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# Remarks on logarithmic Sobolev constants, exponential integrability and bounds on the diameter

By

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Although the first two results we aim to simplify deal with abstract Markov generators on probability spaces, we would like to briefly present the purpose of this note in the setting of the Laplace-Beltrami operator $\Delta$ on a complete connected Riemannian manifold $M$ of finite volume $V$. We will consider the normalized measure $d\mu = \frac{1}{V} dv$ where $dv$ denote the Riemannian measure and let $\nabla$ be the Riemannian gradient on $M$.

For a nonnegative bounded (say) real valued function $f$ on $M$, let $E(f)$ denote the entropy of $f$ with respect to $\mu$ defined by

$$E(f) = \int f \log f d\mu - \int f d\mu \log\left(\int f d\mu\right).$$

We will say that $\Delta$ satisfies a logarithmic Sobolev inequality if there exists $\rho > 0$ such that for all $C^\infty$, compactly supported or bounded, functions $f$ on $M$,

$$\rho E(f^2) \leq 2\int f(-\Delta f) d\mu = 2\int |\nabla f|^2 d\mu.$$

The largest possible value $\rho_0$ for $\rho$ is called the logarithmic Sobolev constant of the Laplacian $\Delta$ on $M$, or simply of $M$.

More generaly, one may consider, following [B], inequalities between entropy and energy of the type

$$E(f^2) \leq \Phi(\| |\nabla f| \|_2^2)$$

for all $C^\infty$ bounded functions $f$ with $\|f\|_2 = 1$ where $\Phi$ is a nonnegative function on $[0, \infty)$.

With these notations, S. Aida, T. Masuda and I. Shigekawa [A-M-S] recently showed that, when $\rho_0 > 0$, whenever $f$ is a function on $M$ such that $\| |\nabla f| \|_\infty \leq 1$ (that is $f$ is Lipschitz with Lipschitz norm less than or equal to 1), then, for every $0 < \alpha < \rho_0/2$,

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$$\int e^{\alpha f^2} d\mu < \infty. \tag{1}$$

Moreover, $\int e^{\alpha f^2} d\mu \leq \exp(\alpha\rho_0(\rho_0 - 2\alpha)^{-1} \| f \|_2^2)$.

D. Bakry and D. Michel (see [B-M] for a special case and [B] for the general result), using refined semigroup minorations, showed that under a general inequality between entropy and energy, the diameter $D$ of $M$ satisfies the quantitative estimate

$$D \leq \int_0^\infty \frac{1}{x^2} \Phi(x^2) dx. \tag{2}$$

In particular, if the latter integral is finite, every Lipschitz function on $M$ is bounded.

Finally, L. Saloff-Coste [SC] recently proved that if the Ricci curvature of $M$ is bounded below, the existence of a logarithmic Sobolev inequality for $\Delta$ forces $M$ to be compact. Moreover, for some constant $C > 0$ only depending on the dimension of $M$ and the lower bound on the Ricci curvature,

$$\frac{D}{\log D} \leq \frac{C}{\rho_0} \tag{3}$$

as soon as $D \geq C$.

As announced, the purpose of this note is to provide short and elementary proofs of these three results. In the proof of (1), we closely follow the argument of [A-M-S] (originally due to I. Herbst and E. B. Davies and B. Simon [D-S]) with some improvements in the exposition and show in particular that the strong exponential integrability (1) also holds under defective logarithmic Sobolev inequalities (see also [A-S]). This method is used to give a very simple proof of the bound (2) on the diameter. In the last part, we establish (3) and actually improve the quantitative estimate by removing the logarithmic factor. As we will see, it then turns out to be sharp.

## 1. Exponential integrability and bounds on the diameter

The framework of the first two results (1) and (2) deals more generally with abstract Markov generators and we now turn to some notation in this respect. On some probability space $(E, \mathscr{B}, \mu)$, let L be a Markov generator with ($L^2$-) domain $\mathscr{D}$. Without going far into the technical details (refering to [B], [D-M-M], [A-M-S]), we will assume that there exists an algebra $\mathscr{A}$ of bounded real valued functions on $E$ contained in $\mathscr{D}$ and dense in $L^2(\mu)$. We moreover assume that $\mathscr{A}$ is stable by L and by the action of $C^\infty$ functions and that L is invariant for $\mu$ on $\mathscr{A}$. One can introduce the "carré du champ" operator by setting

$$\Gamma(f, g) = \tfrac{1}{2} \mathrm{L}(fg) - f\mathrm{L}g - g\mathrm{L}f$$

for every $f, g$ in $\mathscr{A}$. In particular, by invariance, $\int f(-\mathrm{L}f)d\mu = \int \Gamma(f, f)d\mu$ for every $f$ in $\mathscr{A}$. As an additional main assumption on L, we will assume that $\Gamma$ is a derivation in the sense that when $\varphi$ is a real $C^\infty$ function, for every $f, g$ in $\mathscr{A}$,

$$\Gamma(\varphi(f), g) = \varphi'(f)\Gamma(f, g).$$

It is well known that the definition of the operator $\Gamma$ may be extended to all of the domain of the Dirichlet form $\mathscr{E}(f, f) = \int f(-\mathrm{L}f)d\mu < \infty$. For our purposes here, we will simply agree that a function $f$ on $E$ is such that $\|\Gamma(f, f)\|_\infty \leq 1$ if there is a sequence $(f_n)_{n\in\mathbf{N}}$ in $\mathscr{A}$ such that $\|\Gamma(f_n, f_n)\|_\infty \leq 1$ for every $n$ which converges $\mu$-almost surely to $f$. Such a function may be considered as a Lipschitz function with respect to L (or $\Gamma$). For example, for the Laplace operator on a Riemannian manifold $M$ of finite volume, $\Gamma(f, g) = \nabla f \cdot \nabla g$ for smooth functions $f$ and $g$ (for example $C^\infty$ and bounded or constant outside some compact set) and a function $f$ on $M$ such that $\|\Gamma(f, f)\|_\infty \leq 1$ in the preceding sense is simply a Lipschitz function with Lipschitz norm $\| |\nabla f| \|_\infty \leq 1$. One further important example included in this abstract setting is the Ornstein-Uhlenbeck generator on a Wiener space $(E, H, \mu)$ for which $\|\Gamma(f, f)\|_\infty \leq 1$ if and only if the Malliavin derivative $Df$ of $f$ satisfies $\| |Df|_H \|_\infty \leq 1$ (see [A-M-S]). Equivalently, $f$ is 1-Lipschitz in the directions of the Cameron-Martin Hilbert space $H$ [E-S].

In this framework, one may speak of the "diameter" of the space $E$ for L in the following sense (cf. [B-M], [B]). If $f$ is measurable on $E$, set $\tilde{f}(x, y) = f(x) - f(y)$ on $E \times E$. Define then the diameter $D$ of $E$ by

$$D = \sup_{f\in\mathscr{A},\ \|\Gamma(f,f)\|_\infty \leq 1} \|\tilde{f}\|_{\mathrm{L}^\infty(\mu\otimes\mu)}.$$

It is easily seen that this definition coincides with the usual diameter in a Riemannian setting.

To ease the notation, we use sometimes $\langle f \rangle$ to denote the expectation of an integrable function $f$ on $(E, \mathscr{B}, \mu)$. As before, we then say that $\rho_0$ is the logarithmic Sobolev constant of L if for all functions $f$ in $\mathscr{A}$,

$$\rho_0 E(f^2) \leq 2\langle f(-\mathrm{L}f)\rangle = 2\langle \Gamma(f, f)\rangle.$$

One speaks of an inequality between entropy and energy if for all functions $f$ in $\mathscr{A}$ with $\langle f^2 \rangle = 1$,

$$E(f^2) \leq \Phi(\langle \Gamma(f, f)\rangle) \tag{4}$$

where $\Phi$ is nonnegative on $[0, \infty)$. If we assume $\Phi$ to be concave, we can replace $\Phi$ by the family of its tangent lines so that inequality (4) may be expressed equivalently by a family of (generalized) logarithmic Sobolev inequalities (cf. [D])

$$E(f^2) \leq (\Phi(x) - x\Phi'(x))\langle f^2\rangle + \Phi'(x)\langle \Gamma(f, f)\rangle, \qquad x \geq 0.$$

We may now turn to the proof of (1) and (2) which we treat simultaneously. Let first $g \in \mathscr{A}$ be such that $\|\Gamma(g,g)\|_\infty \le 1$. We will apply inequality (4) to the family of functions $f = e^{\lambda g/2}$, $\lambda \in \mathbf{R}$. Let $G(\lambda) = \int e^{\lambda g} d\mu$ be the Laplace transform of $g$ and observe that $G'(\lambda) = \int g e^{\lambda g} d\mu \left( = \frac{1}{\lambda} \int f^2 \log f^2 d\mu \right)$ ($\lambda \neq 0$). Now, since $\Gamma$ is a derivation, $\Gamma(f,f) = (\lambda^2/4) e^{\lambda g} \Gamma(g,g)$ so that inequality (4) yields that, for every $\lambda$,

$$\lambda G'(\lambda) - G(\lambda) \log G(\lambda) \le G(\lambda) \Phi\left(\frac{\lambda^2}{4}\right).$$

Let then $H(\lambda) = \frac{1}{\lambda} \log G(\lambda)$, $\lambda > 0$. The preceding inequality reads

$$H'(\lambda) \le \frac{1}{\lambda^2} \Phi\left(\frac{\lambda^2}{4}\right) \tag{5}$$

for every $\lambda > 0$. Since

$$H(0) = \lim_{\lambda \to 0} \frac{1}{\lambda} \log G(\lambda) = \frac{G'(0)}{G(0)} = \langle g \rangle,$$

it follows that, for every $\lambda > 0$,

$$\frac{1}{\lambda} \log G(\lambda) = H(\lambda) = H(0) + \int_0^\lambda H'(u) du \le \langle g \rangle + \int_0^\lambda \frac{1}{u^2} \Phi\left(\frac{u^2}{4}\right) du.$$

Therefore,

$$\int e^{\lambda(g - \langle g \rangle)} d\mu \le \exp\left(\lambda \int_0^\lambda \frac{1}{u^2} \Phi\left(\frac{u^2}{4}\right) du\right) \tag{6}$$

for every $\lambda \ge 0$.

Now, assume that $C = \int_0^\infty \frac{1}{u^2} \Phi\left(\frac{u^2}{4}\right) du = \frac{1}{2} \int_0^\infty \frac{1}{x^2} \Phi(x^2) dx < \infty$. By the preceding inequality applied to $g$ and $-g$, for every $\lambda \ge 0$ and every $\varepsilon > 0$,

$$\mu(|g - \langle g \rangle| \ge C + \varepsilon) \le \mu(g - \langle g \rangle \ge C + \varepsilon) + \mu(-g - \langle -g \rangle \ge C + \varepsilon) \le 2e^{-\lambda(C+\varepsilon)} e^{\lambda C}.$$

As $\lambda \to \infty$, we get that $\|g - \langle g \rangle\|_\infty \le C$. Inequality (2) then immediately follows by the very definition of $D$.

We turn to (1) and assume thus the existence of a logarithmic Sobolev constant $\rho_0 > 0$. We may therefore take $\Phi(x) = \frac{2}{\rho_0} x$ in (6). Hence, for every $\lambda \ge 0$ (actually every $\lambda$ by replacing $g$ by $-g$),

$$\int e^{\lambda(g - \langle g \rangle)} d\mu \le e^{\lambda^2/2\rho_0}.$$

By Chebyshev's inequality, for every $t \ge 0$ and $\lambda \ge 0$,

$$\mu(g - \langle g \rangle \geq t) \leq e^{-\lambda t} e^{\lambda^2/2\rho_0},$$

and, optimizing in $\lambda$,

$$\mu(g - \langle g \rangle \geq t) \leq e^{-\rho_0 t^2/2}. \tag{7}$$

By applying the result also to $-g$, for every $t \geq 0$,

$$\mu(|g - \langle g \rangle| \geq t) \leq 2e^{-\rho_0 t^2/2}. \tag{8}$$

This inequality thus holds for every function $g$ in $\mathscr{A}$ such that $\|\Gamma(g, g)\|_\infty \leq 1$. Let now $(g_n)_{n \in \mathbf{N}}$ be a sequence in $\mathscr{A}$ which converges $\mu$-almost surely to $g$ and such that $\|\Gamma(g_n, g_n)\|_\infty \leq 1$ for every $n$. Let $m$ be large enough so that $\mu(|g| \leq m) \geq \frac{3}{4}$. Then, for some $n_0$ and every $n \geq n_0$, $\mu(|g_n| \leq m+1) \geq \frac{1}{2}$. Choose furthermore $t_0 > 0$ with $2e^{-\rho_0 t_0^2/2} < \frac{1}{2}$. Since each $g_n$ satisfies (8), it follows, by intersecting the sets $\{|g_n| \leq m + 1\}$ and $\{|g_n - \langle g_n \rangle| \geq t_0\}$, that $|\langle g_n \rangle| \leq t_0 + m + 1$ for every $n \geq n_0$. Moreover, coming back to (8),

$$\mu(|g_n| \geq t + t_0 + m + 1) \leq 2e^{-\rho_0 t^2/2}$$

for every $n \geq n_0$ and every $t \geq 0$. In particular, $\sup_n \int g_n^2 d\mu < \infty$ so that, by uniform integrability, it immediately follows that every Lipschitz function $g$ on $E$ such that $\|\Gamma(g, g)\|_\infty \leq 1$ is integrable and satisfies (7). In particular, $\int e^{\alpha g^2} d\mu < \infty$ for every $\alpha < \rho_0/2$. Let us mention however that the deviation inequality (7) is a much more precise and useful tool than rather only the latter integrability property. This is particular the case in the context of abstract Wiener spaces and Gaussian measures on Banach spaces. For example, if $\mu$ is the canonical Gaussian measure on $\mathbf{R}^n$ and $g$ the Euclidean distance $|\cdot|$ to the origin, then, for every $t \geq 0$,

$$\mu(x \in \mathbf{R}^n; |x| \geq \sqrt{n} + t) \leq e^{-t^2/2}.$$

The deviation inequalities (7) actually belong to the family of concentration inequalities of isoperimetric flavor. We refer to [L-T] and [L2] for more information on this aspect. It should be mentioned also that this approach to deviation inequalities of course requires first the knowledge of a logarithmic Sobolev inequality and that usually the arguments needed to prove a logarithmic Sobolev inequality yield in a direct and shorter way these deviation inequalities (see e.g. [L1] for the Gaussian example).

The preceding strong exponential integrability under logarithmic Sobolev inequalities actually also holds under the so-called defective logarithmic Sobolev inequalities of the type

$$E(f^2) \leq a\langle f^2 \rangle + b\langle \Gamma(f, f) \rangle, \qquad f \in \mathscr{A}, \tag{9}$$

for some $a, b > 0$. Thus, in the preceding notation, $\Phi(x) = a + bx$, $x \geq 0$. Indeed, coming back to (5), for every $g \in \mathscr{A}$ with $\|\Gamma(g, g)\|_\infty \leq 1$, we see that we also have, for every $\lambda \geq 1$,

$$\frac{1}{\lambda}\log G(\lambda) = H(\lambda) = H(1) + \int_1^\lambda H'(u)\,du \le \log G(1) + a + \frac{1}{4}b\lambda.$$

Hence, for $\lambda \ge 1$,

$$\int e^{\lambda g}d\mu \le \left(\int e^{g}d\mu\right)^{\lambda} e^{a\lambda + b\lambda^2/4}. \tag{10}$$

Let us choose first $\lambda = 2$. Then $\int e^{2g}d\mu \le C(\int e^{g}d\mu)^2$ with $C = e^{2a+b}$. Let $m$ be large enough so that $\mu(|g| \ge m) \le 1/4C$. Then $\mu(e^g \ge e^m) \le 1/4C$ and

$$\begin{aligned}\int e^g d\mu &\le e^m + \mu(e^g \ge e^m)^{1/2}\left(\int e^{2g}d\mu\right)^{1/2}\\ &\le e^m + \sqrt{C}\,\mu(e^g \ge e^m)^{1/2}\int e^g d\mu\\ &\le 2e^m.\end{aligned}$$

Returning to (10), for every $\lambda \ge 1$,

$$\int e^{\lambda g}d\mu \le 2^\lambda e^{m\lambda + a\lambda + b\lambda^2/4} \le e^{C'\lambda^2}$$

where $C' = 1 + m + a + b/4$. Using Chebyshev's inequality as before, we see that

$$\mu(g \ge t) \le e^{-t^2/4C'}$$

at least for every $t \ge 0$ large enough (depending on $C'$). The same inequality applies to $-g$. Moreover, the proof was presented in such a way to show that this inequality also applies to any $g$ in $E$ with $\|\Gamma(g,g)\|_\infty \le 1$ since, besides $\|\Gamma(g,g)\|_\infty \le 1$, the only parameter used on $g$ is $m$. In particular, under a defective logarithmic Sobolev inequality (9), $\int e^{\alpha g^2}d\mu < \infty$ for some $\alpha > 0$ for every $g$ with $\|\Gamma(g,g)\|_\infty \le 1$, improving thus upon Theorem 3.2 in [A-M-S]. This result has been also obtained in [A-S] where further integrability properties under only $L^p$-bounds on $\Gamma(g,g)$ are described.

## 2. Logarithmic Sobolev constant in Riemannian manifolds with Ricci curvature bounded below

In the final part of this work, we turn to a proof of (3). As in [SC], our argument is based on the parabolic inequality by P. Li and S.-T. Yau, however at some elementary level. Let thus $M$ be as in the introduction a complete connected (noncompact) Riemannian manifold of dimension $n$ with normalized Riemannian measure $d\mu$. We assume that the Ricci curvature Ric on $M$ is bounded below by $-K$, $K \ge 0$. We denote by $\rho_0$ the logarithmic Sobolev constant of the Laplacian $\Delta$ on $M$ and assume in what follows that $\rho_0 > 0$.

Let $P_t = e^{t\Delta}$, $t \ge 0$, be the heat semigroup on $M$. The Li-Yau inequality

[L-Y] (cf. [D]) indicates that for every positive $C^\infty$ function $f$ on $M$, and every $\alpha > 1$, $t > 0$, at each point of $M$,

$$\frac{|\nabla P_t f|^2}{(P_t f)^2} - \alpha \frac{\Delta P_t f}{P_t f} \le \frac{n\alpha^2}{2t}\left(1 + \frac{Kt}{\alpha - 1}\right).$$

(Note that when $K = 0$ one may choose $\alpha = 1$.) In the sequel, we simply take $\alpha = 2$. We will not try actually to sharpen our choices of constants. Anyway, this could only improve the constant $C$ in (3) but not the dependence on $\rho_0$. In particular therefore, for every $0 < t \le 1$,

$$-\frac{\Delta P_t f}{P_t f} \le \frac{\gamma}{t}$$

where $\gamma = \gamma(n, K) = n(1 + K)$ ($\ge 1$). Fix $f$ positive and smooth on $M$. Letting $F(t) = P_t f$, $0 < t \le 1$, evaluated at some point in $M$, the preceding inequality implies that $\gamma F(t) + tF'(t) \ge 0$ for every $0 < t \le 1$. Therefore, the function $t^\gamma F(t)$ is increasing on $(0, 1]$ so that

$$P_t f \le \frac{1}{t^\gamma} P_1 f, \qquad 0 < t \le 1. \tag{11}$$

Now, by the fundamental relation between logarithmic Sobolev inequalities and hypercontractivity due to L. Gross [G], we know that the heat semigroup $(P_t)_{t \ge 0}$ has hypercontractivity constant $\rho_0$, that is, for every $1 < p < q < \infty$, $\| P_t f \|_q \le \| f \|_p$ as soon as $e^{\rho_0 t} \ge [(q - 1)/(p - 1)]^{1/2}$. We apply this property with $p = 2$ and $t = 1$ so that $q = 1 + e^{2\rho_0}$. Therefore, taking the $L^q$-norms of both sides of (11), we get that, for every $C^\infty$ bounded function $f$ on $M$ (positive or not),

$$\| P_t f \|_q \le \frac{1}{t^\gamma} \| P_1 f \|_q \le \frac{1}{t^\gamma} \| f \|_2, \qquad 0 < t \le 1.$$

It is well-known that such a semigroup estimate is equivalent to a (local) Sobolev inequality. To somewhat keep track of the constants let us briefly recall the steps of the argument. By interpolation, for every $f$ as before,

$$\| P_t f \|_r \le \frac{1}{t^{\theta\gamma}} \| f \|_2, \qquad 0 < t \le 1,$$

where $\frac{1}{\theta}\left(\frac{1}{2} - \frac{1}{r}\right) = \left(\frac{1}{2} - \frac{1}{q}\right)$, $0 < \theta < 1$. Let us choose simply $\theta$ such that $\theta\gamma = \frac{1}{4}$. Now, using that

$$(\mathrm{I} - \Delta)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} P_t \, dt,$$

it rather easily follows (cf. e.g. [D] or [V-SC-C]) that for every smooth function

$f$ on $M$, $\|(\mathrm{I} - \Delta)^{-1/2} f\|_r^2 \le 8\|f\|_2^2$, that is

$$\|f\|_r^2 \le 8(\|f\|_2^2 + \|\,|\nabla f|\,\|_2^2). \tag{12}$$

The numerical constant 8 has no reason to be sharp.) It is rather surprising that a Sobolev inequality of logarithmic type implies a true Sobolev inequality (of power type). This may be considered as an effect of curvature. One could now deduce from this Sobolev inequality (12) that $D$ is bounded and thus that $M$ is compact by various arguments. For example, this inequality could be iterated (very much as in Moser's iteration principle). An alternate argument combines minorations of volumes of balls together with the fact that $M$ has finite volume. Taking into account our first part, we prefer to follow the approach by D. Bakry and D. Michel [B-M] and bound the diameter $D$ of $M$ with (2) and some inequality between entropy and energy. This method seems indeed to yield rather sharp bounds in general (cf. [B]). To this aim, fix a function $f$ with $\|f\|_2 = 1$ and consider the probability measure $dv = f^2 d\mu$. By Jensen's inequality,

$$\log \|f\|_r^2 = \frac{2}{r} \log\left(\int |f|^{r-2} dv\right) \ge \frac{r-2}{r} \int \log f^2 dv = \frac{r-2}{r} E(f^2).$$

Hence (12) implies that

$$E(f^2) \le \Phi_1(\|\,|\nabla f|\,\|_2^2), \qquad \|f\|_2 = 1,$$

with $\Phi_1(x) = \dfrac{r}{r-2} \log(8 + 8x)$, $x \ge 0$. In addition, we also have by the definition of the logarithmic Sobolev constant $\rho_0$ that

$$E(f^2) \le \Phi_2(\|\,|\nabla f|\,\|_2^2), \qquad \|f\|_2 = 1,$$

with $\Phi_2(x) = \dfrac{2}{\rho_0} x$, $x \ge 0$. We therefore get an inequality between entropy and energy (4) with $\Phi = \min(\Phi_1, \Phi_2)$. According to (2),

$$\begin{aligned} D &\le \int_0^\infty \frac{1}{x^2} \Phi(x^2) dx \le \frac{2}{\rho_0} + \int_1^\infty \frac{r}{r-2} \frac{1}{x^2} \log(8 + 8x^2) dx \\ &\le \frac{2}{\rho_0} + C_1 \frac{r}{r-2} \end{aligned}$$

where $C_1 > 0$ is numerical. Now, recall that $q = 1 + \mathrm{e}^{2\rho_0}$, $\gamma = n(1 + K)$ and that

$$\left(\frac{1}{2} - \frac{1}{r}\right) = \frac{1}{4\gamma}\left(\frac{1}{2} - \frac{1}{q}\right).$$

We then simply observe that

$$\frac{r}{r-2} = 4\gamma \frac{q}{q-2} = 4\gamma \frac{1 + \mathrm{e}^{2\rho_0}}{1 + \mathrm{e}^{2\rho_0} - 2} \le \frac{20\gamma}{\min(1, \rho_0)}.$$

Therefore,

$$D \leq \frac{C}{\min(1, \rho_0)} \tag{13}$$

where the constant $C$ only depends on $n$ and $K$. The conclusion follows since when $D > C$, $\rho_0 \leq 1$ and thus $D \leq C/\rho_0$. This bound is sharp since, as is mentioned at the end of [SC], classical arguments from the theory of hypercontractivity show conversely that when $\lambda_1 \geq \varepsilon > 0$, where $\lambda_1$ is the first eigenvalue of $-\Delta$, then

$$\rho_0 \geq \frac{c}{D}$$

where $c > 0$ only depends on $n$, $K$ and $\varepsilon$ Recall also from [SC] that there exist (compact) manifolds of constant negative sectional curvature with arbitrarily large diameter and $\lambda_1$ uniformly bounded away from zero. One consequence of (3) or (13) is thus that the ratio $\rho_0/\lambda_1$ (always $\leq 1$) can be made arbitrarily small. This is in constrast with the case of compact manifolds with nonnegative Ricci curvature for which $\frac{4n}{(n+1)^2}\lambda_1 \leq \rho_0 \leq \lambda_1$ [R]. Notice that, together with (13), we recover in this case a weak form of Cheng's inequality $\lambda_1 \leq n\pi^2/D^2$ [C].

It should be noticed finally that, rather than hypercontractivity, the preceding proof of thecompactness of $M$ under the condition $\rho_0 > 0$ only uses actually that there exist $t_0 > 0$ and $1 \leq p < q \leq \infty$ such that $P_{t_0}$ is a bounded operator from $L^p$ into $L^q$.

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