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Nondiscrete local ramified class field theory

By

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1. Introduction

Let p be a prime, Q_p the field of p-adic numbers, Ω an algebraic closure of Q_p and $\bar{\Omega}$ the (topologic) completion of Ω. Suppose k is an infinite algebraic extension of Q_p with finite residue field and such that the exponent of p in the Steinitz number $[k: Q_p]$ is finite, and $\bar{k}$ its (topological) completion. We study the finite abelian totally ramified extensions of k and $\bar{k}$, in terms of subgroups of norms of $U(k)$ and $U(\bar{k})$ respectively. More precisely, if ℓ is a finite abelian extension of k and $\bar{\ell}$ its completion, then one has the following commutative diagram

$$\begin{array}{ccc} U(\bar{k})/\bar{H} & \xrightarrow{\delta_{\bar{\ell}/\bar{k}}} & \mathrm{Gal}\,(\bar{\ell}/\bar{k})_{\mathrm{ram}} \\ \phi\uparrow & & \uparrow\mathrm{Res}^{-1} \\ U(k)/H & \xrightarrow{\delta_{\ell/k}} & \mathrm{Gal}\,(\ell/k)_{\mathrm{ram}} \end{array}$$

where all the arrows are functorial isomorphisms, and H and $\bar{H}$ are the subgroups of norms of units from ℓ and $\bar{\ell}$ respectively. Moreover, one has a continuous group homomorphism

$$\widehat{U(k)} \xrightarrow{\delta_k} \mathrm{Gal}\,(k_{\mathrm{ab}}/k)_{\mathrm{ram}}$$

(where $\widehat{U(k)}$ is the completion of $U(k)$ with respect to the subgroups of finite index), which is surjective and whose kernel is the subgroup of roots of unity in $U(k)$ of order $q_1 = (q-1, [k: Q_p]_\infty)$.

Throughout the paper use ideas and results of Hazewinkel's ([3]) and Iwasawa's ([5]).

As a remark, here we describe the finite abelian extensions (totally ramified) of an infinite totally ramified extension of a local field with only finite wild ramification, while J.M. Fontaine and J.P. Wintenberger do it for totally ramified extensions of a local field with only finite tame ramification ([2]). Our next goal is to put these two together.

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2. Notations

In what follows p will be a prime number, Q_p the field of p-adic numbers, Ω an algebraic closure of Q_p and $\bar{\Omega}$ the completion of Ω with respect to the unique extension of the p-adic valuation. The valuation on $\bar{\Omega}$ (normalised such that $v(p) = 1$) will be denoted by v. We shall use also the notations: $\bar{M}$ for the (topologic) completion of any subset $M \subset \bar{\Omega}$ and k_v for the residual field of any subfield k of $\bar{\Omega}$. $\mathscr{F}(\Omega/Q_p)$ will denote the set of fields k, $Q_p \subseteq k \subseteq \Omega$, and $\mathscr{F}_c(\bar{\Omega}/Q_p)$ will denote the set of fields K, $Q_p \subseteq K \subseteq \bar{\Omega}$ such that K is complete. If $k \in \mathscr{F}(\Omega/Q_p)$, and ℓ/k is a finite Galois extension then $\mathrm{Gal}(\ell/k)_{\mathrm{ram}} = \mathrm{Gal}(\ell/k_0)$ where k_0 is the maximal unramified subextension of ℓ/k.

Let $Q_p \subseteq k \subseteq \Omega$ such that k_v is finite or is algebraicly closed. Choose a sequence of fields $k_1 \subseteq k_2 \subseteq \cdots \subseteq k$ such that $\bigcup_{i \geq 1} k_i = k$ and all the k_i are finite extensions of Q_p if k_v is finite, and respectively of $(Q_p)_{\mathrm{ur}}$ (the maximal unramified extension of Q_p in Ω) if k_v is algebraicly closed. If ℓ/k is finite and Galois, and $\ell = k(\alpha)$, let $\ell_i = k_i(\alpha)$. Then there exists an $n_0 \in N$ such that ℓ_n/k_n is Galois and $\mathrm{Gal}(\ell_n/k_n) \simeq \mathrm{Gal}(\ell/k)$ for any $n \geq n_0$. Moreover $\ell = \bigcup_{i \geq 1} \ell_i$.

Any field $Q_p \subseteq k \subseteq \Omega$ defines a Steinitz number $[k: Q_p]$ which contains prime factors with finite or infinite exponents. The product of the factors with exponent ∞ will be denoted by $[k: Q_p]_\infty$.

We define similarly $[k: (Q_p)_{\mathrm{ur}}]_\infty$ if $(Q_p)_{\mathrm{ur}} \subseteq k \subseteq \Omega$.

3. Subfields in $\bar{\Omega}$

There is a canonical one-to-one correspondence between $\mathscr{F}(\Omega/Q_p)$ and $\mathscr{F}(\bar{\Omega}/Q_p)$, which is a consequence of the formula giving the distance between conjugates, as given in [1]. We summarize in the following theorem some results regarding it which are used later. For a detailed proof, see [4]

Theorem 3.1. (*a*) *The maps defined by* $\mathscr{F}(\Omega/Q_p) \ni k \mapsto \bar{k} \in \mathscr{F}_c(\bar{\Omega}/Q_p)$ *and* $\mathscr{F}_c(\bar{\Omega}/Q_p) \ni K \mapsto K \cap \Omega \in \mathscr{F}(\Omega/Q_p)$ *are one-to-one and inverse one to the other.*

(*b*) *Let* $k, \ell \in \mathscr{F}(\Omega/Q_p)$ *such that* ℓ/k *is finite and Galois. Then* $\bar{\ell}/\bar{k}$ *is finite and Galois and one has* $\mathrm{Gal}(\ell/k) \simeq \mathrm{Gal}(\bar{\ell}/\bar{k})$, *the isomorphism being the canonical one.*

(*c*) *Let* $K, L \in \mathscr{F}_c(\bar{\Omega}/Q_p)$ *such that* L/K *is finite and Galois. Denote*: $k = K \cap \Omega$ *and* $\ell = L \cap \Omega$. *Then* ℓ/k *is finite and Galois and one has* $\mathrm{Gal}(\ell/k) \simeq \mathrm{Gal}(L/K)$, *the isomorphism being the canonical one.*

The following two theorems can be deduced from general valuation theory, but we give here elementary proofs, for the sake of completness.

Theorem 3.2. *Let* $Q_p \subseteq k \subset \Omega$ *such that* k_v *is finite and* $p \nmid [k: Q_p]_\infty$.

Then any cyclic extension ℓ/k *of prime degree* $q/[k: Q_p]_\infty$ *is inertial* (*i.e.* $[\ell_v: k_v] = q$).

Proof. Let $|k_v| = p^h$.

(a) Suppose that $q \nmid p^h - 1$. Let $Q_p \subseteq k_1 \subseteq k_2 \subseteq \cdots \subseteq k$ be a sequence of finite extensions of Q_p such that $\bigcup_{i \geq 1} k_i = k$. Let $\ell = k(\alpha)$ and $\ell_i = k_i(\alpha)$. Choose an n_0 such that: $[\ell_{n_0}: k_{n_0}] = q$, $(k_{n_0})_v = k_v$ and $m = [k_{n_0+1}: k_{n_0}]$ be divisible by q but not by p.

We may suppose that ℓ_{n_0}/k_{n_0} is totally ramified (if not, then $[(\ell_{n_0})_v: (k_{n_0})_v] = q$, hence $[\ell_v: k_v] = q$). Since k_{n_0+1}/k_{n_0} is also totally ramified, one may choose α and β such that $k_{n_0+1} = k_{n_0}(\beta)$, $\ell_{n_0+1}/k_{n_0}(\alpha)$ and α, β are roots of two polynomials of the form $f = x^q - \pi$ and respectively $g = x^m - \pi$, and π' being uniformizing elements of k_{n_0}.

Let $u = \dfrac{\pi}{\pi'} \in U(k_{n_0})$ and denote by $\bar{u}$ the image of u in k_v. Since $q \nmid p^h - 1$, $X^q - \bar{u}$ has a root in k_v, hence $X^q - u$ has a root in k_{n_0}.

It follows: $\ell_{n_0} = k_{n_0}(\beta^{m/q}) \subseteq k_{n_0+1}$ which is impossible.

(b) Suppose that $q/p^h - 1$. Let, as above, $k_{n_0+1} = k_{n_0}(\beta)$, $l_{n_0}(\alpha) = k_{n_0}(\alpha)$, and $u = \dfrac{\pi}{\pi'} \in U(k_{n_0})$. We may suppose that $u \notin [U(k_{n_0})]^q$.

Let $v = \dfrac{\alpha}{\beta^{m/2}} \in \ell$. One has $v^q = \dfrac{\alpha^q}{\beta^m} = \dfrac{\pi}{\pi'} = u$, hence the image $\bar{v}$ of v in ℓ_v does not lie in k_v. It follows $[(k_{n_0+1} \cdot \ell_{n_0})_v: k_v] = q$ thus $[\ell_v: k_v] = q$.

Theorem 3.3. *Let $(Q_p)_{\mathrm{nr}} \subseteq k \subseteq \Omega$ such that $p \nmid [k: (Q_p)_{\mathrm{nr}}]_\infty$. Then the degree of any finite extension of k is relatively prime with $[k: (Q_p)_{\mathrm{nr}}]_\infty$.*

The proof in the case of cyclic extensions of prime degree is analogous to that of Theorem 3.2(a). The general case reduces immediately to the Galois case, which reduces to the prime cyclic case by the resolubility of the Galois group.

4. The fundamental exact sequence

Let $K, L \subseteq \Omega$ with algebraic closed residual fields.

Theorem 4.1. *Let $\bar{L}/\bar{K}$ be finite and Galois. Then*

$$N_{\bar{L}/\bar{K}}(U(\bar{L})) = U(\bar{K})$$

Proof. For the proof let us first note the following:

Lemma 4.1 ([5], Cap. 2, Lemma 2 and Theorem 1). *Let ℓ/k be a finite Galois extension, where $k \subseteq \ell \subseteq \bar{\Omega}$ are complete, discrete, with algebraicly closed residual fields. Let π', π uniformizing elements of ℓ and k respectively. Then there exists $s \in \mathbf{N}$ such that for any $k \geq s$ and any $u \in k$ with $u \equiv 1 \pmod{\pi^k}$ there exists an $\zeta \in \ell$, $\zeta \equiv 1 \pmod{\pi'^k}$ such that $N_{\ell/k}(\zeta) = u$. From the proof given there it follows that for ℓ/k cyclic of prime degree we may take*

$$s = \frac{v(\pi' - \sigma(\pi'))}{v(\pi')}$$

where σ is a generator of $\mathrm{Gal}(\ell/k)$; *and if*

$$k \subseteq k_1 \subseteq k_2 \subseteq \cdots \subseteq k_n = \ell$$

where k_{i+1}/k_i is cyclic of prime degree for any i, and if s_i is defined as above, then we may take $s = \max_{1 \le i \le n} \{s_i\}$.

Now let K, L satisfying the above hypothesis, let

$$\left.\begin{array}{l}(Q_p)_{\mathrm{nr}} \subseteq k_1 \subseteq k_2 \subseteq \cdots \subseteq K \\ (Q_p)_{\mathrm{nr}} \subseteq \ell_1 \subseteq \ell_2 \cdots \subseteq L\end{array}\right\} \quad \text{as in §2,}$$

and let π_n, π_n' be uniformizing elements of k_n and ℓ_n respectively.

Let s_0 be as in Lemma 4.1. We shall prove that there exists $n_0 \in \mathbf{N}$ and $M \in \mathbf{R}$ such that:

$$s_n v(\pi_n) \le M \qquad \text{for } n \ge n_0$$

Clearly we may reduce to the case when $[L: K] = q$ is a prime. Let n_0 be such that $[\ell_n : k_n] = q$ for $n \ge n_0$, and let $i \ge n_0$. Then $s_i v(\pi_i') = v(\pi_i' - \sigma(\pi_i'))$ where $\langle \sigma \rangle = \mathrm{Gal}(\ell_i/k_i) = \mathrm{Gal}(L/K)$. Let $f(x) = x^q + a_1 x^{q-1} + \cdots a_q$ be the minimal polynomial of π_i' over k_i and let $\pi_{i1}' = \pi_i'$, $\pi_{i2}' \cdots \pi_{iq}'$ be the roots of f. One has:

$$f'(\pi_i') = (\pi_i' - \pi_{i2}') \cdots (\pi_i' \pi_{iq}') = q\pi_i^{q-1} + \cdots + a_{q-1}.$$

It follows:

$$\begin{aligned} v(\pi_i' - \sigma(\pi_i')) &\le \sum_{j=2}^{q} v(\pi_i' - \pi_{ij}') = v(f'(\pi_i')) \\ &= \min\{v(q\pi_i'^{q-1}), \ldots, v(a_{q-1})\} \le v(q\pi_i'^{q-1}) \end{aligned}$$

hence: $s_i v(\pi_i) \le q \cdot [v(q) + (q-1)]$.

Now let $u \in U(\bar{K})$. There exist $a_{n_0}, a_{n_0+1}, \ldots,$ such that

$$\begin{cases} a_n \in k_n & \text{for any } n \\ \prod_{n=n_0}^{\infty} a_n = u. \end{cases}$$

Since $\lim_{n\to\infty} v(a_n - 1) = \infty$ there exists $m_0 \in \mathbf{N}$ such that

$$v(a_n - 1) \ge s_n v(\pi_n) \qquad \text{for } n > m_0$$

From the lemma, there exists $b_n \in \bar{l}_n$ such that $N_{\bar{l}_n/\bar{k}_n}(b_n) = a_n$ and

$$v(b_n - 1) = \frac{v(a_n - 1)}{[L: K]} \qquad \text{for } n > m_0.$$

From the discrete case of Theorem 4.1 which is proved in ([5], Cap. 2, §2.1, Theorem 1) it follows the existence of an $b_{m_0} \in \bar{l}_{m_0}$ such that

$$N_{\bar{l}_{m_0}/\bar{k}_{m_0}}(b_{m_0}) = a_{n_0} \cdot a_{n_0+1} \cdots a_{m_0}.$$

The product $\prod_{n \ge m_0} b_n$ converges in $\bar{L}$ and its limit b satisfies

$$N_{\bar{L}/\bar{K}}(b) = u.$$

K and L being as above, we denote by $V(\bar{L}/\bar{K})$ the subgroup of $U(\bar{L})$ generated by $\left\{\zeta^{\sigma-1} = \dfrac{\sigma(\zeta)}{\zeta}/\zeta \in U(L),\ \sigma \in \mathrm{Gal}\,(\bar{L}/\bar{K})\right\}$.

One has: $N_{\bar{L}/\bar{K}}(V(\bar{L}/\bar{K})) = 1$. Let us suppose that $[K:(Q_p)_{\mathrm{nr}}]_\infty$ is not divisible by p. Theorem 3.4 implies then that $[L: K]$ and $[K:(Q_p)_{\mathrm{nr}}]_\infty$ are relatively prime, hence we may fix an n_0 such that $[k_{i+1}: k_i]$ and $[L: K]$ are relatively prime and $[\ell_i: k_i] = [L: K]$ for any $i \geq n_0$.

For $n \geq n_0$ and $\sigma \in \mathrm{Gal}\,(L/K)$ we define:

$$i(\sigma) = (\pi_n'^{[k_n:k_{n_0}]})^{\sigma-1} \pmod{V(\bar{L}/\bar{K})}$$

where π_n' denotes an uniformizing element of ℓ_n.

It is easy to see that $i(\sigma)$ does not depend on the choice of n and π_n', and that "i" is a homomorphism of groups. Then one has the following sequence of groups:

$$1 \longrightarrow \mathrm{Gal}\,(\bar{L}/\bar{K}) \xrightarrow{\ i\ } U(\bar{L})/V(\bar{L}/\bar{K}) \xrightarrow{N_{L/K}} U(\bar{K}) \longrightarrow 1 \tag{4.1}$$

We shall prove in this section that this is an exact sequence.

Clearly the homomorphism $N_{\bar{L}/\bar{K}} \circ i$ is null and $N_{\bar{L}/\bar{K}}$ is onto (Theorem 4.1).

Proposition 4.1. *If K, L are as above, L/K is abelian and $[K:(Q_p)_{\mathrm{nr}}]_\infty$ is not divisible by p then "i" is a monomorphism.*

Proof. (a) Suppose firstly that $\bar{L}/\bar{K}$ is cyclic and let ρ be a generator of the Galois group. If $a \in \mathbf{Z}$ is such that $i(\rho^a) = 1$ then there exists $\zeta \in U(L)$ such that

$$(\pi_{n_0}')^{\rho^a-1} = \frac{\zeta^\rho}{\zeta}$$

and we get $\rho(\pi_{n_0}' \cdot \zeta^{-1}) = \pi_{n_0}'^a \cdot \zeta^{-1}$. This implies that

$$\alpha = \pi_{n_0}'^a \cdot \zeta^{-1} \in \bar{K}.$$

Thus $a = \dfrac{v(\alpha)}{v(\pi_{n_0}')} = [\bar{L}: \bar{K}] \cdot \dfrac{v(\alpha)}{v(\pi_{n_0})}$ is divisible by $[\bar{L}: \bar{K}]$, hence $\rho^a = 1$ and "i" is a monomorphism.

(b) If $G = \mathrm{Gal}\,(\bar{L}/\bar{K})$ is not cyclic, and if $\sigma \in G$, $\sigma \neq 1_{\bar{L}}$, then there exists a subgroup H of G such that $\sigma \notin H$ and G/H is cyclic. Let $M = L^H = \{x \in L/\tau(x) = x,\ \forall \tau \in H\}$. Then $\bar{M} = \bar{L}^H$ and $[\bar{M}: \bar{K}]$ is relatively prime with $[k_{i+1}: k_i]$ for any $i \geq n_0$, where n_0 is defined as above. We have $\mathrm{Gal}\,(\bar{M}/\bar{K}) = G/H$. Let $\sigma' = \sigma/\bar{M} \neq 1$. Since $i: \mathrm{Gal}\,(\bar{M}/\bar{K}) \to U(\bar{M})/V(\bar{M}/\bar{K})$ is a monomorphism $i(\sigma') \neq 1$. Then:

$$N_{\bar{L}/\bar{M}}(i(\sigma)) = N_{L/M}(\pi_{n_0}'^{\sigma-1}) = N_{L/M}(\pi_{n_0}')^{\sigma'-1} = i(\sigma') \neq 1_M.$$

Hence $i(\sigma) \neq \ell_{\bar{L}}$ and "i" is a monomorphism.

Proposition 4.2. $\operatorname{Ker}(N_{\bar{L}/\bar{K}}) \subseteq \operatorname{Im}(i)$.

Proof. (a) Suppose that $\bar{L}/\bar{K}$ is cyclic and let σ be a generator of the Galois group. If $x \in U(\bar{L})$ satisfies $N_{\bar{L}/\bar{K}}(x) = 1$, then there exists $a \in \bar{L}$ such that $x = a^{\sigma - 1}$. Let $a_1, a_2, \dots, a_n, \dots \in L$ such that $a_n \in \ell_n$ for any n and $\lim_{n \to \infty} a_n = a$. Let $n_0 \in \mathbf{N}$ be such that $[\bar{L}\colon \bar{K}]$ is relative prime with $[k_{i+1} : k_i]$ for any $i \geq n_0$.

There exists $m_0 \geq n_0$ such that $v(a_n) = v(a_{m_0})$ for any $n \geq m_0$. Hence $v(a) = v(a_{m_0})$. Let π'_{m_0} be an uniformizing element of ℓ_{m_0} and let $k \in \mathbf{Z}$ be such that $a\pi'^{k}_{m_0} \in U(\bar{L})$.

Then, since $([\bar{L}\colon \bar{K}], [k_{m_0} : k_{n_0}]) = 1$, there exists $k' \in \mathbf{N}$ such that

$$x \equiv ((\pi'_{m_0})^{[k_{m_0} : k_{n_0}]})^{\sigma k' - 1} (\operatorname{mod} V(\bar{L}/\bar{K})), \text{ hence } x = i(\sigma^{k'}).$$

(b) Let $\bar{L}/\bar{K}$ be abelian, of degree $n = [\bar{L}\colon \bar{K}]$. We shall proceed by induction on n. Let as above $n_0 \in \mathbf{N}$ such that $(n, [k_{i+1} : k_i]) = \ell$ for $i \geq n_0$.

Let $K \subseteq M \subseteq L$ such that M/K be cyclic. Let $\zeta \in U(\bar{L})$ such that $N_{\bar{L}/\bar{K}}(\zeta) = 1$ and denote: $\zeta' = N_{\bar{L}/\bar{M}}(\zeta)$. Then $N_{\bar{M}/\bar{K}}(\zeta') = 1$, hence $\zeta' \equiv (\pi''_{n_0})^{\sigma' - 1} \bmod V(\bar{M}/\bar{K})$, where π''_{n_0} is an uniformizing element of m_{n_0}, $(Q_p)_{\mathrm{nr}} \subseteq m_1 \subseteq m_2 \subseteq \cdots \subseteq M$ being a sequence of discrete valued field as in §2 (one may choose π''_{n_0} such that $\pi''_{n_0} = N_{L/M}(\pi'_{n_0})$, π'_{n_0} being an uniformizing element of ℓ_{n_0}).

Denoting $t = \zeta'^{o}[(\pi''_{n_0})^{\sigma' - 1}]^{-1}$, there exists $\eta \in V(\bar{L}/\bar{K})$ such that $t = N_{\bar{L}/\bar{M}}(\eta)$.

One has:

$$N_{\bar{L}\bar{K}}(\zeta) = \zeta' = [N_{\bar{L}/\bar{M}}(\pi'_{n_0})]^{\sigma' - 1} \cdot N_{\bar{L}/\bar{M}}(\eta) = N_{\bar{L}/\bar{M}}((\pi'_{n_0})^{\sigma - 1} \cdot \eta)$$

where $\sigma|_M = \sigma'$. Let $\lambda = \pi'^{\sigma - 1}_{n_0} \cdot \eta \cdot \zeta^{-1}$. Since $N_{\bar{L}/\bar{M}}(\lambda) = 1$, from the inductive hypothesis there exists $\tau \in \operatorname{Gal}(\bar{L}/\bar{M})$ such that

$$\lambda \equiv \pi'^{\tau - 1}_{n_0} (\operatorname{mod} V(\bar{L}/\bar{M}))$$

Then

$$\zeta \equiv \pi'^{\sigma - 1}_{n_0} \cdot \pi'^{\tau^{-1} - 1}_{n_0} (\operatorname{mod} V(\bar{L}/\bar{K})) \equiv \pi'^{\sigma \cdot \tau^{-1} - 1}_{n_0} (\operatorname{mod} V(\bar{L}/\bar{K})),$$

and $\zeta = i(\sigma\tau^{-1}) \in \operatorname{Im}(i)$.

We have obtained the following:

Theorem 4.2. *If $p \dagger [K : (Q_p)_{\mathrm{nr}}]_\infty$ then the sequence* (1) *is exact.*

Proposition 4.3. *Let $(Q_p)_{\mathrm{nr}} \subseteq K \subseteq L \subseteq \Omega$ such that L/K is abelian and $p/[K : (Q_p)_{\mathrm{nr}}]_\infty$. Then:*

(*a*) *If $p \dagger [\bar{L}\colon \bar{K}]$ then "i" may be defined as above and the sequence* (1) *is exact.*

(*b*) *If $[L\colon K] = p^t$, $t \in \mathbf{N}$, then $N_{\bar{L}/\bar{K}}\colon U(\bar{L})/V(\bar{L}/\bar{K}) \to U(\bar{K})$ is an isomorphism.*

Proof. The proof of (a) is analogous to that of Theorem 4.2. In order to prove (b), one may reduce to the case $\operatorname{Gal}(\bar{L}/\bar{K})$ cyclic. Let σ be a generator of it. If $x \in U(\bar{L})$ satisfies $N_{\bar{L}/\bar{K}}(x) = \ell$ then there exists $a \in \bar{L}$ such that

$x = a^{\sigma - 1}$. Let $a_n \in \ell_n$ for $n \geq \ell$, such that $\lim_{n \to \infty} a_n = a$ and let $n_0 \in \mathbf{N}$ such that $v(a_n) = v(a_n) = v(a_{n_0})$ for any $n \geq n_0$. Then there exists $k \in \mathbf{N}$ such that $x \equiv \pi_{n_0}'^{\sigma - 1} (\mathrm{mod}\ V(\bar{L}/\bar{K}))$. Since $p/[K:(Q_p)_{\mathrm{nr}}]_\infty$ there exists $m > n_0$ such that $p^t/[k_m : k_{n_0}]$. Then

$$x \equiv (\pi_m^{[k_m : k_{n_0}]})^{\sigma - 1} (\mathrm{mod}\ V(\bar{L}/\bar{K})) \equiv \pi_m'^{\sigma[k_m : k_{n_0}] - 1} (\mathrm{mod}\ V(\bar{L}/\bar{K})) \equiv \ell (\mathrm{mod}\ V(\bar{L}/\bar{K}))$$

Remark. If $(Q_p)_{\mathrm{nr}} \subseteq K \subseteq \Omega$, then $\bar{K}$ may have a finite immediate extension Σ, $\bar{K} \subsetneq \Sigma \subseteq \bar{\Omega}$ only if $p/[K:(Q_p)_{\mathrm{nr}}]_\infty$ and $[\Sigma : \bar{K}] = p^t$, $t \in \mathbf{N}^*$.

For a proof one may apply Theorem 3.3.

5. The maximal unramified extension

In this section we consider a field $Q_p \subseteq k \subseteq \Omega$ with finite residual field k_v such that $p \nmid [k : Q_p]_\infty$ and we shall study the maximal unramified extension k_{nr} of k.

Proposition 5.1. *Let k be as above and let k_1/k_v be finite, of degree n. Then there exists a unique extension $k \subseteq \ell \subseteq \Omega$ such that:*

(1) *the residual field of ℓ is k_1,*

(2) $[l : k] = n$.

It follows that ℓ/k is Galois and cyclic.

The proof follows as is the case: k/Q_p finite.

Let $k^{(n)}$ be the unique extension of k given by Proposition 5.1, and let $k_{\mathrm{nr}} = \bigcup_{n \in \mathbf{N}^*} k^{(n)}$.

The extension k_{nr}/k is abelian and one has $k_{\mathrm{nr}} = k(V_\infty)$, where V_∞ denotes the set of all roots of unity of order $q^n - 1$, $n \in \mathbf{N}^*$ and $q = |k_v|$

Proposition 5.2. *Let $K = k_{\mathrm{nr}}$. Then, the residual field K_v of K is the algebraic closure of k_v and one has a canonic topologic isomorphism:*

$$\mathrm{Gal}\,(K/k) \simeq \mathrm{Gal}\,(K_v/k_v).$$

Again, the proof is like in the case: k/Q_p finite.

Now we consider the following automorphism of K_v over k_v:

$$\omega \mapsto \omega^q, \qquad \text{for } \omega \in K_v.$$

This corresponds, by the isomorphism of Proposition 5.2, to an automorphism ψ of K/k, called the Frobenius automorphism of K/k.

The prolongation by continuity of ϕ to $\bar{K}$ will be denoted also by ϕ.

One has the sequence:

$$1 \longrightarrow U(\bar{k}) \xrightarrow{j} U(\bar{K}) \xrightarrow{\phi - 1} U(\bar{K}) \longrightarrow 1 \tag{5.1}$$

where j is the inclusion and $(\phi - 1)(\zeta) = \dfrac{\phi(\zeta)}{\zeta}$ for any $\zeta \in U(\bar{K})$.

Theorem 5.1. *The sequence* (5.1) *is exact.*

Proof. We note firstly that $\operatorname{Im} j \subseteq \ker(\phi - 1)$.

Let $Q_p \subseteq k_1 \subseteq k_2 \subseteq \cdots \subseteq k$ be a sequence of finite extensions of Q_p such that $k = \bigcup_i k_i$. Let $K_i = (k_i)_{\mathrm{nr}}$ for any i. Then:

$$(Q_p)_{\mathrm{nr}} \subseteq K_1 \subseteq K_2 \subseteq \cdots K \text{ and } K = \bigcup_k K_i$$

(a) Let us prove that $\phi - 1$ is onto.

If $a \in U(\bar{K})$ then there exists $a_i \in U(K_i)$ for any $i \geq \ell$ such that $a = \prod_{i=1}^{\infty} a_i$. Since the sequence:

$$1 \longrightarrow U(k_i) \longrightarrow U(\bar{K}_i) \xrightarrow{\phi_i - 1} U(\bar{K}_i) \longrightarrow 1$$

is exact for any i ([5], §4.2 Theorem 2), there exists $\zeta_i \in U(\bar{K}_i)$ such that $\dfrac{\phi(\zeta_i)}{\zeta_i} = a_i$.

Denoting by π_i a uniformizing element of k_i (and thus also of K_i and $\bar{K}_i$) then since $\zeta_i \in \overline{k_i(V_\infty)}$ one has:

$$\zeta_i = \sum_{j=0}^{\infty} \alpha_{ij}\pi_i^j,\ \phi(\zeta_i) = \sum_{j=0}^{\infty} \alpha_{ij}^q \pi_i^j,\ \alpha_{i0} \neq 0,\ \alpha_{ij} \in V_\infty \cup \{0\}.$$

Now, if $n_i \in \mathbf{N} \cup \{\infty\}$ is the exponent of π_i in $(a_i - 1)$ then $\phi(\zeta_i) \equiv \zeta_i (\operatorname{mod} \pi_i^{n_i})$ and we derive: $\alpha_{ij}^q \alpha_{ij}$ and $\alpha_{ij} \in k_i$ for $j = 0, 1, \ldots, n_i - 1$.

Hence $\rho_i = \alpha_{i0} + \alpha_{i1}\pi_i + \cdots + \alpha_{in_i - 1}\pi_i^{n_i - 1} \in U(k_i) = \ker(\phi_i - 1)$ and denoting $\eta_i = \rho_i^{-1}\zeta_i$ one has:

$$\eta_i \equiv 1 (\operatorname{mod} \pi_i^{n_i}) \text{ and } (\phi - 1)\eta_i = (\phi - 1)\zeta_i = a_i \text{ for any } i \geq 1.$$

Then the product $\Pi_{i=1}\, \eta_i$ is convergent and $(\phi - 1)(\prod_{i=1}^{\infty} \eta_i) = \prod_{i=1}^{\infty} a_i = a$.

(b) Let us prove that $\ker(\phi - 1) \subseteq \operatorname{Im} j$.

If $x \in \ker(\phi - 1) \subseteq U(\bar{K})$ then there exists $b_i \in U(K_i)$ such that:

$$x = \lim_{i \to \infty} b_i$$

$\phi(x) = x$ implies $\lim_{i\to\infty} \phi(b_i) = \lim_{i\to\infty} b_i$, hence $\lim_{i\to\infty} v(\phi(b_i) - b_i) = \infty$.

Put, as above: $b_i = \sum_{j=0}^{\infty} \alpha_{ij}\pi_i^j$, $\alpha_{ij} \in V_\infty \cup \{0\}$, $\alpha_{i_0} \neq 0$.

From $\phi(b_i) - b_i = \sum_{i=0}^{\infty} (\alpha_{ij}^q - \alpha_{ij})\pi_i^j$.

We derive: $\alpha_{ij}^q = \alpha_{ij}$ and $\alpha_{ij} \in k_i$ for $j = 0, 1, \ldots t_i - 1$, where $t_i \in \mathbf{N} \cup \{\infty\}$ denote the exponent of π_i in $(\phi(b_i) - b_i)$.

Then, it we put $c_i = \alpha_{i_0} + \alpha_{i_1}\pi_i + \cdots + \alpha_{it_i - 1}\pi_i^{t_i - 1} \in k_i \subseteq k$, we have $v(b_i - c_i)\, i \to \infty$ ∞ hence $x = \lim_{i\to\infty} c_i \in U(\bar{k})$.

6. The fundamental isomorphism

Let k as in §5 and suppose for the moment that $p \dagger [k : Q_p]_\infty$.

Let E be a finite abelian extension of k, $K = k_{\mathrm{nr}}$, $k_0 = K \cap E$ the maximal unramified extension of k in E and let $L = KE = E_{\mathrm{nr}}$. Denote by ϕ_0 and ψ the Frobenius automorphisms of K/k_0 and L/E respectively. One has: $\psi/K = \phi_0$ and $(\psi - 1)V(\bar{L}/\bar{K}) = V(\bar{L}/\bar{K})$. Then the homomorphism $\psi - 1: U(\bar{L}) \to U(\bar{L})$ from §5 induces the (onto) homomorphism, denoted also by $\psi - 1$:

$$\psi - 1: U(\bar{L})/V(\bar{L}/\bar{K}) \to U(\bar{L})/V(\bar{L}/\bar{K})$$

One has the diagram:

$$\begin{array}{ccccccccc} & & & & A & & B & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Gal}(\bar{L}/\bar{K}) & \xrightarrow{i} & U(\bar{L})/V(\bar{L}/\bar{K}) & \xrightarrow{N_{\bar{L}/\bar{K}}} & U(\bar{K}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\ 1 & \longrightarrow & \mathrm{Gal}(\bar{L}/\bar{K}) & \xrightarrow{i} & U(\bar{L})/V(\bar{L}/\bar{K}) & \xrightarrow{N_{\bar{L}/\bar{K}}} & U(\bar{K}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & C & & D & & & & \end{array} \tag{6.1}$$

where γ is the null homomorphism, $\alpha = \psi - 1$, $\beta = \phi_0 - 1$, $A = \ker \alpha$, $B = \ker \beta$, $C = \operatorname{coker} \gamma$, $D = \operatorname{coker} \alpha$.

Also one sees that: $C = \mathrm{Gal}(\bar{L}/\bar{K})$, $D = 1$, $B = U(\bar{k})$ and $A = U(\bar{E}) \cdot V(\bar{L}/\bar{K})/V(\bar{L}/\bar{K})$.

The diagram (6.1) is commutative and has exact rows and columns, hence "$N_{\bar{L}/\bar{K}}$" and "i" define the homomorphisms $A \xrightarrow{N_{\bar{L}/\bar{K}}} B$ and $C \xrightarrow{i} D$ and the "snake lemma" gives a homomorphism $\delta: B \to C$ such that the sequence $A \xrightarrow{N_{\bar{L}/\bar{K}}} B \xrightarrow{\delta} C \xrightarrow{i} D$ is exact.

We get then an induced isomorphism:

$$\delta: U(\bar{k}_0)/N_{\bar{E}/\bar{k}_0}(U(\bar{E})) \to \mathrm{Gal}(\bar{L}/\bar{K}) \simeq \mathrm{Gal}(\bar{E}/\bar{k}_0) \simeq \mathrm{Gal}(\bar{E}/\bar{k})_{\mathrm{ram}}$$

Theorem 6.1. *If $p \dagger [k: Q_p]_\infty$ and E is a finite abelian extension of k, then one has an isomorphism*

$$\delta_{E/k}: U(\bar{k})/N_{\bar{E}/\bar{k}}(U(\bar{E})) \to \mathrm{Gal}(\bar{E}/\bar{k})_{\mathrm{ram}}$$

(*b*) *If $p/[k: Q_p]_\infty$ and E is a finite abelian extension of k such that $p \dagger [\bar{E}: \bar{k}]$ then one has an isomorphism*

$$\delta_{E/k}: U(\bar{k})/N_{\bar{E}/\bar{k}}(U(\bar{E})) \to \mathrm{Gal}(\bar{E}/\bar{k})_{\mathrm{ram}}$$

(*c*) *If $p/[k: Q_p]_\infty$ and E is a finite abelian extension of k such that $[\bar{E}: \bar{k}] = p^t$ then*

$$N_{\bar{E}/\bar{k}}(U(\bar{E})) = U(\bar{k})$$

For the proof we need the following result:

Lemma 6.1. *If k is as above and k' is a finite unramified extension of k then $N_{\bar{k}'/\bar{k}}(U(\bar{k}')) = U(\bar{k})$.*

Proof. For a proof of Lemma 6.1 one may use Lemma 4 §3.3 of [5] and the technique used in this paper.

Now for (a) let $K = k_{\mathrm{nr}}$, $L = E_{\mathrm{nr}}$, $\psi =$ the Frobenius automorphism of K/k, $\phi =$ a prolongation of ϕ to L and $\ell = \{x \in E/\psi(x) = x\}$. Then $\ell_{\mathrm{nr}} = \ell \cdot K = L$, $\ell \cap K = k$, ℓ/k is totally ramified, E/ℓ is unramified, $\bar{\ell}/\bar{k}$ is totally ramified, $\bar{E}/\bar{\ell}$ is unramified and one has:

$$N_{\bar{E}/\bar{\ell}}(U(\bar{E})) = U(\bar{\ell}) \quad \text{and} \quad U(\bar{k})/N_{\bar{\ell}/\bar{k}}(U(\bar{\ell})) \simeq \mathrm{Gal}\,(\bar{L}/\bar{K})$$

But $N_{\bar{\ell}/\bar{k}}(U(\bar{\ell})) = N_{\bar{E}/\bar{k}}(U(\bar{E}))$ hence $U(\bar{k})/N_{\bar{E}/\bar{k}}(U(\bar{E})) \simeq \mathrm{Gal}\,(\bar{L}/\bar{K})$

which proves (a).

A proof of (b) comes in a similar way, by reproducing Lemma 6.1, the diagram [5] and all §5 in the hypothesis stated in (b).

As for (c), let K and L be the maximal inertial extensions of k and E respectively, denote $k_0 = E \cap K$ and let ψ and ϕ_0 be the Frobenius automorphisms of L/E and K/k_0 respectively. Then one has the diagram:

$$\begin{tikzcd}
& A \arrow[d] & B \arrow[d] & \\
1 \arrow[r] & U(I)/V(\bar{L}/\bar{K}) \arrow[r, "N_{\bar{L}/\bar{K}}"] \arrow[d, "\psi - 1"] & U(\bar{K}) \arrow[r] \arrow[d, "\phi_0 - 1"] & 1 \\
1 \arrow[r] & U(\bar{L})/V(\bar{L}/\bar{K}) \arrow[r, "N_{\bar{L}/\bar{K}}"] \arrow[d] & U(\bar{K}) \arrow[r] & 1 \\
& 1 & &
\end{tikzcd}$$

From Proposition 4.3 it follows that: $U(\bar{k}_0) = N_{\bar{E}/\bar{k}_0}(U(\bar{E}))$. Applying Lemma 6.1 which is also true in this case if k' is a finite inertial extension of k, we obtain $U(\bar{k}) = N_{\bar{E}/\bar{k}}(\bar{E})$.

Remark 6.1. The isomorphism $\delta_{E/k}$ defined by Theorem 6.1 (a) and (b) will be called "fundamental isomorphism."

If k and E are as in Theorem 6.1 (a) or (b) and if $Q_p \subseteq k_1 \subseteq k_2 \subseteq \cdots \subseteq k$ is a sequence of finite extensions of Q_p such that $\bigcup_{i \geq 1} k_i = k$, then there exists $n_0 \in \mathbf{N}$ such that $([\bar{K} : \bar{k}], [k_{i+1} : k_i]) = 1$ for any $i \geq n_0$. Let π' be a uniformizing element of E_{n_0}. If $u \in U(\bar{k})$ there exists $u_0 \in U(\bar{k}_0)$ with $N_{\bar{k}/\bar{k}_0}(u_0) = u$. Then there exists $\zeta \in U(\bar{L})$ with $N_{\bar{L}/\bar{K}}(\zeta) = u_0$ and there exists $\sigma \in \mathrm{Gal}\,(\bar{L}/\bar{K})$ such that

$$\zeta^{\psi - 1} \equiv \pi'^{\sigma - 1} \,(\mathrm{mod}\; V(\bar{L}/\bar{K})).$$

The isomorphism $\delta_{E/k}$ is given by:

$$u(\mathrm{mod}\; N_{\bar{E}/\bar{k}}(U(\bar{E}))) \mapsto \sigma \in \mathrm{Gal}\,(\bar{L}/\bar{K}) \simeq \mathrm{Gal}\,(\bar{E}/\bar{k})_{\mathrm{ram}}$$

The isomorphisms $\delta_{E/k}$ has an important property of functoriality. Let k and E be as in Theorem 6.1 (a) or (b) and let $k \subseteq E' \subseteq E$. Then $N_{\bar{E}/\bar{k}}(U(\bar{E})) \subseteq N_{\bar{E}'/\bar{k}}(U(\bar{E}'))$ and one has the diagram:

$$\begin{array}{ccc} U(\bar{k})/N_{\bar{E}/\bar{k}}(U(\bar{E})) & \longrightarrow & \mathrm{Gal}(\bar{E}/\bar{k})_{\mathrm{ram}} \\ \downarrow & & \downarrow \\ U(\bar{k})/N_{\bar{E}'/\bar{k}}(U(\bar{E}')) & \longrightarrow & \mathrm{Gal}(\bar{E}'/\bar{k})_{\mathrm{ram}} \end{array} \tag{6.2}$$

where the vertical homomorphisms are the canonic ones.

Proposition 6.1. *The diagram* (6.2) *is commutative.*

For the proof, see [5], §5.2, Lemma 3, and the above remark.

7. The subgroups of norms

Proposition 7.1. *Let $Q_p \subseteq k \subseteq \Omega$ such that the residual field k_v of k is finite and $p \dagger [k: Q_p]_\infty$. Let $k \subseteq \ell$ be a finite abelian extension and let $\{k_i\}$, $\{\ell_i\}$ be sequences as in* §2. *Denote*: $H_1 = N_{\ell_i/k_i}(U(\ell_i))$, $H = N_{\ell/k}(U(\ell))$ *and* $\tilde{H} = N_{\bar{\ell}/\bar{k}}(U(\bar{\ell}))$. *Then*:

(1) $H_{i+1} = N^{-1}_{k_{i+1}/k_i}(H_i)$ *for any* $i \in \mathbf{N}^*$.
(2) $H_i \subseteq H_{i+1}$ *for any* $i \in \mathbf{N}^*$.
(3) $H = \bigcup_{j \geq i_0} H_j$ *where* $i_0 \in \mathbf{N}$ *is such that* $([k_i: k_{i0}], [\ell: k]) = 1$ *for any* $i \geq i_0$.
(4) $\tilde{H} = \bar{H}$.

Proof. (1) follows from the equality $\ell_{i+1} = k_{i+1} \cdot \ell_i$.
(2) is obvious.
(3) follows from the equalities: $N_{\ell/k/\ell_i} = N_{\ell_i/k_i}$ for any $i \geq i_0$.
(4) If $\bar{x} \in \tilde{H}$ then $\bar{x} = N_{\bar{\ell}/\bar{k}}(\bar{y})$, $\bar{y} \in U(\bar{\ell})$. Since $\bar{y} = \lim_{n\to\infty} y_n$, where $y_n \in U(\ell)$ one has $\bar{x} = N_{\bar{\ell}/\bar{k}}(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} N_{\ell/k}(y_n)$, hence $\tilde{H} \subseteq \bar{H}$.

In order to obtain the other inclusion it is enough to prove that $\tilde{H}$ is closed in $U(\bar{k})$. We shall show that $\tilde{H}$ is an open subgroup of $U(\bar{k})$, hence it is also a closed subgroup of $U(\bar{k})$. Let $\alpha \in H$ and choose $a \in U(\bar{\ell})$ such that $\alpha = N_{\bar{\ell}/\bar{k}}(a)$. Denote by

$$f(x) = x^q + \alpha_1 x^{q-1} + \cdots + \alpha_{q-1} x + \alpha_q$$

the minimal polynomial of a over $\bar{k}$. Then $\alpha = \alpha_q^m$ where $m = [\bar{\ell}: \bar{k}(a)]$. Let $\delta > 0$ and $\beta \in U(\bar{k})$ such that $v(\alpha - \beta) > \delta$.

Let $\beta_q \in \bar{\Omega}$ be a root of $F_\beta(x) = X^m - \beta$ for which $v(\alpha_q - \beta_q)$ is largest. If $\zeta \in \Omega$ denotes a primitive root of 1 of order m, then:

$$v(\alpha_q - \beta_q) \geq \frac{1}{m} v\left(\prod_{i=0}^{m-1} (\alpha_q - \zeta^m \cdot \beta_q)\right) = \frac{1}{m} v(\alpha - \beta) > \frac{\delta}{m}$$

Hence for large δ one has:

$$v(\alpha_q-\beta_q)>\sup_{1\le i\le m-1} v(1-\zeta^i)\ge \sup_{\substack{\sigma\in \mathrm{Gal}(\bar\Omega/\bar k)\\ \sigma(\alpha_q)\ne\alpha_q}} v(\alpha_q-\sigma(\alpha_q))$$

and from Krasner's Lemma we derive: $\bar k(\beta_q)\subseteq \bar k(\alpha_q)$, i.e. β_q is in $\bar k$.

Now let $g(x)=x^q+\alpha_1x^{q-1}+\cdots+\alpha_{q-1}x+\alpha_q$ and denote by $b_1,\dots,b_q$ the roots of $g(x)$ in $\bar\Omega$, arranged such that $v(a-b_1)\ge v(a-b_j)$ for $2\le j\le q$.

Since $v(a-b_1)\ge \dfrac{1}{q}v((a-b_1)\cdots(a-b_q))=\dfrac{1}{q}v(g(a))=\dfrac{1}{q}v(\beta_q-\alpha_q)\ge\dfrac{\delta}{q\cdot m}$, it follows from Krasner's Lemma that for large δ one has $\bar k(a)\subseteq \bar k(b_1)$, hence $g(x)$ is irreducible over $\bar k$, $\bar k(a)=\bar k(b_1)$, and $\tilde H\ni N_{\bar\ell/\bar k}(b_1)=\beta_q^m=\beta$. Thus $\tilde H$ is open in $U(\bar k)$ and this completes the proof of (4).

Let k, $\{k_i\}$ be as in Proposition 6.1. Let $i_0\in\mathbf N$ and H_{i_0} be a subgroup of $U(k_{i_0})$ such that: $|U(k_{i0})/H_{i0}|$ is relatively prime with $[k_i : k_{i0}]$ for any $i\ge i_0$. Denote for $i\ge i_0$: $H_i=N^{-1}_{k_i/k_{i0}}(H_{i0})$ and let $H=\bigcup_{i\ge i_0}H_i$. Denote by $\mathscr H(k)$ the set of subgroups H of $U(k)$ which are obtained in this manner (by varying i_0 and H_{i_0}) and by $\mathscr H(\bar k)$ the set of subgroups $\bar H$ of $U(\bar k)$ where H runs over $\mathscr H(k)$.

Proposition 7.2. *Let $Q_p\subseteq k\subseteq\Omega$ such that $p\nmid[k:Q_p]$ and k_v is finite. For any $H\in\mathscr H(k)$ there exists a finite totally ramified abelian extension ℓ of k such that: $N_{\ell/k}U(\ell)=H$ and $N_{\bar\ell/\bar k}(U(\bar\ell))=\bar H$.*

Proof. For any $i\ge i_0$ let ℓ_i be a totally ramified finite abelian extension of k_i such that $H_i=H_{\ell_i/k_i}(U(\ell_i))$. One has $\ell_{i+1}=k_{i+1}\ell_i$ hence if $\ell_{i_0}=k_{i_0}(\alpha)$ then $\ell_i=k_i(\alpha)$ for any $i\ge i_0$. Now put $\ell=k(\alpha)$ and conclude the proof by applying Proposition 7.1.

Theorem 7.1. *Let k, ℓ, $\{k_i\}$, $\{\ell_i\}$ be as in Proposition 7.1. There exists an isomorphism $\delta_{\ell/k}$ such that the following diagram (where* Res *is the restriction and ϕ is induced by the inclusion $U(k)\subseteq U(\bar k)$) is commutative:*

$$\begin{array}{ccc} U(\bar k)/\bar H & \xrightarrow{\delta_{\bar\ell/\bar k}} & \mathrm{Gal}(\bar\ell/\bar k)_{\mathrm{ram}}\\ \phi\uparrow & & \uparrow \mathrm{Res}^{-1}\\ U(k)/H & \xrightarrow{\delta_{\ell/k}} & \mathrm{Gal}(\ell/k)_{\mathrm{ram}}\end{array}$$

Proof. We have to prove that ϕ is an isomorphism (then we put $\delta_{\ell/k}=\mathrm{Res}\circ\delta_{\bar\ell/\bar k}\circ\phi^{-1}$). Let $\bar\alpha_1,\bar\alpha_2,\dots,\bar\alpha_m\in U(\bar k)$ be a system of representatives for $U(\bar k)/\bar H$.

For any $n\in\mathbf N$ let $\alpha_1^{(n)},\dots,\alpha_m^{(n)}\in H$ be such that $v(\bar\alpha_i-\alpha_i^{(n)})\ge n$ for $i=1,\dots,m$. We assert that for large n the images of $\alpha_1^{(n)},\dots,\alpha_m^{(n)}$ in $U(k)/H$ are distinct. If not, then there exists $i_0\ne j_0$ and an increasing sequence $\{n_t\}_{t\in\mathbf N}$ such that $(\alpha_{i0}^{(n_t)})/(\alpha_{j0}^{(n_t)})\in H$ for any t, and this implies $(\bar\alpha_{i0})/(\bar\alpha_{j0})=\lim_{t\to\infty}((\alpha_{i0}^{(n_t)})/(\alpha_{j0}^{(n_t)}))\in\bar H$, contrary to our assumption.

We have thus the inequality:

$$|U(k)/H| \geq |U(\bar{k})/\bar{H}|$$

If there exist $\beta_1, \ldots, \beta_{m+1} \in U(k)$ which have distinct images in $U(k)/H$ then they also have distinct images in $U(k_i)/H_i$, where "i" is chosen large enough such that $\beta_1, \ldots, \beta_{m+1} \in U(k_i)$. But $|U(k_i)/H_i| = [\ell_i : k_i]_{\mathrm{ram}} = [\bar{\ell} : \bar{k}]_{\mathrm{ram}} = |U(\bar{k})/\bar{H}|$. Therefore:

$$|U(k)/H| = |U(\bar{k})/\bar{H}|.$$

Now let $\alpha \in U(k) \cap \bar{H}$. Fix an $a \in U(\bar{\ell})$ for which $N_{\bar{\ell}/\bar{k}}(a) = \alpha$. Let $f(x) = x^q + \alpha_1 x^{q-1} + \cdots + \alpha_q$ be the minimal polynomial of a over $\bar{k}$, then $\alpha = \alpha_q^m$ where $m = [\bar{\ell} : \bar{k}(\alpha)]$. Let:

$$g(x) = x^q + \beta_1 x^{q-1} + \cdots + \beta_{q-1} x + \alpha_q, \quad \text{where} \quad \beta_i \in k,\ v(\beta_i - \alpha_i) > \delta.$$

If δ is large enough, then from Krasner's Lemma it follows that there exists a root b of $g(x)$ such that $\bar{k}(a) = \bar{k}(b)$. Moreover, $g(x)$ is irreducible over $\bar{k}$ and $N_{\bar{\ell}/\bar{k}}(b) = \alpha_q^m = \alpha$. Since $g(x) \in k(x)$ it follows that $b \in \ell$ and $a \in H$.

This proved ϕ is injective. Hence it is an isomorphism, as asserted.

Theorem 7.2. *Let $Q_p \subseteq k \subseteq \Omega$ such that $p \dagger [k : Q_p]_\infty$ and $|k_v| = q < \infty$. Let $q_1 = (q - 1, [k : Q_p]_\infty)$ and $V_{q_1} =$ the group of roots of 1 of order q_1 in $U(k)$. Then*:

$$\bigcap_{\substack{\ell \supseteq k \\ ab}} N_{\ell/k}(U(\ell)) = V_{q_1}.$$

Proof. Let $a \in V_{q_1}$ and let ℓ be a finite abelian extension of k. Then

$$d_\ell = |U(k)/N_{\ell/k}(U(\ell))| = |\mathrm{Gal}\,(\ell/k)_{\mathrm{ram}}|$$

is prime with $[k : Q_p]_\infty$ hence is prime with q_1. Since the order of $a(\mathrm{mod}\, N_{\ell/k}(U(\ell)))$ is a divisor of both d_ℓ and q_1 it follows that $a \in N_{\ell/k}(U(\ell))$. Thus:

$$V_{q_1} \subseteq \bigcap_{\substack{\ell \supset k \\ ab}} N_{\ell/k}(U(\ell))$$

Now let $a \in U(\ell)$, $a \notin V_{q_1}$. We have to prove the existence of a finite abelian extension ℓ/k such that $a \notin N_{\ell/k}(U(\ell))$. Let $i \in \mathbf{N}$ such that:

(1) $\left(p \cdot \dfrac{q-1}{q_1},\ [k_j : k_i]\right) = 1$ for any $j > i$.

(2) $a \in U(k_i)$.

Let $m \in \mathbf{N}$. Denote $U^m(k_i) = \{u \in k_i / u \equiv 1 \,(\mathrm{mod}\, \pi_i^m)\}$ and $V^m(k_i) = U^m(k_i) \cdot V_{q_1}$. Since $|V^m(k_i)/U^m(k_i)| = q_1$ and $|U(k_i)/U^m(k_i)| = q^m(q-1)$, it follows that $|U(_i)/V^m(k_i)| = \dfrac{q^m(q-1)}{q_1}$ is relatively prime to $[k_j : k_i]$ for any $j > i$.

Let $H_i^m = V^m(k_i)$, $H_j^m = N_{k_j/k_i}^{-1}(H_i^m)$ and $H^m = \bigcup_{j \geq i} H_j^m$. From Proposition 6.2 it follows that there exists a finite abelian extension ℓ_m of k for which $N_{\ell_m/k}(U(\ell_m)) = H^m$.

Since $a \notin V_{q_1}$, there exists $m \in \mathbf{N}$ such that $a \notin V^m(k_i)$. Then one has for any $j \geq i$:

$$N_{k_j/k_i}(a) = a^{[k_j;k_i]} \notin H_i^m, \text{ hence } a \notin H_j^m.$$

Therefore $a \notin H^m$.

Corollary 7.1. *Let $Q_p \subseteq k \subseteq \Omega$ such that $p \dagger [k: Q_p]_\infty$, and $|k_v| = q < \infty$. Then $\bigcap_{\substack{\ell \supseteq k \\ ab}} N_{\ell/k}(U(\ell)) = 1$ if and only if $q-1$ and $[k: Q_p]_\infty$ are relatively prime.*

Theorem 7.3. *The hypothesis and notations being as in Theorem 7.2, let k_{ab} be the maximal abelian extension of k. Then one has:*

$$\mathrm{Gal}\,(k_{\mathrm{ab}}/k)_{\mathrm{ram}} \simeq \widehat{U(k)}/V_{q_1}.$$

(*where $\widehat{U(k)}$ is the completion of $U(k)$ w.r. to the subgroups of finite index*)

Proof. For any finite abelian extension ℓ of k one has the isomorphism:

$$U(k)/N_{\ell/k}(U(\ell)) \xrightarrow{\delta_{\ell/k}} \mathrm{Gal}\,(\ell/k)_{\mathrm{ram}}$$

and if $k \subseteq \ell \subseteq \ell'$, such that ℓ' is finite and abelian, the diagram

$$\begin{array}{ccc} U(k)/N_{\ell'/k}(U(\ell')) & \xrightarrow{\delta_{\ell'/k}} & \mathrm{Gal}\,(\ell'/k)_{\mathrm{ram}} \\ \downarrow & & \downarrow \\ U(k)/N_{\ell/k}(U(\ell)) & \xrightarrow{\delta_{\ell/k}} & \mathrm{Gal}\,(\ell/k)_{\mathrm{ram}} \end{array}$$

is commutative. Then there exists a canonic isomorphism

$$\delta_k \colon \varprojlim U(k)/N_{\ell/k}(U(\ell)) \to \varprojlim \mathrm{Gal}\,(\ell/k)_{\mathrm{ram}}$$

But

$$\varprojlim U(k)/N_{\ell/k}(U(\ell)) \simeq \widehat{U(k)}/\textstyle\bigcap_\ell N_{\ell/k}(U(\ell)) = \widehat{U(k)}/V_{q_1},$$

and

$$\varprojlim \mathrm{Gal}\,(\ell/k)_{\mathrm{ram}} \simeq \mathrm{Gal}\,(k_{ab}/k_{nr}) \simeq \mathrm{Gal}\,(k_{ab}/k)_{\mathrm{ram}}.$$

We conclude this paper with the following result which comes naturally from what was already proved.

Theorem 7.4. *Let $Q_p \subseteq k \subseteq \bar{\Omega}$ such that the residual field of k is finite and $p \dagger [k: Q_p]_\infty$. Then there exists a canonical one-to-one correspondence between $\mathscr{H}(k)$ and the set of finite abelian extensions of k_{nr}, and a canonical one-to-one* *correspondence between $\mathscr{H}(\bar{k})$ and the set of complete finite abelian extensions of $\bar{k}_{nr}$.*

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