J. Math. Kyoto Univ. (JMKYAZ)
35-2 (1995) 299–324

On holomorphic maps between Riemann surfaces which preserve *BMO*

By

Yasuhiro GOTOH

0. Introduction

We say that a nonconstant holomorphic map between plane domains is a *BMO* map if it preserves *BMO*, where *BMO* is the space of functions of bounded mean oscillation with respect to the 2-dimensional Lebesgue measure.

Reimann [16] and Jones [10] showed that *BMO* is invariant under quasiconformal maps. Hence conformal maps are *BMO* maps. Osgood [13] characterized *BMO* maps in the case of universal covering maps of plane domains. In [4] we defined *BMO* space on general Riemann surfaces and extended his result to Riemann surfaces. We also characterized *BMO* maps between plane domains in [5]. Moreover we investigated Blaschke type holomorphic maps between the extended complex planes in [6], and gave an estimate for their operator norms as *BMO* maps. In this paper we treat *BMO* maps between Riemann surfaces in succession.

In §1 we give a characterization of *BMO* maps between plane domains (Theorem 1), which extends our former results in [5]. We give also a characterization of *BMOH* maps between plane domains (Theorem 2), where *BMOH* map is a nonconstant holomorphic map which preserves harmonic *BMO* functions. In particular we show that a covering map between plane domains is a *BMO* map if and only if it is a *BMOH* map (Corollary 6).

In §2 we investigate Hahn metric on Riemann surfaces which is a generalization of the quasihyperbolic metric. We generalize several properties of the quasihyperbolic metric to Hahn metric. In particular we show that the Hahn metrical length of every closed curve which is not homotopic to a point is not less that $\pi/2$ (Proposition 9).

In §3, by using the result in §2, we investigate *BMO* maps between Riemann surfaces. In particular: (1) We give a characterization of *BMO* maps with noncompact targets (Theorem 9); (2) We give a characterization of *BMO* maps in case of covering maps (Theorem 11 and 12); (3) We give several results which indicate an essential difference between *BMO* maps with noncompact targets and *BMO* maps with compact targets (cf. Corollary 17 and 20, Theorem 14). We cannot obtain, however, a characterization of *BMO* maps with compact targets.

Received August 29, 1994, Revised Novemer 7, 1994

1. *BMO* maps between plane domains

Let D be a plane domain. Let $BMO(D)$ be the space of all locally integrable functions g on D such that

$$\|g\|_{*,D} = \sup_{B} |B|^{-1} \int_{B} |g - g_B| \, dxdy < \infty,$$

where the supremum is taken over all disks B in D and g_B denotes the average of g over B and $|B|$ is the 2-dimensional Labesgue measure of B. Let $BMOH(D)$ (resp. $BMOA(D)$) be the space of all harmonic (analytic) functions in $BMO(D)$.

We say a nonconstant holomorphic map $F: D \to D'$ between plane domains is a BMO (resp. $BMOH$, $BMOA$) map if $T_F g = g \circ F \in BMO(D)$ for every $g \in BMO(D')$ ($BMOH(D')$, $BMOA(D')$). The category argument shows that a $BMO(BMOH$, $BMOA)$ map F induces a bounded operator T_F. Let $\|T_F\|_{BMO}$ ($\|T_F\|_{BMOH}$, $\|T_F\|_{BMOA}$) be its operator norm. In this section we investigate BMO and $BMOH$ maps between plane domains.

In the following $\hat{\mathbf{C}}$ denotes the extended complex plane, $d(\cdot, \cdot)$ denotes the Euclidean distance, disk means an open disk, $\mathrm{rad}(B)$ denotes the radius of a disk B, tB, $t > 0$, denotes the disk having the same center as B and $t\,\mathrm{rad}(B)$ as its radius, Δ denotes the unit disk, and a covering map means an unbranched unbounded (not necessarily normal) holomorphic covering map. We say two positive constants a and b are comparable (or $a \approx b$) if $A^{-1} \le a/b \le A$ for some universal constant $A \ge 1$, and $a \lesssim b$ if $a \le Ab$ for some universal constant $A > 0$.

The following remarkable result by Reimann and Jones shows that in spite of the dependence of the definition of $BMO(D)$ on the 2-dimensional Lebesgue measure on D which is not conformally invariant, $BMO(D)$ becomes a function space on a "Riemann surface" D. Using this fact we define BMO on general Riemann surfaces later.

Proposition 1 (quasiconformal invariance)([16], [10]). *Let $F: D \to D'$ be a quasiconforal map between plane domains. Then $g \circ F \in BMO(D)$ for every $g \in BMO(D')$ and*

$$C(K)^{-1} \|g\|_{*,D'} \le \|g \circ F\|_{*,D} \le C(K) \|g\|_{*,D'},$$

where K is the maximal dilatation of F.

Corollary 1. *Let $F: D \to D'$ be a conformal map between plane domains. Then $g \circ F \in BMO(D)$ for every $g \in BMO(D')$ and $\|g \circ F\|_{*,D} \approx \|g\|_{*,D'}$.*

We list up some basic facts about BMO which we need later.

Let $\mathscr{B}_h(D)$ the harmonic Bloch space on D, that is, $\mathscr{B}_h(D)$ is the space of all harmonic functions g on D such that

$$\|g\|_{\mathscr{B}_h(D)} = \sup_{z \in D} d(z, \partial D) |\nabla g(z)| < \infty,$$

and let $\mathscr{B}(D)$ be the standard (analytic) Bloch space on D.

Proposition 2 (cf. [2]). *$BMOH(D) = \mathscr{B}_h(D)$, and $BMOA(D) = \mathscr{B}(D)$ for every plane domain D and $\|g\|_{*,D} \approx \|g\|_{\mathscr{B}_h(D)}$, $g \in BMOH(D)$. In particular $BMOH(\mathbf{C}) = BMOA(\mathbf{C}) = \mathbf{C}$.*

Let $\mathscr{F}_{D,L}$, $L > 0$, denote the set of all disks B on a plane domain D such that $d(B, \partial D) \geq L\mathrm{rad}(B)$.

Proposition 3 (localization theorem)(cf. [17], [7, Lemma 6]). *Let $L \geq 1$ and g a function in $L^1_{loc}(D)$ satisfying the condition $|B|^{-1}\int_B |g - g_B|\,dxdy \leq K$ for every disk $B \in \mathscr{F}_{D,L}$, then $g \in BMO(D)$ and $\|g\|_{*,D} \lesssim LK$.*

Proposition 4 (removability theorem)(cf. [17]). *Let D be a plane domain and E a discrete subset of D such that $\#(E \cap B) \leq N$ for every disk $B \in \mathscr{F}_{D,L}$. Let $D' = D \setminus E$ and $g \in BMO(D')$. Then $g \in BMO(D)$ and $\|g\|_{*,D} \leq C(L, N)\|g\|_{*,D'}$.*

Proof. We note that let Ω be an arbitrary plane domain and $z \in \Omega$, then $BMO(\Omega \setminus \{z\}) = BMO(\Omega)$ and it holds that $\|g\|_{*,\Omega} \lesssim \|g\|_{*,\Omega\setminus\{z\}}$ (cf. [17]). And so the assertion follows from Proposition 3.

Lemma 1. *Let B be a disk, $g \in L^1(B)$, and μ a signed measure on B such that $\Delta g = \mu$ in the distributional sense (, which means g is locally a difference of two superharmonic functions). Let k be a C^2 function on B with compact support. Then*

$$\left|\int_B k\,d\mu\right| \leq \|\Delta k\|_\infty \int_B |g - g_B|\,dxdy.$$

Proof.

$$\left|\int_B k\,d\mu\right| = \left|\int_B g\Delta k\,dxdy\right| = \left|\int_B (g - g_B)\Delta k\,dxdy\right| \leq \|\Delta k\|_\infty \int_B |g - g_B|\,dxdy.$$

Proposition 5 ([3]). *Let s be a superharmonic function on a plane domain D such that $\Delta s = -\mu$ and $\|s\|_{*,D} \leq K$, then μ is a uniformly locally finite measure on D, that is, $\mu(B) \leq C(L)K$ holds for every $L > 0$ and disk $B \in \mathscr{F}_{D,L}$.*

Proof. Let $B \in \mathscr{F}_{D,L}$, z_0 its center, r its radius, and $B' = (L+1)B (\subset D)$. Let $k_0 \geq 0$ be a C^2 function on Δ with compact support such that $k_0 = 1$ on $\{|z| < 1/(L+1)\}$. Let $k(z) = k_0((z - z_0)/((L+1)r))$. Then $k \geq 0$ is a C^2 function on B' with compact support such that $k_0 = 1$ on B. It follows from Lemma 1

$$\mu(B) \leq \int_{B'} k\,d\mu \leq \|\Delta k\|_\infty |B'|\,\|s\|_{*,D} = \frac{\|\Delta k_0\|_\infty}{(L+1)^2 r^2}|B'|\,\|s\|_{*,D} \leq C(L)K.$$

Let $F: \Delta \to \mathbf{C}$ be a holomorphic map. If its Schwarzian derivative

$$S_F(z) = \left(\frac{F''(z)}{F'(z)}\right)' - \frac{1}{2}\left(\frac{F''(z)}{F'(z)}\right)^2$$

satisfy $|S_F(z)| \leq 2/(1-|z|^2)^2$, $z \in \Delta$, then F is univalnt on Δ([12]). Hence we have

Lemma 2. *Let $F: D \to D'$ be a holomorphic map between plane domains such that $|S_F(z)| \leq K/d(z, \partial D)^2$, $z \in D$, then F is univalent on each disk $B \in \mathscr{F}_{D,L}$, where $L = \sqrt{K/2}$.*

We say a nonconstant holomorphic map F: $R \to R'$ between Riemann surfaces is p-valent, if $\# F^{-1}(\zeta) \leq p$, $\zeta \in R'$, where $F^{-1}(\zeta)$ is to be counted with multiplicity.

Lemma 3. *Let $F: \Delta \to R$ be a p-valent holomorphic map into a Riemann surface R, $R \neq \hat{\mathbf{C}}$, and E_F the set of all branch points of F. Then $\#(B \cap E_F) \leq C(\alpha, p)$ for every $\alpha > 0$ and disk $B \subset \Delta$ with hyperbolic radius α, where E_F is to be counted with multiplicity. Moreover if F is locally univalent, then F is univalent on each disk $B \subset \Delta$ with hyperbolic radius $C(p)$.*

Proof. First, assume that there exist a plane domain R, and a sequence of p-valent holomorphic map F_n: $\Delta \to R$ such that $\#(E_{F_n} \cap \{|z| < 1/n\}) \geq p$. We set $\mu_p(f) = \max\{|a_n| \,|\, 0 \leq n \leq p\}$ for a holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on Δ, then the family of all p-valent holomorphic functions f on Δ such that $\mu_p(f) \leq 1$ forms a normal family (cf. [9]). Let $g_n(z) = F_n(z) - F_n(0)$ and $h_n(z) = g_n(z)/\mu_p(g_n)$. Then $\mu_p(h_n) = 1$ and $h_n(0) = 0$, and so some subsequence h_{n_k} converges uniformly on compact set to a nonconstant holomorphic function h on Δ. Then the origin is a branch point of h with multiplicity $\geq p$. Hence h_{n_k} (and so F_{n_k}) is not p-valent for sufficient large k, which is a contradiction. The same argument holds in the case of locally univalent functions.

Next let R be a Riemann surface. Let π: $\tilde{R} \to R$ ($\tilde{R} = \mathbf{C}$ or Δ) be a universal covering map and $\tilde{F}$ a lift of F. Let F be p-valent. Since $\tilde{F}$ is a p-valent function having the same branch points as F, the same estimate holds. Finally let F be locally univalent p-valent then $\tilde{F}$ is also locally univalent p-valent. Hence there exists a constant $0 < l_p < 1$ such that $\tilde{F}$ is univalent on $\{|z| < l_p\}$. We can assume $\tilde{F}(0) = 0$. Let $r = |\tilde{F}'(0)|\, l_p/4$, then $\tilde{F}(\{|z| < l_p\}) \supset \{|w| < r\}$ by Koebe's theorem. Then $\{|w| < r/2p\}$ contains no two equivalent points, indeed if w_1, w_2, $w_1 \neq w_2$, is equivalent points on $\{|w| < r/2p\}$ and let T be the covering transformation such that $T(w_1) = w_2$ then $T^n(w_1) \in \{|w| < r\}$, $0 \leq n \leq p$, which is a contradiction. Applying Koebe's theoem again we have $\tilde{F}(\{|z| < l_p/32p\}) \subset \{|w| < r/2p\}$ and so the assertion follows.

Let D be a plane domain and E a closed subset of D such that $D \setminus E$ is connected. We say E is removable for $BMO(D)$ if every $BMO(D \setminus E)$ function is a restriction of some $BMO(D)$ function. Then there exists a constant $C > 0$ such that for every $g \in BMO(D \setminus E)$ we can choose a function $\tilde{g} \in BMO(D)$, $\tilde{g}\,|(D \setminus E) = g$ so that $\|\tilde{g}\|_{*,D} \leq C\, \|g\|_{*,D \setminus E}$ by the open mapping theorem. In case of $D = \mathbf{C}$, E is removable for $BMO(\mathbf{C})$ if and only if E is a uniform domain (cf. [10]). When D is an arbitrary plane domain, E is removable for $BMO(D)$ if and only if $D \setminus E$ is a "relative" uniform domain (cf. [7]).

Theorem 1 (cf. [5]). *Let $F: D \to D'$ be a nonconstant holomorphic map between plane domains. Then the following conditions are equivalent to each other*:

(1) *F is a BMO map*;

(2) *F is of bounded local valence on D, that is, there exist a constant $L > 0$ and an integer $p > 0$ such that for every disk $B \in \mathscr{F}_{D,L}$, F is p-valent on B*;

(3) *The set E_F of all branch points of F is a removable set for $BMO(D)$ and there exists a constant $L > 0$ such that for every disk $B \in \mathscr{F}_{D \setminus E_F, L}$, F is univalent on B*;

(4) $\log|F'| \in BMO(D)$;

(5) $\sup_{\zeta \in \mathbf{C}} \|\log|F - \zeta|\|_{*,D} < \infty$;

(6) $\sup_{\zeta \in D'} \|\log|F - \zeta|\|_{*,D} < \infty$.

If D' admits Green functions $g_{D'}(\cdot, \zeta)$, $\zeta \in D'$, then the next condition is also equivalent:

(7) $\sup_{\zeta \in D'} \|g_{D'}(F, \zeta)\|_{*,D} < \infty$.

Remark that only the condition (4) is new: This theorem was proved in [5] except for (4).

Proof. We have $(1) \Rightarrow (5) \Rightarrow (6)$ and $(1) \Rightarrow (7)$, since $\log|\cdot| \in BMO(\mathbf{C})$, $g_{D'}(\cdot, \zeta) \in BMO(D')$, and $\sup_{\zeta \in D'} \|g_{D'}(\cdot, \zeta)\|_{*,D'} < \infty$ (cf. [3]). Applying Proposition 5 to superharmonic functions $-\log|F - \zeta|$, $g_{D'}(F, \zeta)$, we have $(6) \Rightarrow (2)$, and $(7) \Rightarrow (2)$. $(2) \Rightarrow (3)$ is a consequence of Lemma 3 and the removability theorem. $(3) \Rightarrow (1)$ follows from the localization theorem.

($(3) \Rightarrow (4)$) Let F satisfy (3). Let $G = D \setminus E_F$. If $G = \mathbf{C}$ then $D = \mathbf{C}$ and $F(z) = az + b$, hence (4) holds. Next let $G \neq \mathbf{C}$. Let $z \in G$ and B' the disk in G around z such that $d(B', \partial G) = L\operatorname{rad}(B')$. Applying Koebe's theorem we have $|F''(z)/F'(z)| \leq 4/\operatorname{rad}(B') \leq C_1/d(z, \partial G)$. Hence by Proposition 2, $\log|F'| \in BMO(G)$. And so $\log|F'| \in BMO(D)$.

($(4) \Rightarrow (3)$) Let $\log|F'| \in BMO(D)$. Applying Proposition 5 to a superharmonic function $-\log|F'|$, we have $\#(B \cap E_F) \leq C_2$ for every disk $B \in \mathscr{F}_{D,1}$. Hence E_F is removable for $BMO(D)$ by the removability theorem. Let $G = D \setminus E_F$. Since $\log|F'|$ is harmonic on G, Proposition 2 shows that $|F''(z)/F'(z)| \leq C_3/d(z, \partial G)$, $z \in G$. Hence $|S_F(z)| \leq C_4/d(z, \partial G)^2$, $z \in G$. And so Lemma 2 shows that F is univalent on each disk $B' \in \mathscr{F}_{G,C_5}$.

Corollary 2. *Let $F: \Delta \setminus \{0\} \to D'$ be a BMO map into a plane domain D'. Then the origin is not an essential singularity of F.*

Proof. By Theorem 1 F is p-valent on each disk $B \subset (1/2)\Delta$ such that $d(B, \{0\}) \geq \operatorname{rad}(B)$. Hence by Lemma 3 there exist a constant $\varepsilon > 0$ and sequences $\{r_n\}_{n=1}^{\infty}$, $2^{-n-1} \leq r_n \leq 2^{-n}$, $\{\theta_n\}_{n=1}^{\infty}$, $0 \leq \theta_n < 2\pi$, such that F is univalent on B_z for each $z \in N$ and $F(\zeta) \neq 0$ on $\bigcup_{z \in N} B_z$, where $B_z = \{\zeta \in \mathbf{C} \mid |\zeta - z| < \varepsilon|z|\}$, and $N = \bigcup_n (\{|\zeta| = r_n\} \cup \{\zeta = re^{i\theta_n} \mid 2^{-n-2} \leq r \leq 2^{-n}\})$. Then by Koebe's theorem

$$|F(z_2)/F(z_1)| \approx 1, \quad z_1, z_2 \in (1/2)B_z, \ z \in N,$$

hence

$$|F(z)| \leq C_1 |F(r_n)|, \quad |z| = r_n, \quad n = 1, 2, \ldots,$$

$$|F(r_{n+1})| \leq C_2 |F(r_n)|, \quad n = 1, 2, \ldots,$$

and so $|F(z)| \leq C_3 C_4^n \leq C_5 r_n^{-c_6}$, $|z| = r_n$, hence the origin is a removable singularity of $z^{[c_6]+1}F(z)$, which implies the assertion.

There exists, however, a holomorphic map $F: \Delta \setminus \{0\} \to \hat{\mathbf{C}}$ having the origin as an essential singularity such that F is p-valent on each disk $B \subset (1/2)\Delta$ satisfying the condition $d(B, \{0\}) \geq \mathrm{rad}\,(B)$. Let $\pi: \mathbf{C} \setminus \{0\} \to T$ be a covering map onto a torus T, and $f: T \to \hat{\mathbf{C}}$ a nonconstant holomorphic map, then $F = f \circ \pi$ is one such function.

Corollary 3. *Let E be a finite set of points on $\mathbf{C}$, $D = \mathbf{C} \setminus E$, D' a plane domain, and $F: D \to D'$ a nonconstant holomorphic map. Then F is a BMO map if and only if F is a rational map.*

Proof. First, a rational map is a *BMO* map since it is a finite valent map. Next let F be a *BMO* map. Let $D_0 = \{z \in \mathbf{C} \mid 0 < |z| < 1, 1/z \notin E\}$, then $G: D_0 \to D'$, $G(z) = F(1/z)$, is a *BMO* map. Hence $E \cup \{\infty\}$ are not essential singularities of F by Corollary 2.

In particular we have

Corollary 4 (cf. [5]). (1) *Let $F: \mathbf{C} \to \mathbf{C}$ be a nonconstant holomorphic map. Then the following conditions are equivalent to each other*:

a) *F is a BMO map*;

b) *$\mathbf{C} \setminus E_F$ is a uniform domain and there exists a constant $L > 0$ such that F is univalent on each $B \in \mathscr{F}_{D \setminus E_F, L}$*;

c) *F is a polynomial.*

(2) *There is no BMO map $F: \mathbf{C} \to \mathbf{C} \setminus \{0\}$.*

Corollary 5. *Let $F: D \to D'$ be a nonconstant quasiregular map between plane domains. Then the following conditions are equivalent to each other*:

(1) *F preserves BMO*;

(2) *F is of bounded local valence on D*;

(3) *The set E_F of all branch points of F is a removable set for $BMO(D)$ and there exists a constant $L > 0$ such that for every disk $B \in \mathscr{F}_{D \setminus E_F, L}$, F is univalent on B*;

(4) $\log J_F \in BMO(D)$, *where J_F is the Jacobian of F*;

(5) $\log |\nabla F| \in BMO(D)$.

Proof. Let $F = F_h \circ F_q$, where F_h is a holomorphic map and F_q is a quasiconformal map. Then $\log J_F = (2 \log |F_h'|) \cdot F_q + \log J_{F_q}$ and $\log J_{F_q} \in BMO(D)$ (see [17]). Since $K^{-1} J_F \leq |\nabla F|^2 \leq K J_F$ where K is the maximal dilatation of F, $\log J_F \in BMO(D)$ if and only if $\log |\nabla F| \in BMO(D)$. Similarly $\log J_{F_q} \in BMO(D)$ if and only if $\log |\nabla F_q| \in BMO(D)$. Moreover let $G: D_1 \to D_2$ be a quasiconformal map with maximal dilatation K, then for every $L > 0$ and every $B \in \mathscr{F}_{D_1, C(K,L)}$

there exists a $B' \in \mathscr{F}_{D_2,L}$ such that $G(B) \subset B'$. And so Theorem 1 and the quasiconformal invariance of BMO imply the assertion.

Next we characterize $BMOH$ maps.

Theorem 2. *Let $F: D \to D'$, $D' \neq \mathbf{C}$, be a nonconstant holomorphic map between plane domains. Then the following conditions are equivalent to each other:*
(1) *F is a $BMOH$ map;*
(2) *$D \neq \mathbf{C}$ and there exists a constant $K > 0$ such that*

$$\frac{|dF(z)|}{d(F(z), \partial D')} \leq K \frac{|dz|}{d(z, \partial D)}, \qquad z \in D;$$

(3) *There exists a constant $L > 0$ such that for every disk $B \in \mathscr{F}_{D,L}$, $F(B)$ does not surround any component of $\partial D'$;*
(4) $\alpha = \sup_{\zeta \in \mathbf{C} \setminus D'} \| \log |F - \zeta| \|_{*,D} < \infty$;
(5) $\beta = \sup_{\zeta \in \partial D'} \| \log |F - \zeta| \|_{*,D} < \infty$.
Moreover $\| T_F \|_{BMOH} \approx \inf K \approx \alpha \approx \beta$ *and* $\inf L \lesssim \inf K \lesssim \inf L + 1$.

Proof. (2) $\Rightarrow$ (1) is a consequence of Proposition 2. Since $\log |\cdot| \in BMO(\mathbf{C})$, (1) $\Rightarrow$ (4) $\Rightarrow$ (5) holds.

((5) $\Rightarrow$ (2)) Let F satisfy (5). Then $D \neq \mathbf{C}$ since $BMOH(\mathbf{C}) = \mathbf{C}$. Let $z \in D$ and ζ a point on $\partial D'$ such that $d(F(z), \partial D') = |F(z) - \zeta|$. Applying Proposition 2 to a function $\log |F - \zeta|$, we have $|F'(z)|/|F(z) - \zeta| \leq C/d(z, \partial D)$.

((2)$\Rightarrow$(3)) Let F satisfy (2). We show that (3) holds for $L > K/\pi$. Otherwise, there exists a disk $B \in \mathscr{F}_{D,L}$ such that $F(B)$ surround a point $w_0 \in \partial D'$. Then there exist two points $z_1, z_2 \in B$, $z_1 \neq z_2$, such that $F(z_1) = F(z_2)$ and $F(\gamma)$ surrounds w_0, where γ is a segment on B joining z_1 to z_2. Let $w - w_0 = re^{i\theta}$, then

$$2\pi \leq \int_{F(\gamma)} \frac{r|d\theta|}{r} \leq \int_{F(\gamma)} \frac{|dw|}{d(w, \partial D')} \leq \int_\gamma \frac{K|dz|}{d(z, \partial D)} < \frac{2K \operatorname{rad}(B)}{d(B, \partial D)} \leq \frac{2K}{L} < 2\pi,$$

which is a contradiction.

((3) $\Rightarrow$ (2)) Let F satisfy (3). If $D = \mathbf{C}$, $\mathbf{C} \setminus F(\mathbf{C})$ contains a (unbounded) continuum, which contradicts with Picard's theorem. Hence $D \neq \mathbf{C}$. Let $z \in D$ and $B \subset D$ a disk around z such that $d(B, D) = L\operatorname{rad}(B)$. Then there exists a simple connected domain G such that $F(B) \subset G \subset D'$. Applying Schwarz lemma we have

$$\begin{aligned} \frac{|dF(z)|}{d(F(z), \partial D')} &\leq \frac{|dF(z)|}{d(F(z), \partial G)} \leq 4\rho_G(F(z))\,|dF(z)| \\ &\leq 4\rho_B(z)\,|dz| = \frac{4|dz|}{\operatorname{rad}(B)} \leq \frac{4(L+1)|dz|}{d(z, \partial D)}, \end{aligned}$$

where ρ_G, ρ_B denote the hyperbolic metric.

Note that if $D' = \mathbf{C}$, $F: D \to D'$ is always a $BMOH$ map since $BMOH(\mathbf{C}) = \mathbf{C}$.

Corollary 6. *Let $\pi: D \to D'$, $D' \neq \mathbf{C}$, be a covering map between plane domains. Then the following conditions are equivalent to each other*:

(1) *π is a BMO map*;

(2) *π is a BMOH map*;

(3) *There exists a constant $L > 0$ such that for every disk $B \in \mathscr{F}_{D,L}$, π is univalent on B*;

(4) *There exists an integer $p > 0$ such that for every disk $B \in \mathscr{F}_{D,1}$, π is p-valent on B*;

(5) *$D \neq \mathbf{C}$ and there exists a constant $K > 0$ such that*

$$\frac{|d\pi(z)|}{d(\pi(z), \partial D')} \leq K \frac{|dz|}{d(z, \partial D)}, \qquad z \in D;$$

(6) $\log|\pi'| \in BMOH(D)$;

(7) $\alpha_1 = \sup_{\zeta \in \mathbf{C}} \|\log|\pi - \zeta|\|_{*,D} < \infty$;

(8) $\alpha_2 = \sup_{\zeta \in D'} \|\log|\pi - \zeta|\|_{*,D} < \infty$;

(9) $\alpha_3 = \sup_{\zeta \in \mathbf{C} \setminus D'} \|\log|\pi - \zeta|\|_{*,D} < \infty$;

(10) $\alpha_4 = \sup_{\zeta \in \partial D'} \|\log|\pi - \zeta|\|_{*,D} < \infty$.

If D' admits Green functions $g_{D'}(\cdot, \zeta)$, $\zeta \in D'$, then the next condition is also equivalent:

(11) $\alpha_5 = \sup_{\zeta \in D'} \|g_{D'}(\pi, \zeta)\|_{*,D} < \infty$.

Moreover $\|T_F\|_{BMO}$, $\|T_F\|_{BMOH}$, $\inf L + 1$, $\inf p$, $\inf K$, $\|\log|\pi'|\|_{*,D} + 1$, *and* α_k, $(k = 1, 2, 3, 4(, 5))$ *are comparable to each other.*

Proof. Since $|dz|/d(z, \partial D) \leq 4|d\pi(z)|/d(\pi(z), \partial D')$, $z \in D$, by Koebe's theorem, we have $\inf K \geq 1/4$. Hence $\|T_F\|_{BMOH}$, $\inf L + 1$, $\inf K$, α_3, and α_4 are comparable by Theorem 2. The proof of Theorem 1 shows $\inf L + 1 \approx \|\log|\pi'|\|_{*,D}$ and $\inf p \lesssim \alpha_k \lesssim \|T_F\|_{BMO} \lesssim \inf L + 1$, $k = 1, 2, 5$. As for $\inf K \lesssim \inf p$ see Corollary 12 below.

Remark that the condition (5) of Corollary 6 implies that the metrics $|dz|/d(z, \partial D)$ and $|d\pi(z)|/d(\pi(z), \partial D')$ on D are comparable to each other.

Let $\pi: \tilde{D} \to D$, $\tilde{D} = \mathbf{C}$ or Δ, be a universal covering map of plane domain D. Let $BMO_0(D)$ be the space of all $L^1_{loc}(D)$ functions g such that $g \circ \pi \in BMO(\tilde{D})$. Let $BMOH_0(D)$ (resp. $BMOA_0(D)$) be the space of all harmonic (holomorphic) $BMO_0(D)$ functions. Then it always holds that $BMO_0(D) \subset BMO(D)$ by the conformal invariance of BMO. Hence π is BMO (resp. $BMOH$, $BMOA$) map if and only if $BMO_0(D) = BMO(D)(BMOH_0(D) = BMOH(D), BMOA_0(D) = BMOA(D))$.

Corollary 7 (cf. [13], [4]). *Let D be a plane domain having $\pi: \Delta \to D$ as a universal covering map. Then the following conditions are equivalent to each other*:

(1) *π is a BMO map*;

(2) *π is a BMOH map*;

(3) *There exists a constant* $\alpha > 0$ *such that for every disk* $B \subset \Delta$ *with hyperbolic radius* α, π *is univalent on* B;

(4) *The hyperbolic metric* $\rho_D(z)|dz|$ *and the quasihyperbolic metric* $|dz|/d(z, \partial D)$ *on* D *are comparable, that is, there exists a constant* $K > 0$ *such that*

$$\frac{|dz|}{d(z, \partial D)} \leq K\rho_D(z)|dz|, \qquad z \in D;$$

(5) $\log|\pi'| \in BMOH(\Delta)$;

(6) $\log \rho_D \in BMO_0(D)$.

Moreover $\|T_\pi\|_{BMO}$, $\|T_\pi\|_{BMOH}$, $\inf(\alpha^{-1}) + 1$, $\inf K$, $\|\log|\pi'|\|_{*,\Delta} + 1$, *and* $\|\log \rho_D \circ \pi\|_{*,\Delta}$ *are comparable to each other.*

Proof. Since $\rho_\Delta(z)|dz|$ and $|dz|/d(z, \partial\Delta)$ are comparable, the equivalence of (1) $\sim$ (5) follows. Next, applying Proposition 5 to the equality $\log \rho_\Delta = \log \rho_D \circ \pi + \log|\pi'|$ we have

$$\|\log \rho_D \circ \pi\|_{*,\Delta} \gtrsim \int_{|z|<1/2} \Delta(\log \rho_\Delta) dxdy \gtrsim 1.$$

And so we have $\|\log|\pi'|\|_{*,\Delta} + 1 \approx \|\log \rho_D \circ \pi\|_{*,\Delta}$, since $\log \rho_\Delta \in BMO(\Delta)$.

It is well known that the condition (3) of the above corollary holds if and only if $\mathbf{C} \setminus D$ is a uniformly perfect set (cf. [1], [15]), and we can replace the condition (3) with

(3') D admits the Green function g_D and there exists a constant $K > 0$ such that the domain $\{\zeta \in D \mid g_D(\zeta, z) > K\}$ is simply connected for every $z \in D$.

It is to be noted that (3) $\Rightarrow$ (3') does not hold in general when D is a Riemann surface even if D admits the Green function (cf. [19]).

Corollary 8. *Let* $F: D \to D'$ *be a nonconstant holomorphic map between plane domains, and* D' *simply connected. Then* F *is a BMOH map and* $\|T_F\|_{BMOH} \lesssim 1$.

Here we give a lower estimation of *BMOH* map norms.

Proposition 6. *Let* $F: D \to D'$, $D' \neq \mathbf{C}$, *be a nonconstant holomorphic map between plane domains. Let* F *be univalent on a subdomain* D_0 *of* D *and* $F(D_0)$ *contains a disk* B *such that* $d(B, \partial D') \leq L\mathrm{rad}(B)$, $L \geq 1$. *Then* $\|T_F\|_{BMOH} \gtrsim 1/L$.

Proof. Let $\zeta \in \partial D'$ such that $d(\zeta, B) \leq L\mathrm{rad}(B)$. Then an easy calculation shows $\|\log|\cdot - \zeta|\|_{*,B} \gtrsim 1/L$. Since *BMO* is conformally invariant, we have

$$\|\log|F - \zeta|\|_{*,D} \geq \|\log|F - \zeta|\|_{*,D_0} \gtrsim \|\log|\cdot - \zeta|\|_{*,F(D_0)} \gtrsim 1/L.$$

Let $F_n: \Delta \to n\Delta$, $F_n(z) = z$. Then $\|T_{F_n}\|_{BMOH} \approx 1/n$. Hence the above estimation is best possible. (Compare this with the fact $\|T_F\|_{BMO} \geq 1$ for every F.)

Next we investigate *BMOA* maps. Let g be an analytic function on a plane domain D. Let $d_g(z)$ be the radius of the largest schlicht disk around $g(z)$ on the Riemann surface of the inverse function of g. Then $g \in \mathscr{B}(D)$ if and only if $\sup_{z\in D} d_g(z) < \infty$ (cf. [14]), hence by Proposition 2 we have

Theorem 3 ([4]). *Every nonconstant holomorphic map $F: D \to D'$ between plane domains is BMOA map and $\| T_F \|_{BMOA} \lesssim 1$.*

Using this fact we can give another proof of Corollary 8 as follows: Let $g \in BMOH(D')$, and f an analytic function on D' such that $\mathrm{Re} f = g$. Since $|\nabla f| = 2|\nabla g|$, Proposition 2 and Theorem 3 shows $\| g \circ F \|_{*,D} \lesssim \| f \circ F \|_{\mathscr{B}(D)} \lesssim \| f \|_{\mathscr{B}(D')} \lesssim \| g \|_{*,D'}$.

Example 1. Let $\Delta^* = \Delta \setminus \{0\}$ and $g(z) = \log|z|$.
(1) Let $F: \Delta \to \Delta^*$ be a universal covering map, then F is not a *BMOH* map by Corollary 7.
(2) Let $F_n: \Delta^* \to \Delta^*$, $F_n(z) = z^n$. Then $\| T_{F_n} \|_{BMO} \approx \| T_{F_n} \|_{BMOH} \approx n$ by Corollary 6. ($\| T_{F_n} \|_{BMOH} \geq n$ is trivial since $g \circ F_n(z) = ng(z)$.)
(3) Let $G_n: \Delta^* \to \Delta$, $G_n(z) = z^n$. Then $\| T_{G_n} \|_{BMOH} \approx 1$ by Corollary 8 and Proposition 6. On the other hand $\| T_{G_n} \|_{BMO} \approx \| T_{F_n} \|_{BMO} \approx n$.
(4) Let $B: \Delta \to \Delta$ be a Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \left(- \frac{z - (1 - 1/n^3)}{1 - (1 - 1/n^3)z} \right)^n,$$

then B is a *BMOH* map and $\| T_B \|_{BMOH} \approx 1$ by Corollary 8 and Proosition 6. On the other hand B is not a *BMO* map by Theorem 1.
(5) Let B is the Blaschke product in (4). Let $D = B^{-1}(\Delta^*)$. then $B_0 = B|D: D \to \Delta$ is a *BMOH* map and $\| T_{B_0} \|_{BMOH} \approx 1$ by Corollary 8 and Proposition 6. On the other hand, $B_1 = B|D: D \to \Delta^*$ is not a *BMOH* map by Corollary 7.

Finally we remark that if a given holomorphic map between plane domains is a *BMO* map, then it is always a *BMO* map independent of the choice of its target by Theorem 1.

2. Hahn metric

Let R be a Riemann surface. We define the Hahn metric $\hat{\rho}_R(z)|dz|$ on R by

$$\hat{\rho}_R(z) = \inf_G \rho_G(z) (\geq \rho_R(z)),$$

where the infimum is taken over all simply connected domains $G \subset R$ containing z and $\rho_G(z)|dz|$ denotes the hyperbolic metric on G. If G does not admit the hyperbolic metric we regard $\rho_G(z)$ as 0. Equivalently, $\hat{\rho}_R(z) = \inf |\phi'(0)|^{-1}$, where the infimum is taken over all conformal maps ϕ of Δ into R such that $\phi(0) = z$. If R is simply connected, Hahn metric and the hyperbolic metric coincide. Hahn metric is a conformally invariant continuous metric and there exists a unique simply connected domain $G \subset R$ for each $z \in R$ such that $\hat{\rho}_R(z) = \rho_G(z)$

([8], [11]). The Hahn metrics on $\mathbf{C}\setminus\{0\}$, $\Delta\setminus\{0\}$, and $\{r_1<|z|<r_2\}$ are explicitly calculated by [11]. By the definition we have

Proposition 7. (1) *Let $S\subset R$ then $\hat{\rho}_R(z)|dz|\le\hat{\rho}_S(z)|dz|$, $z\in S$.*
(2) *Let $\pi\colon\tilde{R}\to R$ be a covering map. Then $\hat{\rho}_R(\pi(z))|d\pi(z)|\ge\hat{\rho}_{\tilde{R}}(z)|dz|$, $z\in\tilde{R}$.*

Hahn metric on a plane domain D, $D\neq\mathbf{C}$, is equivalent to the quasihyperbolic metric $|dz|/d(z,\partial D)$. Indeed we have the following by Koebe's theorem:

Proposition 8 (cf. [4]). $\dfrac{|dz|}{4d(z,\partial D)}\le\hat{\rho}_D(z)|dz|\le\dfrac{|dz|}{d(z,\partial D)}$, $z\in D$.

Hence Hahn metric is a generalization of the quasihyperbolic metric. In this section we investigate some basic properties of Hahn metric.

Let $\pi\colon D\to R$ be a covering map between a plane domain D and a Riemann surface R. Let l_z, $z\in D$, denotes the Euclidean radius of the largest disk on D around z on which π is univalent. We set $l_z=\infty$ if D contains arbitrary large such disks. Then we have

Theorem 4 (cf. [4]). $\dfrac{|dz|}{4l_z}\le\hat{\rho}_R(\pi(z))|d\pi(z)|\le\dfrac{|dz|}{l_z}$, $z\in D$.

Proof. We may assume $l_z<\infty$ since $\hat{\rho}_{\mathbf{C}}=0$. Let $\phi_0(\zeta)=\pi(z+l_z\zeta)$. Since $\phi_0\colon\Delta\to R$ satisfies $\phi_0(0)=\pi(z)$, we have $\hat{\rho}_R(\pi(z))|\pi'(z)|\le|\pi'(z)|/|\phi_0'(0)|=1/l_z$. Next Let $\phi\colon\Delta\to R$ be an arbitrary injective holomorphic map such that $\phi(0)=\pi(z)$. Let G be the component of $\pi^{-1}(\phi(\Delta))$ containing z. Applying Koebe's theorem to the conformal map $g=\pi^{-1}\circ\phi\colon\Delta\to G$ we have $|\pi'(z)|/|\phi'(0)|$ $=1/|g'(0)|\ge 1/4l_z$.

We have $\hat{\rho}_{\hat{\mathbf{C}}}\le\hat{\rho}_{\mathbf{C}}=0$ by Proposition 7. On the other hand, let $R\neq\hat{\mathbf{C}}$, $\mathbf{C}$, and $\pi\colon D\to R$ its universal covering map, then $l_z<\infty$. Hence

Corollary 9 ([11]). *Hahn metric $\hat{\rho}_R(z)|dz|$ degenerates if and only if $R=\hat{\mathbf{C}}$ or $\mathbf{C}$.*

Let R be a Riemann surface having $\pi\colon\Delta\to R$ as its universal covering map. Let r_z, $z\in R$, be the hyperbolic radius of the largest hyperbolic disk around z on which π is univalent. Let L_z be the Euclidean radius of the disk on Δ around the origin with hyperbolic radius r_z. Then since $\hat{\rho}_\Delta=\rho_\Delta$ we have

Corollary 10. *Let R be a Riemann surface having $\pi\colon\Delta\to R$ as its universal covering map. Then*

$$\frac{\rho_R(z)|dz|}{4L_z}\le\hat{\rho}_R(z)|dz|\le\frac{\rho_R(z)|dz|}{L_z},\qquad z\in R.$$

Corollary 11 (cf. [4]). *Let R be a Riemann surface having $\pi\colon\Delta\to R$ as its universal covering map. Then Hahn metric and the hyperbolic metric are comparable to each other if and only if $\inf_{z\in R}r_z>0$.*

Let h_R and $\hat{h}_R$ denote the hyperbolic distance function and Hahn distance function on a Riemann surface R respectively. Lt $z \in R$, $t > 0$, and set

$$B^R_{z,t} = \{\zeta \in R \mid h_R(\zeta, z) < t\}, \qquad \hat{B}^R_{z,t} = \{\zeta \in R \mid \hat{h}_R(\zeta, z) < t\}.$$

Lemma 4. *Let $\pi: D \to R$ be a covering map between a plane domain D and a Riemann surface $R(\neq \mathbf{C}, \hat{\mathbf{C}})$, then*

$$\pi\left(\left\{|\zeta - z| < \frac{l_z t}{2}\right\}\right) \subset \hat{B}^R_{\pi(z),t} \subset \pi(\{|\zeta - z| < 6 l_z t\}), \qquad z \in D,\ 0 < t \leq \frac{1}{12}.$$

$$\frac{t}{8}\rho_{\hat{B}^R_{z,t}}(z)|dz| \leq \hat{\rho}_R(z)|dz| \leq 6t\hat{\rho}_{\hat{B}^R_{z,t}}(z)|dz|, \qquad z \in R,\ 0 < t \leq \frac{1}{12}.$$

Proof. Let $|\zeta - z| \leq l_z/2$ then $l_z/2 \leq l_\zeta \leq 3l_z/2$ hence $|d\zeta|/6l_z \leq \hat{\rho}_R(\pi(\zeta))|d\pi(\zeta)| \leq 2|d\zeta|/l_z$ by Theorem 4, which implies the first inequality. Next let $z \in R$ and $w \in \pi^{-1}(z)$. Using Theorem 4 again we have

$$\rho_{\hat{B}^R_{z,t}}(z)|dz| \leq \hat{\rho}_{\hat{B}^R_{z,t}}(z)|dz| \leq \hat{\rho}_{B_0}(z)|dz| = \rho_{B_0}(z)|dz| = \frac{2}{l_w t}|dw| \leq \frac{8}{t}\hat{\rho}_R(z)|dz|,$$

where $B_0 = \pi(\{|\zeta - w| < l_w t/2\})$. We obtain the inequality $\hat{\rho}_R(z)|dz| \leq 6t\rho_{\hat{B}^R_{z,t}}(z)|dz|$ similarly.

Lemma 5. *Let $R(\neq \hat{\mathbf{C}}, \mathbf{C})$ be a Riemann surface.*
(1) *Let $\phi: \Delta \to R$ be an injective holomorphic map then*

$$\phi(B^\Delta_{z,t}) \subset \hat{B}^R_{\phi(z),t}, \qquad z \in \Delta,\ t > 0.$$

In particular $\phi(\{|\zeta| < t\}) \subset \hat{B}^R_{\phi(0),4t/3}$, $0 < t \leq 1/2$.
(2) *Let $z \in R$. Then there exists an injective holomorphic map $\phi_0: \Delta \to R$, $\phi_0(0) = z$, such that*

$$\hat{B}^R_{z,t/12} \subset \phi_0(\{|\zeta| < t\}), \qquad 0 < t \leq 1.$$

Proof. (1) is trivial since $\{|\zeta| < t\} \subset B^\Delta_{0,4t/3}$, $0 < t \leq 1/2$. Next let $\pi: D \to R$ be a covering map from a plane domain D, $w \in \pi^{-1}(z)$, and set $\phi_0(\zeta) = \pi(w + l_w\zeta/2)$, $\zeta \in \Delta$. Then Lemma 4 shows that $\phi_0(|\zeta| < t\}) = \pi(\{|\zeta - w| < l_w t/2\}) \supset \hat{B}^R_{z,t/12}$.

Proposition 9. *Let γ be a closed curve on Riemann surface R which is not homotopic to a point, then*

$$\int_\gamma \hat{\rho}_R(z)|dz| \geq \frac{\pi}{2}.$$

The equality holds if and only if $R = \mathbf{C}\setminus\{0\}$ and $\gamma = \{|z| = r\}$, $0 < r < \infty$.

Proof. (Dr. T. Sugawa, oral communication) Let R be a Riemann surface and γ a closed curve on R which is not homotopic to a point. Let $\pi: \tilde{R} \to R$, $\tilde{R} = \mathbf{C}$ or Δ, be the universal covering map. Let g be the covering transformation induced by γ, G the cyclic group generated by g, and $R_0 = \tilde{R}/G$. Then

$R_0 = \{a < |z| < b\}$, $0 \le a < b \le \infty$. Let $\tilde{\gamma} \subset R_0$ be the unique closed curve which is a lift of γ under the covering map $\pi_0 \colon R_0 \to R$. Since $\hat{\rho}_{\mathbf{C}\setminus\{0\}}(z) = 1/4|z|$ (see Proposition 8), Proposition 7 and 8 show that

$$\int_\gamma \hat{\rho}_R(z)|dz| \ge \int_{\tilde{\gamma}} \hat{\rho}_{R_0}(z)|dz| \ge \int_{\tilde{\gamma}} \hat{\rho}_{\mathbf{C}\setminus\{0\}}(z)|dz| = \int_{\tilde{\gamma}} \frac{|dz|}{4|z|} \ge \pi/2.$$

Since the equality in Proposition 7 (1) (resp. (2)) holds if and only if $R = S$ ($\tilde{R} = R$) (cf. [11]), the equality above holds if and only if $R = R_0 = \{0 < |z| < \infty\}$ and $\gamma = \tilde{\gamma} = \{|z| = r\}$, $0 < r < \infty$.

In the case of the quasihyperbolic metric on a plane domain D, we have

$$\int_\gamma \frac{|dz|}{d(z, \partial D)} \ge 2\pi,$$

for every closed curve $\gamma \subset D$ which is not homotopic to a point, where the constant 2π is best possible.

Proposition 10. *Let $F \colon R \to R'$ be a nonconstant holomorphic map between Riemann surfaces, where $R, R' \ne \mathbf{C}, \hat{\mathbf{C}}$. Then the following conditions are equivalent:*

(1) *There exists a constant $K > 0$ such that*

$$\hat{\rho}_{R'}(F(z))|dF(z)| \le K\hat{\rho}_R(z)|dz|, \qquad z \in R;$$

(2) *There exists a constant $L > 0$ such that $F(\hat{B}^R_{z,L})$ is contractible on R' for each $z \in R$.*

Moreover $\inf(L^{-1}) \lesssim \inf K \lesssim \inf(L^{-1}) + 1$.

Proof. ((1)$\Rightarrow$(2)) Let F satisfy (1). We show (2) holds if we set $L \le \pi/4K$. Otherwise, there exists $z \in R$ such that $F(\hat{B}^R_{z,L})$ is not contractible. Then there exist two points $z_1, z_2 \in \hat{B}^R_{z,L}$, $z_1 \ne z_2$, $F(z_1) = F(z_2)$, and a curve γ on $\hat{B}^R_{z,L}$ joining z_1 to z_2 such that $\int_\gamma \hat{\rho}_R(\zeta)|d\zeta| < 2L$ and $F(\gamma)$ is a closed curve which is not homotopic to a point. It follows from Proposition 9 that

$$\frac{\pi}{2} \le \int_{F(\gamma)} \hat{\rho}_{R'}(w)|dw| \le K \int_\gamma \hat{\rho}_R(\zeta)|d\zeta| < 2KL \le \frac{\pi}{2},$$

which is a contradiction.

((2)$\Rightarrow$(1)) Let F satisfy (2). We may assume $L \le 1/12$. Let $z \in R$, and $G \subset R'$ a simply connected domain containing $F(\hat{B}^R_{z,L})$. Applying Lemma 4 we obain

$$\begin{aligned}\hat{\rho}_{R'}(F(z))|dF(z)| &\le \hat{\rho}_G(F(z))|dF(z)| \\ &= \rho_G(F(z))|dF(z)| \le \rho_{\hat{B}^R_{z,L}}(z)|dz| \le \frac{8}{L}\hat{\rho}_R(z)|dz|.\end{aligned}$$

Corollary 12. *Let $\pi \colon R \to R'$, $R \ne \hat{\mathbf{C}}, \mathbf{C}$, be a covering map between Riemann*

surfaces. Then the following three conditions are equivalent to each other:
(1) *There exists a constant* $K > 0$ *such that*

$$\hat{\rho}_{R'}(\pi(z))\,|d\pi(z)| \le K\hat{\rho}_R(z)\,|dz|, \qquad z \in R;$$

(2) *There exists a constant* $L > 0$ *such that* π *is univalent on* $\hat{B}^R_{z,L}$ *for each* $z \in R$;
(3) *There exists an integer* $p > 0$ *such that* π *is p-valent on each* $\hat{B}^R_{z,1/24}$, $z \in R$.
Moreover $\sqrt{\inf p} \lesssim \inf(L^{-1}) + 1 \approx \inf K \lesssim \inf p$.

Proof. Since $K \ge 1$, $\inf K$ and $\inf(L^{-1}) + 1$ are comparable by Proposition 10.

Next we show $\inf K \lesssim \inf p$. Let (3) hold. Let $z \in R$, $\zeta = \pi(z)$ and $\pi_0 \colon \tilde{R} \to R$, $\tilde{R} = \mathbf{C}$, or Δ, a universal covering map such that $\pi_0(0) = z$. Let l and l' be the largest Euclidean radii of the disks on $\tilde{R}$ around the origin on which π_0, $\pi \circ \pi_0$ are univalent respectively. Then $\pi \circ \pi_0$ is p-valent on $\{|w| < l/24\}$ by Lemma 4. Then $\{|w| < l/48p\}$ contains no two equivalent points for $\pi \circ \pi_0$ (cf. the proof of Lemma 3). And so $l' \ge l/48p$. Hence Theorem 4 shows $\hat{\rho}_{R'}(\zeta)\,|d\zeta| \le |dw|/l' \le 48p\,|dw|/l \le 192p\hat{\rho}_R(z)\,|dz|$.

Finally we show $\sqrt{\inf p} \lesssim \inf(L^{-1}) + 1$. Let (2) hold. We may assume $L \le 1/12$. Let $\pi_0 \colon \tilde{R} \to R$, $\tilde{R} = \mathbf{C}$, Δ, be a universal covering map. Let $z \in R$ and $w \in \pi_0^{-1}(z)$. Since $\hat{B}^R_{z,1/24} \subset \pi_0(\{|\zeta - w| < l_w/4\})$, Theorem 4 shows that there exist $z_k \in \hat{B}^R_{z,1/24}$, $1 \le k \le k_0$, $k_0 \lesssim L^{-2}$, such that $\hat{B}^R_{z,1/24} \subset \bigcup_k \hat{B}^R_{z_k,L}$. Since each $\hat{B}^R_{z_k,L}$ contains at most one point which is equivalent to z by $\pi \circ \pi_0$, we have $\inf p \lesssim L^{-2}$.

A simple example (see Example 3 below) shows there exists a sequence of covering maps F_n such that $\inf K_n = n$, $\inf p_n \approx n^2$, $n = 1, 2, \ldots$ On the other hand, we show that if R' is noncompact $\inf(L^{-1}) + 1 \approx \inf K \approx \inf p$ holds later (Theorem 11). Here we give one sufficient condition for (1) (and (2), (3)) above.

Lemma 6. *Let* $\pi \colon R \to R'$ *be a covering map between Riemann surfaces satisfying the condition a) or b) below. Then* π *satisfies the condition* (1) *of Corollary* 12.
a) R *is a torus or* $\mathbf{C} \setminus \{0\}$, *and* R' *is a torus.*
b) R' *admits the hyperbolic metric, and* $\inf_{z \in R'} r_z > 0$, *where* r_z *is the hyperbolic radius of the largest hyperbolic disk around* $z \in R'$.
In particular if $\pi \colon R \to R'$, $R \ne \mathbf{C}$, $\hat{\mathbf{C}}$, *is a covering map with compact target, then the condition* (1) *of Corollary* 12 *always holds.*

Proof. (Case a)) The assertion easily follows if we apply Theorem 4 to the universal covering map of R. (Case b)) By Corollary 11 we have

$$\hat{\rho}_{R'}(\pi(z))\,|d\pi(z)| \le C\rho_{R'}(\pi(z))\,|d\pi(z)| = C\rho_R(z)\,|dz| \le C\hat{\rho}_R(z)\,|dz|.$$

Lemma 7. *Let* $F \colon R \to R'$, $R \ne \mathbf{C}$, $\hat{\mathbf{C}}$, $R' \ne \hat{\mathbf{C}}$, *be a nonconstant holomorphic map,* $L > 0$, *and* $z \in R$. *Let* F *be locally univalent p-valent on* $\hat{B}^R_{z,L}$. *Then there exists a constant* $C = C(p, L) > 0$ *such that* F is univalent on $\hat{B}^R_{z,C}$.

Proof. Let $z \in R$. Because of Lemma 5 (2) there exists an injective holomorphic map $\phi_0: \Delta \to R$, $\phi_0(0) = z$, such that $\hat{B}^R_{z,t/12} \subset \phi_0(\{|\zeta| < t\})$, $0 < t \leq 1$. Let $r_1 = \min\{3L/4, 1/2\}$ then $\phi_0(\{|\zeta| < r_1\}) \subset \hat{B}^R_{z,L}$ by Lemma 5 (1). Hence $F \circ \phi_0$ is p-valent on $\{|\zeta| < r_1\}$. Therefore Lemma 3 shows that there exists a constant $r_2 < r_1$ such that $F \circ \phi_0$ is univalent on $\{|\zeta| < r_2\}$. And so F is univalent on $\hat{B}^R_{z,r_2/12}$.

3. *BMO* maps between Riemann surfaces

Let R be a Riemann surface. Let $BMO_*(R)$ be the space of all locally integrable functions g on R such that

$$\|g\|_{**,R} = \sup_{\phi} \pi^{-1} \int_{\Delta} |g \circ \phi - (g \circ \phi)_{\Delta}| dxdy < \infty,$$

where the supremum is taken over all injective holomorphic maps $\phi: \Delta \to R$ ([4]). Equivalently, $\|g\|_{**,R} = \sup_{\phi} \|g \circ \phi\|_{*,\Delta}$, where the supremum is taken over all injective holomorphic maps ϕ of Δ into R, or over all injective holomorphic maps ϕ of a plane domain into R.

Also $\mathscr{B}_{h*}(R)$ be the space of all harmonic functions g on R such that

$$\|g\|_{\mathscr{B}_{h*}(R)} = \sup_{z \in R} |\nabla g(z)| \hat{\rho}_R(z)^{-1} < \infty.$$

In the case that D is a plane domain we have

Proposition 11 ([4]). (1) *$BMO_*(D) = BMO(D)$ and it holds that*

$$\|g\|_{*,D} \leq \|g\|_{**,D} \lesssim \|g\|_{*,D}, \qquad g \in BMO_*(D).$$

(2) *$\mathscr{B}_{h*}(D) = \mathscr{B}_h(D)$ and it holds that*

$$\|g\|_{\mathscr{B}_h} \leq \|g\|_{\mathscr{B}_{h*}} \leq 4\|g\|_{\mathscr{B}_h}, \qquad g \in \mathscr{B}_{h*}(D).$$

(1) is a consequence of the conformal invariance of *BMO* (Corollary 1) and (2) is a consequence of Proposition 8. Hence BMO_* and $\mathscr{B}_{h*}$ are generalizations of *BMO* and $\mathscr{B}_h$ to Riemann surfaces respectively. So in the following we identify BMO_*, $\mathscr{B}_{h*}$ with BMO, $\mathscr{B}_h$ and use the notation $BMO(R)$, $\mathscr{B}_h$ instead of $BMO_*(R)$, $\mathscr{B}_{h*}$ for an arbitrary Riemann surface R for the simplicity, ignoring the ambiguity of universal constant factors of their norms.

Now we investigate *BMO* maps between Riemann surfaces. By the definition we have

Proposition 12. (1) *Let $i: R \to R'$ be an inclusion map. Then i is a BMO map and $\|T_i\|_{BMO} \leq 1$.*
(2) *Let $\pi: R \to R'$ be a covering map. Then $\|g\|_{*,R'} \leq \|T_\pi g\|_{*,R}$, $g \in BMO(R')$.*

Theorem 5 (cf. [4]). *Let R be a Riemann surface, D a plane domain, and $\pi: D \to R$ a covering map. Let $g \in L^1_{\mathrm{loc}}(R)$. Then $g \in BMO(R)$ if and only if*

$$K = \sup |B|^{-1} \int_B |g \circ \pi - (g \circ \pi)_B| \, dxdy < \infty,$$

where the supremum is taken over all disks $B \subset D$ such that π is univalent on B. Moreover it holds that $K \leq \|g\|_{*,R} \lesssim K$.

Proof. $K \leq \|g\|_{*,R}$ is trivial. Next let $\phi\colon \Delta \to R$ be an injective holomorphic map. Let $\tilde{\phi}\colon \Delta \to D$ be a lift of ϕ, then $\tilde{\phi}$ is also injective. Let $D_0 = \tilde{\phi}(\Delta)$. Let $g \in L^1_{loc}(R)$ satisfy $K < \infty$. Since π is injective on each disk $B \subset D_0$, $\|g \circ \pi\|_{*,D_0} \leq K$. Hence we have $\|g \circ \phi\|_{*,\Delta} = \|(g \circ \pi) \circ \tilde{\phi}\|_{*,\Delta} \lesssim \|g \circ \pi\|_{*,D_0} \lesssim K$, by Corollary 1, which implies the assertion.

Theorem 6. *Proposition 2, Theorem 3, and Corollary 8 are true for Riemann surfaces, that is,*

(1) *$BMOH(R) = \mathscr{B}_h(R)$, $BMOA(R) = \mathscr{B}(R)$ hold for every Riemann surface R and $\|g\|_{\mathscr{B}_h(R)} \approx \|g\|_{*,R}$, $g \in BMOH(R)$.*

(2) *Every nonconstant holomorphic map $F\colon R \to R'$ between Riemann surfaces is $BMOA$ map and $\|T_F\|_{BMOA} \lesssim 1$.*

(3) *Let $F\colon R \to R'$ be a nonconstant holomorphic map between Riemann surfaces, and R' simply connected. Then F is a $BMOH$ map and $\|T_F\|_{BMOH} \lesssim 1$.*

Proof. Since the proof is a routine work, we prove only (1). Let g be a harmonic function on R, then by Proposition 2

$$\|g\|_{*,R} = \sup_{\phi} \|g \circ \phi\|_{*,\Delta} \approx \sup_{\phi,z} \frac{|\nabla g(\phi(z))|\,|\phi'(z)|}{\rho_\Delta(z)}$$

$$= \sup_{G,w} \frac{|\nabla g(w)|}{\rho_G(w)} = \sup_{w} \frac{|\nabla g(w)|}{\hat{\rho}_R(w)} = \|g\|_{\mathscr{B}_h(R)},$$

where $\sup_{\phi,z}$ is taken over all injective holomorphic maps $\phi\colon \Delta \to R$ and all points $z \in \Delta$, and $\sup_{G,w}$ is taken over all simply connected subdomains G of R and all points $w \in G$.

Theorem 7 (generalized localiztion theorem) (cf. Proposition 3, cf. [4]). *Let $L \leq 1$, $K > 0$, and g a locally integrable function on a Riemann surface $R(\neq \hat{\mathbf{C}}, \mathbf{C})$ such that $\|g\|_{*,\hat{B}_{z,L}} \leq K$ for every $z \in R$. Then $g \in BMO(R)$ and $\|g\|_{*,R} \lesssim L^{-1}K$.*

Proof. Let $\phi\colon \Delta \to R$ be an injective holomorphic map, and $w \in \Delta$. Then $\phi(B^\Delta_{w,L}) \subset \hat{B}^R_{\phi(w),L}$. Hence $\|g \circ \phi\|_{*,B^\Delta_{w,L}} \leq K$, and so $\|g \circ \phi\|_{*,\Delta} \lesssim L^{-1}K$ by the localization theorem.

Theorem 8 (generalized removability theorem) (cf. Proosition D, cf. [4]).

(1) *Let E be a discrete subset of a Riemann surface $R(\neq \hat{\mathbf{C}}, \mathbf{C})$ such that $\#(\hat{B}^R_{z,L} \cap E) \leq K$, $z \in R$. Let $R_0 = R \setminus E$, and $g \in BMO(R_0)$. Then $g \in BMO(R)$ and $\|g\|_{*,R} \leq C(K, L)\|g\|_{*,R_0}$.*

(2) *Let $E \subset \hat{\mathbf{C}}$ satisfy $\#E \leq K$, and $R_0 = \hat{\mathbf{C}} \setminus E$. Let $g \in BMO(R_0)$. Then $g \in BMO(\hat{\mathbf{C}})$ and $\|g\|_{*,\hat{\mathbf{C}}} \leq C(K)\|g\|_{*,R_0}$. In particular we have $BMO(\mathbf{C}) =$*

$BMO(\hat{\mathbf{C}})$.

Proof. The assertion (2) is trivial by the removability theorem. Let $\phi: \Delta \to R$ be an injective holomorphic map. Then $\phi(B^{\Delta}_{z,L}) \subset \hat{B}^{R}_{\phi(z),L}$, $z \in \Delta$. Hence $\#(B^{\Delta}_{z,L} \cap \phi^{-1}(E)) \leq K$, so by the removability theorem

$$\| g \circ \phi \|_{*,\Delta} \leq C(K, L) \| g \circ \phi \|_{*,\Delta \setminus \phi^{-1}(E)} \leq C(K, L) \| g \|_{*,R_0}.$$

Proposition 13 (cf. Proposition 5). *Let s be a superharmonic function on a Riemann surface $R(\neq \hat{\mathbf{C}}, \mathbf{C})$ such that $\Delta s = -\mu$ and $\| s \|_{*,R} \leq K$, then μ is a uniformly locally finite measure on R with respect to the Hahn metric, that is, $\mu(\hat{B}^{R}_{z,1/24}) \lesssim K$, $z \in R$.*

Proof. Let $z \in R$, and ϕ_0 the map in Lemma 5. Then by Proposition 5

$$\mu(\hat{B}^{R}_{z,1/24}) \leq \mu(\phi_0(\{|\zeta| < 1/2\})) \lesssim \| s \circ \phi_0 \|_{*,\Delta} \lesssim \| s \|_{*,R}.$$

Lemma 8 (cf. [5]). (1) *Let R be a noncompact Riemann surface, and $z \in R$. Then there exists a function p_z on R satisfying the following conditions*:

a) p_z *is harmonic on* $R \setminus \{z\}$;

b) $p_z(\zeta) + \log|\zeta - z|$ *is harmonic near* z;

c) $\| p_z \|_{*,R} \lesssim 1$.

(2) *Let R be a compact Riemann surface, and $z_1, z_2 \in R$, $z_1 \neq z_2$. Then there exists a function $p_{z_1 z_2}$ on R satisfying the following conditions*:

a) $p_{z_1 z_2}$ *is harmonic on* $R \setminus \{z_1, z_2\}$;

b) $p_{z_1 z_2}(\zeta) + \log|\zeta - z_1|$ *is harmonic near* z_1, *and* $p_{z_1 z_2}(\zeta) - \log|\zeta - z_2|$ *is harmonic near* z_2;

c) $\| p_{z_1 z_2} \|_{*,R} \lesssim 1$.

Proof. First let R be noncompact. Let p_z be the Green function with pole z if R admits the Green function, and the Evans-Selberg potential with pole z (cf. [18]) if R does not admit the Green function. Then

$$\int_{p_z = s} |{*dp_z}| \leq 2\pi, \qquad s \in \mathbf{R}.$$

Let $\phi: \Delta \to R \setminus \{z\}$ be an injective holomorphic map and f is an analytic function on Δ such that $\operatorname{Re} f = p_z \circ \phi$. Since the Riemann surface of the inverse function of f does not contain a schicht disk whose radius is larger than π, Proposition 2 shows that $\| p_z \circ \phi \|_{*,\Delta} \lesssim 1$. Hence $\| p_z \|_{*,R \setminus \{z\}} \lesssim 1$, and so $\| p_z \|_{*,R} \lesssim 1$ by the generalized removability theorem.

Next let R be compact. Let $p_{z_1 z_2}$ be the unique (up to constants) harmonic function on $R \setminus \{z_1, z_2\}$ satisfying the condition a) and b). Then $p_{z_1 z_2}$ is an Evans-Selberg potential on $R \setminus \{z_2\}$ with pole z_1. Hence $\| p_{z_1 z_2} \|_{*,R \setminus \{z_2\}} \lesssim 1$ and so $\| p_{z_1 z_2} \|_{*,R} \lesssim 1$ by the generalized removability theorem again.

The next theorem is a generalization of Theorem 1. Combining it with Corollary 4, we obtain a characterization of BMO maps $F: R \to R'$ in the case that R' is not compact.

Theorem 9 (cf. Theorem 1, cf. [4]). *Let $F: R \to R'$, $R \neq \mathbf{C}, \hat{\mathbf{C}}$, be a nonconstant holomorphic map between Riemann surfaces. We consider the following two conditions*:

(1) *F is a BMO map*;

(2) *The set E_F of all branch points of F is a removable set for $BMO(R)$ and there exists a constant $L > 0$ such that for every disk $z \in R \setminus E_F$, F is univalent on $\hat{B}^{R \setminus E_F}_{z,L}$.*

(3) *F is of bounded local valence with respect to Hahn metric, that is, there exist a constant $L > 0$ and an integer $p > 0$ such that F is p-valent on $\hat{B}^R_{z,L}$ for each $z \in R$.*

Then it always holds that $(2) \Rightarrow (1)$, $(3) \Rightarrow (1)$. *In the case of $R' \neq \hat{\mathbf{C}}$,* $(3) \Rightarrow (2)$ *holds. Moreover if R' is noncompact* (1), (2), *and* (3) *are equivalent to each other. In particular if R' admits the Green functions $g_{R'}(\cdot, z)$, $z \in R$, the following condition is also equivalent*:

(4) $\sup_{\zeta \in R'} \| g_{R'}(F, \zeta) \|_{*,R'} < \infty$.

Proof. $(2) \Rightarrow (1)$ follows from the conformal invariance of BMO and the generalized localization theorem.

($(3) \Rightarrow (2)$ if $R' \neq \hat{\mathbf{C}}$) Let $R' \neq \hat{\mathbf{C}}$ and F satisfy (3). Let $z \in R$ and $\phi_0: \Delta \to R$, $\phi_0(0) = z$, be the injective holomorphic map satisfying the condition of Lemma 5 (2). Then $F \circ \phi_0$ is p-valent on $B^{\Delta}_{0,L}$ by Lemma 5 (1). Since $R' \neq \hat{\mathbf{C}}$, Lemma 3 shows that $\#(E_{F \circ \phi_0} \cap B^{\Delta}_{0,C_1}) \leq C_2$. And so $\#(E_F \cap \hat{B}^R_{z,C_3}) \leq C_2$ by Lemma 5 (2). Hence E_F is removable for $BMO(R)$ by the generalized removability theorem. Finally since $R' \neq \hat{\mathbf{C}}$, Lemma 7 shows the assertion.

($(3) \Rightarrow (1)$) Let F satisfy (3). Let $w_0 \in R'$ and set $R'_0 = R' \setminus \{w_0\}$, $R_0 = R \setminus F^{-1}(w_0)$. Then $F | R_0: R_0 \to R'_0$, $R'_0 \neq \hat{\mathbf{C}}$, satisfys the condition (3). Therefore $F | R_0$ satisfys (2), and so $F | R_0$ is a BMO map. Since $F^{-1}(w_0)$ is removable for $BMO(R)$ by the generalized removability theorem, F is a BMO map.

($(1) \Rightarrow (3)$ if R' is noncompact) Let R' be noncompact, and F a BMO map. $\zeta \in R'$, and p_ζ the function in Lemma 8. Then $\| p_\zeta \circ F \|_{*,R} \lesssim \| T_F \|_{BMO}$. Since $\Delta(p_\zeta \circ F) = -2\pi \sum_{w \in F^{-1}(\zeta)} \delta_w$, where δ_w is the dirac measure at w, Proposition 13 shows that $\#(\hat{B}^R_{z,1/24} \cap F^{-1}(\zeta)) \lesssim \| T_F \|_{BMO}$, $z \in R$. This argument shows $(1) \Rightarrow (4) \Rightarrow (3)$ when R' admits the Green function.

If $R' = \hat{\mathbf{C}}$, Theorem 9 $(1) \Rightarrow (3)$, $(1) \Rightarrow (2)$ do not hold in general (see Theorem 14 below). Here we give a necessary condition for a nonconstant holomorphic map with compact targets to be a BMO map. The sets $F^{-1}(\zeta)$, $\zeta \in R'$, are similar to each other in the following sense:

Proposition 14. *Let $F: R \to R'$ be a BMO map between Riemann surfaces. Let R' be compact, and $z_1, z_2 \in R'$, $z_1 \neq z_2$. Let k be a C^2 function on Δ with compact support, and $\phi: \Delta \to R$ an injective holomorphic map. Then*

$$\Big| \sum_{w \in (F \circ \phi)^{-1}(z_1)} k(w) - \sum_{w \in (F \circ \phi)^{-1}(z_2)} k(w) \Big| \lesssim \| T_F \|_{BMO} \, \| \Delta k \|_\infty .$$

Proof. Let $p_{z_1 z_2}$ be the function in Lemma 8 on R'. Since

$$\Delta(p_{z_1 z_2} \circ F \circ \phi) = 2\pi\Big(\sum_{w \in (F \circ \phi)^{-1}(z_2)} \delta_w - \sum_{w \in (F \circ \phi)^{-1}(z_1)} \delta_w\Big),$$

the assertion follows from Lemma 1.

If R' is noncompact we can show that

$$\Big|\sum_{w \in (F \circ \phi)^{-1}(z)} k(w)\Big| \lesssim \| T_F \|_{BMO} \| \Delta k \|_\infty$$

by using the function p_z of Lemma 8 instead of $p_{z_1 z_2}$. If R is compact, however, this estimation does not hold in general. (See Theorem 13 and Corollary 20 below.)

The next corollary is an immediate consequence of Theorem 9 if R is noncompact.

Corollary 13. *Let $F: R \to R'$ be a BMO map between Riemann surfaces, $z \in R$, and $v_F(z)$ the valency of F at z, that is, $F(\zeta) = F(z) + c(\zeta - z)^{v_F(z)} + \dots$, $c \neq 0$, near z. Then $v_F(z) \lesssim \| T_F \|_{BMO}$.*

Proof. Let $\phi: \Delta \to R$ be a injective holomorphic map such that $\phi(0) = z$, $F(\phi(\zeta)) \neq F(\phi(0))$, $\zeta \in \Delta \setminus \{0\}$, and $F(\phi(\Delta)) \neq R'$. Let $w_2 = F(z)$ and $w_1 \in R' \setminus F(\phi(\Delta))$. Let $k \geq 0$ be a C^2 function on Δ with compact support such that $k(0) = 1$. Then by Proposition 14 we have

$$v_F(z) = \sum_{\zeta \in (F \circ \phi)^{-1}(w_2)} k(\zeta) - \sum_{\zeta \in (F \circ \phi)^{-1}(w_1)} k(\zeta) \lesssim \| T_F \|_{BMO} \| \Delta k \|_\infty \lesssim \| T_F \|_{BMO}.$$

We give several consequences of Theorem 9.

Corollary 14. *Let $F: R \to R'$ be a nonconstant p-valent holomorphic map between Riemann surfaces. Then F is a BMO map and $\| T_F \|_{BMO} \leq C(p)$. Especially a nonconstant holomorphic map between compact Riemann surfaces is always a BMO map.*

Proof. Let $w_1, w_2 \in F(R)$, $w_1 \neq w_2$ and set $R'_0 = R' \setminus \{w_1, w_2\}$, $R_0 = R \setminus F^{-1}(\{w_1, w_2\})$. Then $F_0 = F \mid R_0: R_0 \to R'_0$ satisfy the condition (3) of Theorem 9. Hence $\| T_{F_0} \|_{BMO} \leq C_1(p)$. Hence $\| T_F \|_{BMO} \leq C_2(p)$ by the generalized removability theorem.

Corollary 15. *Whether a given nonconstant holomorphic map $F: R \to R'$ is a BMO map or not is independent of the choice of its target, that is, let $i_k: R' \to R''_k$, $k = 1, 2$, be injective holomorphic maps, then $i_1 \circ F$ is a BMO map if and only if $i_2 \circ F$ is a BMO map.*

Proof. It suffices to show that let $F: R \to R'$ be a nonconstant holomorphic map, R'_0 a proper subdomain of R' such that $F(R) \subset R'_0$, then F is *BMO* map if and only if $F_0 = F: R \to R'_0$ is a *BMO* map. In this case R'_0 is noncompact. (Case 1) Let $R = \mathbf{C}$ and $R'_0 = \mathbf{C}$ or $\mathbf{C}\setminus\{0\}$. Then $R' = \hat{\mathbf{C}}$ or $\mathbf{C}$ hence $BMO(\hat{\mathbf{C}}) = BMO(\mathbf{C}) = BMO(\mathbf{C}\setminus\{0\})$ implies the assertion. (Case 2) Let $R \neq \hat{\mathbf{C}}, \mathbf{C}$. Let F_0 be a BMO map. Since R'_0 is noncompact, F_0 is of bounded valence with respect to the Hahan metric by Theorem 9. Therefore F is BMO map by Theorem 9 again.

Corollary 16. *Let $F\colon R \to R'$ and $G\colon R' \to R''$ be both nonconstant holomorphic maps. Let R'' be noncompact, and $G \circ F$ a BMO map. Then F is a BMO map.*

Proof. Note that R, R' are both noncompact. In the case $R \neq \mathbf{C}$ the assertion follow from Theorem 9. In the case $R = \mathbf{C}$ then $R' = \mathbf{C}$ or $\mathbf{C}\setminus\{0\}$ and $R'' = \mathbf{C}$ or $\mathbf{C}\setminus\{0\}$. Hence $G \circ F$ is a polynomial by Corollary 4, and so F is a polynomial, which is BMO map by Corollary 4 again.

Next we generalize Theorem 2. Since its proof is almost the same as that of Theorem 2 except for (2)$\Leftrightarrow$(3), which we have already proved as Proposition 10, we omit its proof.

Theorem 10 (cf. Theorem 2). *Let $F\colon R \to R'$, $R, R' \neq \mathbf{C}, \hat{\mathbf{C}}$, be a nonconstant holomorphic map between Riemann surfaces. We consider the following three conditions:*

(1) *F is a $BMOH$ map;*

(2) *There exists a constant $K > 0$ such that*

$$\hat{\rho}_{R'}(F(z))\,|dF(z)| \le K\hat{\rho}_R(z)\,|dz|, \qquad z \in R;$$

(3) *There exists a constant $L > 0$ such that $F(\hat{B}^R_{z,L})$ is contractible on R' for each $z \in R$.*

Then it always holds (2)$\Leftrightarrow$(3)$\Rightarrow$(1) *and* $\|T_F\|_{BMOH} \lesssim \inf K$, $\inf(L^{-1}) \lesssim \inf K \lesssim \inf(L^{-1}) + 1$. *In particular if R' is a plane domain, these three conditions and the following two conditions are equivalent to each other:*

(4) $\alpha = \sup_{\zeta\in\mathbf{C}\setminus R'} \|\log|F-\zeta|\|_{*,R} < \infty$;

(5) $\beta = \sup_{\zeta\in\partial R'} \|\log|F-\zeta|\|_{*,R} < \infty$;

Moreover $\alpha \approx \beta \approx \|T_F\|_{BMOH} \approx \inf K$.

We note that (1)$\Rightarrow$(2) does not holds in general.

Example 2. Let R, $R \neq \hat{\mathbf{C}}$, be a compact Riemann surface, $z_0 \in R$, and set $R' = R\setminus\{z_0\}$. Let $\{|z - z_0| < 1\} \subset R$ be a local disk around z_0, and set $G = \{0 < |z - z_0| < 1\} \subset R'$. We define a map $F\colon \Delta \to R$ by $F(\zeta) = e^{(\zeta-1)/(\zeta+1)} \in G$, which is a universal covering map onto G. F does not satisfy the condition (2) of Theorem 10 since $\hat{\rho}_G(\zeta)|d\zeta|$ and $\hat{\rho}_{R'}(\zeta)|d\zeta|$ are comparable on $\{0 < |z - z_0| < 1/2\} \subset G$. On the other hand, F is a $BMOH$ map since $BMOH(R') = \mathbf{C}$,

We don't know whether (1)$\Rightarrow$(2) is true or not under the assumption $BMOH(R') \neq \mathbf{C}$. We note that we cannot apply the same method used to prove Theorem 2 to prove this. Indeed, even if $BMOH(R') \neq \mathbf{C}$, there is no family of harmonic functions h_z, $z \in R'$, such that

a) $\|h_z\|_{*,R'} \leq C_1,\ z \in R'$,

b) $|\nabla h_z(z)| \geq C_2 \hat{\rho}_{R'}(z),\ z \in R'$,

in general. Let R, $R \neq \hat{\mathbf{C}}$, be a compact Riemann surface, $z_1, z_2 \in T$, $z_1 \neq z_2$, and set $R' = T \setminus \{z_1, z_2\}$. Then $BMOH(R')$ is the space of all harmonic functions on R' which has at most logarithmic singularities at z_1, z_2, so dim $BMOH(R') = 1$. Hence there exists a $z_0 \in R'$ such that $\nabla h(z_0) = 0$, $h \in BMOH(R')$, and so b) does not holds.

Theorem 11 (cf. Corollary 6). *Let $\pi: R \to R'$, $R \neq \mathbf{C}, \hat{\mathbf{C}}$, be a covering map between Riemann surfaces. Then the following conditions are equivalent to each other*:

(1) *π is a BMO map*;

(2) *There exists a constant $L > 0$ such that π is univalent on $\hat{B}^R_{z,L}$ for each $z \in R$*;

(3) *There exists an integer $p > 0$ such that π is p-valent on each $\hat{B}^R_{z,1/24}$, $z \in R$.*

(4) *There exists a constant $K > 0$ such that*

$$\hat{\rho}_{R'}(\pi(z))\,|d\pi(z)| \leq K \hat{\rho}_R(z)\,|dz|, \qquad z \in R,$$

and we have $\|T_F\|_{BMO} \lesssim \inf K \lesssim \inf p$ and $\inf K \approx \inf(L^{-1}) + 1$. Moreover if R' is noncompact we have $\|T_F\|_{BMO} \approx \inf K \approx \inf(L^{-1}) + 1 \approx \inf p$. In particular if R' admits the Green functions $g_{R'}(\cdot, z)$, $z \in R$, the following condition is also equivalent:

(4) $\alpha = \sup_{\zeta \in R'} \|g_{R'}(F, z)\|_{*,R'} < \infty$,
and $\alpha \approx \|T_F\|_{BMO}$.

Proof. (2)$\Leftrightarrow$(3)$\Leftrightarrow$(4) and $\inf(L^{-1}) + 1 \approx \inf K \lesssim \inf p$ follows from Corollary 12. (2)$\Rightarrow$(1) and $\|T_F\|_{BMO} \lesssim \inf(L^{-1}) + 1$ follows from the generalized localization theorem. If R' is noncompact Theorem 9 (and its proof) shows (1)$\Rightarrow$(3) and $\inf p \lesssim \|T_F\|_{BMO}$. Next if R is compact (3) always holds by Lemma 6.

We note that if R' is compact $p \lesssim \|T_F\|_{BMO}$ does not hold in general (See Example 3 below).

The following Theorem and Theorem 11 completely characterize BMO maps in case of covering maps:

Theorem 12. (1) *Let $\pi = e^z: \mathbf{C} \to \mathbf{C} \setminus \{0\}$. Then π is not a BMOH map. (And so π is not a BMO map.)*

(2) *Let $\pi: \mathbf{C} \to T$ be a uniersal covering map of a torus T with modulus $\tau \in \Omega$, where Ω is a fundamental set $\{z = x + iy \mid y > 0,\ -1/2 \leq x \leq 1/2,\ |z| \geq 1\}$ of Riemann moduli space. Then π is a BMO map and*

$$\|T_\pi\|_{BMO} \approx \operatorname{Im} \tau.$$

(*In other word, $\|T_\pi\|_{BMO} \approx \exp(h([i], [\tau]))$,* $\operatorname{Im} \tau > 0$, *where $h([i], [\tau])$* is the distance between $[i]$ *and* $[\tau]$ *in the Riemann moduli space induced by the Teichnüller metric.) In particular it holds that* $\|g\|_{*,T} \leq \|g \circ \pi\|_{*,\mathbf{C}} \lesssim \operatorname{Im} \tau \|g\|_{*,T}$, $g \in BMO(T)$.

We don't know whether the similar estimation holds or not in case of genus ≥ 2. (1) is a consequence of Theorem 2. To prove (2) we need the following two lemmas.

Lemma 9. *Let T be a torus with modulus i, that is, $T = \mathbf{C}/\{m + ni \mid m, n \in \mathbf{Z}\}$. Then its universal covring map $\pi: \mathbf{C} \to T$ is a BMO map.*

Proof. Let $\Omega = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$ and B_0 its circumscribed disk. Let B be a disk on $\mathbf{C}$. (Case 1) Let $\mathrm{rad}(B) \leq \mathrm{rad}(B_0)$. Then π is 2-valent on B, hence $\|g \circ \pi\|_{*,B} \lesssim \|g\|_{*,T}$ by Corollary 14. (Case 2) Let $\mathrm{rad}(B) > \mathrm{rad}(B_0)$. Then by the periodicity of $g \circ \pi$ and Case 1, we have

$$|B|^{-1} \int_B |g \circ \pi - (g \circ \pi)_{B_0}| \, dxdy \lesssim |\Omega|^{-1} \int_\Omega |g \circ \pi - (g \circ \pi)_{B_0}| \, dxdy$$

$$\lesssim |B_0|^{-1} \int_{B_0} |g \circ \pi - (g \circ \pi)_{B_0}| \, dxdy \lesssim \|g\|_{*,T}.$$

Lemma 10. *Let $p > 0$ be an integer, $\Sigma = \{m + ni \mid m, n \in \mathbf{Z}\}$, $\Sigma_p = \{m/p + ni \mid m, n \in \mathbf{Z}\}$, $T_1 = \mathbf{C}/\Sigma$, and $T_2 = \mathbf{C}/\Sigma_p$. Let $\pi: T_1 \to T_2$ be the canonical p-valent covering map. Then π is a BMO map and $\|T_\pi\|_{BMO} \approx p$.*

Proof. $\|T_\pi\|_{BMO} \lesssim p$ follows from Theorem 11. Next let $\tilde{g}$ be a $L^1_{loc}(\mathbf{C})$ function such that

$$\tilde{g}(z) = \begin{cases} t, & 0 \leq t \leq 1/2, \\ (1 - t), & 1/2 < t < 1, \end{cases}$$

where $t = \mathrm{Im}\, z - [\mathrm{Im}\, z]$. Let g be a $L^1_{loc}(T_2)$ function induced by $\tilde{g}$. Let $B \subset \mathbf{C}$ be a disk which contains no two Σ_p-equivalent points. Then

$$|B|^{-1} \int_B |\tilde{g} - \tilde{g}_B| \, dxdy \leq \sup_{z_1, z_2 \in B} |\tilde{g}(z_1) - \tilde{g}(z_2)| \leq 1/p.$$

Hence $\|g\|_{*,T_2} \lesssim 1/p$ by Theorem 12. Since $\|g \circ \pi\|_{*,T_1}$ is independent of p, we have $\|T_F\|_{BMO} \gtrsim p$.

Proof of Theorem 12 (2). By the quasiconformal invariance of BMO, we can assume $\tau = ip$, $p \in \mathbf{N}$. Let $\Sigma_p = \{m/p + ni \mid m, n \in \mathbf{Z}\}$ then $T = \mathbf{C}/\Sigma_p$. Let $\Sigma = \{m + ni \mid m, n \in \mathbf{Z}\}$, $T_1 = \mathbf{C}/\Sigma$, $\pi_0: T_1 \to T$ the canonical p-valent covering map, and $\pi_1: \mathbf{C} \to T_1$ the universal covering map. Since $\pi = \pi_0 \circ \pi_1$ Lemma 9 and 10 show that $\|T_\pi\|_{BMO} \leq \|T_{\pi_0}\|_{BMO} \|T_{\pi_1}\|_{BMO} \lesssim p$.

Next, by Lemma 10 there exists a $BMO(T)$ function g such that $\|g\|_{*,T} = 1$ and $\|g \circ \pi_0\|_{*,T_1} \gtrsim p$. Hence $\|g \circ \pi\|_{*,\mathbf{C}} = \|g \circ \pi_0 \circ \pi_1\|_{*,\mathbf{C}} \gtrsim p$ by Proposition 12, and so $\|T_\pi\|_{BMO} \gtrsim p$.

Corollary 17 (cf. [6]). *Let $F: \mathbf{C} \to \hat{\mathbf{C}}$ be an elliptic function of order p, and T a torus associated with F. Then F is a BMO map and $\|T_F\|_{BMO} \leq C(T, p)$.*

Proof. Let $\pi: \mathbf{C} \to T$ be a universal covering map. Then $F = F_0 \circ \pi$, where $F_0: T \to \hat{\mathbf{C}}$ is a p-valent holomorphic map. Hence Theorem 12 and Corollary 14 we have $\| T_F \|_{BMO} \le \| T_{F_0} \|_{BMO} \| T_\pi \|_{BMO} \le C_1(p) C_2(T)$.

Corollary 18 (cf. Lemma 10). *Let $F: T_1 \to T_2$ be a nonconstant holomorphic map between tori. Then F is a BMO map and $\| T_F \|_{BMO} \lesssim \operatorname{Im} \tau_2$, where τ_2 is the modulus of T_2 contained in a fundamental set $\{z = x + iy \,|\, y > 0, \ -1/2 \le x \le 1/2, |z| \ge 1\}$ of Riemann moduli space.*

Proof. Let $g \in BMO(T_2)$. Let $\pi_i: \mathbf{C} \to T_i$, $i = 1, 2$, be universal covering maps, and $\tilde{F}: \mathbf{C} \to \mathbf{C}$ a lift of F. Then $\tilde{F}(z) = az + b$ so $T_{\tilde{F}}$ is an isometry of $BMO(\mathbf{C})$. Hence by Proposition 12 and Theorem 12 we have

$$\| g \circ F \|_{*, T_1} \le \| g \circ F \circ \pi_1 \|_{*, \mathbf{C}} = \| g \circ \pi_2 \circ \tilde{F} \|_{*, \mathbf{C}} = \| g \circ \pi_2 \|_{*, \mathbf{C}} \lesssim \operatorname{Im} \tau_2 \| g \|_{*, T_2}.$$

Combining Theorem 11 and 12 we have

Corollary 19. *Let $\pi: R \to R'$ be a covering map betwen Riemann surfces. If R' is compact π is always a BMO map.*

Example 3. Let T be a torus and $F_n: T \to T$ a covering map induced by $\tilde{F}_n: \mathbf{C} \to \mathbf{C}$, $\tilde{F}_n(z) = nz$. Then F_n is n^2-valent, and $\hat{\rho}_T(F_n(z)) |dF_n(z)| = n \hat{\rho}_T(z) |dz|$ by the homogeneity of T. On the other hand $\| T_{F_n} \|_{BMO} \le C(T)$.

Let $B: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a Blaschke type holomorphic map such that

$$B(z) = \prod_{n=1}^{N} \frac{z - z_n}{1 - \bar{z}_n z}, \qquad |z_n| < 1,$$

Let $d\mu_B = \sum_{n=1}^{N} (1 - |z_n|^2) d\delta_{z_n}$, where δ_{z_n} is the dirac measure at z_n. We denote the Carleson constant of a given positive measure μ on Δ by $\operatorname{Carl}(\mu)$. We set $\operatorname{Carl}_*(B) = \sup_{\zeta \in \Delta} \operatorname{Carl}(\mu_{B_\zeta})$, where $B_\zeta(z) = (B(z) - \zeta)/(1 - \bar{\zeta} B(z))$. Then we showed the following:

Theorem 13 ([6]). *Let $\operatorname{Carl}_*(B) \le K$ then $\| T_B \|_{BMO} \le C_1(K)$. Conversely $\| T_B \|_{BMO} \le L$ then $\operatorname{Carl}_*(B) \le C_2(L)$.*

From this, we can easily construct a sequence of rational functions $F_n: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ such that $\| T_{F_n} \|_{BMO} \lesssim 1$ and $\deg F_n \to \infty$, or even construct a Blaschke type holomorphic map preserving $BMO(\hat{\mathbf{C}})$ with essential singularities in the following sense:

Example 4 ([6]). Let F be a Blaschke type function with respect to the upper half plane such that

$$F(z) = \prod_{n=0}^{\infty} \frac{z - 2^{-n} i}{z + 2^{-n} i} \prod_{n=1}^{\infty} \frac{2^n i - z}{2^n i + z}.$$

F is meromorphic on $\mathbf{C} \setminus \{0\}$. Then by repeating the proof of Theorem 13 we can show that $F: \mathbf{C} \setminus \{0\} \to \hat{\mathbf{C}}$ is a BMO map ([6]) which has the origin as an essential singularity. (Compare this with Corollary 2.) Since $BMO(\hat{\mathbf{C}}) = BMO(\mathbf{C}\setminus\{0\})$ we can regard T_F as a bounded operator on $BMO(\hat{\mathbf{C}})$. We can similarly regard an elliptic function as a BMO map between $\hat{\mathbf{C}}$ with an essential singularity at ∞.

Furthermore we have

Theorem 14. *Let D be a plane domain and $\{z_n\}$ a sequence of distinct points on D having no cluster points in D. Then there exists a BMO map $F: D \to \hat{\mathbf{C}}$, $\| T_F \|_{BMO} \lesssim 1$, having $\{z_n\}$ as the set of all poles of F.*

Proof. We show that if $|\varepsilon_n|$, $n = 1, 2, \ldots$, are sufficiently small,

$$F(z) = z + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{z - z_n}$$

is a required map.

Let B_n' be a disk on D around z_n such that $B_n' \cap B_m' = \emptyset$, $n \neq m$. Let $B_n = (1/3)B_n'$, and $D_0 = D\setminus \bigcup_n B_n$. If $|\varepsilon_n|$, $n = 1, 2, \ldots$, are sufficiently small, $F_0 = F|D_0$ is a conformal map which has a quasiconformal extension $\tilde{F}_0$ to D such that the maximal dilatation of $\tilde{F}_0$ is less than 2. Let $g \in BMO(\hat{\mathbf{C}})$. Since BMO is quasiconformally invariant $\| g \circ \tilde{F}_0 \|_{*,D} \lesssim \| g \|_{*,\hat{\mathbf{C}}}$. Since F is 2-valent on each B_n', we have $\| g \circ F \|_{*,B_n'} \lesssim \| g \|_{*,\hat{\mathbf{C}}}$ by Corollary 14. Let B_n'' be a disk tangent to B_n such that $\operatorname{rad}(B_n'') = \operatorname{rad}(B_n)$. Since $B_n'' \subset B_n'$,

$$\begin{aligned} |(g \circ F)_{B_n} - (g \circ \tilde{F}_0)_{B_n}| &\le |(g \circ F)_{B_n} - (g \circ F)_{B_n''}| + |(g \circ \tilde{F}_0)_{B_n''} - (g \circ \tilde{F}_0)_{B_n}| \\ &\lesssim \| g \circ F \|_{*,B_n'} + \| g \circ \tilde{F}_0 \|_{*,D} \lesssim \| g \|_{*,\hat{\mathbf{C}}}. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{B_n} |g \circ F - g \circ \tilde{F}_0| \, dxdy \\ &\quad\le \int_{B_n} |g \circ F - (g \circ F)_{B_n}| \, dxdy + \int_{B_n} |g \circ \tilde{F}_0 - (g \circ \tilde{F}_0)_{B_n}| \, dxdy \\ &\quad\quad + \int_{B_n} |(g \circ F)_{B_n} - (g \circ \tilde{F}_0)_{B_n}| \, dxdy \\ &\quad\lesssim |B_n| \, \| g \circ F \|_{*,B_n'} + |B_n| \, \| g \circ \tilde{F}_0 \|_{*,D} + |B_n| \, \| g \|_{*,\hat{\mathbf{C}}} \lesssim |B_n| \, \| g \|_{*,\hat{\mathbf{C}}}. \end{aligned}$$

Let B be a disk on D.

(Case 1) Let $\operatorname{rad}(B) \le \operatorname{rad}(B_n)$ for some n such that $B \cap B_n \neq \emptyset$. Then $B \subset B_n'$, hence

$$|B|^{-1} \int_B |g \circ F - (g \circ F)_B| \, dxdy \le \| g \circ F \|_{*,B_n'} \lesssim \| g \|_{*,\hat{\mathbf{C}}}.$$

(Case 2) Let $\operatorname{rad}(B) > \operatorname{rad}(B_n)$ for every n such that $B \cap B_n \neq \emptyset$. Since $B_n \subset 3B$ holds for such n, we have $\sum_{B_n \cap B \neq \emptyset} |B_n| \le 9|B|$. Hence

$$\begin{aligned}
&\int_B |g \circ F - (g \circ \tilde{F}_0)_B| \, dxdy \\
&\quad \leq \int_B |g \circ F - g \circ \tilde{F}_0| \, dxdy + \int_B |g \circ \tilde{F}_0 - (g \circ \tilde{F}_0)_B| \, dxdy \\
&\quad \leq \sum_{B_n \cap B \neq \emptyset} \int_{B_n} |g \circ F - g \circ \tilde{F}_0| \, dxdy + |B| \, \|g \circ \tilde{F}_0\|_{*,D} \\
&\quad \lesssim \sum_{B_n \cap B \neq \emptyset} |B_n| \, \|g\|_{*,\hat{\mathbf{C}}} + |B| \, \|g\|_{*,\hat{\mathbf{C}}} \lesssim |B| \, \|g\|_{*,\hat{\mathbf{C}}} .
\end{aligned}$$

Corollary 20. *Let $\{z_n\}_{n=1}^N$ be a finite sequence of distinct points on $\hat{\mathbf{C}}$. Then there exists a BMO map $F: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$, $\| T_F \|_{BMO} \lesssim 1$, having $\{z_n\}_{n=1}^N$ as the set of all poles of F.*

Proof. We can assume $z_1 = \infty$. Then, since $BMO(\mathbf{C}) = BMO(\hat{\mathbf{C}})$, we can show that if $|\varepsilon_n|$, $n = 1, 2, \ldots N$, are sufficiently small,

$$F(z) = z + \sum_{n=2}^{N} \frac{\varepsilon_n}{z - z_n}$$

is a required map similarly.

Department of Mathematics
Kyoto University

Current address:
Department of Mathematics,
National Defense Academy,
Hashirimizu Yokosuka Japan

References

[1] A. F. Beardon and C. Pommerenke, The Poincaré metric of plane domains, J. London Math. Soc., (2) **18** (1978), 475–483.
[2] Y. Gotoh, On *BMO* functions on Riemann surface, J. Math. Kyoto Univ., **25** (1985), 331–339.
[3] Y. Gotoh, On *BMO* property for potentials on Riemann surfaces, J. Math. Kyoto Univ., **27** (1987), 349–365.
[4] Y. Gotoh, On some extension property for *BMO* functions on Riemann surfaces, J. Math. Kyoto Univ., **28** (1988), 141–152.
[5] Y. Gotoh, On the composition of functions of bounded mean oscillation with analytic functions, J. Math. Kyoto Univ., **29** (1989), 309–315.
[6] Y. Gotoh, On the composition of functions of bounded mean oscillation with meromorphic functions, J. Math. Kyoto Univ., **31** (1991), 635–642.
[7] Y. Gotoh, *BMO* extension theorem for relative uniform domains, J. Math. Kyoto Univ., **33** (1993), 171–193.
[8] K. T. Hahn, Some remarks on a new pseudo-differential metric, Ann. Polon. Math., **39** (1981), 71–81.
[9] W. K. Hayman, Multivalent Functions, Cambridge Tracts in Mathematics and Mathematical Physics, no 48, Cambridge Univ. Press, 1967.
[10] P. Jones, Extension theorems for *BMO*, Indiana Univ. Math. J., **29** (1980), 41–66.

[11] C. D. Minda, The Hahn metric on Riemann surfaces, Kodai Math. J., **6** (1983), 57–69.

[12] Z. Nehari, The Schwarzian derivatives and schlicht functions, Bull. Amer. Math. Soc., **55** (1949), 545–551.

[13] B. G. Osgood, Some properties of f''/f' and the Poincaré metric, Indiana Univ. Math. J., **31** (1982), 449–461.

[14] C. Pommerenke, On Bloch functions, J. London Math. Soc. (2), **2** (1970), 689–695.

[15] C. Pommerenke, Uniformly perfect sets and the Poincaré metric, Arch. Math., **32** (1979), 192–199.

[16] H. M. Reimann, Functions of bounded mean oscillation and quasiconformal mappings, Comm. Math. Helv., **49** (1974), 260–276.

[17] H. M. Reimann and T. Rychener, Funktionen beschränkter mittelerer Oszillation, Lecture Notes in Math. 489, Springer, 1975.

[18] L. Sario and M. Nakai, Classification theory of Riemann surfaces, Springer, 1970.

[19] M. Taniguchi, An example of an open Riemann surface not uniformly large with respect to Green's functions, Kodai Math. J., **10** (1987), 209–213.