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Tame extension operators for Hermite's interpolation problem on algebraic varieties

By

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1. Introduction

Let V be an algebraic variety in $\mathbf{C}^n$ and let $H(\mathbf{C}^n)$ and $H(V)$ denote respectively the space of entire functions on $\mathbf{C}^n$ and the space of holomorphic functions on V. These spaces will be regarded with the topology of uniform convergence on compact subsets, then they are nuclear Fréchet spaces. In the following we shall consider these spaces as graded (i.e. with fixed system of norms) Fréchet spaces with the system of norms

$$\|F\|_r = \sup_{|z| \le r} |F(z)|, \qquad r \ge 1, \ F \in H(\mathbf{C}^n),$$

and

$$\|f\|_r = \sup_{|z| \le r} |f(z)|, \qquad r \ge 1, \ f \in H(V).$$

We shall consider also as graded the space $H(V)^{k+1}$, $k = 1, 2, \ldots$, of all $(k+1)$-tuples $f = (f_0, \ldots, f_k)$, where $f_0, \ldots, f_k \in H(V)$, with the system of norms

$$\|f\|_r = \max\{\|f_j\|_r;\ 0 \le j \le k\}, \qquad r \ge 1.$$

Suppose L is a first order linear differential operator in $\mathbf{C}^n$ with polynomial coefficients. For any $k = 0, 1, 2, \ldots$ we study the following

Problem. For given $f_0, \ldots, f_k \in H(V)$ find an entire function F such that the restrictions of $F, LF, \ldots, L^k F$ on V coincide with $f_0, f_1, \ldots, f_k$ respectively.

By analogy with the classical case we call this problem **Hermite's Interpolation Problem.** Our aim here is to study this interpolation problem from the point of view of linear functional analysis. We improve our previous results (see [4])

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on existence of linear extension operators solving the Hermite's interpolation problem showing that these operators are linearly tame in the sense of the following

Definition. Suppose $(E, \|\cdot\|_\gamma)$ and $(F, \|\cdot\|_s)$ are two graded Fréchet spaces. An operator $T: (E, \|\cdot\|_r) \to (F, \|\cdot\|_s)$ is said to be linearly tame if there exists a constant a such that

$$\forall r \exists C_r: \|Tx\|_r \leq C_r \|x\|_{ar} \qquad \forall x \in E.$$

The notion of tame operator appeared in the paper of Sergeraert [15]; see also Dubinsky and Vogt [5].

We obtain even a little stronger "tameness" (see Theorem 3) by proving that for our extension operators constants $C_r = Cr^d$, where C, d do not depend on r. Using this fact we show that the extension operators act between some spaces of holomorphic functions with radial bounds on the growth. As a consequence we get that the Hermite's interpolation problem has for data with bounds on the growth a solution with almost the same bounds on the growth.

Our proofs are based on the structure results for ideals, generated by polynomials, obtained in [3]. For a convenience of the readers we formulate here these results in an appropriate form (see [3], Corollary 4 and its proof).

Theorem 0. *Let $Q = \{Q_k(z)\}_{k=1}^m$ be a finite set of polynomials of variables $z = (z_1, \ldots, z_n)$ and let*

$$J(Q) = \{\sum_{k=1}^m G_k Q_k: G_k \in H(\mathbf{C}^n)\}$$

be the ideal, generated by Q in the algebra of the entire functions $H(\mathbf{C}^n)$. Then there exist linear operators

$R_k: H(\mathbf{C}^n) \to H(\mathbf{C}^n)$, $0 \leq k \leq p$, such that:

(*a*) $F = R_0 F + \sum_{k=1}^m (R_k F) Q_k$ *for all* $F \in H(\mathbf{C}^n)$;

(*b*) $\ker R_0 = J(Q)$, $R_0^2 = R_0$, $R_0(H(\mathbf{C}^n))$ *is a coordinate subspace of* $H(\mathbf{C}^n)$ *with respect to the natural basis* $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\alpha \in \mathbf{Z}_+^n$;

(*c*) $\|R_k F\|_r \leq C_1 r^{d_1} \|F\|_{a_1 r}$, $k = 0, 1, \ldots, m$, $F \in H(\mathbf{C}^n)$, *where the constants* C_1, a_1, d_1 *depend only on* Q.

It is well known by the theory of functions of several complex variables (see [7] or [8]) that any holomorphic function on V can be extended to an entire function on $\mathbf{C}^n$. Moreover, using so called "cohomologies with bounds" (see [14] or [6], [8]) one can prove the following

Lemma 1. *For any $f \in H(V)$ there exists an extension $F \in H(\mathbf{C}^n)$ such that*

$$\|F\|_r \leq C_2 r^{d_2} \|f\|_{a_2 r} \qquad \text{for } r \geq 1,$$

where the constants C_2, d_2, a_2 depend only on the algebraic variety V.

Sketch of the proof of this lemma, based on cohomology technique is given by Mitiagin [13].

2. Spaces of admissible data

The $(k+1)$-tuple $(f_0, f_1, \ldots, f_k) \in H(V)^{k+1}$ is said to be "admissible data" for the Hermite's interpolation problem if there exists a solution for these data. It is easy to see that in general case not any $(k+1)$-tuple $(f_0, \ldots, f_k)$ gives admissible data. For example, if $k = 1$ and L is a tangent operator for V in the point $z \in V$, then we have $LF(z) = Lf_0(z) = f_1(z)$ for any solution F of the Hermite interpolation problem with data f_0, f_1, i.e. if $Lf_0(z) \neq f_1(z)$ then the pair (f_0, f_1) does not give us admissible data.

Let X_k be the set of all $(k+1)$-tuples which are admissible data; then X_k is a linear subspace of the space $H(V)^{k+1}$. Further we give a characterization of the spaces X_k, $k = 1, 2, \ldots$.

If $G_1, \ldots, G_s$ are entire functions we denote by $J[G_1, \ldots, G_m]$ the ideal, generated by them in the algebra $H(\mathbf{C}^n)$. It is known (see e.g. [3]) that the ideal

$$J_0 = \{F \in H(\mathbf{C}^n) : F|_V = 0\}$$

is finitely generated by polynomials. Let $P_1, \ldots, P_m$ be polynomials such that $J_0 = J[P_1, \ldots, P_m]$. Then we have

Theorem 1. *Suppose $f_0, f_1 \in H(V)$; then $(f_0, f_1) \in X_1$ if and only if*

$$F_1 - LF_0 \in J[P_1, \ldots, P_m, LP_1, \ldots, LP_m] \tag{1}$$

for all entire functions F_0 and F_1 which are extensions of the functions f_0 and f_1 respectively.

Proof. Suppose that $(f_0, f_1) \in X_1$ and the entire functions F_0 and F_1 are extensions of f_0 and f_1 respectively. Then

$$(f_0, f_1) = (F_0|_V, LF_0|_V) + (0, [F_1 - LF_0]_V)$$

and therefore $(0, [F_1 - LF_0]_V) \in X_1$. This means that there exists an entire function G, such that $G|_V = 0$, $LG|_V = [F_1 - LF_0]_V$. Since $G|_V = 0$ we have $G \in J_0 = J[P_1, \ldots, P_m]$, i.e. $G = \sum_{i=1}^{m} G_i P_i$, where G_i, $i = 1, \ldots, m$, are entire functions. Obviously we have

$$LG = \sum_{i=1}^{m} LG_i \cdot P_i + \sum_{i=1}^{m} G_i \cdot LP_i \in J[P_1, \ldots, P_m, LP_1, \ldots, LP_m].$$

On the other hand $[F_1 - LF_0 - LG]_V = 0$, therefore $F_1 - LF_0 - LG \in J[P_1, \ldots, P_m, LP_1, \ldots, LP_m]$, hence (1) holds.

Conversely, if (1) holds, then there exist entire functions g_i, h_i, $i = 1, \ldots, m$, such that

$$F_1 - LF_0 = \sum_{i=1}^{m} g_i P_i + \sum_{i=1}^{m} h_i LP_i.$$

Obviously we have

$$[F_1 - LF_0]_V = [\sum_{i=1}^{m} h_i LP_i]_V.$$

Consider the entire function

$$H = \sum_{i=1}^{m} h_i P_i.$$

Since

$$LH|_V = [\sum_{i=1}^{m} Lh_i \cdot P_i + \sum_{i=1}^{m} h_i \cdot LP_i]_V = [\sum_{i=1}^{m} h_i \cdot LP_i]_V,$$

we obtain $LH|_V = [F_1 - LF_0]_V$, hence the entire function $F_0 + H$ is a solution of the Hermite interpolation problem with data f_0, f_1.

To give an analogous characterization of the spaces X_k, $k = 2, 3, \ldots$, we need the following lemma.

Lemma 2. *The ideals*

$$J_k = \{G \in H(\mathbf{C}^n) \colon G|_V = 0,\ LG|_V = 0, \ldots, L^k G|_V = 0\}$$

are finitely generated by polynomials.

Proof. We shall use an induction with respect to k. It is known that the ideal $J_0 = \{f \in H(\mathbf{C}^n) \colon f|_V = 0\}$ is generated by polynomials $P_1, \ldots, P_m$. Assume that for some $k = 0, 1, 2, \ldots$ the ideal J_k is generated by polynomials $Q_1, \ldots, Q_s \in J_k$. Since

$$J_{k+1} = \{f \in J_k \colon L^{k+1} f|_V = 0\},$$

we have

$$J_{k+1} = \{\sum_{j=1}^{s} g_j Q_j \colon g_j \in H(\mathbf{C}^n),\ L^{k+1}(\sum_{j=1}^{s} g_j Q_j)|_V = 0\}.$$

It is easy to check (using $Q_j \in J_k$), that

$$[L^{k+1}(\sum_{j=1}^{s} g_j Q_j)]_V = [\sum_{j=1}^{s} g_j L^{k+1} Q_j]_V,$$

therefore

$$J_{k+1} = \{\sum_{j=1}^{s} g_j Q_j \colon g_j \in H(\mathbf{C}^n), [\sum_{j=1}^{s} g_j L^{k+1} Q_j]_V = 0\}.$$

Consider the polynomial homomorphism

$$\psi\colon H(\mathbf{C}^n)^{s+m+1} \to H(\mathbf{C}^n)^2,$$

where

$$\psi(h_0, h_1, \dots, h_m, g_1, \dots, g_s) = (h_0 - \sum_{j=1}^{s} g_j Q_j, \sum_{i=1}^{m} h_i P_i - \sum_{i=1}^{s} g_j L^{k+1} Q_j).$$

The module $\ker \psi$ is generated by a finite number of $s+m+1$-tuples with polynomial components (see [8, Lemma 7.6.3]). Suppose these generators are $(G_{0,\ell}, G_{1,\ell}, \dots, G_{s+m+1,\ell})$, $\ell = 1, \dots, r$. Then it is easy to see that the polynomials $G_{0,\ell}$, $\ell = 1, 2, \dots, r$ generate the ideal J_{k+1}.

Suppose the polynomials $Q_1, \dots, Q_s$ are generators of the ideal J_k, i.e. $J_k = J[Q_1, \dots, Q_s]$. Then the following theorem is true:

Theorem 2. *Suppose* $f_0, \dots, f_k, f_{k+1} \in H(V)$; *then* $(f_0, \dots, f_k, f_{k+1}) \in X_{k+1}$ *if and only if* $(f_0, \dots, f_k) \in X_k$ *and*

$$F_{k+1} - L^{k+1} F \in J[P_1, \dots, P_m, L^{k+1} Q_1, \dots, L^{k+1} Q_s] \tag{2}$$

for any solution F of the Hermite's interpolation problem with data $f_0, \dots, f_k$ *and for any entire function* F_{k+1}, *which is an extension of the function* f_{k+1}.

Proof. If $(f_0, \dots, f_k, f_{k+1}) \in X_{k+1}$, then obviously $(f_0, \dots, f_k) \in X_k$. Suppose F is a solution of the Hermite's interpolation problem with data $f_0, \dots, f_k$ and F_{k+1} is an entire function, such that $F_{k+1}|_V = f_{k+1}$. Then we have

$$(f_0, \dots, f_k, f_{k+1}) = (F|_V, \dots, L^k F|_V, L^{k+1} F|_V) + (0, \dots, 0, [F_{k+1} - L^{k+1} F]_V)$$

and certainly $(0, \dots, 0, [F_{k+1} - L^{k+1} F]_V) \in X_{k+1}$. Therefore there exists an entire function $H \in J_k$, such that $L^{k+1} H|_V = [F_{k+1} - L^{k+1} F]_V$. Since $J_k = J[Q_1, \dots, Q_s]$ we have

$$H = \sum_{j=1}^{s} h_j Q_j$$

for some $h_j \in H(\mathbf{C}^n)$. It is easy to see that

$$L^{k+1} H|_V = [\sum_{j=1}^{s} h_j L^{k+1} Q_j]_V,$$

thus $L^{k+1} H \in J[P_1, \dots, P_m, L^{k+1} Q_1, \dots, L^{k+1} Q_s]$. On the other hand $[F_{k+1} - L^{k+1} F - L^{k+1} H]_V = 0$, i.e.

$$F_{k+1} - L^{k+1} F - L^{k+1} H \in J_0 \subset J[P_1, \dots, P_m, L^{k+1} Q_1, \dots, L^{k+1} Q_s],$$

hence (2) holds.

Conversely, if (2) holds, then there exist entire functions g_i, $i = 1, \dots, m$, and h_j, $j = 1, \dots, s$, such that

$$F_{k+1} - L^{k+1}F = \sum_{i=1}^{m} g_i P_i + \sum_{j=1}^{s} h_j L^{k+1} Q_j. \tag{3}$$

Consider the function $H = \sum_{j=1}^{s} h_j Q_j$. It is easy to see that

$$L^{k+1}H|_V = [\sum_{j=1}^{s} h_j L^{k+1} Q_j]_V = [F_{k+1} - L^{k+1}F]_V,$$

hence the entire function $F + H$ is a solution of Hermite's interpolation problem with data $f_0, \dots, f_k, f_{k+1}$.

3. Tame extension operators

Suppose as in the previous section that V is an algebraic variety in $\mathbf{C}^n$ and the ideal $J_0 = \{f \in H(\mathbf{C}^n) : f|_V = 0\}$ is generated by polynomials $P_1, \dots, P_m$. Let $f \in H(V)$ and $F \in H(\mathbf{C}^n)$ be any extension of the function f. Applying Theorem 0 to the ideal $J_0 = J[P_1, \dots, P_m]$ we obtain corresponding operators R_i, $i = 0, \dots, m$. Then the function $R_0 F$ does not depend on the choice of an extension F (since $\ker R_0 = J_0$), therefore the operator

$$S_0 : H(V) \to H(\mathbf{C}^n), \qquad S_0 f = R_0 F,$$

is a correctly defined linear continuous operator (see [3]). Moreover as we show in the next lemma the operator S_0 is tame.

Lemma 3. *There exist constants C, a, d depending only on V such that*

$$\| S_0 f \|_r \le C r^d \| f \|_{ar}, \qquad r > 1.$$

Proof. We fix an arbitrary function $f \in H(V)$. By Lemma 1 there exists an entire function F such that $F|_V = f$ and $|F|_r \le C_2 r^{d_2} |f|_{a_2 r}$, hence we get by (c) of Theorem 0 that

$$\| S_0 f \|_r = \| R_0 F \|_r \le C_1 r^{d_1} \| F \|_{a_1 r}$$
$$\le C_1 C_2 r^{d_1} r^{d_2} \| f \|_{a_1 a_2 r} = C r^d \| f \|_{ar},$$

where $C = C_1 C_2$, $d = d_1 + d_2$, $a = a_1 a_2$.

Let us note that the existence of a tame linear extension operator $S_0 : H(V) \to H(\mathbf{C}^n)$ follows by the results of the paper [3] as it was observed by Aytuna [1]. Indeed, by ([3], Cor. 3) it is known that operators R_i, $0 \le i \le p$, act continuously in the spaces $H(\Delta_{rB})$, $r > 1$, where $B = (B_1, \dots, B_n)$ is a vector with positive coordinats and $\Delta_{rB} = \{z \in \mathbf{C}^n : |z_i| \le r B_i,\ 1 \le i \le n\}$. Let us consider also the space $H(V \cap \Delta_{rB})$, $r > 1$, and put

$$|F|_{\rho B} = \sup_{\Delta_{\rho B}} |F(z)|, \qquad F \in H(\Delta_{rB}), \qquad 1 < \rho < r;$$
$$|f|_{\rho B} = \sup_{V \cap \Delta_{\rho B}} |f(z)|, \qquad f \in H(V \cap \Delta_{rB}), \quad 1 < \rho < r.$$

As above (corresponding to any function $g \in H(V \cap \Delta_{rB})$ the function $R_0 G$, where

$G \in H(\Delta_{rB})$ is any extension of g) we obtain a linear continuous operator

$$S_0^{(r)}: H(V \cap \Delta_{rB}) \to H(\Delta_{rB}).$$

Obviously if $f \in H(V)$ then $S_0^{(r)} f = S_0 f|_{\Delta_{rB}}$, therefore the operator S_0 act continuously from the space $H(V \cap \Delta_{rB})$ into the space $H(\Delta_{rB})$, i.e.

$$\forall \rho < r \ \exists \rho_1 < r, \ C_\rho : |S_0 f|_{\rho B} \leq C_\rho |f|_{\rho_1 B}.$$

Taking $\rho = r/2$ we obtain

$$|S_0 f|_{\frac{r}{2} B} \leq C_r |f|_{\rho_1 B} \leq C_r |f|_{rB} \qquad \text{for any } r > 1,$$

hence

$$\| S_0 f \|_r \leq C_r \| f \|_{br}, \qquad r > 1,$$

where $b = 2 \max(B_1, \ldots, B_n)/\min(B_1, \ldots, B_n)$. However in such a way we get no information about the constant C_r. For us it is important to know how C_r depends on r because we apply Lemma 2 to spaces of functions with bounds on the growth.

Consider for any $k = 1, 2, \ldots$ the operator

$$T_k : H(\mathbf{C}^n) \to H(V)^{k+1}, \qquad T_k F = (F|_V, LF|_V, \ldots, L^k F|_V).$$

It is easy to see that the operator T_k is continuous and $X_k = T_k(H(\mathbf{C}^n))$, $J_k = \ker T_k$. Of course the following natural question arises: Does there exist a linear continuous right inverse operator for the operator T_k, $k = 1, 2, \ldots$? Obviously if there exists such an operator, then it corresponds a solution to every admissable data for the Hermite's interpolation problem. Therefore we call it "extension operator" for the Hermite's interpolation problem.

Existence (or nonexistence) of continous linear operators solving many linear problems of complex analysis was proved by Mitiagin and Henkin [12]. Vogt [17] found conditions for splitting of exact sequences of nuclear Fréchet spaces, i.e. for existence of continuous linear right inverse operators. By his technique have been obtained many results on existence of continuous linear operators, solving some problems for holomorphic or C^∞-functions -see Bierstone and Schwarz [2], Meise and Taylor [9], Meise and Vogt [10], Meise, Taylor and Vogt [11], Taylor [16], Vogt [18]. Here we prove the existence of tame right inverse operators of operators T_k directly, using the results of the paper [3].

Theorem 3. *For any* $k = 1, 2, \ldots$, *there exists a linear operator* $S_k : X_k \to H(\mathbf{C}^n)$, *which is right inverse to the opeator* T_k *and tame in the sense*

$$\| S_k f \|_r \leq C_k \cdot r^{d_k} \| f \|_{a_k r}, \qquad r \geq 1, \tag{4}$$

where the constants C_k, d_k, a_k *depend only on* k *and* V.

Proof. Suppose that for some $k = 0, 1, 2, \ldots$ there exists an operator S_k, which is right inverse to the operator T_k and satisfies (4). Take $(f_0, \ldots, f_k, f_{k+1}) \in$

X_{k+1}: then by Theorem 2 $(f_0,\ldots,f_k)\in X_k$ and the relation (2) holds. Put $F = S_k(f_0,\cdots,f_k)$, $F_{k+1} = S_0(f_{k+1})$. By the proof of Theorem 2 it is known that the Hermite's interpolation problem with data $f_0,\ldots,f_{k+1}$ has a solution of the form $F + H = F + \sum_{j=1}^{s} h_j Q_j$, where the entire functions h_j are determined by (3).

On the other hand it is known by Theorem 0 that for any ideal in $H(\mathbf{C}^n)$, generated by polynomials, there exists a tame linear splitting with respect to generators. In particular, for the ideal

$$J = J(P_1,\ldots,P_m, L^{k+1}Q_1,\ldots,L^{k+1}Q_s)$$

there exist tame linear operators $R_i\colon J \to H(\mathbf{C}^n)$, $i = 1,\ldots,m$ and $R_j'\colon J \to H(\mathbf{C}^n)$, $j = 1,\ldots,s$, such that we have for every $G\in J$

$$G = \sum_{i=1}^{m} R_i(G)P_i + \sum_{j=1}^{s} R_j'(G)L^{k+1}Q_j.$$

If we put $G = F_{k+1} - L^{k+1}F$ we get (3) with $h_j = R_j'(F_{k+1} - L^{k+1}F)$. Therefore the entire function

$$F + \sum_{j=1}^{s} R_j'(F_{k+1} - L^{k+1}F)\cdot Q_j$$

is a solution of the Hermite's interpolation problem with data $f_0,\ldots,f_{k+1}$ and we obtain that the operator

$$S_{k+1}(f_0,\ldots,f_{k+1}) = S_k(f_0,\ldots,f_k) + \sum_{j=1}^{s} R_j'(S_0(f_{k+1}) - L^{k+1}S_k(f_0,\ldots,f_k))Q_j$$

is a right inverse operator for T_{k+1}. Using the obtained explicit formula for S_{k+1} we get immediately by induction with respect to k that the operator S_{k+1}, $k = 0, 1, 2,\ldots$, is linear and continuous as a composition of linear continuous operators. Moreover, we get that the operator S_{k+1}, $k=0, 1,\ldots$, is tame. Indeed, it is easy to see that the operator L is tame in the sense

$$\| LF \|_r \leq Cr^\gamma \| F \|_{2r},$$

where the constants C, γ depend only on L. If the operator S_k satisfies (4) then using the estimate (c) of Theorem 0 and Lemma 3 we obtain for the operator S_{k+1} the estimate

$$\| S_{k+1} f \|_r \leq C_{k+1} r^{d_{k+1}} \| f \|_{a_{k+1}r}, \qquad r \geq 1,$$

where the constants C_{k+1}, d_{k+1}, a_{k+1} depend only on L and V.

4. Application to functions with bounds on the growth

Using the tameness of operators S_k from Theorem 3 one can obtain that the Hermite's interpolation problem for data with bounds on the growth has a solution with "similar" bounds on the growth.

Let for any $\rho > 0$, $\sigma > 0$, $H_{\rho,\sigma}(\mathbf{C}^n)$ be the Banach space of all entire functions of order ρ and type $\leq \sigma$ (i.e. such that $\| F \|_r \leq C \exp(\sigma r^\rho)$ for some $C > 0$) with the norm

$$\| F \|_{\rho,\sigma} = \sup \{ \| F \|_r \exp(-\sigma r^\rho), \quad r > 1 \}.$$

We denote by $H_\rho(\mathbf{C}^n)$, $H^\rho(\mathbf{C}^n)$ the algebras of all entire functions of order ρ, which are of minimal or finite type respectively, i.e.

$$H_\rho(\mathbf{C}^n) = \lim \operatorname{proj}_{\sigma \to 0} H_{\rho,\sigma}(\mathbf{C}^n), \quad H^\rho(\mathbf{C}^n) = \lim \operatorname{ind}_{\sigma \to \infty} H_{\rho,\sigma}(\mathbf{C}^n).$$

Analogously we consider the Banach spaces $H_{\rho,\sigma}(V)$ of all holomorphic functions of order ρ and type $\leq \sigma$ on the algebraic variety $V \subset \mathbf{C}^n$ with the norm

$$\| f \|_{\rho,\sigma} = \sup \{ \| f \|_r \exp(-\sigma r^\rho), \quad r > 1 \}$$

and correspondingly the algebras of holomorphic functions on V of order ρ, which are minimal or finite type respectively:

$$H_\rho(V) = \lim \operatorname{proj}_{\sigma \to 0} H_{\rho,\sigma}(V), \quad H^\rho(V) = \lim \operatorname{ind}_{\sigma \to \infty} H_{\rho,\sigma}(V).$$

Theorem 4. *If $f_0, \ldots, f_k$ are holomorphic functions on V of order ρ and finite type such that $(f_0, \ldots, f_k)$ is admissible data for the Hermite's interpolation problem then there exists a solution of order ρ and finite type. Moreover extension operators S_k from Theorem 3 act continuously as follows:*

$$S_k \colon X_k \cap H_\rho(V)^{k+1} \to H_\rho(\mathbf{C}^n), \quad S_k \colon X_k \cap H^\rho(V)^{k+1} \to H^\rho(\mathbf{C}^n).$$

Proof. By Theorem 3 we obtain

$$\begin{aligned} \| S_k(f) \|_r &\leq C_k r^{d_k} \| f \|_{a_k r} \leq C_k r^{d_k} \| f \|_{\rho,\sigma} \exp(\sigma (a_k r)^\rho) \\ &\leq \tilde{C}_k \| f \|_{\rho,\sigma} \exp(\sigma_1 r^\rho), \end{aligned}$$

where $\sigma_1 = 2\sigma a_k^\rho$, $\tilde{C}_k = C_k \sup \{ r^{d_k} \exp(-\sigma a_k^\rho r^\rho), \ r > 1 \}$.

Therefore $\| S_k(f) \|_{\rho,\sigma_1} \leq \tilde{C}_k \| f \|_{\rho,\sigma}$, i.e. the operator S_k acts continuously from the space $X_k \cap H_{\rho,\sigma}^{k+1}(V)$ into the space $X_k \cap H_{\rho,\sigma_1}(\mathbf{C}^n)$. Since σ_1 is proportional to σ the operator S_k acts continuously from $X_k \cap H_\rho(V)^{k+1}$ into $H_\rho(\mathbf{C}^n)$ and from $X_k \cap H^\rho(V)^{k+1}$ into $H^\rho(\mathbf{C}^n)$.

The statement of Theorem 4 can be easily generalized if one consider instead of majorants of the kind $\exp(\sigma r^\rho)$ another system of majorants. Suppose the system M of increasing functions $M_p(r) \colon (1, \infty) \to (1, \infty)$, $p = 1, 2, \ldots$, satisfies the condition

(a) $\quad \forall d, D, p \ \exists p', C \colon r^d M_{p'}(Dr) \leq C M_p(r), \quad r > 1,$

or

(b) $\quad \forall d, D, p \ \exists p', C \colon r^d M_p(Dr) \leq C M_{p'}(r), \quad r > 1.$

Consider in the case (a) the spaces

$$H_M(\mathbf{C}^n) = \lim \operatorname{proj}_p H_{M_p}(\mathbf{C}^n), \quad H_M(V) = \lim \operatorname{proj}_p H_{M_p}(V),$$

and in the case (b) the spaces

$$H^M(\mathbf{C}^n) = \lim \operatorname{ind}_p H_{M_p}(\mathbf{C}^n), \quad H^M(V) = \lim \operatorname{ind}_p H_{M_p}(V),$$

where $H_{M_p}(\mathbf{C}^n)$ and $H_{M_p}(V)$ are respectively the Banach space of all entire functions F such that

$$\|F\|_{M_p} = \sup_{r>1} (\|F\|_r / M_p(r)) < \infty,$$

and the Banach space of all holomorphic functions f on V such that

$$\|f\|_{M_p} = \sup_{r>1} (\|f\|_r / M_p(r)) < \infty.$$

By Theorem 3 we obtain in the case (a)

$$\begin{aligned} \forall p \exists p', C\colon \|S_k(f)\|_r \le C_k r^{d_k} \|f\|_{a_k r} &\le C_k r^{d_k} \|f\|_{M_{p'}} \cdot M_{p'}(a_k r) \\ &\le C_k C \|f\|_{M_{p'}} \cdot M_p(r), \end{aligned}$$

i.e. the operator S_k acts continuously from the space $X_k \cap H_{M_{p'}}(V)^{k+1}$ into the space $H_{M_p}(\mathbf{C}^n)$. Analogously we get that $\forall p \exists p', C$ such that the operator

$$S_k\colon X_k \cap H_{M_p}(V)^{k+1} \to H_{M_{p'}}(\mathbf{C}^n)$$

is continuous. Hence we have the following.

Theorem 5. *Under above notations if the system of majorants $M = (M_p)$ satisfies (a) (respectively (b)) then the operator S_k from Theorem 3 act continuously as follows:*

$$S_k\colon X_k \cap H_M(V)^{k+1} \to H_M(\mathbf{C}^n),$$

or respectively

$$S_k\colon X_k \cap H^M(V)^{k+1} \to H^M(\mathbf{C}^n).$$

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