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Measure-valued branching diffusions: immigrations, excursions and limit theorems

By

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1. Introduction

Measure-valued branching diffusion processes (MBD processes) have been extensively studied concerning various problems such as ergodic behaviors [2], [17], sample path properties [4], [24], historical processes [5], [9], entrance laws [7] and so on.

In the present paper we focus upon the immigration structure of the MBD process and discuss the following problems: The first is to characterize the immigration structure associated with a given MBD process. We do this by establishing a one to one correspondence between immigration diffusion processes of the MBD process and entrance laws of its basic Markov process. The immigration process is ordinarily determined by an immigration measure supported by the state space of the basic process. However, when the basic process is an absorbing Brownian motion in a smooth domain (in this case we call the associated MBD process a super absorbing Brownian motion or simply a super ABM following Dynkin), the immigration structure consists of two parts, one is a measure supported by the interior domain and the other is a measure supported by the boundary. In particular, the latter one involves excursions of the absorbing Brownian motion from the boundary.

Secondly we discuss the immigration diffusion process of the super ABM over $(0, \infty)$, for which we derive a stochastic partial differential equation (an SPDE). When the immigration measure has compact support, so does the immigration process. We shall present a limit theorem for the range of the immigration process.

The third one is to discuss central limit theorems of immigration processes. Assuming that the basic Markov process is a Lévy process in $\mathbf{R}^d$, one can observe a "clustering-diffusive dichotomy" in the central limit theorems. More precisely, if the symmetrization of the basic process is recurrent, then the limiting Gaussian field is spatially uniform, while if the symmetrization is transient, the limiting Gaussian field is spatially fluctuating.

We remark that Dynkin [7] obtained a characterization of entrance laws of

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the MBD processes in terms of entrance laws of the basic spatial Markov processes. Our first result may be regarded as an extention of Dynkin's result to the immigration processes. We mention also [21], [22] where are discussed the law of large numbers for immigration processes and the convergence from particle systems.

1.1. MBD **processes.** Given a locally compact seperable topological space S, let $C_0(S)$ be the Banach space of continuous functions vanishing at infinity equipped with the supremum norm. Note that if S is compact, $C_0(S)$ coincides with $C(S)$, the totality of continuous functions on S. Let $M_F(S)$ denote the space of finite Borel measures on S equipped with the topology of weak convergence. Throughout this paper we use $\mu(f)$ to denote the integral of the function f relative to the measure μ.

Let $(T_t)_{t\geq 0}$ be a strongly continuous conservative Feller semigroup on $C_0(S)$, and let A be the strong generator of (T_t) defined on $D(A)\subset C_0(S)$. Let $C([0,\infty)$, $M_F(S))$ be the space of all continuous paths from $[0,\infty)$ to $M_F(S)$ with the coordinate process denoted by $(w_t)_{t\geq 0}$ and the natural filtration $(\mathscr{G}, \mathscr{G}_t)$. To every $\mu\in M_F(S)$ there corresponds a unique probability measure $\mathbf{P}_\mu$ on $C([0,\infty), M_F(S))$ such that for each $f\in D(A)$,

$$M_t(f) := w_t(f) - \mu(f) - \int_0^t w_s(Af)\,\mathrm{d}s, \qquad t\geq 0, \tag{1.1}$$

is a $(\mathscr{G}_t)$-martingale starting at 0 with quadratic variation process

$$\langle M(f)\rangle_t = \int_0^t w_s(f^2)\,\mathrm{d}s, \qquad t\geq 0. \tag{1.2}$$

The probability measure $\mathbf{P}_\mu$ is the distribution on $C[(0,\infty), M_F(S))$ of the MBD process $(X_t, \mathbf{P}_\mu)$ driven by (T_t) with state space $M_F(S)$. See [11] or [27] for the above results.

In this paper we do *not* assume the conservativeness of the basic driving semigroup (T_t). We shall discuss the MBD processes in a broader state spacce rather than $M_F(S)$, which is formulated by introducing a reference function given by

Condition [A]. There are a bounded strictly positive function $\rho\in D(A)$ and a constant $c>0$ such that $T_t\rho\leq c\rho$ for all $0\leq t\leq 1$.

Let $M_\rho(S)$ denote the space of Borel measures μ on S satisfying $\mu(\rho)<\infty$, and denote by $C_\rho(S)$ the space of continuous functions $f\in C_0(S)$ such that $|f|\leq \text{const.}\,\rho$. We equip $M_\rho(S)$ with the following topology: $\mu_n\to\mu$ in $M_\rho(S)$ if and only if $\mu_n(f)\to\mu(f)$ for all $f\in C_\rho(S)$. Under the condition [A], the state space of the MBD process can be enlarged to $M_\rho(S)$. In this case the MBD process is characterized by a martingale problem on space $C([0,\infty), M_\rho(S))$, the $M_\rho(S))$-valued continuous path space; see e.g. [20]. The MBD process can also be characterized by the Laplace functional of its transition law: Let $C_\rho(S)^+$ denote the subspace of non-negative elements of $C_\rho(S)$. Then

$$\mathbf{P}_\mu \exp\{-X_t(f)\} = \exp\{-\mu(V_t f)\}, \qquad f \in C_\rho(S)^+, \tag{1.3}$$

where $\mathbf{P}_\mu$ denotes the conditional expectation given $X_0 = \mu$, and $V_t f$ is the mild solution of the evolution equation

$$\begin{aligned} \frac{\partial V_t f}{\partial t} &= A V_t f - \frac{1}{2}(V_t f)^2, \\ V_0 f &= f. \end{aligned} \tag{1.4}$$

More precisely, $V_t f$ is the unique bounded positive solution of the integral equation

$$V_t f = T_t f - \frac{1}{2}\int_0^t T_{t-s}[(V_s f)^2]\,ds, \qquad t \geq 0,\ f \in C_\rho(S)^+. \tag{1.5}$$

We here introduce some further notations for later use. $C_\kappa(S)$ stands for the subspace of $C_\rho(S)$ whose elements have compact supports. If $S = D$ is a smooth domain in the d-dimensional Euclidean space $\mathbf{R}^d$, we use the superscript 'm' to indicate the order of continuous differentiability, e.g., $C_\rho^1(\bar{D})^+$, $C_\kappa^2(D)$. $\mathscr{D}(\mathbf{R}^d) = C_\kappa^\infty(\mathbf{R}^d)$, and $\mathscr{D}'(\mathbf{R}^d)$ is the space of Schwartz distributions on $\mathbf{R}^d$.

1.2. Immigration processes. Let an MBD process $(X_t, \mathbf{P}_\mu)$ be fixed. Following [19] and [30], we introduce the notion of an immigraion process.

Definition 1.1. An $M_\rho(S)$-valued diffusion process $(Y_t, \mathbf{Q}_\mu)$ is called an immigration diffusion process of the MBD process $(X_t, \mathbf{P}_\mu)$ if $(Y_t)_{t\geq 0}$ under $\mathbf{Q}_\mu$ and $(X_t + Y_t)_{t\geq 0}$ under $\mathbf{P}_\mu \times \mathbf{Q}_0$ have identical laws in $C([0, \infty), M_\rho(S))$ for every $\mu \in M_\rho(S)$.

By Definition 1.1, the transition law $\{\mathbf{Q}_\mu: \mu \in M_\rho(S)\}$ is uniquely determined by $\{\mathbf{P}_\mu: \mu \in M_\rho(S)\}$ and $\mathbf{Q}_0$. In the sequel, we call $(Y_t, \mathbf{Q}_\mu)$ simply an immigration process of $(X_t, \mathbf{P}_\mu)$ instead of an immigration diffusion process since we are only concerned with diffusion processes in this paper. Furthermore we impose the following technical condition:

Condition [M1]. The first moment $\mathbf{Q}_0\{Y_t(\rho)\}$ is finite for every $t \geq 0$.

Definition 1.2. A family of σ-finite measures $(\kappa_t)_{t>0}$ on S is called a locally ρ-integrable entrance law of (T_t) if $\kappa_{r+t} = \kappa_r T_t$ for all $r, t > 0$, and if the integral $\int_0^t \kappa_s(\rho)\,ds$ is finite for all $t > 0$.

It is easy to check by equation (1.5) that for a locally ρ-integrable entrance law (κ_t) of (T_t), we have

$$\kappa_{0^+}(V_t f) := \lim_{\varepsilon \to 0^+} \kappa_\varepsilon(V_t f)$$

$$= \kappa_t(f) - \frac{1}{2}\int_0^t \kappa_{t-s}((V_s f)^2)\,\mathrm{d}s, \qquad t > 0, f \in C_\rho(S)^+. \tag{1.6}$$

In particular, if (κ_t) has the form $\kappa_t = mT_t$ for some measure $m \in M_{\rho_1}(S)$, where $\rho_1 = \int_0^1 T_s\rho\,\mathrm{d}s$, then $\kappa_{0^+}(V_t f) = m(V_t f)$.

Our first result establishes a one to one correspondence between the immigration processes of a given MBD process and the entrance laws of its basic semigroup.

Theorem 1.1. *If* $(Y_t, \mathbf{Q}_\mu)$ *is an immigration process of the MBD process* $(X_t, \mathbf{P}_\mu)$ *satisfying* [M1], *then there exists a unique locally* ρ*-integrable entrance law* (κ_t) *of* (T_t) *such that*

$$\mathbf{Q}_\mu \exp\{-Y_t(f)\} = \exp\left\{-\mu(V_t f) - \int_0^t \kappa_{0^+}(V_s f)\mathrm{d}s\right\}, \tag{1.7}$$
$$t \geq 0,\ f \in C_\rho(S)^+,\ \mu \in M_\rho(S).$$

Conversely, for each locally ρ*-integrable entrance law* (κ_t) *of* (T_t), *there is a unique immigration process* $(Y_t, \mathbf{Q}_\mu)$ *of* $(X_t, \mathbf{P}_\mu)$ *such that* (1.7) *and* [M1] *are fulfilled.*

Next we give a martingale characterization to the immigration process. Let $D_\rho(A) = \{f \in D(A)\colon f, Af \in C_\rho(S)\}$. By Lemma 2.5 of the section 2, the limit $\kappa_{0^+}(f) := \lim_{\varepsilon \to 0^+} \kappa_\varepsilon(f)$ exists for all $f \in D_\rho(A)$. Recall the notion of martingale measure from [31]. Then we obtain

Theorem 1.2. *Let* $(Y_t, \mathbf{Q}_\mu)$ *denote the immigration process associated with the locally* ρ-integrable entrance law (κ_t) *given by* (1.7). *Then there is a unique orthogonal martingale measure* $M(\mathrm{d}s\mathrm{d}x)$ *on* $[0, \infty) \times S$ *having quadratic variation measure* $\langle M\rangle(\mathrm{d}s\mathrm{d}x) = \mathrm{d}s\, Y_s(\mathrm{d}x)$ *such that*

$$Y_t(f) - Y_0(f) = \int_0^t [Y_s(Af) + \kappa_{0^+}(f)]\,\mathrm{d}s + \int_0^t\int_S f(x)M(\mathrm{d}s\mathrm{d}x), \quad f \in D_\rho(A). \tag{1.8}$$

Moreover it holds that

$$Y_t(f) - Y_0(T_t f) = \int_0^t \kappa_s(f)\mathrm{d}s + \int_0^t\int_S T_{t-s}f(x)M(\mathrm{d}s\mathrm{d}x), \quad f \in C_\rho(S). \tag{1.9}$$

1.3. Excursion laws of MBD **processes.** In this paragraph we present some construction of the immigration process by integration of excusion paths by means of Poisson random measures, which has been developed in [29]. It was shown in [7] that for a locally ρ-integrable entrance law (κ_t) of (T_t),

$$\int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(f)})K_t(\mathrm{d}\nu) = \kappa_{0^+}(V_t f), \qquad f \in C_\rho(S), \tag{1.10}$$

defines an entrance law $(K_t)_{t>0}$ of the MBD process $(X_t, \mathbf{P}_\mu)$. By a general theory of Markov processes, there is a σ-finite measure $\mathbf{P}_K$ on $W_\rho^+(S):=C((0,\infty), M_\rho(S))$ such that under $\mathbf{P}_K$ the coordinate spocess $(w_t)_{t>0}$ is a Markov process with the same transition law as $(X_t, \mathbf{P}_\mu)$ and one-dimensional marginal distributions $(K_t)_{t>0}$. Indeed, it holds that for $\mathbf{P}_K$-almost all $w\in W_\rho^+(S)$, $w_t\to 0$ as $t\to 0^+$ and $w_t=0$ for all $t\geq\sigma(w):=\inf\{t>0: w_t=0\}$ (cf. (3.5)).

Let $N(\mathrm{d}s\mathrm{d}w)$ be a Poisson random measure on $[0,\infty)\times W_\rho^+(S)$ with intensity $\mathrm{d}s\times\mathbf{P}_K(\mathrm{d}w)$. Define a measure-valued process $(\bar{Y}_t)_{t\geq 0}$ by

$$\bar{Y}_t=\int_0^t\int_{W_\rho^+(S)} w_{t-s}N(\mathrm{d}s\mathrm{d}w). \tag{1.11}$$

Theorem 1.3. *The process $(\bar{Y}_t)_{t\geq 0}$ defined by* (1.11) *is an $M_\rho(S)$-valued diffusion that is equivalent to the immigration process with initial value* 0 *and the transition function given by* (1.7).

In order to obtain a more explicit form of the right hand side of (1.7), let us consider the following condition.

Condition [E]. Every locally ρ-integrable entrance law (κ_t) of (T_t) has the form $\kappa_t=mT_t$, $t>0$, for some measure $m\in M_{\rho_1}(S)$.

We remark that the condition [E] is satisfied in the following two cases:

(i) (T_t) is conservative ad ρ is a positive constant function;

(ii) (T_t) is the semigroup of a Brownian motion in $\mathbf{R}^d$ and ρ is a positive C^2-function on $\mathbf{R}^d$ satisfying $\rho(x)=\mathrm{e}^{-|x|}$ for $|x|>1$.

By Theorem 1.1, under the condition [E], $(Y_t, \mathbf{Q}_\mu)$ is an immigration process of the MBD process if and only if there is some measure $m\in M_{\rho_1}(S)$ such that

$$\mathbf{Q}_\mu \exp\{-Y_t(f)\}=\exp\left\{-\mu(V_tf)-\int_0^t m(V_sf)\mathrm{d}s\right\}, \tag{1.12}$$
$$t\geq 0,\ f\in C_\rho(S)^+,\ \mu\in M_\rho(S).$$

It would be intuitively plausible that the immigration measure m appearing in (1.12) gives the distribution of the location where the immigrants enter. To justify this intuition we introduce a space of excursion paths of the MBD process and discuss some excursion laws on this space. Let $x\in S$ be fixed. We call $w\in C([0,\infty), M_F(S))$ an $M_F(S)$-valued excursion starting at x if

(i) $w_0=0$, $\sigma(w)>0$ and $w_t=0$ for all $t\geq\sigma(w)$,

(ii) $w_t(1)^{-1}w_t\to\delta_x$ as $t\to 0^+$.

Let $W_x^e(S)$ be the totality of excursion paths starting at x, and let $W^e(S)=\bigcup_{x\in S}W_x^e(S)$. $(\mathscr{G}, \mathscr{G}_t)$ stands for the natural filtration of $W^e(S)$. We then have some excursion laws on $W^e(S)$:

Theorem 1.4. *There is a unique family of σ-finite measure kernels $\{\Lambda^x(\mathrm{d}w)\}$ on $(W^e(S), \mathscr{G})$ such that*

(*i*) *$x\mapsto\Lambda^x$ is continuous*;

(*ii*) *Λ^x is supported by $W^e_x(S)$;*

(*iii*) *$\Lambda^x[w_t(1)] \to 1$ as $t \to 0^+$; and*

(*iv*) *$(W^e(S), \mathscr{G}, \mathscr{G}_t, w_t, \Lambda^x)_{t>0}$ is a Markov process with the same transition laws as the MBD process X.*

Let $N(\mathrm{d}t, \mathrm{d}(x, w))$ be a Poisson point process on $S \hat{\otimes} W^e(S) := \{(x, w) : x \in S, w \in W^e_x(S)\}$ with characteristic measure $m(\mathrm{d}x)\Lambda^x(\mathrm{d}w)$. Set

$$\bar{Y}_t = \int_0^t \int_{S \hat{\otimes} W^e(S)} w_{t-s} N(\mathrm{d}s, \mathrm{d}(x, w)). \tag{1.13}$$

The intuitive meaning of the expression (1.13) is quite clear. At each occurrence time of the Poisson point process, an (x, w) is chosen randomly according to the measure $m(\mathrm{d}x)\Lambda^x(\mathrm{d}w)$, so w is an excursion path starting at x, after that this path grows up as a path of the MBD process. Summing up all those excursion paths we get the immigration process:

Corollary 1.5. *The process $(\bar{Y}_t)_{t \geq 0}$ defined by* (1.13) *is a diffusion realization of the immigration process with initial value* 0 *and the transition function given by* (1.12).

1.4. Super absorbing Brownian motions. In this paragraph, we let (T_t) be the transition semigroup of an absorbing Brownian motion in a smooth domain D. In this case (T_t) does not satisfy the condition [E], unless $D = \mathbf{R}^d$. Indeed for each $x \in \partial D$ there corresponds an extremal entrance law (κ^x_t) defined by

$$\kappa^x_t(f) = \mathbf{D} T_t f(x), \qquad f \in C_0(D) \cap C^1(\bar{D}), \tag{1.14}$$

where $\mathbf{D} := \dfrac{\partial}{\partial \mathbf{n}}$ denotes the inward normal derivative operator at the boundary. (It is known that $T_t f \in C_0(D) \cap C^1(\bar{D})$ for $f \in C_0(D) \cap C^1(\bar{D})$, e.g. [13], p. 65, so that (1.14) is well defined.) In this case the condition [E] is replaced by

Lemma 1.1. *Suppose either of the following two conditions:*

(*i*) *D is bounded;*

(*ii*) *$D = H^d := \{(x_1, \cdots, x_d) \in \mathbf{R}^d : x_1 > 0\}$ and $\rho(x) = \rho_1(x_1)\rho_2(x_2) \cdots \rho_2(x_d)$, where $\rho_1 \in C^2_0((0, \infty))^{++}$ such that $\rho_1(x) = x$ for $0 < x < 1$ and $= \mathrm{e}^{-x}$ for $x > 2$, and $\rho_2 \in C^2_0(\mathbf{R})^{++}$ such that $\rho_2(x) = \mathrm{e}^{-|x|}$ for $|x| > 1$.*

Then every locally ρ-integrable entrance law (κ_t) of T_t has representation

$$\kappa_t = m T_t + \int_{\partial D} l(\mathrm{d}x) \kappa^x_t \tag{1.15}$$

for some $m \in M_{\rho_1}(D)$ and $l \in M_F(\partial D)$.

By virtue of Theorem 1.1 we have

Theorem 1.6. *Suppose that either of the two conditions of Lemma* 1.1 *is fulfilled. Then $(Y_t, \mathbf{Q}_\mu)$ is an immigration process of X satisfying* [M1], *if and* *only if*

$$\mathbf{Q}_\mu \exp\{-Y_t(f)\} = \exp\left\{-\mu(V_t f) - \int_0^t [m(V_s f) + l(\mathbf{D} V_s f)]\,\mathrm{d}s\right\}, \tag{1.16}$$

$$t \geq 0,\ \mu \in M_\rho(D),\ f \in C_\rho^2(D),$$

for some $m \in M_{\rho_1}(D)$ *and* $l \in M_F(\partial D)$.

1.5. An SPDE for an immigration process of the super ABM over $(0, \infty)$. Applying Theorem 1.6 to $D = (0, \infty)$ we see that every immigration process $(Y_t, \mathbf{Q}_\mu)$ of the super ABM on $(0, \infty)$ is expressed by

$$\mathbf{Q}_\mu \exp\{-Y_t(f)\} = \exp\left\{-\mu(V_t f) - \int_0^t [m(V_s f) + c\mathbf{D}_0 V_s f]\,\mathrm{d}s\right\}, \tag{1.17}$$

$$t \geq 0,\ \mu \in M_\rho((0, \infty)),\ f \in ((0, \infty)),$$

for some $m \in M_{\rho_1}((0, \infty))$ and $c \geq 0$, where $\mathbf{D}_0 V_s f = \frac{\partial}{\partial x} V_s f(0^+)$. Recall that the reference function $\rho \in C_0^2((0, \infty))$ has been chosen such that $\rho(x) = x$ for $0 < x < 1$ and $= e^{-x}$ for $x > 2$. Then it indeed holds that $M_{\rho_1}(0, \infty)) = M_\rho((0, \infty))$. It is well-known that the sample path of the super Brownian motion over $\mathbf{R}$ has a continuous density, which solves an SPDE, cf. [20]. One should expect analogous results for the immigration process of the super ABM over $(0, \infty)$. Here we obtain

Theorem 1.7. *Let* $(Y_t, \mathbf{Q}_\mu)$ *be an immigration process of the super ABM defined by* (1.17). *Then there exists a continuous two parameter process* $Y_t(x)$, $(t, x) \in (0, \infty) \times (0, \infty)$ *such that* $Y_t(\mathrm{d}x) = Y_t(x)\mathrm{d}x$ and $Y_t(0^+) = 2c$ *for all* $t > 0$ $\mathbf{Q}_\mu$-*almost surely. Moreover the density process* $Y_t(x)$ *solves the following SPDE*:

$$\frac{\partial}{\partial t} Y_t(x) = \sqrt{Y_t(x)}\,\dot{W}_t(x) + \frac{1}{2}\Delta^* Y_t(x) + \frac{m(\mathrm{d}x)}{\mathrm{d}x} - c\delta_0', \tag{1.18}$$

where $\dot{W}_t(x)$ *is a time-space white noise,* Δ^* *denotes the adjoint operator of the Laplacian in* $(0, \infty)$ *with Dirichlet boundary condition at* 0, *and* δ_0' *is the derivative of the Dirac* δ-*function with test functions* $C_{0\kappa}^2([0, \infty)) := \{f \in C^2([0, \infty)) : f(0) = 0$ *and supp* (f) *is bounded*$\}$. *More precisely, the equation* (1.18) *should be understood in the sense of distribution, i.e.*

$$\begin{aligned}\int_0^\infty Y_t(x) f(x)\mathrm{d}x - \int_0^\infty f(x) Y_0(\mathrm{d}x) &= \int_0^t \int_0^\infty \sqrt{Y_s(x)} f(x) \dot{W}_s(x)\mathrm{d}s\mathrm{d}x \\ &+ \frac{1}{2}\int_0^t \int_0^\infty Y_s(x) f''(x)\mathrm{d}s\mathrm{d}x + tm(f) + ctf'(0)\end{aligned} \tag{1.19}$$

for every $f \in C_{0\kappa}^2([0, \infty))$.

1.6. A limit theorem for the range of the immigration process of the super ABM over $(0, \infty)$. It is well-known that a super Brownian motion over $\mathbf{R}^d$ has compact support property and the distribution of the total range up to extinction can be seeked explicitly, cf [18]. We here present a limit theorem for the range up to time t for the immigration process of the super ABM as $t \to \infty$. For $\mu \in M_\rho((0, \infty))$, $S(\mu)$ stands for the support of μ.

Theorem 1.8. *Let* $(Y_t, \mathbf{Q}_\mu)$ *denote an immigration process given by* (1.17). *Then* $S(Y_t)$ *is bounded for all* $t \geq 0$ $\mathbf{Q}_\mu$*-almost surely if and only if both* $S(\mu)$ *and* $S(m)$ *are bounded. In this case, let* R_t *denote the range of* (Y_t) *up to time* $t > 0$, *i.e.,* $R_t = \bigcup_{0 \leq s \leq t} S(X_s)$, *and let* $\bar{R}_t = \sup\{x > 0 : x \in R_t\}$. *Then* $t^{-1/3}\bar{R}_t$ *converges in distribution as* $t \to \infty$, *and the limit distribution is the so-called Fréchet distribution* (*cf.* [14]) *given by* $F(z) = \mathrm{e}^{-\gamma z^{-3}} (z > 0)$, *where*

$$\gamma = \frac{1}{18}\left(\frac{\Gamma(1/3)\Gamma(1/6)}{\Gamma(1/2)}\right)^3 \left(c + \int_0^\infty x m(\mathrm{d}x)\right). \tag{1.20}$$

1.7. Clustering-diffusive dichotomy in the central limit theorems for immigration processes. In this paragraph, we assume that (T_t) is the transition semigroup of an irreducible Lévy process in $\mathbf{R}^d$ acting on $C_0(\mathbf{R}^d)$. We fix a $\beta > 0$ and a nontrivial function $\phi \in C_\kappa(\mathbf{R}^d)^+$. Define the reference function ρ by

$$\rho(x) = G_\beta \phi(x) := \int_0^\infty \mathrm{e}^{-\beta t} T_t \phi(x) \mathrm{d}t. \tag{1.21}$$

It is obvious tht ρ satisfies the condition [A], so we have an MBD process X associated with (T_t), which we shall call a super Lévy process.

Let $(Y_t, \mathbf{Q}_\mu)$ be an immigration process of the super Lévy process X given by (1.12). We are here concerned with central limit theorems for this immigration process. These provide us a new example of "clustering diffusive dichotomy" since the recurrence of symmetrition of the basic process yields spatial uniformity, while the transience yields spatial fluctuation. The dichotomy phenomenon is often observed in the study of interacting particle systems, cf. [15], [23], etc. We first assume

$$m = c\lambda, \qquad (c > 0.\ \lambda = \text{Lebesgue measure on } \mathbf{R}^d) \tag{1.22}$$

Theorem 1.9. *i) If the symmetrized Lévy process is transient, then the distribution of*

$$Z_t := \frac{Y_t - tm}{\sqrt{t}}, \qquad t > 0, \tag{1.23}$$

under $\mathbf{Q}_0$ *converges as* $t \to \infty$ *to that of a centered Gaussian field* Z *over* $\mathbf{R}^d$ *with covariance functional*:

$$\mathbf{Cov}(Z(f), Z(g)) = \frac{c}{2}\lambda(f\hat{G}g), \qquad f, g \in \mathscr{D}(\mathbf{R}^d), \tag{1.24}$$

where $\hat{G}$ is the potential operator of the symmetrized Lévy process.

ii) If the symmetrized Lévy process is recurrent, there exists an $h(t)$ such that $t^{-1}h(t) \to \infty$ as $t \to \infty$, and the distribution of

$$Z_t := \frac{Y_t - tm}{\sqrt{h(t)}}, \qquad t > 0, \tag{1.25}$$

under $\mathbf{Q}_0$ converges as $t \to \infty$ to that of $\eta \cdot \lambda$, where η is a centered Gaussian random variable with variance c.

In Theorem 1.9 we assumed (1.22), which makes the proof extremely simple since m is an invariant measure of the basic Markov process. However, the dichotomy result does not really depend on the immigration measure. Next we consider a more general immigration measure in the case where (T_t) is a Brownian semigroup. Assume that

$$m(\mathrm{d}x) = \gamma(x)\lambda(\mathrm{d}x), \tag{1.26}$$

where γ is a locally bounded measurable function satisfying

$$\lim_{r \to \infty} r^{-2\alpha}\gamma(rx) = a(x) \tag{1.27}$$

uniformly in $x \in S^{d-1} := \{x \in \mathbf{R}^d : |x| = 1\}$ for a constant $\alpha \geq 0$ and a nontrivial continuous function a on S^{d-1}. Let λ_S denote the surface element of S^{d-1} for $d \geq 2$ and $\lambda_S = \delta_1 + \delta_{-1}$ for $d = 1$. Then $\lambda_S(1) = 2\pi^{d/2}\Gamma(d/2)^{-1}$. Define the constants c_d, $d = 1, 2, \cdots$, by

$$\begin{aligned} c_d &= \frac{\Gamma(\alpha + 1/2)}{\pi(2\alpha + 3)} \int_0^1 r^{-1/2}(2 - r)^\alpha \mathrm{d}r \lambda_S(a) \qquad \text{for } d = 1, \\ &= \frac{2^{\alpha-2}\Gamma(\alpha + 1)}{\pi^2(\alpha + 1)} \lambda_S(a) \qquad \text{for } d = 2, \\ &= \frac{2^{\alpha-1}\Gamma(\alpha + d/2)}{\pi^{d/2}(1 + \alpha)} \lambda_S(a) \qquad \text{for } d \geq 3. \end{aligned} \tag{1.28}$$

Let

$$\begin{aligned} h(M) &= c_1 M^{\alpha+3/2} \qquad \text{for } d = 1, \\ &= c_2 M^{\alpha+1} \log M \qquad \text{for } d = 2, \\ &= c_d M^{\alpha+1} \qquad \text{for } d \geq 3. \end{aligned} \tag{1.29}$$

Normalizing (Y_t), we define a centered $\mathscr{D}'(\mathbf{R}^d)$-valued process

$$Z_t^{(M)} = \frac{Y_t^M - \int_0^{tM} m T_s \mathrm{d}s}{\sqrt{h(M)}}, \qquad M > 0, t > 0. \tag{1.30}$$

Then we have our second central limit theorem as follows.

Theorem 1.10. *As $M \to \infty$, any finite dimensional distributions of $(Z_t^{(M)})_{t>0}$ under $\mathbf{Q}_0$ converge to those of the $\mathscr{D}'(\mathbf{R}^d)$-valued centered Gaussian process $(Z_t)_{t>0}$ that is characterized by*

i) for $d = 1$, $Z_t \equiv \eta_t \lambda$, where $(\eta_t)_{t>0}$ is a continuous centered Gaussian process with covariance $\mathrm{E}\eta_s\eta_t = \kappa(s, t)$, where $\kappa(t, t) = \frac{1}{2} t^{\alpha+3/2}$, $t > 0$ and

$$\kappa(s, t) = \gamma s^{2(\alpha+1)} \int_0^1 \int_0^1 \frac{u^{\alpha+1}[t - s + s(2 - v)uv]^\alpha}{(t - s + 2suv)^{\alpha+1/2}} \mathrm{d}u\mathrm{d}v, \qquad t > s > 0; \tag{1.31}$$

with

$$\gamma = 2^{\alpha-3/2}(2\alpha + 3)\left(\int_0^1 r^{-1/2}(2 - r)^\alpha \mathrm{d}r\right)^{-1}; \tag{1.32}$$

ii) for $d = 2$, $Z_t \equiv \eta_t \lambda$, where (η_t) are independent centered Gaussian random variables with $\mathrm{E}\eta_t^2 = \frac{1}{2} t^{\alpha+1}$;

iii) for $d \geq 3$, (Z_t) are independent $\mathscr{D}'(\mathbf{R}^d)$-valued centered Gaussian random variables with

$$\mathbf{Cov}\,(Z_t(f), Z_t(g)) = \frac{t^{\alpha+1}}{2} \lambda(f G g), \quad f, g \in \mathscr{D}(\mathbf{R}^d), \tag{1.33}$$

where G denotes the potential operator of the Brownian motion.

Finally let us remark that in the special case $\alpha = 0$ and $a(x) \equiv 1$, we have

$$c_d = \frac{4}{3\sqrt{\pi}} \text{ for } d = 1, \ = \frac{1}{2\pi} \text{ for } d = 2, \ = 1 \text{ for } d \geq 3, \tag{1.34}$$

and (1.31) turns into

$$\kappa(s, t) = \frac{\sqrt{2}}{4} [(t + s)^{3/2} - (t - s)^{3/2} - 3s\sqrt{t - s}], \qquad t \geq s > 0. \tag{1.35}$$

The rest of the paper is organized as follows: Section 2 contains the proofs of Theorems 1.1 and 1.2. The proofs of Theorems 1.3, 1.4 and Corollary 1.5 are given in Section 3. Super ABMs are discussed in Section 4, where the proofs of Lemma 1.1 and Theores 1.6 through 1.8 are given. Theorems 1.9 and 1.10 are proved in Section 5.

2. Immigration processes

For an immigration process $Y = (Y_t, \mathbf{Q}_\mu)$ given by Definition 1.1, we set

$$J_t(f) = -\log \mathbf{Q}_0\{\exp - Y_t(f)\}, \qquad f \in C_\rho(S)^+. \tag{2.1}$$

For a σ-finite measure K supported by $M_\rho(S) \setminus \{0\}$, we define a modification of the Laplace functional as in [7],

$$R_K(f) = \int_{M_\rho(S)\setminus\{0\}} (1 - e^{-\nu(f)})K(d\nu), \tag{2.2}$$

under a subsidiary condition

$$\int_{M_\rho(S)\setminus\{0\}} 1 \wedge \nu(\rho)K(d\nu) < \infty. \tag{2.3}$$

Let $\bar{S} := S \cup \{\Delta\}$ denote the one point compactification of S if it is not compact, and let $\bar{S} = S$ if it is compact. Denote by $C(\bar{S})^{++}$ the space of strictly positive continuous functions on $\bar{S}$. Choose a countable dense subset $\mathbf{C}$ of $C(\bar{S})^{++}$ containing all constant functions of positive integers, and let $\mathbf{H} = \{g\rho : g \in \mathbf{C}\}$. It follows from Lemmas 2.1–2.4 below that any immigration process of the MBD process satisfying the condition [M1] is characterized by formula (1.7), which proves the former part of our Theorem 1.1. The converse assertion in Theorem 1.1 is a consequence of Theorem 1.3, which will be proved in section 3. The following Lemma 2.1 is a modification of Lemma 2.3 of [7], of which proof is omitted since it is quite similar to the one given in [7].

Lemma 2.1. *Suppose that K_n, $n = 1, 2, \cdots$, is a sequence of σ-finite measures on $M_\rho(S)\setminus\{0\}$ satisfying* (2.3). *If*

$$R_{K_n}(f) \to R(f), \qquad f \in \mathbf{H}, \tag{2.4}$$

and if

$$\liminf_{n\to\infty} R(\rho/n) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} R(n\rho)/n = 0, \tag{2.5}$$

then there exists a unique σ-finite measure K on $M_\rho(S)\setminus\{0\}$ satisfying (2.3) *such that $R_K(f) = R(f)$ for all $f \in \mathbf{H}$.*

Lemma 2.2. *Suppose that $Y = (Y_t, \mathbf{Q}_\mu)$ is an immigration process of the MBD process, and that $J_t(f)$ is given by* (2.1). *Then there is a family of non-negative functionals (I_t) on $C_\rho(S)^+$ such that*

$$J_t(f) = \int_0^t I_s(f)ds, \quad f \in C_\rho(S)^+, \ t \geq 0. \tag{2.6}$$

Proof. The Chapman-Kolmogorov equation implies that

$$J_{r+t}(f) = J_t(f) + J_r(V_t f), \quad f \in C_\rho(S)^+, \ r, t \geq 0. \tag{2.7}$$

Then for every $f \in C_\rho(S)^+$, $J_t(f)$ is non-decreasing in $t \geq 0$. Fix $M > 0$ and choose a constant c such that $T_t\rho \leq c\rho$ for all $0 \leq t \leq M$. Let $0 \leq c_1 < d_1 < c_2 < d_2 < \cdots < c_n < d_n \leq M$, and let $\sigma(n) = \sum_{k=1}^n (d_k - c_k)$. Using (2.7) one can show by induction that

$$\sum_{k=1}^{n} [J_{d_k}(f) - J_{c_k}(f)] \leq J_{\sigma(n)}(c\rho). \tag{2.8}$$

Because, for $n = 1$ (2.8) follows from (2.7). Assuming that it is true for $n - 1$, by (2.7) we have

$$\begin{aligned}
&\sum_{k=1}^{n} [J_{d_k}(f) - J_{c_k}(f)] \\
&\leq J_{\sigma(n-1)}(c\rho) + J_{d_n}(f) - J_{c_n}(f) \\
&\leq J_{\sigma(n-1)}(c\rho) + J_{d_n - c_n}(V_{\sigma(n-1)}(c\rho)) \\
&\leq J_{\sigma(n)}(c\rho),
\end{aligned}$$

which proves (2.8) for all $n \geq 1$. By Definition 1.1, $Y_t(c\rho) \to 0$ as $t \to 0$ $\mathbf{Q}_0$-almost surely. Thus by (2.1),

$$J_t(c\rho) \to 0 \quad \text{as } t \to 0^+. \tag{2.9}$$

Then the absolute continuity of $J_t(f)$ in $t \geq 0$ follows by (2.8) and (2.9).

Lemma 2.3. *Under the condition on Lemma* 2.2, *there is a family of σ-finite measures $(K_t)_{t>0}$ supported by $M_\rho(S) \setminus \{0\}$ such that*

$$J_t(f) = \int_0^t \mathrm{d}s \int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(f)}) K_s(\mathrm{d}\nu), \quad f \in C_\rho(S)^+, \ t \geq 0, \tag{2.10}$$

and that

$$\int_{M_\rho(S)\setminus\{0\}} K_r(\mathrm{d}\mu) \mathbf{P}_\mu(X_t \in \mathrm{d}\nu) = K_{r+t}(\mathrm{d}\nu), \quad r, t > 0. \tag{2.11}$$

Proof. By Lemma 2.2, there is a null set $N \subset (0, \infty)$ such that for all $s \in N^c$ and $f \in \mathbf{H}$,

$$\begin{aligned}
I_s(f) &= \lim_{r \to 0^+} r^{-1}[J_{s+r}(f) - J_s(f)] \\
&= \lim_{r \to 0^+} r^{-1}[1 - \exp\{-J_r(V_s f)\}].
\end{aligned}$$

Setting

$$K_s^{(r)}(\mathrm{d}\nu) = r^{-1}\mathbf{Q}_0\{\mathbf{P}_{Y_r}(X_s \in \mathrm{d}\nu)\},$$

we get

$$I_s(f) = \lim_{r \to 0^+} \int_{M_\rho(S)\setminus\{0\}} (1 - \exp\{-\nu(f)\}) K_s^{(r)}(\mathrm{d}\nu).$$

Since $\lim_{n\to\infty} J_t(\rho/n) = 0$ by (2.1), we can enlarge the Lebesgue null set N and assume $\lim_{n\to\infty} I_s(\rho/n) = 0$ for all $s \in N^c$. By Jensen's inequality, for $0 < \delta < t$,

$$\int_\delta^t I_s(f)\mathrm{d}s = J_\delta(V_{t-\delta}f) \le \mathbf{Q}_0\{Y_\delta(V_{t-\delta}f)\}. \tag{2.12}$$

It is easy to see that $\lim_{n\to\infty} n^{-1} V_r(n\rho) = 0$, so applying the dominated convergence theorem together with Fatou's lemma to (2.12) we see that,

$$\int_\delta^t \liminf_{n\to\infty} n^{-1} I_s(n\rho)\mathrm{d}s \le \liminf_{n\to\infty} \int_\delta^t n^{-1} I_s(n\rho)\mathrm{d}s = 0.$$

Thus Lemma 2.1 is applicable to I_s for almost every $s \ge 0$, and J_t has representation (2.10). Now (2.7) implies that for $r, t > 0$ and $f \in C_\rho(S)^+$,

$$\int_0^r \mathrm{d}s \int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(f)}) K_{t+s}(\mathrm{d}\nu)$$
$$= \int_0^r \mathrm{d}s \int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(V_t f)}) K_s(\mathrm{d}\nu).$$

By Fubini's theorem there are null subsets N and $N(s)$ of $(0, \infty)$ such that

$$\int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(f)}) K_{t+s}(\mathrm{d}\nu) = \int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(V_t f)}) K_s(\mathrm{d}\nu) \tag{2.13}$$

for all $s \in N^c$, $t \in N(s)^c$ and $f \in \mathbf{H}$. For $f \in C_\rho(S)^+$, the right side of (2.13) is continuous in t, so we can modify the definition of $(K_t)_{t>0}$ to make (2.11) be satisfied.

Lemma 2.4. *Under the condition of Lemma 2.2, (J_t) has the representation*

$$J_t(f) = \int_0^t \kappa_{0^+}(V_s f)\mathrm{d}s, \tag{2.14}$$

where (κ_t) is a locally ρ-integrable entrance law of the basic process (T_t).

Proof. Combining (2.1) and (2.10) we get

$$\mathbf{Q}_\mu\{\exp - Y_t(f)\} = \exp\left\{ - \mu(V_t f) - \int_0^t \mathrm{d}s \int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(f)}) K_s(\mathrm{d}\nu)\right\}. \tag{2.15}$$

By Theorem 1.3 of [7], the above (K_t) can be expressed as follows.

$$\int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(f)}) K_t(\mathrm{d}\nu) = \kappa_{0^+}(V_t f) + \int (1 - \mathrm{e}^{-\eta_{0^+}(V_t f)}) F(\mathrm{d}\eta), \tag{2.16}$$

where (κ_t) is a locally ρ-integrable entrance law of (T_t) and F is a σ-finite measure on the set of locally ρ-integrable entrance laws of (T_t). We shall see that the diffusion assumption on Y forces $F \equiv 0$. It follows from (2.15), (2.16) and the condition [M1] that for each $t \ge 0$ and $f \in C_\rho(S)^+$,

$$\mathbf{Q}_0\{Y_t(f)\} = \int_0^t \mathrm{d}s \int_{M_\rho(S)\setminus\{0\}} \nu(f) K_s(\mathrm{d}\nu)$$

(2.17)

$$= \int_0^t \left[\kappa_s(f) + \int \eta_s(f) F(\mathrm{d}\eta) \right] \mathrm{d}s < \infty.$$

Fix $\alpha > 0$, and notice that $\phi := T_\alpha \rho \in D(A)$. Using the condition [A] one sees

(2.18)

$$\lim_{s \to 0^+} \int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(\phi)}) K_s(\mathrm{d}\nu) = \kappa_{0^+}(\phi) + \int (1 - \mathrm{e}^{-\eta_{0^+}(\phi)}) F(\mathrm{d}\eta)$$
$$= \kappa_\alpha(\rho) + \int (1 - \mathrm{e}^{-\eta_\alpha(\rho)}) F(\mathrm{d}\eta) < \infty.$$

Now we claim that

(2.19)

$$\mathrm{e}^{-Y_t(\phi)} - \mathrm{e}^{-Y_0(\phi)} - \int_0^t \mathrm{e}^{-Y_s(\phi)} \left[Y_s\left(-A\phi + \frac{1}{2}\phi^2 \right) - \kappa_{0^+}(\phi) \right] \mathrm{d}s$$
$$+ \int_0^t \mathrm{d}s \int \mathrm{e}^{-Y_s(\phi)} (1 - \mathrm{e}^{-\eta_{0^+}(\phi)}) F(\mathrm{d}\eta)$$

is a $\mathbf{Q}_\mu$-martingale. To see this it is enough to prove that for each $G \in \sigma\{Y_s : 0 \le s \le r\}$, $\mathbf{Q}_\mu\{1_G \mathrm{e}^{-Y_t(\phi)}\}$ is a differentiable function of t with continuous derivative

$$\mathbf{Q}_\mu\left\{ 1_G \left[Y_t\left(-A\phi + \frac{1}{2}\phi^2 \right) - \kappa_{0^+}(\phi) - \int (1 - \mathrm{e}^{-\eta_{0^+}(\phi)}) F(\mathrm{d}\eta) \right] \mathrm{e}^{-Y_t(\phi)} \right\}.$$

By Markov property and (2.15),

$$\mathbf{Q}_\mu\{1_G \mathrm{e}^{-Y_t(\phi)}\}$$
$$= \mathbf{Q}_\mu\left\{ 1_G \exp\left\{ -Y_r(V_{t-r}\phi) - \int_0^{t-r} \mathrm{d}s \int_{M_\rho(S)\setminus\{0\}} (1 - \mathrm{e}^{-\nu(\phi)}) K_s(\mathrm{d}\nu) \right\} \right\}$$

is continuously differentiable as a function of t, so it suffices to calculate the right derivative:

(2.20)

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-1} \mathbf{Q}_\mu\{1_G[\mathrm{e}^{-Y_{t+\varepsilon}(\phi)} - \mathrm{e}^{-Y_t(\phi)}]\}$$
$$= \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \mathbf{Q}_\mu\{1_G[\mathrm{e}^{-Y_t(V_\varepsilon\phi)} - \mathrm{e}^{-Y_t(\phi)}]\}$$
$$+ \lim_{\varepsilon \to 0^+} \varepsilon^{-1} \mathbf{Q}_\mu\left\{ 1_G \mathrm{e}^{-Y_t(V_\varepsilon\phi)} \left[\exp\left\{ -\int_0^\varepsilon \mathrm{d}s \int (1 - \mathrm{e}^{-\nu(\phi)}) K_s(\mathrm{d}\nu) \right\} - 1 \right] \right\}.$$

Then (2.18) and the dominated convergence theorem yields the desired result.

If $Y_t(\phi)$ is continuous, then by applying Itô's formula to $(\mathrm{e}^{-Y_s(\phi)})^2 = \mathrm{e}^{-2Y_s(\phi)}$ one sees that the martingale (2.19) has quadratic variation process

$$\int_0^t e^{-2Y_s(\phi)}\left[Y_s(\phi^2)+\int(1-2e^{-\eta_{0^+}(\phi)}+e^{-2\eta_{0^+}(\phi)})F(d\eta)\right]ds.$$

Using Itô's formula again one sees,

$$e^{-3Y_t(\phi)}-e^{-3Y_0(\phi)}$$

$$= \text{martingale} + 3\int_0^t e^{-3Y_s(\phi)}\left[Y_s\left(-A\phi+\frac{3}{2}\phi^2\right)+\kappa_{0^+}(\phi)\right]ds \tag{2.21}$$

$$+\,3\int_0^t ds\int e^{-3Y_s(\phi)}(e^{-2\eta_{0^+}(\phi)}-e^{-\eta_{0^+}(\phi)})F(d\eta).$$

Comparing (2.19) and (2.21) we see the increasing process

$$\int_0^t ds\int e^{-3Y_s(\phi)}(1-e^{-\eta_{0^+}(\phi)})^3F(d\eta),\qquad t\geq 0,$$

is a continuous martingale, which forces $F\equiv 0$.

Next we proceed to the proof of Theorem 1.2 which gives a martingale characterization for the immigration process. The following two simple properties of the set $D_\rho(A)$ will be useful.

Lemma 2.5. *i) For each $f\in D_\rho(A)$, the the limit $\kappa_{0^+}(f)=\lim_{t\to 0^+}\kappa_t(f)$ exists.*

ii) For each $f\in C_\rho(S)$ there is a sequence $\{f_n\}$ in $D_\rho(A)$ such tht $\rho^{-1}f_n$ converges as $n\to\infty$ to $\rho^{-1}f$ boundedly and pointwise.

Proof. i) Let $\alpha>0$ be large enough so that

$$T_t\rho<e^\alpha\rho\qquad\text{for }\ 0\leq t\leq 1. \tag{2.22}$$

For $f\in D_\rho(A)$ let $h=\alpha f-Af$. Then we have

$$f=\int_0^\infty e^{-\alpha s}T_sh\,ds. \tag{2.23}$$

Using (2.22) and (2.23) one sees easily

$$\lim_{t\to 0^+}\kappa_t(f)=\int_0^\infty e^{-\alpha s}\kappa_s(h)ds<\infty. \tag{2.24}$$

ii) For any $f\in C_\rho(S)$ one can check that $f_n:=n\int_0^{n^{-1}}T_sf\,ds$ satisfies the requirements.

Proof of Theorem 1.2. As in the proof of Lemma 2.4, for $f\in D_\rho(A)^+$,

$$N_t(f):=e^{-Y_t(f)}-e^{-Y_0(f)}-\int_0^t e^{-Y_s(f)}\left[Y_s\left(-Af+\frac{1}{2}f^2\right)-\kappa_{0^+}(f)\right]ds \tag{2.25}$$

is a martingale with quadratic variation process

$$\langle N(f)\rangle_t = \int_0^t e^{-2Y_s(f)} Y_s(f^2)\mathrm{d}s. \tag{2.26}$$

Then

$$M_t(f) := Y_t(f) - Y_0(f) - \int_0^t [Y_s(\mathrm{A}f) + \kappa_{0^+}(f)]\,\mathrm{d}s \tag{2.27}$$

is a martingale with quadratic variation process

$$\langle M(f)\rangle_t = \int_0^t Y_s(f^2)\mathrm{d}s. \tag{2.28}$$

Note that for $f \in C_\rho(S)^+$, there is a sequence $\{f_n\}$ from $D_\rho(A)^+$ such that $\rho^{-1}f_n \to \rho^{-1}f$ pointwise and boundedly as $n \to \infty$. In view of this fact together with (2.27) and (2.28), there exists a unique orthogonal martingale measure $M(\mathrm{d}s\mathrm{d}x)$ such that

$$M_t(f) = \int_0^t \int_S f(x)M(\mathrm{d}s\mathrm{d}x), \qquad f \in C_\rho(S),$$

and that

$$\langle M(f), M(g)\rangle_t = \int_0^t Y_s(fg)\mathrm{d}s, \qquad f, g \in C_\rho(S),$$

so (1.8) holds. Next we prove (1.9). To simplify the presentation we assume $Y_0 = 0$, since modifications to the general situation are trivial. A routine computation based on (1.7) shows that for any $t > 0$ and $f, g \in C_\rho(S)^+$,

$$\mathbf{Q}_\mu\{Y_t(f)\} = \mu(T_t f) + \int_0^t \kappa_s(f)\mathrm{d}s, \tag{2.29}$$

$$\begin{aligned} \mathbf{Q}_0\{Y_r(g)Y_t(f)\} &= \int_0^r \kappa_s(g)\mathrm{d}s \int_0^t \kappa_s(f)\mathrm{d}s \\ &\quad + \int_0^r \mathrm{d}s \int_0^s \kappa_u(T_{s-u}g\,T_{t-r+s-u}f)\mathrm{d}u, \\ &= \int_0^r \kappa_s(g)\mathrm{d}s \int_0^t \kappa_s(f)\mathrm{d}s \\ &\quad + \int_0^r \mathrm{d}s \int_0^s \kappa_u(T_{r-s}g\,T_{t-s}f)\mathrm{d}u. \end{aligned} \tag{2.30}$$

Using these one sees that for $f \in D_\rho(A)^+$,

$$\mathbf{Q}_0\left\{\int_0^t Y_t(f)Y_s(Ag)\mathrm{d}s\right\} = \int_0^t \kappa_s(f)\mathrm{d}s \int_0^t [\kappa_s(g) - \kappa_{0^+}(g)]\mathrm{d}s$$

$$+\int_0^t ds\int_0^s \kappa_u(T_{t-s}f[T_{t-s}g-g]))du,$$

and

$$\mathbf{Q}_0\{Y_t(f)M_t(g)\}=\int_0^t ds\int_0^s \kappa_u(gT_{t-s}f)du.$$

Let $t_i=it/n$ for $i=0, 1, 2,\cdots$, and $n=1, 2,\cdots$. By the continuity of (T_t) and the Markov property of $(Y_t, \mathbf{Q}_0)$,

(2.31)

$$\mathbf{Q}_0\left\{Y_t(f)\int_0^t\int T_{t-s}g(x)M(dsdx)\right\}$$
$$=\lim_{n\to\infty}\sum_{i=1}^n \mathbf{Q}_0\left\{Y_t(f)\int_{t_{i-1}}^{t_i}\int T_{t-t_i}g(x)M(dsdx)\right\}$$
$$=\lim_{n\to\infty}\sum_{i=1}^n \mathbf{Q}_0\{Y_t(f)[M_{t_i}(T_{t-t_i}g)-M_{t_{i-1}}(T_{t-t_i}g)]\}$$
$$=\lim_{n\to\infty}\sum_{i=1}^n [\mathbf{Q}_0\{Y_{t_i}(T_{t-t_i}f)M_{t_i}(T_{t-t_i}g)\}-\mathbf{Q}_0\{Y_{t_{i-1}}(T_{t-t_{i-1}}f)M_{t_{i-1}}(T_{t-t_i}g)\}]$$
$$=\lim_{n\to\infty}\sum_{i=1}^n \int_{t_{i-1}}^{t_i}\int_0^s \kappa_u(T_{t-s}fT_{t-t_i}g)du$$
$$=\int_0^t ds\int_0^s \kappa_u(T_{t-s}fT_{t-s}g)du.$$

It is easy to see that

$$\mathbf{Q}_0\left[\int_0^t T_{t-s}f(x)M(dsdx)\right]^2=\int_0^t ds\int_0^s \kappa_u((T_{t-s}f)^2du. \tag{2.32}$$

Summing up (2.27), (2.28), (2.31) and (2.32), one gets

$$\mathbf{Q}_0\left[Y_t(f)-\int_0^t \kappa_{t-s}(f)ds-\int_0^t\int T_{t-s}f(x)M(dsdx)\right]^2=0,$$

yielding (1.9).

3. Excursion laws of MBD processes

We first give the proof of Theorem 1.3 which asserts that the process $(\bar{Y}_t)_{t\geq 0}$ defined by (1.11) is a diffusion realization of the immigration process starting at 0. The method used in the following is essentially the same as the one of [29], that is, to combine a semi-continuity argument with some moment estimates. Suppose that the Poisson random measure $N(dsdw)$ is defined on a probability space $(\Omega, \mathscr{A}, \mathbf{Q})$. Let us begin with some estimates for the moments of the immigration process.

Lemma 3.1. *For each $M > 0$ and each $f \in D_\rho(A)^+$, there exists a constant $C(M, f) > 0$ such that for all $0 \le r < t \le M$,*

$$\mathbf{Q}\left\{\left(\bar{Y}_t(f) - \bar{Y}_r(f) - \int_r^t \kappa_s(f)\mathrm{d}s\right)^4\right\} \le (t-r)^2 C(M, f). \tag{3.1}$$

Accordingly, for each $f \in D(A) \cap C_\rho(S)^+$, the process $(\bar{Y}_t(f), t \ge 0)$ has a continuous modification.

Proof. We set that $w_t = 0$ for $t \le 0$ by convention. First note that

$$\bar{Y}_t(f) - \bar{Y}_r(f) - \int_r^t \kappa_s(f)\mathrm{d}s = \int_0^t \int_{W_\rho^+(S)} [w_{t-s}(f) - w_{r-s}(f)]\tilde{N}(\mathrm{d}s\mathrm{d}w),$$

where $\tilde{N}(\mathrm{d}s\mathrm{d}w) = N(\mathrm{d}s\mathrm{d}w) - \mathrm{d}s\mathbf{P}_K(\mathrm{d}w)$. By a moment calculation of Poisson random measures, we get

(3.2)

$$\begin{aligned}
&\mathbf{Q}\left\{\left[\bar{Y}_t(f) - \bar{Y}_r(f) - \int_r^t \kappa_s(f)\mathrm{d}s\right]^4\right\} \\
&= \int_0^t \mathbf{P}_K\{|w_{t-s}(f) - w_{r-s}|^4\}\mathrm{d}s + 3\left[\int_0^t \mathbf{P}_K\{|w_{t-s}(f) - w_{r-s}(f)|^2\}\mathrm{d}s\right]^2 \\
&= \int_0^r \mathrm{d}s \int \mathbf{P}_\nu\{|\mathrm{X}_{t-r}(f) - \nu(f)|^4\} K_{r-s}(\mathrm{d}\nu) + \int_r^t \mathrm{d}s \int \nu(f)^4 K_{t-s}(\mathrm{d}\nu) \\
&\quad + 3\left[\int_0^r \mathrm{d}s \int \mathbf{P}_\nu\{|\mathrm{X}_{t-r}(f) - \nu(f)|^2\} K_{r-s}(\mathrm{d}\nu) + \int_r^t \mathrm{d}s \int \nu(f)^2 K_{t-s}(\mathrm{d}\nu)\right]^2
\end{aligned}$$

Recall the moment estimate of the MBD process from [20],

$$\begin{aligned}
&\mathbf{P}_\mu|\mathrm{X}_t(f) - \mu(f)|^2 \le 2t\,\|f\|\,\mu(T_t f) + 2\mu(|T_t f - f|)^2, \\
&\mathbf{P}_\mu|\mathrm{X}_t(f) - \mu(f)|^4 \\
&\le \text{const.}\,[t^3\,\|f\|^3\mu(T_t f) + t^2\,\|f\|^2\mu(T_t f)^2 + \mu(|T_t f - f|)^2].
\end{aligned}$$

Then using (1.10) together with Lemma 2.1 of [20] one can easily get

$$\int_{M_\rho(S)\setminus\{0\}} \nu(f)^2 K_{r-s}(\mathrm{d}\nu) \le t\,\|f\|\,\kappa_t(f)$$

$$\int_{M_\rho(S)\setminus\{0\}} \nu(f)^4 K_{r-s}(\mathrm{d}\nu) \le 3t^3\,\|f\|^3\kappa_t(f).$$

Substituting these estimates into (3.2) we get (3.1).

Proof of Theorem 1.3. Take an increasing sequence of $\mathbf{P}_K$-measurable sets

$W_\rho^{(n)}(S)$ of $W_\rho^+(S)$ such that $\bigcup_{n=1}^\infty W_n(S) = W_\rho^+(S)$ and $\mathbf{P}_K(W_n(S)) < \infty$ for all n, and set

$$\bar{Y}_t^{(n)} = \int_0^t \int_{W_n(S)} w_{t-s} N(\mathrm{d}s\mathrm{d}w). \tag{3.3}$$

Note that $\rho \in D(A) \cap C_\rho(S)^+$. By the same way as for (3.2) one sees easily

$$\begin{aligned}
&\mathbf{Q}\left[\bar{Y}_t^{(n)}(\rho) - \bar{Y}_r^{(n)}(\rho) - \int_r^t \mathrm{d}s \int_{W_\rho^{(n)}(S)} w_{t-s}(\rho)\mathbf{P}_K(\mathrm{d}w)\right]^4 \\
&= \int_0^t \mathrm{d}s \int_{W_\rho^{(n)}(S)} |w_{t-s}(f) - w_{r-s}(f)|^4 \mathbf{P}_K(\mathrm{d}w) \\
&\quad + 3\left[\int_0^t \mathrm{d}s \int_{W_\rho^{(n)}(S)} |w_{t-s}(f) - w_{r-s}(f)|^2 \mathbf{P}_K(\mathrm{d}w)\right]^2 \\
&\le \mathbf{Q}\left[\bar{Y}_t(\rho) - \bar{Y}_r(\rho) - \int_r^t \kappa_{t-s}(\rho)\mathrm{d}s\right]^4 \\
&\le C(M, \rho)(t-r)^2,
\end{aligned} \tag{3.4}$$

for all n and $0 \le r \le t \le M$, from which it follows that $\bar{Y}_t^{(n)}(\rho)$ has a continuous modification. However, by the expression (3.3), $\bar{Y}_t^{(n)}(\rho)$ is left continuous since $N([0, M] \times W_\rho^{(n)}(S)) < \infty$ for every finite M. Thus it follows that $\bar{Y}_t^{(n)}(\rho)$ is indeed continuous in $t \ge 0$ $\mathbf{Q}$-almost surely.

We here notice that (3.3) together with the continuity of $\bar{Y}_t^{(n)}(\rho)$ for all $n \ge 1$ implies that

$$\lim_{t \to 0^+} w_t(\rho) = 0 \qquad \text{for } \mathbf{P}_K\text{-almost all } w \in W_\rho^+(S), \tag{3.5}$$

thus w_t is an $M_\rho(S)$-valued continuous function of $t \in (-\infty, \infty)$ for $\mathbf{P}_K$-almost all $w \in W_\rho^+(S)$.

Since $\bar{Y}_t(\rho)$ is the increasing limit of the sequence $\bar{Y}_t^{(n)}(\rho)$, it is lower semicontinuous. Now by Lemma 3.1, $\bar{Y}_t(\rho)$ admits a continuous modification $\tilde{Y}_t(\rho)$, which clearly satisfies

$$\bar{Y}_t^{(n)}(\rho) \le \bar{Y}_t(\rho) \le \tilde{Y}_t(\rho), \qquad \text{for all } t \geqq 0,$$

$\mathbf{Q}$-almost surely. By (3.4), $\{\tilde{Y}_t(\rho) - \bar{Y}_t^{(n)}(\rho) : n = 1, 2, \cdots\}$ is a tight family in the space $C([0, \infty), \mathbf{R})$ converging to the zero process, so that for each $M > 0$,

$$\sup_{0 \le t \le M} |\tilde{Y}_t(\rho) - \bar{Y}_t(\rho)| \le \sup_{0 \le t \le M} |\tilde{Y}_t(\rho) - \bar{Y}_t^{(n)}(\rho)| \to 0$$

in probability under $\mathbf{Q}$. Therefore $\tilde{Y}_t(\rho) = \bar{Y}_t(\rho)$ for all $t \ge 0$ $\mathbf{Q}$-almost surely, that is, $\bar{Y}_t(\rho)$ is a continuous process.

Now let $f \in C_\rho(S)^+$ be fixed. Then $g := c\rho - f \in C_\rho(S)^+$ for some $c > 0$. Notice that by (1.11) both $Y_t(f)$ and $Y_t(g)$ are lower semi-continuous functions of t and

$$Y_t(f) + Y_t(g) = c Y_t(\rho)$$

is continuous in t. This implies that $Y_t(f)$ and $Y_t(g)$ are continuous in t, which yields the $M_\rho(S)$-valued continuity of Y_t.

Next we prove Theorem 1.4. By Dynkin's result in [7] for each $x \in S$,

$$\int_{M_\rho(S)\setminus\{0\}} (1 - e^{-\nu(f)}) K_t^x(d\nu) = V_t f(x), \qquad f \in C_0(S), \tag{3.6}$$

defines an entrance law $(K_t^x)_{t>0}$ of the MBD process $(X_t, \mathbf{P}_\mu)$ with state space $M_\rho(S)$. Thus there is a σ-finite measure Λ^x on space $W_\rho^+(S) := C((0, \infty), M_\rho(S))$ such that, under Λ^x, the coordinate process $(w_t)_{t>0}$ of $W_\rho^+(S)$ is a Markov process with the same transition probability as $(X_t, \mathbf{P}_\mu)$ and one dimensional marginal distributions $(K_t^x)_{t>0}$. It can be easily checked that $x \mapsto \Lambda^x$ satisfies the requirements (i) and (iii) of Theorem 1.4. Accordingly what we need to show is that Λ^x is supported by W_x^e. To see this we rely upon Perkins's result in [25] which asserts that a conditional MBD process is a modified Fleming-Viot diffusion.

In the following Lemmas 3.2–3.5, we assume that A is the generator of a strongly continuous conservative Feller semigroup. Let $M^1(S)$ denote the subspace of $M_\rho(S)$ comprising Borel probability measures on S. Fix $r > 0$, and let $\hat{\Omega}(r) = C([0, r], M^1(S))$ with canonical process $(\hat{X}_t)_{0 \le t \le r}$ and natural filtration $(\hat{\mathscr{F}}_t)_{0 \le t \le r}$. Then for each $h \in C([0, r], (0, \infty))$ and $\hat{\mu} \in M^1(S)$, there is a unique probability measure $\hat{\mathbf{P}}_{\mu,h}$ on $\hat{\Omega}(r)$ such that for each $f \in D(A)$,

$$\hat{M}_t(f) = \hat{X}_t(f) - \hat{\mu}(f) - \int_0^t \hat{X}_s(Af)ds, \qquad 0 \le t \le r, \tag{3.7}$$

is a $(\hat{\mathscr{F}}_t, \hat{\mathbf{P}}_{\mu,h})$-martingale starting at 0 with quadratic process

$$\langle \hat{M}(f) \rangle_t = \int_0^t h_s^{-1}[\hat{X}_s(f^2) - \hat{X}_s(f)^2]ds, \qquad 0 \le t \le r. \tag{3.8}$$

$(\hat{X}_t, \hat{\mathscr{F}}_t, \hat{\mathbf{P}}_{\hat{\mu},h})$ defines a time-inhomogeneous diffusion process which is called a modified Fleming-Viot diffusion.

Let $(X_t, \mathbf{P}_\mu)$ be an $M_F(S)$-valued MBD process associated with the generator A. It is well-known that the total mass process $X_t(1)$ is equivalent to the one-dimensional diffusion $(z_t, \tilde{\mathbf{P}}_{\mu(1)})$ in $[0, \infty)$ generated by $\frac{1}{2} x \frac{d^2}{dx^2}$. Hereafter we assume $(z_t, \tilde{\mathbf{P}}_{\mu(1)})$ is realized on the canonical space $\tilde{\Omega} := C([0, \infty), [0, \infty))$ with the natural filtration $(\tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t)$.

Lemma 3.2. ([25], Theorem 3) *For every $\hat{\mathscr{F}}_r$-measurable function $F(\hat{\omega}.)$, every $\tilde{\mathscr{F}}_r$-measurable function $G(z.)$ and every $\mu \in M_F(S) \setminus \{0\}$, it holds that*

$$\begin{aligned} &\mathbf{P}_\mu\{F(X.(1)^{-1}X.)G(X.(1));\ X_r(1) > 0\} \\ &= \tilde{\mathbf{P}}_{\mu(1)}\{G(z.)\hat{\mathbf{P}}_{\hat{\mu},z.}F(\hat{X}.);\ z_r > 0\}. \end{aligned} \tag{3.9}$$

We shall also need the following fact concerning the entrance law (K_t^x).

Lemma 3.3. *For* $t > 0$, $\lambda > 0$ *and* $f \in C_0(S)$,

$$\int_{M_F(S)\setminus\{0\}} (1 - e^{-\lambda\nu(1)})\hat{\nu}(f) K_t^x(d\nu) = \frac{2\lambda}{2 + t\lambda} T_t f(x). \tag{3.10}$$

Proof. Although one can prove (3.10) by elementary calculations based on (3.6), we here use Lemma 3.2 which implies that

$$\begin{aligned}\int_{M_F(S)\setminus\{0\}} (1 - e^{-\lambda\nu(1)})\hat{\nu}(f) K_t^x(d\nu) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \mathbf{P}_{\varepsilon\delta_x} \{(1 - \exp(-\lambda X_t(1))\hat{X}_t(f)\} \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \tilde{\mathbf{P}}_\varepsilon \{1 - \exp(-\lambda z_t)\} \hat{\mathbf{P}}_{\delta_x}\{\hat{X}_t(f)\} \\ &= \frac{2\lambda}{2 + t\lambda} T_t f(x).\end{aligned}$$

Lemma 3.4. *For* $r > 0$, $\eta > 0$ *and* $f \in D(A)$,

$$\lim_{b\to 0^+} \lim_{a\to 0^+} \Lambda^x\{ \sup_{a\le t\le b} |\hat{w}_t(f) - \hat{w}_a(f)| > \eta;\ w_r \neq 0\} = 0. \tag{3.11}$$

Proof. Using (3.9), Markov property and Chebyshev's inequality we have for $0 < a < b \le r$,

$$\begin{aligned}&\Lambda^x\{ \sup_{a\le t\le b} |\hat{w}_t(f) - \hat{w}_a(f)| > \eta;\ w_r \neq 0\} \\ &= \int_{M(S)\setminus\{0\}} K_a^x(d\nu)\tilde{\mathbf{P}}_{\nu(1)}\{\hat{\mathbf{P}}_{\hat{\nu},z.}[\sup_{0\le t\le b-a} |\hat{X}_t(f) - \hat{\nu}(f)| > \eta];\ z_{r-a} > 0\} \\ &\le \frac{2}{\eta}\int_{M(S)\setminus\{0\}} K_a^x(d\nu)\tilde{\mathbf{P}}_{\nu(1)}\left\{\hat{\mathbf{P}}_{\hat{\nu},z.}\left[\int_0^{b-a} \hat{X}_s(|Af|)ds;\ z_{r-a} > 0\right]\right\} \\ &\quad + \int_{M(S)\setminus\{0\}} K_a^x(d\nu)\tilde{\mathbf{P}}_{\nu(1)}\{\hat{\mathbf{P}}_{\hat{\nu},z.}[\sup_{0\le t\le b-a} |\hat{M}_t(f)| > \eta/2];\ z_{r-a} > 0\}\end{aligned}$$

We denote the last two terms by I_1 and I_2, respectively. Using

$$\tilde{\mathbf{P}}_z\{z_t > 0\} = 1 - e^{-2z/t},$$

and (3.10) we get

$$\begin{aligned}I_1 &\le \frac{2}{\eta}\int_{M_F(S)\setminus\{0\}} (1 - e^{-2\nu(1)/(r-a)}) K_a^x(d\nu) \, \| Af \| \, (b - a) \\ &= \frac{4}{r\eta} \| Af \| (b - a) \to 0 \qquad \text{as } a \to 0^+ \text{ and } b \to 0^+.\end{aligned}$$

For I_2 we use a martingale inequality to see that

$$I_2 \le \frac{2}{\eta} \int_{M_F(S)\setminus\{0\}} K_a^x(\mathrm{d}\nu)\tilde{\mathbf{P}}_{\nu(1)}\{\hat{\mathbf{P}}_{\hat{\nu},z.}[|\hat{M}_{b-a}(f)|;\ z_{r-a} > 0]\},$$

where

$$\hat{\mathbf{P}}_{\hat{\nu},z.}|\hat{M}_{b-a}(f)| \le \int_0^{b-a} \hat{\mathbf{P}}_{\hat{\nu},z.}\hat{X}_t(|Af|)\mathrm{d}s + \hat{\mathbf{P}}_{\hat{\nu},z.}|\hat{X}_{b-a}(f) - \hat{\nu}(f)|$$
$$\le \|Af\|(b-a) + \hat{\nu}(T_{b-a}|f - f(x)|) + \hat{\nu}(|f - f(x)|).$$

Therefore,

$$I_2 \le \frac{4}{r\eta}\|Af\|(b-a) + \frac{4}{r\eta}[T_b|f - f(x)|(x) + T_a|f - f(x)|(x)] \to 0$$

as $a \to 0^+$ and $b \to 0^+$, completing the proof.

Lemma 3.5. *For Λ^x-almost all w,*

$$\lim_{t\to 0^+} w_t(1) = 0 \quad \textit{and} \quad \lim_{t\to 0^+} \hat{w}_t = \delta_x, \tag{3.12}$$

so Λ^x is supported by W_x^e.

Proof. By (3.10)

$$\int_{M_F(S)\setminus\{0\}} \nu(1)^2 K_a^x(\mathrm{d}\nu) = a.$$

Since it holds that

$$\tilde{\mathbf{P}}_z\{\sup_{0\le t\le b} z_t^2\} \le 4(z^2 + zb),$$

we obtain

$$\Lambda^x\{\sup_{a\le t\le b} w_t(1)^2\} \le \int_{M_F(S)\setminus\{0\}} 4[\nu(1)^2 + \nu(1)t(b-a)]K_a^x(\mathrm{d}\nu) = 4b,$$

hence the first assertion follows. For the second assertion note that for $0 < b \le r$ and $\eta > 0$,

$$\begin{aligned}
&\Lambda^x\{\sup_{0<t\le b} |\hat{w}_t(f) - f(x)| > \eta;\ w_r \ne 0\}\\
&= \lim_{a\to 0^+} \Lambda^x\{\sup_{a\le t\le b} |\hat{w}_t(f) - f(x)| > \eta;\ w_r \ne 0\}\\
&\le \lim_{a\to 0^+} \Lambda^x\{\sup_{a\le t\le b} |\hat{w}_t(f) - \hat{w}_a(f)| > \eta/2;\ w_r \ne 0\}\\
&\quad + \lim_{a\to 0^+} \Lambda^x\{|\hat{w}_a(f) - f(x)| > \eta/2;\ w_r \ne 0\}.
\end{aligned} \tag{3.13}$$

Using (3.10) we have

$$\lim_{a\to 0^+} \Lambda^x\{|\hat{w}_a(f) - f(x)| > \eta/2;\ w_r \neq 0\}$$

$$= \lim_{a\to 0^+} \int_{\{|\hat{\nu}(f) - f(x)| > \eta\}} (1 - e^{-2\nu(1)/(r-a)}) K_a^x(d\nu)$$

$$\leq \lim_{a\to 0^+} \eta^{-1} \int_{M_F(S)\setminus\{0\}} (1 - e^{-2\nu(1)/(r-a)})\hat{\nu}(|f - f(x)|) K_a^x(d\nu)$$

$$= \lim_{a\to 0^+} \frac{2}{r\eta} T_a|f - f(x)|(x) = 0,$$

then the desired assertion follows from (3.11) and (3.13).

For a non-conservative (T_t), the proof can be reduced to the conservative situation in the following way. Extend (T_t) to a conservative semigroup $(\bar{T}_t)$ on the enlarged state space $\bar{S} = S \cup \{\Delta\}$ by adding an extra point Δ as a trap. For $(\bar{T}_t)$ we denote the associated cumulant semigroup and the entrance law by $(\bar{V}_t)$ and $(\bar{K}_t^x)$, respectively. Obviously $\bar{V}_t f(x) = V_t f(x)$ holds for $x \in S$ if $f(\Delta) = 0$, so K_t^x is indeed the restriction of $\bar{K}_t^x$ to $M_F(S)$. Then it is easy to see that (w_t, Λ^x) is equivalent to $(\bar{w}_t|_S, \bar{\Lambda}^x)$. Since $\bar{\Lambda}^x$ is supported by $\bar{W}^x$, it is obvious that Λ^x is supported by W^x.

Finally, we show the uniqeness assertion of Theorem 1.4.

Lemma 3.6. *Any σ-finite measures Λ^x, $x \in S$, satisfying the requirements (i)–(iv) of Theorem* 1.4 *is uniquely determined.*

Proof. Since $\hat{w}_r \to \delta_x$ as $r \to 0^+$, by the continuity of (V_t) we have for Λ^x-almost all w,

$$\lim_{s\to 0^+} \hat{w}_s(V_{t-s}f) = V_t f(x), \qquad f \in C_0(S)^+.$$

By Markov property, for each $r > 0$ and $\varepsilon > 0$,

$$\lim_{s\to 0^+} \Lambda^x\{w_s(1);\ |\hat{w}_s(V_{t-s}f) - V_t f(x)| < \varepsilon,\ w_s(1) < \varepsilon\}$$

$$\geq \lim_{s\to 0^+} \Lambda^x\{w_s(T_{r-s}1);\ |\hat{w}_s(V_{t-s}f) - V_t f(x)| < \varepsilon,\ w_s(1) < \varepsilon\}$$

$$= \lim_{s\to 0^+} \Lambda^x\{w_r(1);\ |\hat{w}_s(V_{t-s}f) - V_t f(x)| < \varepsilon,\ w_s(1) < \varepsilon\}$$

$$= \Lambda^x\{w_r(1)\},$$

hence by the condition (iii) we see

$$\lim_{s\to 0^+} \Lambda^x\{w_s(1);\ |\hat{w}_s(V_{t-s}f) - V_t f(x)| < \varepsilon,\ w_s(1) \leq \varepsilon\} = 1.$$

Using this and Markov property again we get

$$\Lambda^x(1 - e^{w_t(f)})$$

$$= \lim_{s \to 0^+} \Lambda^x(1 - \exp\{-w_s(V_{t-s}f)\};\ |\hat{w}_s(V_{t-s}f) - V_t f(x)| < \varepsilon,\ w_s(1) < \varepsilon)$$

$$\le \sup_{0<a<\varepsilon} a^{-1}(1 - \exp\{-a[V_t f(x) + \varepsilon]\})$$

$$\times \lim_{s \to 0^+} \Lambda^x\{w_s(1);\ |\hat{w}_s(V_{t-s}f) - V_t f(x)| < \varepsilon,\ w_s(1) < \varepsilon\}$$

$$\to V_t f(x) \qquad \text{as } \varepsilon \to 0^+.$$

A similar argument applies to get

$$\Lambda^x(1 - e^{w_t(f)}) \ge V_t f(x),$$

thus the marginal distributions of Λ^x is uniquely determined. Therefore the uniqueness of Λ^x follows.

Proof of Corollary 1.6. This is almost the same as that of Theorem 1.3, and therefore omitted.

4. Super absorbing Brownian motions

In this section we discuss immigration processes of super ABMs. Let us first give the proofs of Lemma 1.1 and Theorem 1.6 as follows.

Proof of Lemma 1.1 *and Theorem* 1.6. Let D be a bounded smooth domain in $\mathbf{R}^d$ and let (T_t) be the semigroup of an absorbing Brownian motion in D acting on $C_0(D)$. Suppose (κ_t) is a ρ-locally integrable entrance law of (T_t). Noting that $\kappa_1(\rho) < \infty$ we introduce a time inhomogeneous Markov semigroup $(\tilde{T}_t^s)_{0 \le s \le t \le 1}$ and a probability entrance law $(\tilde{\kappa}_t)_{0 < \ \le 1}$ of $(\tilde{T}_t^s)$ by

$$\tilde{T}_t^s f(x) = [T_{1-s}\rho(x)]^{-1} T_{t-s}[f T_{1-t}\rho](x), \tag{4.1}$$

and

$$\tilde{\kappa}_t(f) = \kappa_1(\rho)^{-1}\kappa_t(f T_{1-t}\rho). \tag{4.2}$$

Since $T_t f \in C_0(D) \cap C^1(\bar{D})$ for $f \in C_0(D) \cap C^1(\bar{D})$ (cf. [13], p. 65), $\tilde{T}_t^s f$ is extended to a continuous function on $\bar{D}$ such that

$$\tilde{T}_t^s f(z) = \kappa_{1-s}^z(\rho)^{-1}\kappa_{t-s}^z(f T_{1-t}\rho) \qquad \text{for } z \in \partial D. \tag{4.3}$$

Choose $r_n \to 0^+$ such that $\gamma := \lim_n \kappa_{r_n}$ defines a probability measure γ on $\bar{D}$. By (4.1)–(4.3)

$$\begin{aligned} \kappa_1(\rho)^{-1}\kappa_t(f) &= \tilde{\kappa}_t(f[T_{1-t}\rho]^{-1}) \\ &= \int_D \gamma(dz)[T_1\rho(z)]^{-1} T_t f(z) + \int_{\partial D} \gamma(dz)[\kappa_1^z(\rho)]^{-1}\kappa_t^z(f), \end{aligned} \tag{4.4}$$

which proves Lemma 1.1 in the case that D is bounded. When D is unbounded, the limit $\gamma := \lim_n \tilde{\kappa}_{r_n}$ defines a probability measure γ on $\bar{D} \cup \{\infty\}$. But for $D = \mathbf{H}^d$

we claim that

(4.5) $$\tilde{T}_t^s f(x) \to 0 \text{ as } |x| \to \infty \qquad \text{for } x \in D.$$

Once (4.5) is proved, Lemma 1.1 follows for $D = \mathbf{H}^d$ in the same way as above. Note that

(4.6) $$T_t f(x) = \int_{H^d} [g_t(x_1 - y_1) - g_t(x_1 + y_1)] \prod_{i=2}^{d} g_t(x_i - y_i) f(y) \mathrm{d}y_1 \cdots \mathrm{d}y_n,$$

where

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}, \qquad x \in \mathbf{R},\ t > 0.$$

Then it holds that for $c < a < b < d$,

(4.7) $$\lim_{|x| \to \infty} \int_a^b g_t(x - y) \mathrm{d}y \left(\int_c^d g_t(x - y) \mathrm{d}y \right)^{-1} = 0,$$

so that for $0 < c < a < b < d$,

$$\lim_{x \to \infty} \int_a^b [g_t(x - y) - g_t(x + y)] \mathrm{d}y \left(\int_c^d [g_t(x - y) - g_t(x + y)] \mathrm{d}y \right)^{-1} = 0.$$

Since $T_{1-t}\rho$ is a bounded strictly positive continuous function on $\mathbf{H}^d$, (4.5) follows from (4.1), (4.6) and (4.7).

Now Theorem 1.6 is immediate since by (1.5) for every $t > 0$ and $f \in C_0(D) \cap C^1(\bar{D})$, $\kappa_{0^+}^z(V_t f) = \mathbf{D} V_t f(z)$ holds for $z \in \partial D$.

Next we give the proof of Theorems 1.7. Let $(Y_t, \mathbf{Q}_0)$ be the immigration process of the super ABM over $(0, \infty)$ given by (1.19). By Theorem 1.2, for each $f \in C_\rho((0, \infty))^+$, $\mathbf{Q}_0$-almost surely,

(4.8) $$Y_t(f) = \int_0^t [m(T_{t-s} f) + c\kappa_{t-s}^o(f)] \mathrm{d}s + \int_0^t \int_0^\infty T_{t-s} f(x) M(\mathrm{d}s\mathrm{d}x),$$

where $M(\mathrm{d}s\mathrm{d}x)$ is an orthogonal martingale measure on $[0, \infty) \times (0, \infty)$ with quadratic variation measure $\langle M \rangle (ds, dx) = Y_s(\mathrm{d}x)\mathrm{d}s$ and (κ_t^o) is the entrance law of the ABM in $(0, \infty)$ defined by

(4.9) $$\kappa_t^o(\mathrm{d}x) = \kappa_t^o(x)\mathrm{d}x = \sqrt{\frac{2}{\pi t}} \frac{x}{t} \mathrm{e}^{-x^2/2t} \mathrm{d}x, \qquad t > 0,\ x > 0.$$

Let

(4.10) $$Z_t(y) = \int_0^t \int_0^\infty p_{t-s}(x, y) M(\mathrm{d}s\mathrm{d}x),$$

and let

(4.11) $$Y_t(y) = Z_t(y) + \int_0^t \left[\int_0^\infty p_{t-s}(x, y) m(\mathrm{d}x) + c\kappa^o_{t-s}(y) \right] \mathrm{d}s,$$

where

$$p_t(x, y) = g_t(x - y) - g_t(x + y), \qquad t, x, y > 0.$$

By a stochastic Fubini theorem, $Y_t(\mathrm{d}x) = Y_t(x)\mathrm{d}x$ holds $\mathbf{Q}_0$-almost surely. (See e.g. [32]).

Now we prove Theorem 1.7 by a series of lemmas. Since the arguments are quite similar to those given in [20], we here present only an outline. Recall that ρ is a function in $C_0^2(0, \infty)^{++}$ such that $\rho(x) = x$ for $0 < x < 1$ and $= e^{-x}$ for $x > 2$. The proof of the following Lemma 4.1 is omitted since it is quite elmentary.

Lemma 4.1. *For $M > 0$ and $n \geq 0$ there is a constants $C(n, M) > 0$ such that*

(4.12) $$\int_0^M \mathrm{d}s \int_0^\infty g_s(x - y)^2 e^{ny} \mathrm{d}y \leq C(n, M) \cdot e^{nx}$$

for $x > 0$, and

(4.13) $$\int_0^t \mathrm{d}s \int_0^\infty [p_{t-s}(y, z) - p_{r-s}(x, z)]^2 \mathrm{d}z \leq C(0, M) \cdot (\sqrt{t - r} + |y - x|) \cdot (e^{nx} + e^{ny})$$

for $0 \leq r \leq t \leq M$ and $x, y > 0$.

Lemma 4.2. *For $M > 0$ and $n \geq 0$ there is a constant $C(n, M) > 0$ such that*

(4.14) $$\mathbf{Q}_0\{Z_t(x)^{2n}\} \leq C(n, M) \cdot e^{nx}$$

for $0 \leq t \leq M$ and $x > 0$. Moreover,

(4.15) $$\mathbf{Q}_0\{Z_t(x)^2\} \to 0 \qquad \text{as } x \to 0^+.$$

Proof. Since $m \in M_\rho((0, \infty))$, it is easy to check that for each $M > 0$ there exists $C(M) > 0$ such that

(4.16) $$\int_0^M \mathrm{d}s \int_0^\infty p_s(x, y) m(\mathrm{d}y) \leq C(M) \cdot e^x.$$

for all $x > 0$. Moreover it holds that

(4.17) $$\sup_{x>0} \int_0^t \kappa^o_s(x) \mathrm{d}s = \lim_{x \to 0^+} \int_0^t \kappa^o_s(x) \mathrm{d}s = 2$$

for all $t > 0$. Now notice that

(4.18)
$$\begin{aligned}\mathbf{Q}_0\{Z_t(x)^2\} &= \int_0^t \mathbf{Q}_0\{Y_s(p_{t-s}^2(x,\cdot))\}\mathrm{d}s \\ &\le \int_0^t \frac{\mathrm{d}s}{\sqrt{2\pi(t-s)}} \int_{t-s}^t [m(p_u(x,\cdot)) + c\kappa_u^o(x)]\mathrm{d}u.\end{aligned}$$

from which (4.15) follows. Next note that (4.14) holds for $n=1$ by (4.16)–(4.18). Assuming that (4.14) holds for $n \le m$ we shall show it for $n=2m$, which will yields (4.14) for all n. Under the induction assumption, we have $\mathbf{Q}_0\{Y_t(x)^{2m}\} \le C(m, M)\cdot \mathrm{e}^{mx}$. Then by making use of Burkholder-Davis-Gundy's and then Hölder's inequalities,

$$\begin{aligned}&\mathbf{Q}_0\{Z_t(x)^{4m}\} \le C_1(2m, M)\cdot \mathbf{Q}_0\left[\int_0^t \mathrm{d}s \int p_{t-s}^2(x,y)Y_s(y)\mathrm{d}y\right]^{2m} \\ &\le C_1(2m, M)\cdot \mathbf{Q}_0\left\{\int_0^t \mathrm{d}s\int_0^\infty p_{t-s}^2(x,y)Y_s^{2m}(y)\mathrm{d}y\right\}\cdot\left[\int_0^t \mathrm{d}s\int_0^\infty p_{t-s}^2(x,y)\mathrm{d}y\right]^{2m-1} \\ &\le C_1(2m, M)\cdot \mathrm{e}^{2mx},\end{aligned}$$

thus (4.14) holds for $n=2m$.

Proof of Theorem 1.7. Using Lemma 4.1 with a similar argument as in the proof of (4.14), we get that

(4.19)
$$\mathbf{Q}_0\{|Z_t(y) - Z_r(x)|^{2n}\} \le C(n, M)\cdot(\mathrm{e}^{nx} + \mathrm{e}^{ny})(\sqrt{t-r} + |y-x|)^{n-1}$$

for $0 \le r \le t \le M$ and $x, y > 0$. Therefore, $\{Z_t(x): t \ge 0, x > 0\}$ has a continuous modification vanishing as $x \to 0^+$ by (4.15), hence $\{Y_t(x): t \ge 0, x > 0\}$ has a continuous modification satisfying $Y_t(0^+) = 2c$ by (4.17). Tracing the arguments of [20] one can define a time-space white noise $\dot{W}_t(x)$ on an extension of the original probability space such that $M(\mathrm{d}s\mathrm{d}x) = \sqrt{Y_s(x)}\,\dot{W}_s(x)\mathrm{d}s\mathrm{d}x$. Then by (1.11) the density process $\{Y_t(x): t \ge 0, x > 0\}$ satisfies the SPDE (1.21).

Now we proceed to the proof of Theorem 1.8. Let us start with the following nonlinear equation:

(4.20)
$$\begin{cases} u_t(t,x) = \frac{1}{2}u_{xx}(t,x) - \frac{1}{2}u(t,x)^2 + \frac{1}{2}\theta^2 1_{[a,\infty)}(x), & t>0, x>0, \\ u(0,x) = f(x), \quad x>0, \\ u(t,0) = 0, \quad t>0, \\ u(t,\cdot) \text{ is uniformly bounded on each finite time interval.} \end{cases}$$

Lemma 4.3. *For bounded non-negative* $f \in C_0((0,\infty)) \cap C^1([0,\infty))$, *the equation* (4.20) *has a unique solution in* $C^{0,1}([0,\infty)\times[0,\infty)) \cap C^{1,2}((0,\infty)\times[(0,\infty)\setminus\{a\}])$.

Proof. Recall that (T_t) denotes the transition semigroup of the absorbing Brownian motion in $(0,\infty)$. It is known that the evolution equation

(4.21) $$u(t, x) = T_t f(x) + \int_0^t T_{t-s}\left[\frac{1}{2}\theta^2 1_{[a,\infty)} - \frac{1}{2}u(s)^2\right](x)\mathrm{d}s, \qquad t \geq 0,$$

has a unique positive solution $u(t, x)$ bounded on each finite time interval; see [12], [16], etc. Then it is a routine task to check that $u(\cdot, \cdot) \in C^{0,1}([0, \infty) \times [0, \infty)) \cap C^{1,2}((0, \infty) \times [(0, \infty)\setminus\{a\}])$ and it solves (4.20), hence we omit the details.

Hereafter we denote by $u^a(t, x; \theta)$ the solution to (4.20) with $f \equiv 0$. Then it holds that

(4.22) $$0 \leq u^a(t, x; \theta) \leq \theta, \qquad t \geq 0,\ x \geq 0,$$

and that

(4.23) $$a^2 u^a(a^2 t, ax; \theta) = u^1(t, x; a^2\theta), \qquad t \geq 0,\ 0 \leq x \leq 1;$$

see [16], [18].

Lemma 4.4. *The limit*

(4.24) $$u^a(t, x) = \lim_{\theta \to \infty} u^a(t, x; \theta)$$

exists in $C^{0,1}([0, \infty) \times [0, a)) \cap C^{1,2}((0, \infty) \times (0, a))$ *and it satisfies*

(4.25) $$\begin{cases} u_t(t, x) = \frac{1}{2}u_{xx}(t, x) - \frac{1}{2}u(t, x)^2 & t > 0,\ 0 < x < a, \\ u(0, x) = 0, \quad 0 < x < a, & \\ u(t, 0) = 0, \quad t > 0, & \end{cases}$$

Moreover $u^a(t, x)$ *has the following scaling property*:

(4.26) $$a^2 u^a(a^2 t, ax) = u^1(t, x), \qquad t \geq 0,\ 0 \leq x \leq 1.$$

Proof. Let $0 < b < a$ and let $g_t^{0b}(x, y)$ denote the transition density of the absorbing Brownian motion in $(0, b)$. Then $u^a(t, x; \theta)$ satisfies

(4.27) $$\begin{aligned} u^a(t, x; \theta) = &-\frac{1}{2}\int_0^t \mathrm{d}s \int_0^b g_{t-s}^{0b}(x, y)u(s, y; \theta)^2 \mathrm{d}y \\ &-\frac{1}{2}\int_0^t \frac{\partial}{\partial y} g_{t-s}^{0b}(x, b)u(s, b; \theta)^2 \mathrm{d}s, \end{aligned}$$

so $u^a(t, x)$ satisfies the same integral equation. Using this one checks that $u^a(t, x)$ is in $C^{0,1}([0, \infty) \times [0, a)) \cap C^{1,2}([0, \infty) \times (0, a))$ and it satisfies (4.25). The scaling property (4.26) follows from (4.23).

Lemma 4.5. *The limit*

(4.28) $$u^a(x : \theta) = \lim_{t \to \infty} u^a(t, x; \theta)$$

exists in $C^1([0,\infty))\cap C^2((0,\infty)\setminus\{a\})$, *which is the unique solution of*

$$(4.29)\qquad \begin{cases} u_{xx}(x)=u(x)^2, & 0<x<a,\\ u_{xx}(x)=u(x)^2-\theta^2, & x>a,\\ 0\le u(x)\le\theta, & x\ge 0,\\ u(0)=0. \end{cases}$$

Proof. Since $u^a(t,x;\theta)$ is non-decreasing in $t\ge 0$, the limit (4.28) exists. Note that $u^a(t,x;\theta)$ satisfies

$$u^a(r+t,x;\theta)=T_tu^a(r,\cdot;\theta)(x)+\int_0^t T_{t-s}\left[\frac{1}{2}\theta^2 1_{[a,\infty)}-\frac{1}{2}u^a(r+s;\theta)^2\right](x)\mathrm{d}s.$$

Letting $r\to\infty$ in the above equation we get

$$u^a(x;\theta)=T_tu^a(\cdot,\theta)(x)+\int_0^t T_{t-s}\left[\frac{1}{2}\theta^2 1_{[a,\infty)}-\frac{1}{2}u^a(\cdot,\theta)^2\right](x)\mathrm{d}s,\qquad t\ge 0,$$

from which it follows that $u^a(x;\theta)$ lies in $C^1([0,\infty))\cap C^2((0,\infty)\setminus\{a\})$ and differentiating in t gives (4.29).

To see the uniqueness of the solutions of (4.29) first note that any solution $u(x)$ of (4.29) is concave in (a,∞), so $u(\infty)=\theta$ and $u'(\infty)=0$. If $u(x)$ and $v(x)$ are two solutions of (4.29), then $w(x):=u(x)-v(x)$ vanishes at $x=0$ and ∞. Suppose that $w(x)$ is not identically equal to 0, we may assume $w(x_0)=\max_x w(x)>0$ for some $x_0>0$. Since

$$w''(x)=[u(x)-v(x)]w(x),\qquad x>0,\ x\ne a,$$

we have

$$w(x)-w(x_0)=\int_{x_0}^x \mathrm{d}y\int_{x_0}^y [u(z)-v(z)]w(z)\mathrm{d}z>0$$

when $|x-x_0|$ is small, which is absurd.

Lemma 4.6. *The limit* $u^a(x):=\lim_{t\to\infty}u^a(t,x)$ *exists in* $C^1([0,a))\cap C^2((0,a))$, *which is the unique solution of*

$$(4.30)\qquad \begin{cases} u_{xx}(x)=u(x)^2, & 0<x<a,\\ u(0)=0, & u(a^-)=\infty. \end{cases}$$

Moreover it holds that

$$(4.31)\qquad u_x^a(0)=\lim_{t\to\infty}u_x^a(t,0).$$

Proof. Since $u^a(t,x)$ is non-decreasing in $t\ge 0$, the limit (4.30) exists. Letting $t\to\infty$ in (4.27) we obtain

$$u^a(x)=-\frac{1}{2}\int_0^\infty \mathrm{d}s\int_0^b g_s^{0b}(x,y)u^a(y)^2\mathrm{d}y-\frac{1}{2}\int_0^\infty \frac{\partial}{\partial y}g_s^{0b}(x,b)u^a(x)^2\mathrm{d}s.$$

This implies that $u^a(x) \in C^1([0, a)) \cap C^2((0, a))$ and

$$\begin{cases} u_{xx}(x) = u(x)^2, & 0 < x < a, \\ u(0) = 0. \end{cases}$$

Next we claim that for $u^a(x; \theta)$ given by (4.28),

$$\lim_{\theta \to \infty} u^a(a; \theta) = \infty. \tag{4.32}$$

Once (4.32) is proved, since

$$u^a(a^-) = \lim_{x \to a^-} \lim_{t \to \infty} u^a(t, x) \geq \lim_{x \to a^-} \lim_{t \to \infty} u^a(t, x; \theta) = u^a(x; \theta),$$

it will follow that $u^a(a^-) = \infty$ and $u^a(x)$ will solve (4.30). To see (4.32) note by the first equation of (4.29) that

$$\frac{1}{2} u_x^a(x; \theta)^2 - \frac{1}{3} u^a(x; \theta)^3 = \frac{1}{2} u_x^a(0; \theta)^2, \qquad 0 \leq x \leq a.$$

By this and the second quation of (4.29),

$$\begin{aligned} &\frac{1}{2} u_x^a(x; \theta)^2 - \frac{1}{3} u^a(x; \theta)^3 + \theta^2 u^a(x; \theta) \\ &= \frac{1}{2} u_x^a(a; \theta)^2 - \frac{1}{3} u^a(a; \theta)^3 + \theta^2 u^a(a; \theta) \\ &= \frac{1}{2} u_x^a(0; \theta)^2 + \theta^2 u_x^a(a; \theta), \qquad x \geq a. \end{aligned} \tag{4.33}$$

Letting $x \to \infty$ in (4.33) we see

$$\frac{2}{3} \theta^3 = \frac{1}{2} u_x^a(0; \theta)^2 + \theta^2 u^a(a; \theta).$$

which yields (4.32) since $u^a(x; \theta) \leq u^a(x)$ implies $u_x^a(0; \theta) \leq u_x^a(0)$.

Finally we show (4.31). For small $x > 0$,

$$\begin{aligned} &t^{-1} \int_0^t [u^a(s, x) - x u_x^a(s, 0)] \mathrm{d}s \\ &= t^{-1} \int_0^t \mathrm{d}s \int_0^x \mathrm{d}y \int_0^y u_{xx}^a(s, z) \mathrm{d}z \\ &= t^{-1} \int_0^x \mathrm{d}y \int_0^y \left[2u^a(t, z) + \int_0^t u^a(s, z)^2 \mathrm{d}s \right] \mathrm{d}z. \end{aligned}$$

Taking $t \to \infty$ we get

$$u^a(x) - x \lim_{t \to \infty} u_x^a(t, 0) = \int_0^x \mathrm{d}y \int_0^y u^a(z)^2 \mathrm{d}z,$$

which yields the desired conclusion.

Lema 4.7. *Let* $(Y_t, \mathbf{Q}_\mu)$ *denote the immigration process given by* (1.19). *Then for* $a > 0$ *and* $\theta > 0$,

$$\begin{aligned}\mathbf{Q}_0 \exp\left\{-\frac{1}{2}\theta^2 \int_0^t Y_s([a, \infty))\mathrm{d}s\right\} \\ = \exp\left\{-\int_0^t [m(u^a(s;\theta)) + cu_x^a(s, 0; \theta)]\mathrm{d}s\right\},\end{aligned} \tag{4.34}$$

where $u^a(s, x; \theta)$ *is the solution of* (4.20) *with* $f = 0$.

Proof. Note that for $f(t, x) \in C^2([0, \infty) \times [0, \infty))$,

$$M_t^f := Y_t(f(t)) - \int_0^t [Y_s(f_s(s) + f_{xx}(s)/2) + m(f(s)) + cf_x(s, 0)]\mathrm{d}s$$

is a martingale with quadratic variation process

$$\langle M^f \rangle_t := \int_0^t Y_s(f(s)^2)\mathrm{d}s.$$

Applying this to $f(t, x) = u^a(r - t, x; \theta)$, with some approximating argument, we see that

$$\begin{aligned}&\exp\left\{-Y_t(u^a(r-t;\theta)) - \frac{1}{2}\theta^2 \int_0^t Y_s([a, \infty))\mathrm{d}s\right\} - 1 \\ &= -\int_0^t [m(u^a(r-s;\theta)) + cu_x^a(r-s, 0; \theta)] \\ &\qquad \exp\left\{-Y_s(u^a(r-s;\theta)) - \frac{1}{2}\theta^2 \int_0^s Y_u([a, \infty))\mathrm{d}u\right\}\mathrm{d}s \\ &\qquad + \text{martingale},\end{aligned}$$

from which the desired relation follows.

Proof of Teorem 1.8. Since the immigration process has a jointly continuous desity $Y_t(x)$, applying Lemmas 4.7 and 4.4 we have

$$\begin{aligned}\mathbf{Q}_0\{\bar{R}_t \le a\} &= \mathbf{Q}_0\left\{-\int_0^t Y_s([a, \infty))\mathrm{d}s = 0\right\} \\ &= \exp\left\{-\int_0^t [m(u^a(s)) + cu_x^a(s, 0)]\mathrm{d}s\right\}.\end{aligned} \tag{4.35}$$

By the scaling property (4.26),

$$\mathbf{Q}_0\{t^{-1/3}\bar{R}_t \le a\}$$

(4.36)
$$= \exp\left\{-\int_0^{t^{1/3}a^{-2}} \mathrm{d}s \int_0^\infty u^1(s, t^{-1/3}a^{-1}x)m(\mathrm{d}x) + t^{-1/3}a^{-1}\int_0^{t^{1/3}a^{-2}} cu_x^1(s, 0)\mathrm{d}s\right\}.$$

Since $u(t, x)$ solves (4.25), as in the proof of Lemma 4.6,

$$\int_0^t [u^1(s, x) - xu_x^1(s, 0)]\mathrm{d}s$$
$$= \int_0^x \mathrm{d}y \int_0^y \left[2u^1(t, z) + \int_0^t u^1(s, z)^2 \mathrm{d}s\right]\mathrm{d}z$$
$$\leq x^2 u^1(x)\left[1 + \frac{t}{2}u^1(x)\right].$$

Using this and Lemma 4.6 we obtain

$$\lim_{t\to\infty}\int_0^{t^{1/3}a^{-2}} \mathrm{d}s \int_0^\infty u^1(s, t^{-1/3}a^{-1}x)m(\mathrm{d}x)$$
(4.37)
$$= \lim_{t\to\infty} t^{-1/3}a^{-1}\int_0^{t^{1/3}a^{-2}} u_x^1(s, 0)\mathrm{d}s \int_0^\infty xm(\mathrm{d}x)$$
$$= a^{-3}u_x^1(0)\int_0^\infty xm(\mathrm{d}x).$$

Hence from (4.36) and (4.37) it follows that

$$\lim_{t\to\infty} \mathbf{Q}_0\{t^{-1/3}\bar{R}_t \leq a\} = \exp\left\{-a^{-3}u_x^1(0)\left[c + \int_0^\infty xm(\mathrm{d}x)\right]\right\}.$$

The explicit value of $u_x^1(0)$ can be found by a similar argument as [18].

5. Central limit theorems for the immigration processes

Let $(Y_t, \mathbf{Q}_\mu)$ be an immigration process associated with an immigration measure $m \in M_{\rho_1}(S)$. By Theorem 1.2, we have $\mathbf{Q}_\mu$-almost surely,

(5.1)
$$Y_t(f) = Y_0(T_t f) + \int_0^t m(T_{t-s}f)\mathrm{d}s + \int_0^t\int_S T_{t-s}f(x)M(\mathrm{d}s\mathrm{d}x), \qquad f \in C_\rho(S),$$

where $M(\mathrm{d}s\mathrm{d}x)$ is an orthogonal martingale measure on $[0, \infty) \times S$ having quadratic variation measure

(5.2)
$$\langle M\rangle(\mathrm{d}s\mathrm{d}x) = \mathrm{d}s\, Y_s(\mathrm{d}x).$$

Our proof of the first central limit theorem is based on the following

Theorem 5.1. *Suppose that m is (T_t)-invariant and that for each $f \in C_\kappa(S)^+$ there is a constant $C(f) > 0$ such that*

$$\int_0^t \| T_s f \| \mathrm{d}s \leq \sqrt{t}\, C(f). \tag{5.3}$$

Then for each $f \in C_\kappa(S)^+$ the distribution of

$$\frac{Y_t(f) - tm(f)}{\sqrt{\mathbf{Var}\,(Y_t(f))}} \tag{5.4}$$

under $\mathbf{Q}_0$ converges as $t \to \infty$ to the normal distribution $N(0, 1)$.

For the proof we need a simple fact on martingales.

Lemma 5.1. *Suppose for each $t \geq 0$ we have a continuous martingale $(M_u^{(t)}, u \geq 0)$ with $M_0^{(t)} = 0$. If there exists $u(t) \to \infty$ such that $\langle M^{(t)} \rangle_{u(t)}$ converges as $t \to \infty$ to some constant $\sigma \geq 0$ in probability, then the distribution of $M_{u(t)}^{(t)}$ converges as $t \to \infty$ to the normal distribution $N(0, \sigma)$.*

Proof. Note that for each $t \geq 0$, $(M_u^{(t)})$ is a time change of a standard Brownian motion $B^{(t)}(u)$, i.e., $M_u^{(t)} = B^{(t)}(\langle M^{(t)} \rangle_u)$. Then for $\theta \in \mathbf{R}$ and $\varepsilon > 0$,

$$\begin{aligned}
&\mathbf{E}\,|\exp\{i\theta M_{u(t)}^{(t)}\} - \exp\{i\theta B^{(t)}(\sigma)\}| \\
&= \mathbf{E}\,|\exp\{i\theta B^{(t)}(\langle M^{(t)} \rangle_{u(t)})\} - \exp\{i\theta B^{(t)}(\sigma)\}| \\
&\leq |\theta|\,\mathbf{E}\sup\{|B^{(0)}(u) - B^{(0)}(\sigma)| : |u - \sigma| \leq \varepsilon\} + 2\mathbf{P}\{|\langle M^{(t)} \rangle_{u(t)} - \sigma| > \varepsilon\},
\end{aligned}$$

which yields the desired conclusion.

Proof of Theorem 5.1. Let $f \in C_\kappa(\mathbf{R}^d)^+$ be fixed. Note that by (5.1), (5.2) and the (T_t)-invariance of m we get

$$V(t) := \mathbf{Var}\,(Y_t(f)) = \int_0^t (t - s) m((T_s f)^2) \mathrm{d}s. \tag{5.5}$$

In order to apply Lemma 5.1 we set

$$M_u^{(t)}(f) = \frac{1}{\sqrt{V(t)}} \int_0^u \int_S T_{t-s} f(x) M(\mathrm{d}s\mathrm{d}x). \tag{5.6}$$

where $T_s f = 0$ for $s < 0$ by convention. Then for every fixed f and t, $\{M_u^{(t)}(f), u \geq 0\}$ is a continuous martingale and

$$\begin{aligned}
\langle M^{(t)}(f) \rangle_t &= V(t)^{-1} \int_0^t Y_s((T_{t-s} f)^2) \mathrm{d}s \\
&= 1 + V(t)^{-1} \int_0^t \mathrm{d}s \int_S \int_r^t T_{s-r}(T_{t-s} f)^2(x) \mathrm{d}s M(\mathrm{d}r\mathrm{d}x).
\end{aligned}$$

Combining this with (5.1)–(5.6) and the present assumption we get

$$
\begin{aligned}
&\mathbf{Q}_0|\langle M^{(t)}(f)\rangle_t - 1|^2 \\
&= \frac{1}{V(t)^2}\mathbf{Q}_0\left\{\int_0^t \mathrm{d}r \int_S \left(\int_r^t T_{s-r}(T_{t-s}f)^2(x)\mathrm{d}s\right)^2 Y_r(\mathrm{d}x)\right\} \\
&\leq \frac{1}{V(t)^2}\int_0^t \mathrm{d}r\left(\int_r^t \|T_{t-s}f\|\,\mathrm{d}s\right)^2 \mathbf{Q}_0\{Y_r((T_{t-r}f)^2)\} \\
&\leq \frac{C(f)^2}{V(t)^2}\int_0^t r(t-r)m((T_rf)^2)\mathrm{d}r \\
&\leq \frac{tC(f)^2}{V(t)},
\end{aligned}
$$

which vanishes as $t\to\infty$ if $\int_0^\infty m((T_sf)^2)\mathrm{d}s=\infty$. On the other hand, $\int_0^\infty m((T_sf)^2)\mathrm{d}s < \infty$ implies that $V(t)\sim$ const. t as $t\to\infty$, so that

$$
\begin{aligned}
&\frac{1}{t^2}\int_0^t s(t-s)m((T_sf)^2)\mathrm{d}s \\
&\leq \varepsilon\int_0^{\varepsilon t} m((T_sf)^2)\mathrm{d}s + \int_{\varepsilon t}^t m((T_sf)^2)\mathrm{d}s,
\end{aligned}
$$

which vanishes as $t\to\infty$ and $\varepsilon\to 0$. Thus we have $\lim_{t\to\infty}\mathbf{Q}_0|\langle M^{(t)}(f)\rangle_t - 1|^2 = 0$, completing the proof of Theorem 5.1 by virtue of Lemma 5.1.

Proof of Theorem 1.9. We first note that for any irreducible Lévy semigroup (T_t) the estimate (5.3) is known; see Theorem 4.3 of [28]. Suppose that the symmetrized semigroup $(\hat{T}_t)$ is transient. Then we have for $f\in C_\kappa(\mathbf{R}^d)^+$,

$$
\int_0^\infty m((T_sf)^2)\mathrm{d}s = \frac{1}{2}m(f\hat{G}f) < \infty,
$$

and

$$
\mathbf{Var}\,(Y_t(f) \sim m(f\hat{G}f)t/2 \qquad \text{as } t\to\infty.
$$

By Theorem 5.1, the distribution of

$$
[Y_t(f) - tm(f)]/\sqrt{t} \tag{5.7}
$$

converges to $N(0, m(f\hat{G}f)t/2)$. Next suppose that the symmetrized semigroup $(\hat{T}_t)$ is recurrent. Recalling that $m = c\lambda$, we fix some $\phi\in C_\kappa(\mathbf{R}^d)^+$ with $\lambda(\phi)=1$ and let

$$
h(t) = \int_0^t (t-s)\lambda((T_s\phi)^2)\mathrm{d}s. \tag{5.8}
$$

Then $\int_0^\infty m((T_sf)^2)\mathrm{d}s = \infty$ implies that $t^{-1}h(t)\to\infty$ as $t\to\infty$. By Theorem 5.3 of [26], if $f, g\in C_\kappa(\mathbf{R}^d)^+$ with $\lambda(g)>0$, then

$$\lim_{t\to\infty}\int_0^t \hat{T}_s f(x)ds\left(\int_0^t \hat{T}_s g(x)ds\right)^{-1} = \lambda(f)/\lambda(g) \tag{5.9}$$

uniformly for x and y in each compact set. Using this we obtain that for every $f\in C_\kappa(\mathbf{R}^d)^+$,

$$\lim_{t\to\infty} h(t)^{-1}\,\mathbf{Var}\,\{Y_t(f)\}$$

$$= \lim_{t\to\infty} c\int_0^t (t-s)\lambda((T_s f)^2)ds\left(\int_0^t (t-s)\lambda((T_s\phi)^2)ds\right)^{-1} = c\lambda(f)^2.$$

By Theorem 5.1 and (5.9) the distribution of (5.7) converges to $N(0, c\lambda(f)^2)$. Since $f\in C_\kappa(\mathbf{R}^d)^+$ was arbitary, it is a routine task to see the convergence of (5.7) in the sense of distributions in $\mathscr{D}'(\mathbf{R}^d)$, and the theorem is proved.

Next we proceed to the proof of Theorem 1.10. Recall that now (T_t) is the standard Brownian semigroup on $\mathbf{R}^d$ and the immigration measure is $m(dx) = \gamma(x)\lambda(dx)$ with γ satsfying (1.27). By (2.30) and the symmetry of (T_t) we see that for $f\in C_\kappa(\mathbf{R}^d)^+$,

$$\mathbf{Q}_0\{Y_t(f)\} = \int_0^t \lambda(\gamma T_u f)du, \tag{5.10}$$

$$\mathbf{Var}\,\{Y_t(f)\} = \int_0^t dr\int_0^r \lambda(T_u\gamma\cdot(T_{r-u}f)^2)du, \tag{5.11}$$

$$\mathbf{Cov}\,(Y_s(g), Y_t(f)) = \int_0^s dr\int_0^r \lambda(T_u\gamma\cdot T_{r-u}g\cdot T_{t-s+r-u}f)du \tag{5.12}$$

Lemma 5.2. *If* $f\in C_\kappa(\mathbf{R}^d)^+$, *then as* $t\to\infty$,

$$\mathbf{Q}_0\{Y_t(f)\} = \lambda(f)b_t + o(b_t), \tag{5.13}$$

where b_t *is given by*

$$b_t = \frac{2^{\alpha-1}\Gamma(\alpha+d/2)}{\pi^{d/2}(1+\alpha)}\lambda_S(a)t^{1+\alpha} \tag{5.14}$$

Proof. Note that (1.27) implies $\gamma(x)\le \text{const.}\,(1+|x|^{2\alpha})$ Therefore

$$T_t\gamma(x)\le \text{const.}\,(1+t^\alpha+|x|^{2\alpha}). \tag{5.15}$$

Furthermore,

$$\lim_{t\to\infty} t^{-\alpha}T_t\gamma(x)$$

$$= \lim_{t\to\infty}(2\pi)^{-d/2}\int t^{-\alpha}\gamma(x+\sqrt{t}z)\exp\{-|z|^2/2\}dz$$

$$(5.16)\qquad = (2\pi)^{-d/2}\int |z|^{2\alpha}a(z/|z|)\exp\{-|z|^2/2\}\mathrm{d}z$$

$$= (2\pi)^{-d/2}\int_0^\infty r^{2\alpha}\lambda_S(a)\exp\{-r^2/2\}r^{d-1}\mathrm{d}r$$

$$= 2^{\alpha-1}\pi^{-d/2}\Gamma(\alpha+d/2)\lambda_S(a).$$

By (5.10), (5.15) and (5.16) we obtain

$$\lim_{t\to\infty} t^{-\alpha-1}\mathbf{Q}_0\{Y_t(f)\}$$

$$= \lim_{t\to\infty} t^{-\alpha-1}\int_0^t \mathrm{d}s\int \gamma(x)T_sf(x)\mathrm{d}x$$

$$= \lim_{t\to\infty}\int_0^1 \mathrm{d}r\int f(x)t^{-\alpha}T_{rt}\gamma(x)\mathrm{d}x$$

$$= \int_0^1 \mathrm{d}s\int f(x)\lim_{t\to\infty} t^{-\alpha}T_{rt}\gamma(x)\mathrm{d}x$$

$$= \frac{2^{\alpha-1}\Gamma(\alpha+d/2)}{\pi^{d/2}(1+\alpha)}\lambda_S(a)\lambda(f).$$

Lemma 5.3. *If* $f\in C_\kappa(\mathbf{R}^d)^+$, *then as* $t\to\infty$,

$$(5.17)\qquad \begin{aligned} \mathbf{Var}\{Y_t(f)\} &= \lambda(f)^2h(t)+o(h(t)), &&\text{for } d=1 \text{ and } 2,\\ &= \lambda(fGf)h(t)+o(h(t)), &&\text{for } d\ge 3,\end{aligned}$$

where $h(t)$ *is defined by* (1.29).

Proof. Since $T_tf(x)\le \text{const.}(1\wedge t^{-d/2})$, it follows by (5.15) that

$$(5.18)\qquad \int t^{-\alpha}T_{t-u}\gamma(x)(T_uf)^2(x)\mathrm{d}x \le \text{const.}(1\wedge u^{-d/2}).$$

Then for $d\ge 3$, by (5.11), (5.16) and (5.18) we get

$$\lim_{t\to\infty} t^{-\alpha-1}\mathbf{Var}\{Y_t(f)\} = \lim_{t\to\infty} t^{-\alpha-1}\int_0^t \mathrm{d}r\int_0^r \mathrm{d}u\int T_{r-u}\gamma(x)T_uf(x)^2\mathrm{d}x$$

$$= \lim_{t\to\infty}\int_0^1 \mathrm{d}s\int_0^\infty \mathrm{d}u\int t^{-\alpha}T_{st-u}\gamma(x)T_uf(x)^2\mathrm{d}x$$

$$(5.19)\qquad = \int_0^1 \mathrm{d}s\int_0^\infty \mathrm{d}u\int \lim_{t\to\infty} t^{-\alpha}T_{st-u}\gamma(x)T_uf(x)^2\mathrm{d}x$$

$$= \frac{2^{\alpha-1}\Gamma(\alpha+d/2)}{\pi^{d/2}(1+\alpha)}\lambda_S(a)\int_0^\infty \lambda((T_uf)^2)\mathrm{d}u$$

$$= \frac{2^{\alpha-2}\Gamma(\alpha+d/2)}{\pi^{d/2}(1+\alpha)}\lambda_S(a)\lambda(f\hat{G}f),$$

proving (5.17) in the case $d \geq 3$. To get it for $d = 1$ and 2, we use the following relation for the Brownian transition density $g_t(x, y) = g_t(x - y)$:

$$g_s(x, y)g_t(x, z) = g_{st/(s+t)}(x, (ty + sz)/(s + t))g_{s+t}(y, z) \tag{5.20}$$

to find that

$$\begin{aligned} &\int T_r\gamma(x)T_sg(x)T_tf(x)\mathrm{d}x \\ &= \int \mathrm{d}x \iint T_r\gamma(x)g_s(x, y)g_t(x, z)g(y)f(z)\mathrm{d}y\mathrm{d}z \\ &= \iint g_{s+t}(y, z)g(y)f(z)\mathrm{d}y\mathrm{d}z \int T_r\gamma(x)g_{st/(s+t)}(x, (s + t)^{-1}(ty + sz))\mathrm{d}x \\ &= \iint T_{r+st/(s+t)}\gamma((ty + sz)/(s + t))g_{s+t}(y, z)g(y)f(z)\mathrm{d}y\mathrm{d}z. \end{aligned} \tag{5.21}$$

When $d = 1$, using (5.11), (5.21) and (5.16),

$$\begin{aligned} &\lim_{t\to\infty} t^{-\alpha-3/2}\,\mathbf{Var}\,\{Y_t(f)\} \\ &= \lim_{t\to\infty} t^{-\alpha-3/2}\int_0^t \mathrm{d}r\int_0^r \mathrm{d}u \iint T_{r-u/2}\gamma((y + z)/2) \\ &\qquad g_{2u}(y, z)f(y)f(z)\mathrm{d}y\mathrm{d}z, \qquad r = st,\ u = vst, \\ &= \lim_{t\to\infty} \frac{1}{2\sqrt{\pi}}\int_0^1 \sqrt{s}\,\mathrm{d}s\int_0^1 v^{-1/2}\mathrm{d}v \iint t^{-\alpha}T_{s(2-v)t/2}\gamma((y + z)/2) \\ &\qquad f(y)f(z)\exp\{-|y - z|^2/4vst\}\mathrm{d}y\mathrm{d}z \\ &= \frac{\Gamma(\alpha + 1/2)}{2\pi(2\alpha + 3)}\lambda_S(a)\lambda(f)^2\int_0^1 v^{-1/2}(2 - v)^{\alpha}\mathrm{d}v. \end{aligned} \tag{5.22}$$

Similar techniques give the analogous result for $d = 2$,

$$\begin{aligned} &\lim_{t\to\infty} t^{-\alpha-1}\log^{-1}t\,\mathbf{Var}\,\{Y_t(f)\} \\ &= \lim_{t\to\infty} \frac{1}{t^{\alpha+1}\log t}\int_0^t \mathrm{d}r\int_0^r \mathrm{d}u \iint T_{r-u/2}\gamma((y + z)/2) \\ &\qquad g_{2u}(y, z)f(y)f(z)\mathrm{d}y\mathrm{d}z \\ &= \lim_{t\to\infty} \frac{1}{t^{\alpha}\log t}\int_0^1 \mathrm{d}s\int_1^{st} \mathrm{d}u \iint T_{st-u/2}\gamma((y + z)/2) \\ &\qquad g_{2u}(y, z)f(y)f(z)\mathrm{d}y\mathrm{d}z, \qquad u = (st)^{1-r}, \\ &= \lim_{t\to\infty} \frac{1}{t^{\alpha}\log t}\int_0^1 \mathrm{d}s\int_0^1 (st)^{1-r}\log(st)\mathrm{d}r \end{aligned}$$

$$\frac{1}{4\pi(st)^{1-r}}\iint T_{st-u/2}\gamma((y+z)/2)f(y)f(z)\exp\{-|y-z|^2/4u\}\,dydz$$

$$= \lim_{t\to\infty}\frac{1}{4\pi}\int_0^1 ds\int_0^1 dr\iint t^{-\alpha}T_{st-u/2}\gamma((y+z)/2)$$

$$f(y)f(z)\exp\{-|y-z|^2/4u\}\,dydz$$

$$= \frac{2^{\alpha-3}\Gamma(\alpha+1)}{\pi^2(\alpha+1)}\lambda_S(a)\lambda(f)^2.$$

The proof is complete.

Lemma 5.4. *If* $g, f\in C_\kappa(\mathbf{R}^d)^+$ *and* $t>s>0$, *then as* $M\to\infty$,

(5.23) $$\mathbf{Cov}(Y_{sM}(g), Y_{tM}(f)) = \lambda(g)\lambda(f)\kappa(s,t)h(M)+o(h(M)) \quad \textit{for } d=1,$$
$$= O(\log^{-1}M)h(M) \quad \textit{for } d=2,$$
$$= O(M^{1-d/2})h(M) \quad \textit{for } d\ge 3,$$

where $h(M)$ *is given by* (1.29), *and* $\kappa(s,t)$ *by* (1.31).

Proof. Noting that f has compact support, we observe by (5.12) and (5.15)

$$h(M)^{-1}\,\mathbf{Cov}(Y_{sM}(g), Y_{tM}(f))$$
$$= h(M)^{-1}\int_0^{sM} dr\int_0^r du\int T_{r-u}\gamma(x)T_ug(x)T_{(t-s)M+u}f(x)dx$$
$$\le \text{const.}\,h(M)^{-1}\int_0^{sM} dr\int_0^{sM}[(t-s)M+u]^{-d/2}du\int g(x)T_r\gamma(x)dx$$
$$\le \text{const.}\,h(M)^{-1}M^{\alpha+1}\int_0^{sM}[(t-s)M+u]^{-d/2}du$$
$$\le \text{const.}\,M^{1-d/2} \quad \text{for } d\ge 3,$$
$$\le \text{const.}\,(\log M)^{-1} \quad \text{for } d=2.$$

For $d=1$, setting $r=psM$ and $u=pqsM$, and using (5.21),

$$\mathbf{Cov}(Y_{sM}(g), Y_{tM}(f))$$
$$= \int_0^{sM} dr\int_0^r du\int T_{r-u}\gamma(x)T_ug(x)T_{(t-s)M+u}f(x)dx$$
$$= \int_0^1 sM\,dp\int_0^1 psM\,dq\int T_{(1-q)psM}\gamma(x)T_{pqsM}g(x)T_{(t-s+pqs)M}f(x)dx$$
$$= s^2M^2\int_0^1 p\,dp\int_0^1 dq\iint T_{r(\cdot)M}\gamma(m(\cdot))g_{(t-s+2pqs)M}(y,z)g(y)f(z)dydz,$$

where

$$r(\cdot) = \frac{ps[t-s+pqs(2-q)]}{t-s+2pqs},$$

and

$$m(\cdot) = \frac{(t-s+pqs)y+pqsz}{t-s+2pqs}.$$

Then by (5.15)

$$\lim_{M\to\infty} M^{-\alpha-3/2}\,\mathbf{Cov}\,(Y_{sM}(g),\,Y_{tM}(f))$$

$$= \lim_{M\to\infty} s^2\int_0^1 p\mathrm{d}p\int_0^1 \mathrm{d}q\iint M^{-\alpha}T_{r(\cdot)M}\gamma(m(\cdot))$$

$$\frac{g(y)f(z)}{\sqrt{2\pi(t-s+2pqs)}}\exp\left\{-\frac{|y-z|}{2(t-s+2pqs)M}\right\}\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{\sqrt{2}}s^2\lambda(g)\lambda(f)\int_0^1 p\mathrm{d}p\int_0^1\frac{r(\cdot)^\alpha 2^{\alpha-1}\Gamma(\alpha+1/2)\lambda_S(a)}{\pi\sqrt{(t-s+2pqs)}}\mathrm{d}q$$

$$= 2^{\alpha-3/2}\pi^{-1}s^{2(\alpha+1)}\Gamma(\alpha+1/2)\lambda_S(a)\lambda(g)\lambda(f)$$

$$\int_0^1\int_0^1\frac{M^{\alpha+1}[t-s+pqs(2-q)]^\alpha}{(t-s+2pqs)^\alpha\sqrt{t-s+2pqs}}\mathrm{d}p\mathrm{d}q$$

$$= \kappa(s,\,t)\lambda_S(a)\lambda(g)\lambda(f)\frac{\Gamma(\alpha+1/2)}{\pi(2\alpha+3)}\int_0^1 r^{-1/2}(2-r)^\alpha\mathrm{d}r.$$

The lemma is proved.

Proof of Theorem 1.10. It suffices to show that for all $f_1,\cdots,f_n\in C_\kappa(\mathbf{R}^d)^+$ and $0<t_1<\cdots<t_n$,

$$\sum_{i=1}^n Z_{t_i}^{(M)}(f_i)\to\sum_{i=1}^n Z_{t_i}(f_i)$$

in distribution as $M\to\infty$. Note that

$$\sum_{i=1}^n Z_{t_i}^{(M)}(f_i) = N_{t_nM}^{(M)}(f_1,\cdots,f_n),$$

where $N_t^{(M)}(f_1,\cdots,f_n)$ is a continuous martingale in $t\geq 0$ defined by

$$N_t^{(M)}(f_1,\cdots,f_n) = h(M)^{-1/2}\int_0^t\int\sum_{i=1}^n T_{t_iM-s}f_i(x)M(\mathrm{d}s\mathrm{d}x).$$

By (5.1) and (5.2),

$$\langle N^{(M)}(f_1,\cdots,f_n)\rangle_{t_nM}$$

$$= h(M)^{-1}\int_0^{t_n M}\int\Big(\sum_{i=1}^{n} T_{t_i M-s}f_i(x)\Big)^2 Y_s(\mathrm{d}x)\mathrm{d}s$$

$$= h(M)^{-1}\int_0^{t_n M}\mathrm{d}s\int_0^{s}\mathrm{d}r\int T_{s-r}\gamma(x)\Big(\sum_{i=1}^{n} T_{t_i M-s}f_i(x)\Big)^2\mathrm{d}x$$

$$+ h(M)^{-1}\int_0^{t_n M}\mathrm{d}s\int_0^{s}\int T_{s-r}\Big(\sum_{i=1}^{n} T_{t_i M-s}f_i\Big)^2(x)M(\mathrm{d}r\mathrm{d}x).$$

Using Lemmas 5.3 and 5.4 we have

$$\lim_{M\to\infty} h(M)^{-1}\int_0^{t_n M}\mathrm{d}s\int_0^{s}\mathrm{d}r\int T_{s-r}\gamma(x)\Big(\sum_{i=1}^{n} T_{t_i M-s}f_i(x)\Big)^2\mathrm{d}x$$

$$= \frac{1}{2}\sum_i \lambda(f_i)^2 t_i^{\alpha+3/2} + \sum_{i<j}\lambda(f_i)\lambda(f_j)\kappa(t_i, t_j) \qquad \text{for } d = 1$$

$$= \frac{1}{2}\sum_i \lambda(f_i)^2 t_i^{\alpha+3/2} \qquad \text{for } d = 2$$

$$= \frac{1}{2}\sum_i \lambda(f_i G f_i) t_i^{\alpha+3/2} \qquad \text{for } d \geq 3.$$

On the other hand, by (5.2),

$$h(M)^{-2}\mathbf{Q}_0\Big[\int_0^{t_n M}\mathrm{d}s\int_0^{s}\int T_{s-r}\Big(\sum_{i=1}^{n} T_{t_i M-s}f_i\Big)^2(x)M(\mathrm{d}r\mathrm{d}x)\Big]^2$$

$$= h(M)^{-2}\int_0^{t_n M}\mathrm{d}r\int_0^{t_n M}\mathrm{d}u\int T_{r-u}\gamma(x)\Big[\int_0^{t_n M} T_{s-r}\Big(\sum_{i=1}^{n} T_{t_i M-s}f_i\Big)^2(x)\mathrm{d}s\Big]^2\mathrm{d}x.$$

Using (5.15) repeatedly we see that the above value is bounded by

$$\text{const.}\, h(M)^{-2}M^{\alpha+1}\Big(\int_0^{t_n M} 1\wedge s^{-d/2}\mathrm{d}s\Big)^3,$$

which goes to zero as $M\to\infty$ by (1.29). Then the desired conclusion follows by Lemma 5.1.

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