of $\mathbf{Q}$-homology planes.

Let X be a smooth affine surface and $X \subset Z$ be a smooth projective compactification with $D := Z - X$.

Lemma 5. *Assume that the irregularity* $q(Z) = 0$. *Then X is a* $\mathbf{Q}$*-homology plane if and only if the irreducible components of D generate* $H_2(Z; \mathbf{Q})$ *freely and* $H_1(D; \mathbf{Q}) = 0$.

Proof. We use the long exact cohomology sequence with $\mathbf{Q}$-coefficients of the pair (X, D). By Poincaré duality, $H^i(Z, D; \mathbf{Q}) = H_{4-i}(X)$. Hence $H_i(X) = 0$ for $i > 0$ if and only if the restriction map $H^i(Z; \mathbf{Q}) \to H^i(D; \mathbf{Q})$ is an isomorphism for $i < 4$. Since $H_1(Z; \mathbf{Q}) = H_3(Z; \mathbf{Q}) = 0$ by assumption, it follows that X is a $\mathbf{Q}$-homology plane if and only if $H_1(D; \mathbf{Q}) = 0$ and the irreducible components of D generate $H_2(Z; \mathbf{Q})$ freely.

Now let X be an affine surface with either a $\mathbf{C}$-fibration or a $\mathbf{C}^*$-fibration, $\phi: X \to B$. For a suitable smooth compactification $X \subset Z$ we get a $\mathbf{P}^1$-fibration $\Phi: Z \to \bar{B}$, where $\bar{B}$ is a smooth compactification of B. We will need the following result due to Gizatullin.

Lemma 6. *Let F be a scheme-theoretic fibre of* Φ. *Then we have;*

(1) F_{red} *is a connected normal crossing divisor all whose irreducible components are isomorphic to* $\mathbf{P}^1$.

(2) *If F is not isomorphic to* $\mathbf{P}^1$, *then* F_{red} *contains a* (-1)*-curve. If a* (-1)*-curve occurs with multiplicity 1 in F, then* F_{red} *contains another* (-1)*-curve.*

Note that from (1) it follows that a (-1)-curve in F_{red} meets atmost two other irreducible components of F.

Let ϕ: $X \to B$ be a $\mathbf{C}^*$-fibration and Φ: $Z \to \overline{B}$ be an extension as above. Then D contains either one or two irreducible components which map onto $\overline{B}$ by Φ. We will call these components as *horizontal*. All other irreducible components of D are contained in the fibres of Φ. An irreducible component of D will be called a *D-component* for the sake of brevity. We say that ϕ is *twisted* if there is only one horizontal D-component (in [F], such a fibration is called a *gyoza*). Otherwise we say that ϕ is *untwisted* (in [F], such a fibration is called a *sandwitch*). In the untwisted case the horizontal D-components are cross-sections of Φ and in the twisted case the horizontal D-component is a 2-section.

The next result follows by an easy counting argument using the fact that the irreducible components of the divisor at infinity in a smooth compactification of a $\mathbf{Q}$-homology plane generate the Picard group, $Pic(X)$, freely over $\mathbf{Q}$.

Lemma 7. (cf. [G-M], Lemma 3.2). *Let ϕ: $X \to B$ be a $\mathbf{C}^*$-fibration on a $\mathbf{Q}$-homology plane X. Then we have;*

(1) If ϕ is twisted, then $B \cong \mathbf{C}$, all the fibres of ϕ are irreducible, there is a unique fibre F_0 of ϕ such that $F_{0_{red}}$ is isomorphic to $\mathbf{C}$ and all other fibres are isomorphic to $\mathbf{C}^*$, if taken with reduced structure.

(2) If ϕ is untwisted and $B \cong \mathbf{P}^1$, then all the properties of the fibres of ϕ are the same as (1) above.

(3) If ϕ is untwisted and $B \cong \mathbf{C}$, then ϕ has exactly one fibre F_0 with two irreducible components and all the other fibres are isomorphic to $\mathbf{C}^*$, if taken with reduced structure. Either both the components of F_0 are isomorphic to $\mathbf{C}$ which intersect transversally in one point or they are disjoint with one isomorphic to $\mathbf{C}$ and the other one isomorphic to $\mathbf{C}^*$.

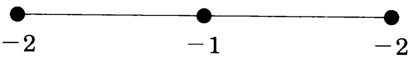
In order to avoid repetitive arguments in the proof of the theorem, we give detailed proof of the next result and use such arguments without details later on.

Lemma 8. *Let X be a $\mathbf{Q}$-homology plane with $\bar{\kappa}(X) = 0$ and ϕ: $X \to B$ be a $\mathbf{C}^*$-fibration. Let F_0 be the reducible fibre of ϕ (cf. lemma 7) which contains a contractible irreducible curve C. Consider a smooth completion $Z \supset X$ with $D := Z - X$ an n.c. divisor and Φ: $Z \to \mathbf{P}^1$ a $\mathbf{P}^1$-fibration which extends ϕ.*

(1) Suppose ϕ is twisted.

If $\bar{\kappa}(X-C) = 0$, then the morphism $X - C \to \mathbf{C}^*$ has no singular fibres. If $\bar{\kappa}(X-C) = 1$, then the morphism $X - C \to \mathbf{C}^*$ has at least one multiple fibre.

In both the cases, the fibre over the point $p_\infty := \mathbf{P}^1 - B$ can be assumed to have the dual graph



and the horizontal component D_h intersects the (-1)-curve transversally in a single point.

(2) *Suppose ϕ is untwisted and $B \cong \mathbf{C}$.*

Then the fibre F_∞ over p_∞ is a regular fibre of Φ and the two horizontal D-components meet this fibre in two distinct points. The morphism $X - C \to \mathbf{C}$ has at least one multiple fibre.

(3) *Suppose ϕ is untwisted and $B \cong \mathbf{P}^1$.*

If $\bar{\kappa}(X-C)=0$, then ϕ': $X-C \to \mathbf{C}$ has at least one and at most two multiple fibres. If ϕ' has two multiple fibres, then their multiplicities are 2 each. If $\bar{\kappa}(X-C)=1$, then ϕ' has at least two multiple fibres.

Proof. (1) Let $\phi' = \phi|_{X-C}$. Suppose ϕ' has a multiple fibre, say $m_1 F_1$, with $m_1 \geq 2$. Denote by p_0, p_1 the points $\phi(C)$, $\phi(F_1)$ respectively. Using lemma 9, we can construct a finite ramified covering τ: $A \to \mathbf{C}$, ramified only over p_0, p_1 such that the ramification index over p_i is m_i for $i=0,1$, where m_0 is a large integer. Then the normalization of the fibre product $A \times_{\mathbf{C}} X$ contains a Zariski-open subset U which is a finite étale covering of $X-C$. Since $\bar{\kappa}(A)=1$ for large m_0, by lemma 2, $\bar{\kappa}(U)=1$. But then $\bar{\kappa}(X-C)=1$, since $\bar{\kappa}$ does not change under finite étale coverings by a result of Iitaka (cf. [I]). This contradiction shows that ϕ' has no multiple fibre, if $\bar{\kappa}(X-C)=0$. Hence ϕ' has no singular fibre.

If ϕ' has no multiple fibre, then $X-C$ has a 2-sheeted étale cover which is isomorphic to $\mathbf{C}^* \times \mathbf{C}^*$. Hence $\bar{\kappa}(X-C)=0$.

The assertion about the fibre F_∞ is proved by Fujita in [F], lemma 7.5(2).

(2) The assertion about F_∞ is proved in [F], lemma 7.6(1). If ϕ' has no multiple fibre, then $X-C$ is isomorphic to $\mathbf{C} \times \mathbf{C}^*$, contradicting the assumption that $\bar{\kappa}(X)=0$.

(3) Suppose $\bar{\kappa}(X-C)=0$. If ϕ' has no multiple fibre, then $X-C$ is isomorphic to $\mathbf{C} \times \mathbf{C}^*$, a contradiction. If ϕ' has two multiple fibres m_1F_1, m_2F_2, then letting p_i be the points $\phi(F_i)$ for $i=0,1,2$, we can construct a finite galois covering τ: $A \to \mathbf{P}^1$ which is ramified only over p_i and the ramification index at any point over p_i is m_i for $i=0,1,2$. If one of the m_1, m_2 is strictly bigger than 2, then for large m_0, A is non-rational. But then we see that $\bar{\kappa}(X-C) \geq 1$. Hence $m_1 = m_2 = 2$.

The proof for the case $\bar{\kappa}(X-C)=1$ is similar.

The next result follows from R. H. Fox's solution of Fenchel's conjecture (cf. [Fo] and [C]).

Lemma 9. *Let $a_1, \ldots, a_r$ be distinct points in $\mathbf{P}^1$ with $r \geq 3$ and $m_1, \ldots, m_r$ be integers ≥ 2. Then there is a finite Galois covering τ: $B \to \mathbf{P}^1$ such that the rami-*

fication index at the point a_i *is* m_i *for* $1 \leq i \leq r$. *There is also a similar assertion if* $r=2$ *and* $m_1=m_2$.

Lemma 10. *Let* C_1, C_2 *be two distinct contractible curves on a* **Q**-*homology plane* X *with* $\bar{\kappa}(X) \geq 0$. *Then* $C_1 \cap C_2 \neq \phi$ *and if the intersection is a single point then it is transverse.*

Proof. Since $e(X-C_1)=0$, by lemma 4 $\bar{\kappa}(X-C_1) \leq 1$. Clearly, $\bar{\kappa}(X-C_1) \geq 0$.

Consider the case $\bar{\kappa}(X-C_1)=0$. Since $Pic(X)$ is finite, there exists a regular function f of X such that $(f)=mC_1$ for some integer m. We can assume that the morphism given by f: $X-C_1 \rightarrow \mathbf{C}^*$ has connected general fibres. Then by lemma 2, a general fibre of this morphism is isomorphic to $\mathbf{C}^*$. Thus, X has a $\mathbf{C}^*$-fibration such that C_1 is contained in a fibre. Suppose $C_1 \cap C_2=\phi$. Since C_2 does not contain any non-constant units, the image of C_2 is a point. This contradicts lemma 7.

Suppose $\bar{\kappa}(X-C_1)=1$. If $C_1 \cap C_2=\phi$, then $e(X-(C_1 \cup C_2))=-1$ and hence by lemma 4, $\bar{\kappa}(X-(C_1 \cup C_2))=1$. Then by lemma 3 we see that $X-(C_1 \cup C_2)$ has a C^*-fibration. Since X does not contain any complete curves, this morphism extends to a $\mathbf{C}^*$-fibration on X. Then C_1 and C_2 are mapped to points, otherwise the fibration is a $\mathbf{C}$-fibration. Again by lemma 7, both C_1, C_2 lie in the same fibre and hence C_1, C_2 intersect transversally in a single point by part (3) of lemma 7.

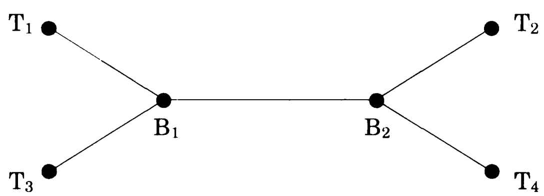
Now we know that $C_1 \cap C_2 \neq \phi$. Suppose $C_1 \cap C_2$ is a single point. Then $e(C_1 \cup C_2)=1$, $e(X-C_1 \cup C_2)=0$, and hence $\bar{\kappa}(X-C_1 \cup C_2) \leq 1$ by lemma 4. Arguing as above, we see that X admits a $\mathbf{C}^*$-fibration such that $C_1 \cup C_2$ is contained in a single fibre and hence they intersect transversally in a single point, again by lemma 7.

3. Fujita's clssification

In this section we describe the classification of NC-minimal **Q**-homology planes with $\bar{\kappa}=0$ due to Fujita (cf. [F], 8.64). There are four types of such surfaces. We also describe Fujita's surfaces $H[-1, 0, -1]$, which are NC-minimal surfaces with $\bar{\kappa}=0$, $e=0$ and $b_1=1$.

Type 1 (cf. [F], §8.26). $H[k, -k]$ with $k \geq 1$

The dual graph of the divisor D at infinity for an m.n.c. compactification is given by



Here $B_1^2=k$, $B_2^2=-k$ and $T_i^2=-2$ for all i. There is a (-1)-curve E_1 meeting the tips T_1, T_2 transversally in a single point and no other point of D. Similarly, there is a (-1)-curve E_2 meeting T_3 and T_4 transversally in a single point and no other point of D. The divisor $F_1=T_1+2E_1+T_2$ is a fibre of a $\mathbf{P}^1$-fibration Φ on $\overline{X}$ and $F_2=T_3+2E_2+T_4$ is another fibre of Φ. The curves B_1 and B_2 are cross sections of Φ. Let F_0 be the fibre of Φ through $B_1\cap B_2$. Clearly $C:=F_0-(B_1\cap B_2)\cong\mathbf{C}$, hence C is a contractible curve in X.

Lemma 11. $\bar{\kappa}(X-C)=0$.

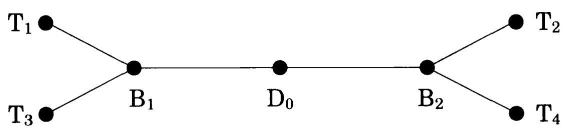
Proof. The $\mathbf{C}^*$-fibration ϕ: $X-C\to\mathbf{C}$ has exactly two multiple fibres corresponding to $2E_1$ and $2E_2$. Let $p_i=\Phi(F_i)$ for $i=0, 1, 2$. Using lemma 9 we can construct a degree 2 galois covering τ: $B\to\mathbf{P}^1$ such that the ramification index over p_i is 2 for each i. By Riemann-Hurwitz formula, $B\cong\mathbf{P}^1$. Then $\overline{X\times_{\mathbf{P}^1}B}\to B$ is a $\mathbf{C}^*$-fibration and $\overline{X\times_{\mathbf{P}^1}B}-\tilde{\tau}^{-1}(C)$ is an étale cover of $X-C$ isomorphic to $\mathbf{C}^*\times\mathbf{C}^*$. Hence $\bar{\kappa}(X-C)=0$.

Types 2, 3 and 4 are denoted by $Y[3, 3, 3]$, $Y[2, 4, 4]$ and $Y[2, 3, 6]$ respectively by Fujita (§8.37, 8.53, 8.54, 8.59, 8.61). The dual graphs of each of these have a unique branch point. There are three maximal twigs T_1, T_2 and T_3 for each of them and $\sum_{i=1}^{3}1/d(T_i)=1$, where $d(T_i)$ is the absolute value of the determinant of the intersection matrix of T_i.

Fujita has shown that $\pi_1(X)$ is a finite cyclic group for any NC-minimal $\mathbf{Q}$-homology plane with $\bar{\kappa}(X)=0$. This result will be used effectively in the next section.

Now we will describe the surfaces $H[-1, 0, -1]$ (cf. [F], §8.5).

The dual graph of an m.n.c. divisor at infinity is given by



Here, $B_1^2=B_2^2=-1$, $D_0^2=0$ and $T_i^2=-2$.

4. Proof of the Theorem (Non NC-minimal case)

Let X be a $\mathbf{Q}$-homology plane with $\bar{\kappa}(X)=0$. In this section we prove the following.

Proposition. *Suppose X does not have an NC-minimal compactification, then X contains a unique contractible curve.*

Proof. Suppose L is a contractible curve in X. Then $\bar{\kappa}(X-L)\leq 1$ and there is a $\mathbf{C}^*$-fibration ϕ': $X-L\to B^1$ which extends to a $\mathbf{C}^*$-fibration ϕ: $X\to B$

and $\phi(L)$ is a point (cf. proof of lemma 10). We choose a smooth compactification $X \subset Z$ such that $D:=Z-X$ is a normal crossing divisor and ϕ extends to a $\mathbf{P}^1$-fibration Φ: $Z \rightarrow \mathbf{P}^1$. We now consider the three cases given by lemma 7.

Case 1. ϕ is twisted. By lemma 7(1), $B \cong \mathbf{C}$ and every fibre of ϕ is irreducible. The fibre $F_\infty := \Phi^{-1}(p_\infty)$ has the dual graph as described in lemma 8 (1) and the 2-section D_h meets the (-1)-curve in F_∞ transversally in a single point.

First consider the case $\bar{\kappa}(X-L)=0$. The surface $X-L$ has the following properties.

(i) $X-L$ is affine

(ii) $\bar{\kappa}(X-L)=0$

(iii) $e(X-L)=b_2(X-L)=0$ and $b_1(X-L)=1$

(iv) $X-L$ is NC-minimal.

The property (iii) follows from the long exact cohomology sequence with compact support of the pair (X, L) and duality. The property (iv) follows from the observation that if $X-L$ is not NC-minimal, then by lemma 1, $X-L$ contains a curve $C \cong \mathbf{C}$. But then C is closed in X and disjoint from L, contradicting lemma 10.

Now the surface $X-L$ is isomorphic to $H[-1, 0, -1]$. Let F_0 be the fibre of Φ containing L. We may assume that any (-1)-curve in D contained in F_0 meets at least two other D-components in F_0. Since D is a connected tree of $\mathbf{P}^1$'s, either $F_{0red}=\bar{L}$ or the horizontal component D_h meets an irreducible component D_0 of D which occurs with multiplicity 2 in F_0 (observe that $F_0-\bar{L}$ is connected). Suppose $D_1 \subset D$ is a (-1)-curve in F_0 which is disjoint from D_h. Then by lemma 6 (1), D_1 meets at most two other D-components contained in F_0. Hence we can contract D_1 to a smooth point and get another compactification Z_1 which satisfies the same properties as Z. Repeating this argument we can assume that $\bar{L}$ and D_0 are the only possible (-1)-curves in F_0. Moreover, if D_0 is a (-1)-curve then it meets two other D-components. We claim that D_h is not a (-1)-curve. Otherwise, the m.n.c. divisor obtained from $D \cup \bar{L}$ by succession of contractions of (-1)-curves cannot be of the type described by Fujita. Now we see that D is an m.n.c. divisor.

Since X is not NC-minimal and D is m.n.c., there exists a (-1)-curve $\bar{C}$ given by lemma 1. Let $C=\bar{C} \cap X$. If $\bar{C} \neq \bar{L}$ then $\bar{C}$ is horizontal as it has to meet L. Hence $\bar{C}$ meets one of the tip components T_i of F_∞. As above, $X-C$ is also of the type $H[-1, 0, -1]$. By contracting C and then the image of T_i, we obtain a compactification divisor of $X-C$ which is not of type $H[-1, 0, -1]$. Hence $C=L$.

By lemma 8(1), $\bar{\kappa}(X-L)=1$ if and only if ϕ has at least one multiple fibre other than L. Now assume that $\bar{\kappa}(X-L)=1$. Then we can see that D_h

meets at least three D-components and hence D can be assumed to be m.n.c. as above. By lemma 1, there is a (-1)-curve $\bar{C}$ in Z satisfying the properties stated there. We arrive at a contradiction as above by first contracting C and then T_i.

Case 2. ϕ is untwisted and $B\cong\mathbf{C}$. Now ϕ has a unique fibre which contains two irreducible components, say L and L'. Any other fibre of ϕ is isomorphic to $\mathbf{C}^*$, if taken with reduced structure. The fibre F_∞ is a smooth fibre of ϕ and the two horizontal components of D meet F_∞ in distinct points. The divisor D may not be m.n.c., but it is obtained from an m.n.c. divisor by successive blow-ups. By lemma 8 (2), the morphism $X-L\rightarrow\mathbf{C}$ has at least one multiple fibre. From this we can see as above that D can be assumed to be m.n.c. Again since X is not NC-minimal, we get a (-1)-curve $\bar{C}\cong\mathbf{P}^1$ on Z which meets only a twig component of D. If $\bar{C}\neq\bar{L}$, then we get a contradiction as above.

Case 3. ϕ is untwisted and $B\cong\mathbf{P}^1$. Then every fibre of ϕ is irreducible. Any fibre of ϕ other than L is isomorphic to $\mathbf{C}^*$, if taken with reduced structure. By lemma 7.6 of [F], we can assume that every fibre of Φ other than the fibre F_0 containing L is a linear chain such that the two horizontal components of D meet the tip components of the fibre. From the connectivity of D we see that the union of D-components in F_0 is connected. Denote by D_1, D_2 the horizontal components. Let D_0 be a D-component contained in F_0 which meets D_1 or D_2. Then D_0 occurs with multiplicity 1 in F_0. If D_0 is a (-1)-curve it can meet at most one more D-componet in F_0. Hence we can contract D_0 to get a smaller compactification of X. Consequently we can assume that $\bar{L}$ is the unique (-1)-curve in F_0.

Now $(K_Z+D)\cdot\bar{L}=0$. On the other hand, if $K_Z+D\approx P+N$ is the Zariski-Fujita decomposition then $P\approx 0$ by the properties of the Zariski decomposition. Hence $N\cdot\bar{L}=0$. From the assumption that X is not NC-minimal, we know that there exists a curve $C\subset X$ such that $C\cong\mathbf{C}$ and its closure $\bar{C}$ occurs in N. But by lemma 10 if $L\neq C$ then $L\cdot C>0$.

If $\bar{\kappa}(X-L)=1$, then by lemma 8, the morphism $X-L\rightarrow\mathbf{C}$ has at least two multipe fibres. Then both D_1 and D_2 are branch points for the dual graph of D and hence D is m.n.c. The curve $\bar{\mathrm{C}}$ above can be assumed to be a (-1)-curve. Since $\bar{C}\cdot\bar{L}>0$, the intersection form on the subspace of Pic $Z\otimes_{z}\mathbf{Q}$ generated by $\bar{C}$ and $\bar{L}$ is not negative definite. Hence $\bar{L}$ does not occur in N and $N\cdot\bar{L}>0$ as $\bar{C}\subset N$, a contradiction. If $\bar{\kappa}(X-L)=0$, then we have a morphism $X\rightarrow\mathbf{C}$ with one fibre mL and general fibre isomorphic to $\mathbf{C}^*$, as in the proof of lemma 10. This is a twisted fibration by lemma 7. Then we are reduced to the case 1 and hence L is the unique contractible curve. This completes the proof of the proposition.

5. Proof of the Theorem (NC-minimal case)

We begin with the following general result.

Lemma 12. *Let Γ be a connected normal crossing divisor on a smooth projective surface Y. Assume the following conditions.*

(i) *Every irreducible component of Γ is isomorphic to $\mathbf{P}^1$.*

(ii) *The dual graph of Γ has at most one branch point.*

(iii) *If the dual graph has a branch point, then Γ has exactly three maximal twigs T_1, T_2 and T_3 and $\sum 1/d(T_i) > 1$.*

(iv) *Γ supports a divisor G with $G \cdot G > 0$.*

Then $\bar{\kappa}(Y - \Gamma) = -\infty$.

Proof. Suppose that $\bar{\kappa}(Y-\Gamma) \geq 0$. We will give the proof when Γ has a branch point. Then $K_Y + \Gamma$ has a Zariski-decomposition $P+N$. First assume that (Y, Γ) is NC-minimal. Then $N = Bk(\Gamma)$. Let C_1, C_2 and C_3 be the irreducible components of the maximal twigs T_1, T_2 and T_3 respectively meeting C_0, the Γ-component corresponding to the branch point. By lemma 6.16 of [F], the coefficients of C_i in $Bk(\Gamma)$ are $1/d(T_i)$. Hence $P = K_Y + C_0 + \sum_{i=1}^{3}(1 - \frac{1}{d(T_i)}) C_i + \ldots$. But then $P \cdot C_0 = -2 + \sum (1 - 1/d(T_i)) < 0$, contradicting the fact that P is nef.

If (Y, Γ) is not NC-minimal, by lemma 1 we can reduce to the case when there is a (-1)-curve E on Y which occurs in N, E is not contained in Γ and $E \cdot \Gamma = 1$, where E meets a component of $Bk(\Gamma)$. Then $\bar{\kappa}(Y-\Gamma) = \bar{\kappa}(Y - \Gamma \cup E)$. By contracting E and any (-1)-curves in the maximal twigs successively we reduce to the situation when either the image of Γ becomes linear or a maximal twig has a vertex with non-negative weight or the NC-minimal case occurs. If a maximal twig has a vertex with non-negative weight then by lemma 6.13 of [F], we get $\bar{\kappa}(Y-\Gamma) = -\infty$, a contradiction. This proves the result.

Let X be an NC-minimal $\mathbf{Q}$-homology plane with $\bar{\kappa}(X) = 0$. Then $\pi_1(X)$ is a finite cyclic group by Fujita.

Lemma 13. *Assume that X contains a contractible curve C. Then X is of type $H[k, -k]$, $k \geq 1$.*

Proof. As before, there is a $\mathbf{C}^*$ fibration ϕ: $X \to B$ with $\phi(C)$ a point and $B \cong \mathbf{C}$ or $\mathbf{P}^1$. We consider the three cases depending on the type of ϕ.

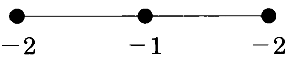
Case 1. ϕ is twisted.

Then $B \cong \mathbf{C}$ and all the fibres of ϕ are irreducible. We claim that ϕ has at most one multiple fibre. Let $p_1, \ldots, p_r$ be the points in B corresponding to the multiple fibres and $p_\infty = \mathbf{P}^1 - B$. If $r \geq 2$, then we can construct a suitable non-cyclic covering $A \to \mathbf{P}^1$, ramified over $p_1, \ldots, p_r, p_\infty$. Then we get a connected étale cover $\widetilde{X} \to X$ with non-cyclic galois group. This is not possible.

Hence $r \leq 1$.

As before, ϕ extends to a $\mathbf{P}^1$-fibration Φ: $Z \to \mathbf{P}^1$ on a smooth compatification Z of X. Let $D := Z - X$. As in lemma 8, we see that $\bar{\kappa}(X-C)=0$ if the morphism $X-C \to \mathbf{C}^*$ has no multiple fibre. Let F_0 be the fibre of Φ containing C.

Using the lemma 12, we now see that the dual graph of D has at least one branch point. But the fibre F_∞ has the form



by lemma 8(1). Hence by lemma 12 again D has at least two branch points and D is obtained from an NC-minimal divisor of the form $H[k, -k]$ for $k \geq 1$.

If the morphism $X-C \to \mathbf{C}^*$ has a multiple fibre with multiplicity $m>1$ and $F_0 \neq C$ then the divisor D is *m.n.c* and the 2-section D_h meets at least four other curves in D. This contradicts Fujita's classification. Hence either the morphism $X-C \to \mathbf{C}^*$ has no multiple fibre or $\bar{C}=F_0$. In the later case, $X-C \to \mathbf{C}^*$ has one multiple fibre by lemma 12 and $\bar{\kappa}(X-C)=1$. Further, D_h is a branch point of D.

Case 2. ϕ is untwisted and $B \cong \mathbf{C}$.

We claim that this case does not occur. First we observe that the fibre F_∞ is a regular fibre of Φ and the two horizontal components meet F_∞ in two distinct points. It is easy to see that D cannot be obtained from any of the surfaces Fujita has described by a finite succession of blowing-ups.

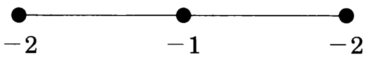
Case 3. ϕ is untwisted and $B \cong \mathbf{P}^1$

The fibration ϕ has at most two multiple fibres by lemma 8. The curve $F_0 - \bar{C}$ is connected. The morphism ϕ': $X-C \to \mathbf{C}$ has at least one multiple fibre by lemma 8(3). If ϕ' has only one multiple fibre, then $X-C$ contains $\mathbf{C}^* \times \mathbf{C}^*$ as a Zariski open subset and hence $\bar{\kappa}(X-C)=0$. Suppose ϕ' has two multiple fibres. Then D is m.n.c. and we see that the horizontal D-components D_1 and D_2 intersect in a point on $\bar{C}$. This shows that X is of type $H[k, -k]$. Further, the multiple fibres have multiplicity 2 each (otherwise D cannot be of type $H[-1, 0, -1]$) and $\bar{\kappa}(X-C)=0$, as in the proof of lemma 8(3).

Next we prove the following.

Lemma 14. *Let X be of type $H[k, -k]$* and X contains a contractible curve L with $\bar{\kappa}(X-L)=1$. *Then $k=1$.*

Proof. From the proof of lemma 10, we know that there is a twisted $\mathbf{C}^*$-fibration ϕ: $X \to \mathbf{C}$ with $\phi(L)$ a point. Further, ϕ' has exactly one multiple fibre, where ϕ': $X-L \to \mathbf{C}^*$ is the restriction. The horizontal component D_h is a branch point for D and the fibre F_∞ has the dual graph,



$\bar{L}$ is a reduced fibre of ϕ by the proof of case 1 of lemma 13. Using lemma 6 repeatedly we see that $\bar{L}$ can be assumed to be the full fibre of ϕ. From Fujita's description of D, we see that $k=1$ because the branch points intersect and one of them is a (-1)-curve.

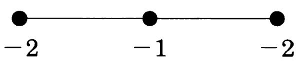
To complete the proof of the theorem, it remains to prove the following result.

Lemma 15. (*1*) *On the surface X of type $H[k, -k]$, there is a unique contractible curve C with $\bar{\kappa}(X-C)=0$.*

(*2*) *On $H[1, -1]$ there is a unique contractible curve L with $\bar{\kappa}(X-L)=1$.*

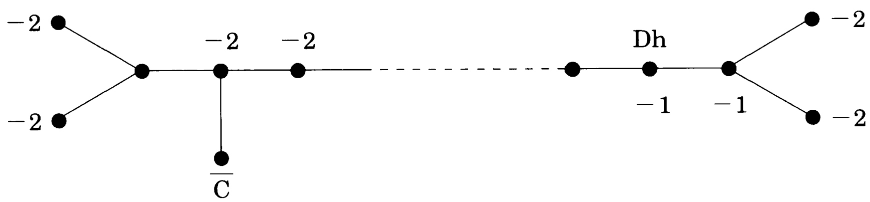
(*3*) *If $k=1$ and C and L are the contractible curves as above then $C \cdot L=2$ and they meet transversally.*

Proof. (1) Let C be a contractible curve on X with $\bar{\kappa}(X-C)=0$. There is a $\mathbf{C}^*$-fibration ϕ: $X \to \mathbf{C}$ such that for some $m \geq 1$, mC is a fibre of ϕ. Then ϕ is a twisted fibration. Let $X \subset Z$ be a smooth projective compactification such that ϕ extends to a $\mathbf{P}^1$-fibration Φ: $Z \to \mathbf{P}^1$. By lemma 8(1) there is no multiple fibre for the map $X \to C \to \mathbf{C}^*$. The fibre F_∞ has the dual graph,



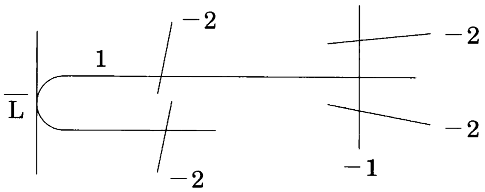
and D_h meets the (-1)-curve in F_∞. Let F_0 be the fibre of ϕ containing $\bar{C}$ and D_0 be the D-component of F_0 that meets D_h. We claim that D_0 meets only one other D-component in F_0. If not, D_0 is a branch point of D and from Fujita's classification, we deduce that D_h is a (-1)-curve and after contracting D_h, we get an NC-minimal completion of X. But this is not of type $H[k, -k]$ with $k \geq 1$. Hence we may even assume that D_0 is not a (-1)-curve.

As before, we may assume that $\bar{C}$ is the only (-1)-curve in F_0. Since an NC-minimal completion of X is obtained from contracting suitable (-1)-curves in D, we conclude that D_h is a (-1)-curve. Then D_0 is a (-2)-curve. By repeating this argument, we infer that the dual graph of $\bar{C} \cup D$ is



By successive contractions of (-1)-curves starting with D_k, we get an m.n.c. compactification divisor of X such that the dual graph of the image of $\bar{C}\cup D$ looks like $H[k, -k]$, with the image of $\bar{C}$ passing through the intersection of the two branching curves. From this it is easy to see that the curve C is unique.

(2) Let L be a contractible curve on X with $\bar{\kappa}(X-L)=1$. By the proof of case 1 of lemma 13 and lemma 14, we can assume that $\bar{L}\cup D$ looks like



Clearly, $\bar{L}$ is a full fibre of the $\mathbf{P}^1$-fibration on Z given by the linear system $|T_2+2B_2+T_4|$. Therefore L is unique.

(3) We have seen that $\bar{C}$ passes through the intersection of B_1 and B_2 and meets transversally with both. Hence $\bar{C}\cdot\bar{L}=2$. Now by lemma 10, $C\cap L$ consists of 2 distinct points as $\bar{L}$ does not pass through $B_1\cap B_2$. This completes the proof of the theorem.

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