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The initial boundary value problem for linear symmetric hyperbolic systems with boundary characteristic of constant multiplicity

By

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§ 0. Introduction

This paper is devoted to the study of the initial boundary value problem for the first order symmetric hyperbolic systems with characteristic boundary of constant multiplicity. We shall show the existence and the differentiability of solutions. Although we study the linear theory in this paper, the main result is stated in such a way that it can be applied to the proof of the convergence of iteration scheme in studying the quasi-linear initial boundary value problem.

Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded open set lying on one side of its smooth boundary Γ. We shall treat differential operators of the form

$$L(v) = A_0(v)\partial_t + \sum_{j=1}^{n} A_j(v)\partial_j + B(v),$$

where $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, and $v = {}^t(v_1(t, x), v_2(t, x), \ldots, v_l(t, x))$ is a given smooth function of the time t and the space variable $x = (x_1, x_2, \ldots, x_n)$. It is assumed that $A_j(\cdot)$, $0 \leq j \leq n$, and $B(\cdot)$ are real $l \times l$ matrices depending smoothly on their arguments. Therefore $A_j(v)$, $0 \leq j \leq n$, and $B(v)$ are smoothly varying real $l \times l$ matrices defined for $(t, x) \in [0, T] \times \bar{\Omega}$. We shall study the mixed initial boundary value problem

$$L(v)u = F \quad \text{in} \quad [0, T] \times \Omega, \tag{0.1}$$

$$Mu = 0 \quad \text{on} \quad [0, T] \times \Gamma, \tag{0.2}$$

$$u(0, x) = f(x) \quad \text{for} \quad x \in \Omega, \tag{0.3}$$

where the unknown function $u = u(t, x)$ is a vector-valued function with l components and where $M(x)$ is an $l \times l$ real matrix depending smoothly on $x \in \Gamma$. We assume that M is of constant rank everywhere on Γ. The inhomogeneous term F of the equation and the initial data f are given vector-valued functions defined on $[0, T] \times \bar{\Omega}$ and $\bar{\Omega}$, respectively. Let $v = (v_1,$

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$\nu_2, \ldots, \nu_n)$ be the outward unit normal to Γ. Then, $A_\nu(v) = \sum_{j=1}^{n} \nu_j A_j(v)$ is called the boundary matrix. If the boundary matrix $A_\nu(v)$ is invertible everywhere on Γ, then the boundary Γ is said to be non-characteristic. If it is not invertible but it has a constant rank on Γ, then the boundary Γ is said to be characteristic of constant multiplicity.

A general theory for the case where the boundary is non-characteristic has been developed by Friedrichs [6], Lax-Phillips [11], Rauch-Massey III [21], and others. The case where the boundary is characteristic has been discussed also by several authors. In particular, the existence of solutions and the well-posedness in the L^2-sense have been proved by Lax-Phillips [11]. In studying the regularity theory for this case, a difficulty which is termed the loss of derivatives in the normal directions has been observed by Tsuji [25] and others. A regularity theory has been given by [14], by assuming that there is an extension of the outward unit normal vector field to a C^0-function defined on a neighborhood of Γ such that the corresponding extension of the boundary matrix $A_\nu(v)$ has a constant rank there. However, for many physical problems, this hypothesis fails to hold. The existence of solutions and the well-posedness in L^2-sense were shown by Rauch [20] under a weaker assumption that the boundary matrix $A_\nu(v)$ is of constant rank only on Γ. (Note that the maximal nonnegativity of the boundary subspace is assumed always.) He obtained also the regularity of solutions in the tangential directions.

The results obtained so far do not seem to be sufficient to handle the quasilinear initial boundary value problem with characteristic boundary. One reason is that the assumptions on the coefficient matrices are too stringent. When we concern ourselves with the quasi-linear problem, the entries of these matrices must lie in the function space in which the solutions are supposed to exist. Even from the view point of the linear theory, the function space $H^m_{\tan}(\Omega)$, in which only the tangential derivatives are taken account, seems to be somewhat simple. (For the definition of $H^m_{\tan}(\Omega)$, see [2].) It has been recognized in the study of the characteristic initial boundary value problem, that the normal differentiability of order one results from the tangential differentiability of ordr two. This seems to be a suitable interpretation of the loss of derivatives in the normal directions. The function space $H^m_*(\Omega)$, that we use in this paper, embodies the above mentioned observation. It is suitable for constructing a linear theory in the sense that not only the a priori estimates of solutions are obtained in this norm but the compatibility condition can be given an appropriate meaning in this function space. However we do not enter into detail here. We note that the function space $H^m_*(\Omega)$ was used in the works of Chen Shuxing [4] and Yanagisawa-Matsumura [26]. It should be remarked that even in the case of the characteristic initial boundary value problem there is an important class of physical problems for which one can get the full regularity ([1], [5], [22], [23]) in the sense that the regularity theory is stated in terms of the usual Sobolev space $H^m(\Omega)$. General criteria for characteristic initial boundary value problems having such property have been given by Ohkubo [16] and also by Kawashima-Yanagisawa-Shizuta [9].

The content of this paper is as follows. In § 1, we give the definitions of $H_*^m(\Omega)$ and the related function spaces, that will be used in this paper. We state the main theorem in § 2. Some remarks are also given. We shall prove our main theorem in § 3, assuming that any data satisfying the compatibility condition of certain order can be approximated by smoother data which satisfy the compatibility condition of higher order and that the uniform estimate for solutions to the approximate problem (see (3.34)–(3.36)) holds. In § 4, the existence of an approximate sequence of data that was assumed in the preceding section will be shown. In § 5, the approximate problem is reduced to the case of a half space. This is a preliminary to the next section. In § 6, the proof of the uniform estimate assumed in § 3 will be given. In Appendices, we shall prove several lemmas used in this paper. The main result of this paper was announced in [18].

§ 1. Function spaces and notations

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index and let $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We write

$$\partial_x = (\partial_1, \dots, \partial_n), \quad \partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}, \qquad 1 \le i \le n,$$

$$\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}.$$

$H^m(\Omega)$, $m \ge 0$, denotes the usual Sobolev space of order m. The norm is

$$\| f \|_m = \Big(\sum_{|\alpha| \le m} \| \partial_x^\alpha f \|^2 \Big)^{\frac{1}{2}}.$$

Here $\| \cdot \|$ denotes the L^2-norm. We recall that a vector field $\Lambda \in C^\infty(\bar{\Omega}; \mathbf{C}^n)$ is said to be tangential if $\langle \Lambda(x), \nu(x) \rangle = 0$ for all $x \in \Omega$.

When $\Omega \subset \mathbf{R}^n$ is a bounded open set with smooth boundary, $H_*^m(\Omega)$, $m \ge 0$, is defined as the set of functions having the following properties:

i) $u \in L^2(\Omega)$.

ii) Let $\Lambda_1, \Lambda_2, \dots, \Lambda_j$ be tangential vector fields and let $\Lambda'_1, \Lambda'_2, \dots, \Lambda'_k$ be non-tangential vector fields. Then $\Lambda_1 \Lambda_2 \cdots \Lambda_j \Lambda'_1 \Lambda'_2 \cdots \Lambda'_k u \in L^2(\Omega)$, if $j + 2k \le m$.

$H_*^m(\Omega)$ is normed as follows. We choose as usual an open covering of Γ, diffeomorphisms, and cut off functions, say, $\mathcal{O}_i, \tau_i, \chi_i$, $1 \le i \le N$. Then $u^{(i)} = (\chi_i u) \circ \tau_i^{-1}$ has as its natural domain $\mathscr{B}_+ = \{x \mid |x| < 1, \ x_1 > 0\}$ with Γ corresponding to $x_1 = 0$. The tangential vector fields given by ∂_k, $k = 2, \dots, n$, in local coordinates are linearly independent. One sees that any tangential vector field can be written in a neighborhood of a point on Γ as a linear combination of $x_1 \partial_1, \partial_2, \dots, \partial_n$ with C^∞-coefficients. It is assumed that the normal vector field ∂_ν corresponds to $-\partial_1$ in local coordinates. Let Ω_δ be the set $\{x \in \Omega \mid \operatorname{dist}(x, \Gamma) > \delta\}$. Let χ_0 be a cut off function such that $\chi_0 = 0$ on a neighborhood of Γ

and let $\chi_0 = 1$ on some Ω_δ. We may assume that $\sum_{i=0}^N \chi_i^2 = 1$ on $\bar{\Omega}$. Then the norm in $H_*^m(\Omega)$ is

$$\|u\|_{m,*}^2 = \|\chi_0 u\|_m^2 + \sum_{i=1}^N \|\chi_i u\|_{m,*}^2, \tag{1.1}$$

$$\|\chi_i u\|_{m,*}^2 = \sum_{|\alpha|+2k\le m} \|\partial_{\tan}^\alpha \partial_1^k u^{(i)}\|_{L^2(\mathscr{B}_+)}^2, \tag{1.2}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$\partial_{\tan}^\alpha = (x_1 \partial_1)^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}.$$

Note that $\partial_{\tan}^\alpha$ in (1.2) can be replaced by

$$\partial_*^\alpha = x_1^{\alpha_1} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n},$$

because the corresponding norms are equivalent to each other. We shall use the same notation for these norms. We notice also that the norms arising from different choices of $\mathcal{O}_i$, τ_i, χ_i are equivalent.

Let us introduce another function space, which is quite analogous to $H_*^m(\Omega)$. We consider the following property:

ii)′ Let $\Lambda_1, \Lambda_2, \dots, \Lambda_j$ be tangential vector fields and let $\Lambda_1', \Lambda_2', \dots, \Lambda_k'$ be non-tangential vector fields. Then $\Lambda_1 \Lambda_2 \cdots \Lambda_j \Lambda_1' \Lambda_2' \cdots \Lambda_k' u \in L^2(\Omega)$, if $j + 2k \le m+1$ and in addition $j + k \le m$.

The set of functions having the properties i), ii)′ is denoted by $H_{**}^m(\Omega)$, where $m \ge 0$. The norm in this space is given by

$$\|u\|_{m,**}^2 = \|\chi_0 u\|_m^2 + \sum_{i=1}^N \|\chi_i u\|_{m,**}^2, \tag{1.3}$$

$$\|\chi_i u\|_{m,**}^2 = \sum_{\substack{|\alpha|+2k\le m+1 \\ |\alpha|+k\le m}} \|\partial_{\tan}^\alpha \partial_1^k u^{(i)}\|_{L^2(\mathscr{B}_+)}^2. \tag{1.4}$$

We have in general a continuous imbedding $H^m(\Omega) \subsetneq H_{**}^m(\Omega) \subsetneq H_*^m(\Omega)$.

Let X be a Hilbert space and let $I \subset \mathbf{R}$ be a closed finite interval. Then, $C(I; X)$ denotes the space of strongly continuous functions on I taking values in X. Similarly, we denote by $C_w(I; X)$ the space of weakly continuous functions on I with values in X. $C(I; X)$ is a Banach space under the maximum norm. The topology of $C_w(I; X)$ is the uniform weak convergence topology. Let $\{u_k\}$ be a sequence in $C_w(I; X)$ and let $u \in C_w(I; X)$. If $u_k(t)$ converges to $u(t)$ in the weak topology of X uniformly in $t \in I$, we say that the sequence $\{u_k\}$ converges to u in $C_w(I; X)$. We note that for any $u \in C_w(I; X)$ we have

$$\sup_{t\in I} \|u(t)\|_X < \infty.$$

If otherwise, there is a convergent sequence $\{t_i\}$ such that $\|u(t_i)\| \to \infty$ ad $i \to \infty$. This contradicts the resonance theorem. In this sense, $C_w(I; X)$ may be regarded as a closed subspace of $L^\infty(I; X)$.

Let $m \geq 0$. We define $X^m([0, T]; \Omega)$ to be the space of functions such that

$$\partial_t^j u \in C([0, T]; H^{m-j}(\Omega)), \qquad 0 \leq j \leq m.$$

Here $\partial_t^j u$, $0 \leq j \leq m$, are the derivatives of u in the distribution sense. Let $u \in X^m([0, T]; \Omega)$. We set

$$|||u(t)|||_m^2 = \sum_{j=0}^{m} \|\partial_t^j u(t)\|_{m-j}^2$$

for $t \in [0, T]$. The norm in $X^m([0, T]; \Omega)$ is given by

$$|||u|||_{X^m([0,T];\Omega)} = \max_{0 \leq t \leq T} |||u(t)|||_m .$$

$X^m([0, T]; \Omega)$ is a Banach space under this norm.

Similarly, $X_*^m([0, T]; \Omega)$, $m \geq 0$, is defined as the space of functions such that

$$\partial_t^j u \in C([0, T]; H_*^{m-j}(\Omega)), \qquad 0 \leq j \leq m.$$

The norm in $X_*^m([0, T]; \Omega)$ is

$$|||u|||_{X_*^m([0,T];\Omega)} = \max_{0 \leq t \leq T} |||u(t)|||_{m,*},$$

$$|||u(t)|||_{m,*}^2 = \sum_{j=0}^{m} \|\partial_t^j u(t)\|_{m-j,*}^2 .$$

It is seen that $X_*^m([0, T]; \Omega)$ is a Banach space under this norm. Let us recall that we used an open covering of Γ, diffeomorphisms, and cut off functions, that is, $\mathcal{O}_i$, τ_i, χ_i, $1 \leq i \leq N$, in defining the norm in $H_*^m(\Omega)$. Let $u^{(i)}(t) = (\chi_i u(t)) \circ \tau_i^{-1}$. Then we have

$$|||u(t)|||_{m,*}^2 = |||\chi_0 u(t)|||_m^2 + \sum_{i=1}^{N} |||\chi_i u(t)|||_{m,*}^2, \tag{1.5}$$

$$|||\chi_i u(t)|||_{m,*}^2 = \sum_{|\gamma| + 2k \leq m} \|D_{\tan}^{\gamma} \partial_1^k u^{(i)}(t)\|_{L^2(\mathscr{B}_+)}^2, \tag{1.6}$$

where $\gamma = (j, \alpha)$, $|\gamma| = j + |\alpha|$, and

$$D_{\tan}^{\gamma} = \partial_t^j \partial_{\tan}^{\alpha} = \partial_t^j (x_1 \partial_1)^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}.$$

We note that $D_{\tan}^{\gamma}$ in (1.6) may be replaced by

$$D_*^{\gamma} = \partial_t^j \partial_*^{\alpha} = \partial_t^j x_1^{\alpha_1} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n},$$

because the corresponding norms in $X_*^m([0, T]; \Omega)$ are equivalent to each other. We shall denote both norms by the same notation.

Let $m \geq 0$. We define $Y_*^m([0, T]; \Omega)$ to be the space of functions such that

$$\partial_t^j u \in C_w([0, T]; H_*^{m-j}(\Omega)), \qquad 0 \leq j \leq m.$$

Let $\{u_k\}$ be a sequence in $Y_*^m([0, T]; \Omega)$ and let $u \in Y_*^m([0, T]; \Omega)$. We say that

u_k converges to u as $k \to \infty$ if, for any $0 \leq j \leq m$, $\partial_t^j u_k(t)$ converges to $\partial_t^j u(t)$ as $k \to \infty$ in the weak topology of $H_*^{m-j}(\Omega)$ uniformly in $t \in [0, T]$. This defines the topology of $Y_*^m([0, T]; \Omega)$. We denote by $Z_*^m(0, T; \Omega)$, $m \geq 0$, the space of functions such that

$$\partial_t^j u \in L^\infty(0, T; H_*^{m-j}(\Omega)), \qquad 0 \leq j \leq m.$$

The norm in $Z_*^m(0, T; \Omega)$ is defined by

$$|||u|||_{Z_*^m(0,T;\Omega)} = \max_{0 \leq j \leq m} \operatorname*{ess\,sup}_{0 \leq t \leq T} \|\partial_t^j u(t)\|_{m-j,*}.$$

Then $Z_*^m(0, T; \Omega)$ is a Banach space under this norm.

We define $\mathscr{H}^m(\Omega; P)$, $m \geq 0$, to be the space of functions such that

$$u \in H_*^m(\Omega), \qquad \tilde{P}u \in H_{**}^m(\Omega).$$

Here $\tilde{P} = \tilde{P}(x)$, $x \in \Omega$, is a smooth extension of $P = P(x)$, $x \in \Gamma$, that is, the orthogonal projection onto $\mathscr{N}(x)^\perp$ which will be described later in condition vi) of Theorem 2.1. We introduce a norm in $\mathscr{H}^m(\Omega; P)$ by

$$\|u\|_{\mathscr{H}^m(\Omega;P)}^2 = \|u\|_{m,*}^2 + \|\tilde{P}u\|_{m,**}^2.$$

$\mathscr{H}^m(\Omega; P)$ endowed with this norm is a Hilbert space. Different choice of $\tilde{P}$ yields an equivalent norm.

Let $W_*^m(0, T; \Omega)$, $m \geq 0$, be the space of functions such that

$$\partial_t^j u \in L^2(0, T; H_*^{m-j}(\Omega)), \qquad 0 \leq j \leq m.$$

If we define on this space a norm by

$$|||u|||_{W_*^m(0,T;\Omega)}^2 = \int_0^T |||u(t)|||_{m,*}^2 \, dt,$$

then $W_*^m(0, T; \Omega)$ is a Hilbert space under this norm. It is seen that, if $u \in W_*^m(0, T; \Omega)$, then we have

$$\partial_t^j u \in C([0, T]; H_*^{m-1-j}(\Omega)), \qquad 0 \leq j \leq m-1.$$

We define $V_*^m(0, T; \Omega)$, $m \geq 1$, to be the space of functions such that

$$u \in W_*^m(0, T; \Omega)$$

and

$$\partial_t^j u(0) \in H^{m-1-j}(\Omega), \qquad 0 \leq j \leq m-1.$$

By defining a norm on $V_*^m(0, T; \Omega)$ by

$$\|u\|_{V_*^m(0,T;\Omega)}^2 = \|u\|_{W_*^m(0,T;\Omega)}^2 + \sum_{j=0}^{m-1} \|\partial_t^j u(0)\|_{m-1-j}^2,$$

$V_*^m(0, T; \Omega)$ is a Hilbert space.

The above notations for function spaces will be used also for vector-valued function spaces.

Finally, when X and Y are Banach spaces, we denote by $\mathscr{L}(X, Y)$ the space of bounded linear operators from X into Y. If $X = Y$, we write simply $\mathscr{L}(X)$ instead of $\mathscr{L}(X, X)$.

§ 2. The existence and differentiability theorem

Before stating our main result, we recall two notions. One is the maximal nonnegativity of the boundary condition and the other is the compatibility condition. $\operatorname{Ker} M(x)$ is said to be a maximal nonnegative subspace of $A_\nu(v)$ if $A_{\nu(x)}(v(t, x))$ is positive semidefinite on $\operatorname{Ker} M(x)$ but not on any subspace containing $\operatorname{Ker} M(x)$ as a proper subspace for $(t, x) \in [0, T] \times \Gamma$. When $\operatorname{Ker} M(x)$ is maximal nonnegative, we say also that the boundary condition is maximal nonnegative. The compatibility condition of order $m-1$ is stated as follows. Given the system (0.1) and the initial data (0.2), we define f_p, $p \geq 1$, successively by formally taking derivatives of order up to $p-1$ of the system with respect to the time variable, solving for $\partial_t^p u$ and evaluating at $t = 0$. Thus f_p is written as a sum of the derivatives (with respect to the space variables) of f of order at most p and the derivatives (with respect to the space and the time variables) of F of order at most $p-1$. A concrete expression for f_p will be given in § 4. We set $f_0 = f$. Then the compatibility condition of order $m-1$ is that

$$M f_p = 0 \quad \text{on } \Gamma,\ 0 \leq p \leq m-1. \tag{2.1}$$

We shall write sometimes $\Delta_p(L(v); f, F)$ instead of f_p in this paper, since f_p is determined by $L(v)$, f, and F.

The main theorem of this paper is the following

Theorem 2.1. *Let* $m \geq 1$ *be an integer and let* $\mu = \max\left(m, 2\left[\dfrac{n}{2}\right] + 6\right)$. *Then the initial boundary value problem* (0.1), (0.2), (0.3) *has a unique solution* u *in* $X_*^m([0, T]; \Omega)$, *provided that the following conditions are satisfied*:

i) $\Omega \subset \mathbf{R}^n$ *is a bounded open set with boundary* Γ *of* C^∞*-class.*

ii) $M(x)$ *is a real matrix valued function of* C^∞*-class defined on* Γ *and* $\dim \operatorname{Ker} M(x)$ *is constant on* Γ.

iii) v *lies in* $X_*^\mu([0, T]; \Omega)$ *and takes values in* $\mathbf{R}^l$. *Furthermore,* $\partial_j^i v(0) \in H^{2\mu+2-i}(\Omega)$, $0 \leq i \leq \mu$.

iv) $v(t, x)$ *lies in* $\operatorname{Ker} M(x)$ *for* $(t, x) \in [0, T] \times \Gamma$.

v) $A_j(v(t, x))$, $j = 0, 1, \ldots, n$, *are real symmetric matrices for* $(t, x) \in [0, T] \times \bar{\Omega}$, *if* v *lies in* $C([0, T] \times \bar{\Omega})$ *and takes values in* $\mathbf{R}^l$. *In addition,* $A_0(v(t, x))$ *is positive definite for* $(t, x) \in [0, T] \times \bar{\Omega}$, *if* v *satisfies the same assumption.*

vi) *There exists a subspace* $\mathscr{N}(x)$ *of* $\mathbf{C}^l$, *defined for* $x \in \Gamma$, *such that we have* $\operatorname{Ker} A_{\nu(x)}(v(t, x)) = \mathscr{N}(x)$ *for* $(t, x) \in [0, T] \times \Gamma$ *if* v *lies in* $C([0, T] \times \bar{\Omega})$, *satisfies* iv), *and if it takes values in* $\mathbf{R}^l$. *Here* $\mathcal{N}(x)$ *is independent of* v.

vii) $\dim \mathcal{N}(x)$ *is constant on* Γ *and* $0 < \dim \mathcal{N}(x) < l$.

viii) $\operatorname{Ker} M(x)$ *is a maximal nonnegative subspace of* $A_{\nu(x)}(v(t, x))$ *for* $(t, x) \in [0, T] \times \Gamma$, *if* v *satisfies the same assumption as in* vi).

ix) $F \in W_*^m(0, T; \Omega)$, $\partial_t^i F(0) \in H^{m-1-i}(\Omega)$, $0 \le i \le m-1$, *and* $f \in H^m(\Omega)$.

x) *The data* f, F *satisfy the compatibility condition of order* $m-1$ *for the initial boundary value problem* (0.1), (0.2), (0.3).

The solution u *obeys the estimate*

$$\text{(2.2)} \qquad |||u(t)|||_{m,*} \le C(M_{\mu-1}^*, K_{\mu-1})\{\|f\|_m + |||F(0)|||_{m-1}\} e^{C(M_\mu^*)t} + C(M_\mu^*) \int_0^t e^{C(M_\mu^*)(t-\tau)} |||F(\tau)|||_{m,*} d\tau,$$

for $t \in [0, T]$, *where* $K_{\mu-1}$ *and* M_r^*, $r = \mu - 1, \mu$, *are constants such that* $|||v(0)|||_{\mu-1} \le K_{\mu-1}$ *and* $|||v|||_{X_*^r([0,T];\Omega)} \le M_r^*$, $r = \mu-1, \mu$, *respectively.* $C(\cdot)$ *and* $C(\cdot, \cdot)$ *are increasing functions of each single variable with positive values.*

Moreover, the solution u *has an extra regularity in the following sense. Let* $P = P(x)$, $x \in \Gamma$, *be the orthogonal projection onto* $\mathcal{N}(x)^\perp$ *and let* $\tilde{P} = \tilde{P}(x)$, $x \in \Omega$, *be an arbitrary smooth extension of* P. *Then* $\tilde{P}u$ *lies in* $X_{**}^m([0, T]; \Omega)$. $|||\tilde{P}u(t)|||_{m,**}$ *is bounded by the right hand side of* (2.2) *for* $t \in [0, T]$.

Remark 1. The case where $m = 1$ is covered essentially by the result of Rauch [20]. The function space used there is $H_{\tan}^m(\Omega)$. Since only the tangential derivatives are taken into account in this function space, we have in general a continuous imbedding $H_*^m(\Omega) \subsetneq H_{\tan}^m(\Omega)$. However, when $m = 1$, these function spaces coincide with each other. Namely, $H_*^1(\Omega) = H_{\tan}^1(\Omega)$. We refer the reader for the case $m = 0$ to Theorem 9 and for the case $m = 1$ to Theorem 10 in [20].

Remark 2. Condition ix) for the set of data f, F seems to be somewhat stringent. But, by a limit argument, we can obtain a more general condition for the data f, F leading to solutions in $X_*^m([0, T]; \Omega)$. In this connection, we point out that the necessary condition for the existence of the solution in $X_*^m([0, T]; \Omega)$ is that

$$f \in \mathscr{H}^m(\Omega; P), \quad f_p \in \mathscr{H}^{m-p}(\Omega; P), \qquad 1 \le p \le m,$$

and the compatibility condition of order $m-1$ is satisfied, provided that $F \in W_*^m(0, T; \Omega)$. The proof of this fact and a sufficient condition for the existence of the solution in $X_*^m([0, T]; \Omega)$ mentioned above will be given in a forthcoming paper.

Remark 3. Instead of condition vii), we may assume that $\dim \mathcal{N}(x)$ is constant on each component of Γ, although it is not identically zero on Γ. In this case, condition ii) may be weakened so that $\dim \operatorname{Ker} M(x)$ is constant on each component of Γ.

§ 3. Proof of the main theorem

First we shall show the existence of aproximate systems and approximate initial data for which the compatibility condition of order m is satisfied.

Lemma 3.1. *Let f, F, and v be as in Theorem 2.1. Then there exist sequences $\{f_k\}$, $\{F_k\}$, and $\{v_k\}$ having the following properties:*

i) $f_k \in H^{m+2}(\Omega)$, $k \geq 1$, *and* $f_k \to f$ *in* $H^m(\Omega)$.

ii) $F_k \in H^{m+2}([0, T] \times \Omega)$, $k \geq 1$, *and* $F_k \to F$ *in* $W_*^m(0, T; \Omega)$. *Furthermore,* $\partial_t^i F_k(0) \to \partial_t^i F(0)$ *in* $H^{m-1-i}(\Omega)$ *for* $0 \leq i \leq m-1$.

iii) $v_k \in X^{\mu+1}([0, T]; \Omega)$, $k \geq 1$, *and* $v_k \to v$ *in* $X_*^\mu([0, T]; \Omega)$, $\partial_t^i v_k(0) \to \partial_t^i v(0)$ *in* $H^{\mu-i}(\Omega)$ *for* $0 \leq i \leq \mu$. *In addition,* $v_k(t, x) \in \mathrm{Ker}\, M(x)$ *for* $(t, x) \in [0, T] \times \Gamma$, $k \geq 1$.

iv) *For the initial boundary value problem*

$$L(v_k)u = F_k \quad \text{in} \quad [0, T] \times \Omega, \tag{3.1}$$

$$Mu = 0 \quad \text{on} \quad [0, T] \times \Gamma, \tag{3.2}$$

$$u(0, x) = f_k(x) \quad \text{for} \quad x \in \Omega, \tag{3.3}$$

the data f_k and F_k satisfy the compatibility condition of order m, that is,

$$M\Delta_p(L(v_k); f_k, F_k) = 0 \quad \text{on} \quad \Gamma,\ 0 \leq p \leq m. \tag{3.4}$$

In order to show Lemma 3.1 we need the following Lemmas 3.1 A and 3.1 B.

Lemma 3.1 A. *Let f, F, and v be as in Theorem 2.1. Then there exist sequences $\{f_k\}$ and $\{F_k\}$ satisfying* i), ii) *of Lemma 3.1 and, furthermore, the compatibility condition of order m for the initial boundary value problem*

$$L(v)u = F_k \quad \text{in} \quad [0, T] \times \Omega, \tag{3.5}$$

$$Mu = 0 \quad \text{on} \quad [0, T] \times \Gamma, \tag{3.6}$$

$$u(0, x) = f_k(x) \quad \text{for} \quad x \in \Omega, \tag{3.7}$$

that is,

$$M\Delta_p(L(v); f_k, F_k) = 0 \quad \text{on} \quad \Gamma,\ 0 \leq p \leq m. \tag{3.8}$$

Lemma 3.1 B. *Let $v \in X_*^s([0, T]; \Omega)$ and let $\partial_t^i v(0) \in H^{2s+2-i}(\Omega)$, $0 \leq i \leq s$, where s is an integer such that $2s \geq \left[\frac{n}{2}\right]$. Let furthermore $v(t, x) \in \mathrm{Ker}\, M(x)$ for $(t, x) \in [0, T] \times \Gamma$. Then there exists a sequence $\{v_k\}$ having the following properties:*

i) $v_k \in X^{2s+2}([0, T]; \Omega)$, $k \geq 1$.

ii) $v_k \to v$ *in* $X_*^s([0, T]; \Omega)$.

iii) $\partial_t^i v_k(0) \to \partial_t^i v(0)$ *in* $H^{s-i}(\Omega)$ *for* $0 \le i \le s$.
iv) $v_k(t, x) \in \operatorname{Ker} M(x)$ *for* $(t, x) \in [0, T] \times \Gamma$, $k \ge 1$.

Assuming for the moment that these lemmas are true, we complete the proof of Lemma 3.1.

Proof of Lemma 3.1. By Lemma 3.1A, there exist sequences $\{f_k\}$ and $\{F_k\}$ satisfying i), ii) of Lemma 3.1 and (3.8). By means of these sequences $\{f_k\}$ and $\{F_k\}$, we construct a sequence $\{U_k\} \subset X^{m+2}([0, T]; \Omega)$ satisfying

$$\partial_t^p U_k(0) = \Delta_p(L(v); f_k, F_k) \quad \text{in } \Omega,\ 0 \le p \le m. \tag{3.9}$$

Set $h_{p,k} = \Delta_p(L(v); f_k, F_k)$, $0 \le p \le m$, $k \ge 1$. Since $\partial_t^i v(0) \in H^{2\mu+2-i}(\Omega)$, $0 \le i \le \mu$, $f_k \in H^{m+2}(\Omega)$, $k \ge 1$, and since $\partial_t^i F_k(0) \in H^{m+1-i}(\Omega)$, $0 \le i \le m-1$, $k \ge 1$, it is shown by Lemma C.1 i) and Lemma C.3 in Appendix C that $h_{p,k} \in H^{m+2-p}(\Omega)$. Let $\tilde{h}_{p,k} \in H^{m+2-p}(\mathbf{R}^n)$ be an extension of $h_{p,k}$ so that there exists a constant C such that $\|\tilde{h}_{p,k}\|_{H^{m+2-p}(\mathbf{R}^n)} \le C\|h_{p,k}\|_{m+2-p}$. Now we use an argument given in [12], pp. 31–32. Let L_0 be a scalar, strictly hyperbolic operator of order $m+1$ with constant coefficients. Let us consider the following Cauchy problem,

$$L_0 \tilde{U}_k = 0 \quad \text{in } [0, T] \times \mathbf{R}^n,$$
$$\partial_t^p \tilde{U}_{k(0)} = \tilde{h}_{p,k} \quad \text{in } \mathbf{R}^n,\ 0 \le p \le m,$$

where the unknown $\tilde{U}_k$ is a vector-valued function with l components. The standard existence theorem shows that there exists a unique solution $\tilde{U}_k \in X^{m+2}([0, T]; \mathbf{R}^n)$ of this Cauchy problem satisfying the usual energy estimate. Then the desired sequence $\{U_k\} \subset X^{m+2}([0, T]; \Omega)$ is given by setting $U_k = \tilde{U}_k|_{[0,T]\times\Omega}$. We have

$$|||U^k|||_{X^{m+2}([0,T];\Omega)} \le C \sum_{p=0}^{m} \|h_{p,k}\|_{m+2-p}. \tag{3.10}$$

Let C_k be a positive constant such that

$$|||U_k|||_{X^{m+1}([0,T];\Omega)} \le C_k. \tag{3.11}$$

We assume in what follows that $C_k \to \infty$, because in general the left hand side of (3.11) is not uniformly bounded in k.

By Lemma 3.1B with $s = \mu$, there exists a sequence $\{v_k\}$ such that

$$\begin{cases} v_k \in X^{2\mu+2}([0, T]; \Omega), & k \ge 1, \\ v_k \to v \text{ in } X_*^{\mu}([0, T]; \Omega), \\ \partial_t^i v_k(0) \to \partial_t^i v(0) \text{ in } H^{\mu-i}(\Omega),\ 0 \le i \le \mu, \\ v_k(t, x) \in \operatorname{Ker} M(x) \text{ for } (t, x) \in [0, T] \times \Gamma,\ k \ge 1. \end{cases} \tag{3.12}$$

Lemma A.3 in Appendix A combined with Lemma C.4 in Appendix C guarantees the existence of the subsequence $\{v_{k_l}\}$ such that

$$(3.13)\quad \begin{cases} |||A_j(v_{k_l}) - A_j(v)|||_{X_*^{\mu}([0,T];\Omega)} \le \dfrac{1}{C_{k_l}^2}, \; 0 \le j \le n, \\ |||B(v_{k_l}) - B(v)|||_{X_*^{\mu}([0,T];\Omega)} \le \dfrac{1}{C_{k_l}^2}, \\ \|\partial_t^i A_j(v_{k_l})(0) - \partial_t^i A_j(v)(0)\|_{\mu-i} \le \dfrac{1}{C_{k_l}^2}, \; 0 \le j \le n, \; 0 \le i \le m-1, \\ \|\partial_t^i B(v_{k_l})(0) - \partial_t^i B(v)(0)\|_{\mu-i} \le \dfrac{1}{C_{k_l}^2}, \; 0 \le i \le m-1. \end{cases}$$

We denote this subsequence $\{v_{k_l}\}$ again by $\{v_k\}$ by abuse of notation. Now let us consider the initial boundary value problem

$$(3.14)\qquad L(v_k)u = F_k' \quad \text{in} \quad [0, T] \times \Omega,$$

$$(3.15)\qquad Mu = 0 \quad \text{on} \quad [0, T] \times \Gamma,$$

$$(3.16)\qquad u(0, x) = f_k(x) \quad \text{for} \quad x \in \Omega,$$

where

$$F_k' = F_k + (A_0(v_k) - A_0(v))\partial_t U_k + \sum_{j=1}^{n} (A_j(v_k) - A_j(v))\partial_j U_k + (B(v_k) - B(v))U_k.$$

Recalling the definitions of f_k, F_k, v_k, U_k, and condition iii) of Theorem 2.1, we see by (3.9) and (3.12) that

$$(3.17)\qquad F_k' \in X_*^m([0, T]; \Omega),\ k \ge 1, \text{ and } \partial_t^i F_k'(0) \in H^{m-i}(\Omega), \quad 0 \le i \le m,\ k \ge 1.$$

By Lemma A.1, Lemma C.1 i), (3.11), and (3.13), it holds that

$$(3.18)\qquad \begin{cases} F_k' \to F \text{ in } W_*^m(0, T; \Omega), \\ \partial_t^i F_k'(0) \to \partial_t^i F(0) \text{ in } H^{m-1-i}(\Omega) \text{ as } k \to \infty, 0 \le i \le m-1. \end{cases}$$

Making use of (3.9), we have

$$(3.19)\qquad \Delta_p(L(v_k); f_k, F_k') = \Delta_p(L(v); f_k, F_k) \text{ in } \Omega,\ 0 \le p \le m.$$

Utilizing (3.19) and (3.8), we obtain

$$(3.20)\qquad M\Delta_p(L(v_k); f_k, F_k') = 0 \quad \text{on } \Gamma,\ 0 \le p \le m.$$

By (3.12), (3.17), and (3.20), f_k, F_k' and v_k satisfy the assumption of Lemma 3.1 A. Hence for any k there exist sequences $\{f_{k,l}\}$ and $\{F_{k,l}\}$ having the following properties:

$$(3.21)\quad \begin{cases} f_{k,l}\in H^{m+2}(\Omega),\ k,\ l\geq 1\\ f_{k,l}\to f_k \text{ in } H^m(\Omega) \text{ as } l\to\infty,\ k\geq 1,\\ F_{k,l}\in H^{m+2}([0,\ T]\times\Omega),\ k,\ l\geq 1,\\ F_{k,l}\to F'_k \text{ in } W^m_*(0,\ T;\ \Omega) \text{ as } l\to\infty,\\ \partial^i_t F_{k,l}(0)\to\partial^i_t F'_k(0) \text{ in } H^{m-1-i}(\Omega) \text{ as } l\to\infty,\ 0\leq i\leq m-1,\ k\geq 1,\\ M\Delta_p(L(v_k);\ f_{k,l},\ F_{k,l})=0 \text{ on } \Gamma,\ k,\ l\geq 1,\ 0\leq p\leq m. \end{cases}$$

We choose a suitable l for each k, say $l(k)$, so that $\{f_{k,l(k)}\}$ and $\{F_{k,l(k)}\}$ are desired sequences. This completes the proof of Lemma 3.1.

The proof of Lemma 3.1A will be given in §4. Now we give a proof of Lemma 3.1B.

Proof of Lemma 3.1B. We construct a sequence $\{w_k\}$ having the following properties:

$$(3.22)\quad \begin{cases} w_k\in C^\infty([0,\ T];\ H^s_*(\Omega)),\ k\geq 1,\\ w_k\to v \text{ in } X^s_*([0,\ T];\ \Omega) \text{ as } k\to\infty,\\ \partial^i_t w_k(0)\in H^{2s+1}(\Omega),\ 0\leq i\leq s,\ k\geq 1,\\ \partial^i_t w_k(0)\to\partial^i_t v(0) \text{ in } H^{2s+1-i}(\Omega) \text{ as } k\to\infty,\ 0\leq i\leq s,\\ w_k(t,\ x)\in \operatorname{Ker} M(x) \text{ for } (t,\ x)\in[0,\ T]\times\Gamma,\ k\geq 1. \end{cases}$$

To this end, we choose $V\in X^{2s+1}((-\infty,0];\ \Omega)$ such that $\partial^i_t V(0)=\partial^i_t v(0)$, $0\leq i\leq s$, $V(t,\ x)\in \operatorname{Ker} M(x)$ for $(t,\ x)\in(-\infty,\ 0]\times\Gamma$, and set

$$\tilde{v}(t,\ x)=\begin{cases} v(t,\ x) & \text{in } [0,\ T]\times\Omega,\\ V(t,\ x) & \text{in } (-\infty,\ 0]\times\Omega. \end{cases}$$

We can construct such a function V, if we use Lemma 3.1C after replacing s by $2s$ and setting $g_i=\partial^i_t v(0)$, $0\leq i\leq s$, $g_i=0$, $s<i\leq 2s-1$. We have $\tilde{v}\in X^s_*((-\infty,\ T];\ \Omega)$ and $\tilde{v}\in X^{2s+1}((-\infty,\ 0];\ \Omega)$.

Let ρ be in $C^\infty(\mathbf{R})$ and let the support of ρ be contained in $[0,\ 1]$. Assume that $\int\rho(t)dt=1$ and that $\rho(t)\geq 0$. Set

$$w_k(t,\ x)=(\rho_{1/k}\mathop{*}_t\tilde{v})(t,\ x)$$

where $\rho_{1/k}(t)=k\rho(kt)$. Then we find that $\{w_k\}$ is the desired sequence.

It is easy to see that there exists a sequence $\{w_{k,l}\}$ such that

$$(3.23)\quad \begin{cases} w_{k,l}\in C^\infty([0,\ T];\ H^p(\Omega)),\ p\geq s,\ k,\ l\geq 1,\\ w_{k,l}\to w_k \text{ in } C^q([0,\ T];\ H^s_*(\Omega)) \text{ as } l\to\infty,\ q\geq 1,\ k\geq 1,\\ \partial^i_t w_{k,l}(0)\to\partial^i_t w_k(0) \text{ in } H^{2s+1}(\Omega) \text{ as } l\to\infty,\ 0\leq i\leq s,\ k\geq 1. \end{cases}$$

Such a sequence can be constructed by using a mollifier in the x variable mentioned in the proof of Lemma B.3 in Appendix B. We define a sequence $v_{k,l}$ by

$$v_{k,l}(t) = w_{k,l}(t) - R_{4s+4}(P_{(\mathrm{Ker}\,M)^\perp}\gamma w_{k,l}(t), \underbrace{0,\dots,0}_{2s+1\text{ times}}), \ t\in[0, T], \tag{3.24}$$

where $P_{(\mathrm{Ker}\,M)^\perp}$ is the orthogonal projection onto $(\mathrm{Ker}\,M(x))^\perp$ for $x\in\Gamma$ and γ is the trace operator on Γ, and where R_{4s+4} denotes the operator from $\prod_{j=0}^{2s} H^{4s+3-2j}(\Gamma)$ to $H_*^{4s+4}(\Omega)$, that was defined in Lemma B.2 i), ii) in Appendix B. By using (3.23), it is seen that

(3.25)
$$\begin{cases} \gamma\partial_t^i w_{k,l}(t) \to \gamma\partial_t^i w_k(t) \text{ in } H^{s-1}(\Gamma) \text{ as } l\to\infty \text{ uniformly on } [0, T],\ i\ge 0,\ k\ge 1, \\ \gamma\partial_t^i w_{k,l}(0) \to \gamma\partial_t^i w_k(0) \text{ in } H^{2s}(\Gamma) \text{ as } l\to\infty,\ 0\le i\le s,\ k\ge 1. \end{cases}$$

Furthermore, by Lemma B.2 ii), we have

$$\begin{aligned} &\partial_t^i R_{4s+4}(P_{(\mathrm{Ker}\,M)^\perp}\gamma w_{k,l}(t), \underbrace{0,\dots,0}_{2s+1\text{ times}}) \\ &= \partial_t^i R_{4s+4,s}(P_{(\mathrm{Ker}\,M)^\perp}\gamma w_{k,l}(t), \underbrace{0,\dots,0}_{[\frac{s}{2}]-1\text{ times}}) \\ &= R_{4s+4,s}(P_{(\mathrm{Ker}\,M)^\perp}\gamma\partial_t^i w_{k,l}(t), \underbrace{0,\dots,0}_{[\frac{s}{2}]-1\text{ times}}) \to 0 \text{ in } H_*^s(\Omega) \text{ as } l\to\infty \\ &\qquad\qquad \text{uniformly on } [0, T],\ i\ge 0,\ k\ge 1, \end{aligned}$$

because $P_{(\mathrm{Ker}\,M)^\perp}\gamma\partial_t^i w_{k,l}(t) \to P_{(\mathrm{Ker}\,M)^\perp}\gamma\partial_t^i w_k(t) = 0$ in $H^{s-1}(\Gamma)$ as $l\to\infty$ uniformly on $[0, T]$ for $i\ge 0$, $k\ge 1$. Recall that, by the last property of (3.22), we have $\partial_t^i w_k(t, x)\in \mathrm{Ker}\,M(x)$ for $(t, x)\in[0, T]\times\Gamma$ and $k\ge 1$, $i\ge 0$. Similarly,

$$\begin{aligned} &\partial_t^i R_{4s+4}(P_{(\mathrm{Ker}\,M)^\perp}\gamma w_{k,l}(t), \underbrace{0,\dots,0}_{2s+1\text{ times}})|_{t=0} \\ &= \partial_t^i R_{4s+4,2s+1}(P_{(\mathrm{Ker}\,M)^\perp}\gamma w_{k,l}(t), \underbrace{0,\dots,0}_{s-1\text{ times}})|_{t=0} \\ &= R_{4s+4,2s+1}(P_{(\mathrm{Ker}\,M)^\perp}\gamma\partial_t^i w_{k,l}(0), \underbrace{0,\dots,0}_{s-1\text{ times}}) \to 0 \text{ in } H_*^{2s}(\Omega) \\ &\qquad\qquad \text{as } l\to\infty,\ 0\le i\le s,\ k\ge 1, \end{aligned}$$

because $P_{(\mathrm{Ker}\,M)^\perp}\gamma\partial_t^i w_{k,l}(0) \to P_{(\mathrm{Ker}\,M)^\perp}\gamma\partial_t^i w_k(0) = 0$ in $H^{2s}(\Gamma)$ as $l\to\infty$ for $0\le i\le s$, $k\ge 1$. Here we used again the last property of (3.22). These observations in conjunction with the properties (3.23) yield

$$\begin{cases} v_{k,l}\in X_*^{4s+4}([0, T]; \Omega),\ k, l\ge 1, \\ v_{k,l}\to w_k \text{ in } C^q([0, T]; H_*^s(\Omega)) \text{ as } l\to\infty,\ q\ge 1,\ k\ge 1, \\ \partial_t^i v_{k,l}(0)\to\partial_t^i w_k(0) \text{ in } H_*^{2s}(\Omega)\subset H^s(\Omega) \text{ as } l\to\infty,\ 0\le i\le s,\ k\ge 1, \\ v_{k,l}(t, x)\in\mathrm{Ker}\,M(x) \text{ for } (t, x)\in(0, T]\times\Gamma,\ k, l\ge 1. \end{cases} \tag{3.26}$$

We choose a suitable subsequence of l, say $l(k)$, so that $v_{k,l(k)}$ has the following properties:

$$(3.27)\qquad \begin{aligned}&v_{k,l(k)}\in X^{2s+2}([0, T]; \Omega),\ k\geq 1,\\ &v_{k,l(k)}\to v \text{ in } X^s_*([0, T]; \Omega) \text{ as } k\to\infty,\\ &\partial_t^i v_{k,l(k)}(0)\to\partial_t^i v(0) \text{ in } H^s(\Omega) \text{ as } k\to\infty,\ 0\leq i\leq s,\\ &v_{k,l(k)}(t, x)\in \operatorname{Ker} M(x) \text{ for } (t, x)\in[0, T]\times\Gamma,\ k\geq 1.\end{aligned}$$

This is seen by combining (3.26) with (3.22). The proof of Lemma 3.1B is complete.

Lemma 3.1C. *Let* $g_i\in H^{s+2-i}(\Omega)$, $0\leq i\leq s-1$, *where* $s\geq\left[\frac{n}{2}\right]$ *is an integer. Assume that* $g_i(x)\in\operatorname{Ker} M(x)$ *for* $x\in\Gamma$, $0\leq i\leq s-1$. *Then there exists* $V\in X^{s+1}((-\infty, 0]; \Omega)$ *such that* $\partial_t^i V(0)=g_i$, $0\leq i\leq s-1$, *and* $V(t, x)\in\operatorname{Ker} M(x)$ *for* $(t, x)\in(-\infty, 0]\times\Gamma$.

Proof. We consider the following initial boundary value problem.

$$(3.28)\qquad \partial_t U+(A_\nu(g_0)+\varepsilon I)\partial_\nu U=G \quad \text{in } [0, T]\times\Omega,$$

$$(3.29)\qquad MU=0 \quad \text{on } [0, T]\times\Gamma,$$

$$(3.30)\qquad U(0, x)=g_0(x) \quad \text{for } x\in\Omega.$$

Here ν is a smooth vector field on $\bar{\Omega}$, which extends the outward unit normal vector to the boundary Γ. The matrices A_j, $j=1,\dots,n$, and M are those which appear in the original initial boundary value problem (0.1), (0.2), (0.3). Recall that $A_\nu=\sum_{j=1}^n \nu_j A_j$, $\partial_\nu=\sum_{j=1}^n \nu_j\partial_j$, and that U is the unknown function. We impose the following condition on G,

$$(3.31)\qquad \begin{cases} G\in X^{s+1}([0, T]; \Omega),\\ \partial_t^i G(0)=g_{i+1}+A_\nu^\varepsilon(g_0)\partial_\nu g_i,\ 0\leq i\leq s-1,\end{cases}$$

where $A_\nu^\varepsilon(g_0)=A_\nu(g_0)+\varepsilon I$. We assume for the moment that such a G exists. We claim that for $\varepsilon>0$ small,

i) the boundary matrix $A_\nu^\varepsilon(g_0)$ is nonsingular on Γ,

ii) $\operatorname{Ker} M(x)$ is a maximal nonnegative subspace of $A_{\nu(x)}^\varepsilon(g_0(x))$ for $x\in\Gamma$,

iii) the data g_0, G satisfy the compatibility condition of $s-1$ for the initial boundary value problem (3.28), (3.29), (3.30).

The properties i), ii) are checked easily. We show the property iii). It is proved by induction on i that

$$(3.32)\qquad \mathring{\partial}_t^i U(0)=g_i \quad \text{in } \Omega,\ 0\leq i\leq s-1.$$

The left hand side of (3.32) denotes $\Delta_i(L_0, g_0, G)$ in the notation introduced in §2, where $L_0=\partial_t+(A_\nu(g_0)+\varepsilon I)\partial_\nu$. Obviously, (3.32) holds by definition when $i=0$. If (3.32) is valid for $i=k$, then we have

$$\mathring{\partial}_t^{k+1}U(0)=\partial_t^k G(0)-A_\nu^\varepsilon(g_0)\partial_\nu\mathring{\partial}_t^k U(0)$$

$$= g_{k+1} + A_\nu^\varepsilon(g_0)\partial_\nu g_k - A_\nu^\varepsilon(g_0)\partial_\nu g_k$$
$$= g_{k+1}.$$

This proves (3.32) for $i = k + 1$. It follows that

$$M\mathring{\partial}_t^i U(0) = Mg_i = 0 \qquad \text{in } \Omega,\ 0 \le i \le s-1.$$

Therefore the compatibility condition of order $s - 1$ is satisfied for the initial boundary value problem (3.28), (3.29), (3.30). We use here Theorem A.1 in [22] and conclude the existence of the solution $U \in X^{s+1}([0, T]; \Omega)$. It is seen by (3.32) that

$$\partial_t^i U(0) = g_i \qquad \text{in } \Omega,\ 0 \le i \le s-1. \tag{3.33}$$

Let χ be a smooth function defined on $[0, \infty)$ with support contained in $[0, T]$ and let $\chi(t) = 1$ for t near 0. Set

$$V(t, x) = \begin{cases} \chi(t)U(-t, x) & \text{for } -T \le t \le 0, \\ 0 & \text{for } t \le -T. \end{cases}$$

Then V has the desired properties.

Finally we construct the inhomogeneous term G. By definition $g_{i+1} \in H^{s+1-i}(\Omega)$, $0 \le i \le s-2$. Since $g_0 \in H^{s+2}(\Omega)$, we have by Lemma C.3 $A_\nu^\varepsilon(g_0) \in H^{s+2}(\Omega)$. We observe that

$$\min\left\{s+2,\ s+1-i,\ (s+2)+(s+1-i)-\left(\left[\frac{n}{2}\right]+1\right)\right\} \ge s+1-i.$$

Then by Lemma C.1 i), it is seen that $A_\nu^\varepsilon(g_0)\partial_\nu g_i \in H^{s+1-i}(\Omega)$, $0 \le i \le s-1$. Therefore

$$\partial_t^i G(0) \in H^{s+1-i}(\Omega), \qquad 0 \le i \le s-2.$$

Using the same method as in the proof of Lemma 3.1, we obtain $G \in X^{s+1}([0, T]; \Omega)$ that satisfies the condition (3.31). This completes the proof of Lemma 3.1C.

Remark. Let $C_b((-\infty, 0]; Y)$ be the space of continuous and bounded functions defined on $(-\infty, 0]$ taking values in a Banach space Y. Then $X^s((-\infty, 0]; \Omega)$ denotes the space of functions such that

$$\partial_t^j u \in C_b((-\infty, 0]; H^{s-j}(\Omega)), \qquad 0 \le j \le s.$$

The norm is

$$|||u|||_{X^s((-\infty,0];\Omega)} = \sup_{t \le 0} |||u(t)|||_s.$$

Similarly, $X_*^s((-\infty, 0]; \Omega))$ is defined by replacing $H^{s-j}(\Omega)$ by $H_*^{s-j}(\Omega)$.

To prove the main Theorem, we proceed as follows. Let $\{f_k\}$, $\{F_k\}$, and $\{v_k\}$ be the sequences whose existence is guaranteed by Lemma 3.1. Let

$U_k \in X^{m+2}([0, T]; \Omega)$ satisfy $\partial_t^p U_k(0) = \Delta_p(L(v_k); f_k, F_k)$ in Ω, $0 \le p \le m$. Such a sequence $\{U_k\}$ can be found by the same argument as in the first part of the proof of Lemma 3.1. Let C_k be a constant such that $|||U_k|||_{X^{m+1}([0,T];\Omega)} \le C_k$. Without loss of generality, we may assume that $C_k \to \infty$. We define a smooth function v on $\bar{\Omega}$ as follows. For x in a suitable neighborhood of Γ, we put $v = v(x'(x))$, where $x'(x)$ denotes the point on the boundary Γ nearest to x. For $x \in \bar{\Omega}$ not belonging to this neighborhood, $v(x)$ may be chosen arbitrarily. Let

$$L(v_k; v, k) = A_0(v_k)\partial_t + \sum_{j=1}^{n} \left(A_j(v_k) + \frac{v_j}{C_k^2} \right) \partial_j + B(v_k).$$

We consider the initial boundary value problem

$$L(v_k; v, k)u = F_k'' \quad \text{in} \quad [0, T] \times \Omega, \tag{3.34}$$

$$Mu = 0 \quad \text{on} \quad [0, T] \times \Gamma, \tag{3.35}$$

$$u(0, x) = f_k(x) \quad \text{for} \quad x \in \Omega, \tag{3.36}$$

where

$$F_k'' = F_k + \frac{1}{C_k^2} \sum_{j=1}^{n} v_j \partial_j U_k.$$

For this initial boundary value problem we prove the following lemma.

Lemma 3.2. *Let k be a sufficiently large integer. Then we have*

i) *The boundary Γ is non-characteristic for the system* (3.34).

ii) *The boundary subspace, that is,* Ker $M(x)$ *is still maximal nonnegative on* $[0, T] \times \Gamma$ *for the system* (3.34).

iii) *For the initial boundary value problem* (3.34), (3.35), (3.36), *the data f_k and F_k'' satisfy the compatibility condition of order m.*

Proof. The statement i) is shown by straightforward calculations. The proof of ii) proceeds along the line of [22], pp. 67–68. The statement iii) readily follows from the definition of U_k and Lemma 3.1 iv).

These observations lead us to the following result.

Proposition 3.3. *Let $m \ge 1$ be an integer. Then the initial boundary value problem* (3.34), (3.35), (3.36) *has a unique solution u^k in $X^{m+1}([0, T]; \Omega)$, which obeys the estimate*

$$\begin{aligned} |||u^k(t)|||_{m,*} \le\ & C(M_{[\frac{n}{2}]+2})\, |||u^k(0)|||_{m,*}\, e^{C(M_\mu^*)t} \\ & + \frac{1}{C_k^2} |||U_k|||_{X^{m+1}([0,T];\Omega)} e^{C(M_\mu^*)t} \\ & + C(M_\mu^*) \int_0^t e^{C(M_\mu^*)(t-\tau)} |||F_k(\tau)|||_{m,*}\, d\tau, \end{aligned} \tag{3.37}$$

for $t \in [0, T]$, *where* $M_{[\frac{n}{2}]+2}$ *and* M_μ^* *are constants independent of* k *such that* $|||v_k|||_{X^{[\frac{n}{2}]+2}([0,T];\Omega)} \le M_{[\frac{n}{2}]+2}$ *and* $|||v_k|||_{X_*^\mu([0,T];\Omega)} \le M_\mu^*$ *for* $k \ge 1$, *respectively, and where* $C(\cdot)$ *is an increasing function of its argument with positive values.*

Moreover, $\tilde{P}u^k$ *lies in* $X^m_{**}([0, T]; \Omega)$, *where* $\tilde{P}$ *is the smooth matrix valued function on* Ω *defined in Theorem* 2.1. *The following estimate holds for* $t \in [0, T]$.

$$
\begin{aligned}
(3.38) \qquad & |||\tilde{P}u^k(t)|||_{m,**} \\
&\le C(M^*_{\mu-1})\{|||u^k(0)|||_{m,*} + |||F_k(0)|||_{m-1,*}\}e^{C(M^*_\mu)t} \\
&\quad + \frac{C(M^*_{\mu-1})}{C_k^2}|||U_k|||_{X^{m+1}([0,T];\Omega)}e^{C(M^*_\mu)t} + C(M^*_\mu)\int_0^t e^{C(M^*_\mu)(t-\tau)}|||F_k(\tau)|||_{m,*}d\tau.
\end{aligned}
$$

Proof. The existence of the solution $u^k \in X^{m+1}([0, T]; \Omega)$ is shown by applying Theorem A.1 in [22], which is the existence theorem for the initial boundary value problems with non-characteristic boundary. (See also [3].) Note that we need Lemma 3.2 to apply the theorem to our situation. The estimates (3.37), (3.38) will be proved later in §6.

The existence of solutions stated in Theorem 2.1 is proved in several steps. We prepare for this purpose the following propositions and lemmas. We observe from (3.37) and the definitions of f_k, F_k, and U_k that $|||u^k|||_{X_*^m([0,T];\Omega)}$ is bounded by a constant independent of k, that is,

$$\sup_k |||u^k|||_{X_*^m([0,T];\Omega)} < \infty.$$

Therefore, $\{u^k\}$ is contained in a ball in $Z_*^m(0, T; \Omega)$ centered at the origin. Then, by a weak* compactness argument, we can choose a subsequence $\{u^{k_i}\}$ such that, for any $0 \le j \le m$, $\{\partial_t^j u^{k_i}\}$ converges in the weak* topology of $L^\infty(0, T; H_*^{m-j}(\Omega))$ as $i \to \infty$. Let u be the limit of $\{u^{k_i}\}$. Then the limit of $\{\partial_t^j u^{k_i}\}$ is $\partial_t^j u$ for $1 \le j \le m$. This is seen by the fact that the limit of $\{\partial_t^j u^{k_i}\}$ in the distribution sense equals $\partial_t^j u$. We denote this subsequence still by $\{u^k\}$ in the following.

Lemma 3.4. *Let* $m \ge 1$. *Let* $\{u^k(t)\}$ *be the subsequence described above. Then* $\{u^k(t)\}$ *converges as* $k \to \infty$ *in the weak topology of* $H_*^m(\Omega)$ *for any* $t \in [0, T]$. *The limit* $u(t)$ *coincides with the limit of* $\{u^k(t)\}$ *in the weak* topology of* $L^\infty(0, T; H_*^m(\Omega))$ *for a.e.* $t \in [0, T]$.

Proof. Let $\varepsilon > 0$ and let ϕ of $L^2(\Omega)$. We fix $t \in [0, T/2]$ and choose $\delta > 0$ small enough. By the assumption $\{u^k\}$ converges as $k \to \infty$ in the weak* topology of $L^\infty(0, \infty; H_*^m(\Omega))$. Since $(1/\delta)\chi_{[t,t+\delta]}\phi \in L^1(0, \infty; L^2(\Omega))$, we have

$$(3.39) \qquad \left|\frac{1}{\delta}\int_t^{t+\delta}(\phi, u^k(s))ds - \frac{1}{\delta}\int_t^{t+\delta}(\phi, u^l(s))ds\right| < \varepsilon \qquad \text{for } k, l \ge N,$$

where N is an integer depending on ε, δ, t, and ϕ. The left hand side of (3.39) is rewritten as

$$\left|(\phi, u^k(t)) - (\phi, u^l(t)) + \frac{1}{\delta}\int_t^{t+\delta}(\phi, u^k(s) - u^k(t))ds - \frac{1}{\delta}\int_t^{t+\delta}(\phi, u^l(s) - u^l(t))ds\right|.$$

Since

$$u^k(s) - u^k(t) = \int_t^s \frac{\partial}{\partial\tau}u^k(\tau)d\tau,$$

we have

$$\|u^k(s) - u^k(t)\| \leq M(s-t),$$

where

$$M = \sup_k |||u^k|||_{X_*^m([0,T];\Omega)}.$$

Similarly,

$$\|u^l(s) - u^l(t)\| \leq M(s-t).$$

Hence, we see that

$$|(\phi, u^k(t)) - (\phi, u^l(t))| - \frac{\delta}{2}M\|\phi\| - \frac{\delta}{2}M\|\phi\| < \varepsilon \qquad \text{for } k, l \geq N.$$

Let us choose δ such that $\delta M\|\phi\| < \varepsilon$. Then we have

$$|(\phi, u^k(t)) - (\phi, u^l(t))| < 2\varepsilon \qquad \text{for } k, l \geq N,$$

where N is an integer depending on ε, t, and ϕ. Replacing $\chi_{[t,t+\delta]}$ by $\chi_{[t-\delta,t]}$, we repeat the same argument as above for $t \in [T/2, T]$. Thus we see that $\{u^k(t)\}$ converges weakly in $L^2(\Omega)$ for any $t \in [0, T]$. On the other hand, we have a uniform estimate for $u^k(t)$, that is,

$$\sup_k \|u^k(t)\|_{m,*} \leq M.$$

It follows from these observations that $\{u^k(t)\}$ converges weakly in $H_*^m(\Omega)$ for every $t \in [0, T]$. The last assertion in the lemma is easily seen by the uniqueness of the limit. This completes the proof of Lemma 3.4.

Proposition 3.5. *Let $m \geq 1$. Then the limit u of the subsequence $\{u^k\}$ which lies in $Z_*^m(0, T; \Omega)$ satisfies* (0.1), (0.2), (0.3).

Proof. Let

$$M = \sup_k |||u^k|||_{X_*^m([0,T];\Omega)}.$$

We recall that each of the u^k's satisfies (3.34). Then we obtain for any $\phi \in C_0^1((0, T) \times \Omega)$

$$(3.40)\qquad \langle A_0(v_k)\partial_t u^k, \phi\rangle + \sum_{j=1}^{n} \langle A_j(v_k)\partial_j u^k, \phi\rangle + \frac{1}{C_k^2}\sum_{j=1}^{n} \langle v_j\partial_j u^k, \phi\rangle + \langle B(v_k)u^k, \phi\rangle = \langle F_k, \phi\rangle + \frac{1}{C_k^2}\sum_{j=1}^{n} \langle v_j\partial_j U_k, \phi\rangle$$

where

$$\langle f, g\rangle = \int_0^T \int_\Omega f\cdot \bar{g}\, dx dt.$$

Integrating the above by parts, we have

$$(3.41)\qquad -\langle A_0(v_k)u^k, \partial_t\phi\rangle - \langle(\partial_t A_0(v_k))u^k, \phi\rangle - \sum_{j=1}^{n} \langle A_j(v_k)u^k, \partial_j\phi\rangle - \sum_{j=1}^{n} \langle(\partial_j A_j(v_k))u^k, \phi\rangle - \frac{1}{C_k^2}\sum_{j=1}^{n} \langle u^k, \partial_j(v_j\phi)\rangle + \langle B(v_k)u^k, \phi\rangle = \langle F_k, \phi\rangle + \frac{1}{C_k^2}\sum_{j=1}^{n} \langle v_j\partial_j U_k, \phi\rangle.$$

The convergence of the first term on the left hand side of (3.41) is seen as follows. We have

$$(3.42)\qquad |\langle A_0(v_k)u^k - A_0(v)u, \partial_t\phi\rangle| \le |\langle(A_0(v_k) - A_0(v))u^k, \partial_t\phi\rangle| + |\langle A_0(v)(u^k - u), \partial_t\phi\rangle|.$$

The first term on the right hand side of (3.42) is bounded by

$$C \,|||\, A_0(v_k) - A_0(v) \,|||_{X_*^{2[\frac{n}{2}]+2}([0,T];\Omega)} \sup_k \max_{0\le t\le T} \| u^k(t)\| \max_{0\le t\le T} \|\partial_t\phi(t)\|,$$

which in turn is estimated by

$$CKM \,|||\, v_k - v \,|||_{X_*^{2[\frac{n}{2}]+2}([0,T];\Omega)} \max_{0\le t\le T} \|\partial_t\phi(t)\|$$

by using Lemma A.3. Here K is a constant depending on $\sup_k |||\, v_k \,|||_{X_*^{2[\frac{n}{2}]+2}([0,T];\Omega)}$.

This shows that the first term on the right hand side of (3.42) becomes smaller than arbitrarily given $\varepsilon > 0$ for sufficiently large k. The second term on the right hand side of (3.42) is rewritten as

(3.43) $$\left| \int_0^T (u^k(t) - u(t), A_0(v(t))^* \partial_t \phi(t)) dt \right|.$$

It is easily seen that $A_0(v(t))^* \partial_t \phi \in C([0, T]; L^2(\Omega))$. Hence

$$\max_{0 \le t \le T} \| A_0(v(t))^* \partial_t \phi(t) \| < \infty.$$

We have also

$$\sup_k \max_{0 \le t \le T} \| u^k(t) \| + \sup_{0 \le t \le T} \| u(t) \| \le 2M.$$

On the other hand, $u^k(t)$ converges to $u(t)$ weakly in $L^2(\Omega)$ for each $t \in [0, T]$ by Lemma 3.4. Therefore, by Lebesgue's dominated convergence theorem, the second term on the right hand side of (3.42) converges to 0 as $k \to \infty$. Next, since

$$\frac{1}{C_k^2} \sum_{j=1}^n |\langle u^k, \partial_j(v_j \phi) \rangle| \le \frac{MT}{C_k^2} \sum_{j=1}^n \max_{0 \le t \le T} \| \partial_j(v_j \phi(t)) \|,$$

we have

(3.44) $$\frac{1}{C_k^2} \sum_{j=1}^n \langle v_j \partial_j u^k, \phi \rangle \to 0$$

as $k \to \infty$. By the definition of C_k, we have also

(3.45) $$\frac{1}{C_k^2} \sum_{j=1}^n \langle v_j \partial_j U_k, \phi \rangle \to 0$$

as $k \to \infty$. Since the other terms on the left hand side of (3.41) are treated more or less in a similar way, we omit the details. We have finally

(3.46) $$\begin{aligned} &- \langle A_0(v)u, \partial_t \phi \rangle - \langle (\partial_t A_0(v))u, \phi \rangle \\ &- \sum_{j=1}^n \langle A_j(v)u, \partial_j \phi \rangle - \sum_{j=1}^n \langle (\partial_j A_j(v))u, \phi \rangle + \langle B(v)u, \phi \rangle \\ &= \langle F, \phi \rangle. \end{aligned}$$

Integrating (3.46) by parts, we have

(3.47) $$\langle A_0(v)\partial_t u + \sum_{j=1}^n A_j(v)\partial_j u + B(v)u - F, \phi \rangle = 0.$$

Since ϕ is arbitrary, we obtain (0.1). We show that u satisfies the boundary condition (0.2). Let us assume for the moment that $m \ge 2$. Since $u \in C_w([0, T]; H_*^m(\Omega))$ by Proposition 3.9, we have

$$\sup_{0 \le t \le T} \| u(t) \|_1 < \infty.$$

Note that $H_*^m(\Omega) \subsetneq H^1(\Omega)$ if $m \ge 2$. We have also

$$\sup_k \max_{0 \le t \le T} \|u^k\|_1 < \infty$$

by Proposition 3.3. Then, combining the interpolation theorem for the Sobolev spaces and Lemma 3.6, we conclude that

$$\sup_{0 \le t \le T} \|u^k(t) - u(t)\|_{1-\varepsilon} \to 0$$

as $k \to \infty$ for $\varepsilon > 0$ small. Hence $u^k(t)|_\Gamma \to u(t)|_\Gamma$ in $H^{\frac{1}{2}-\varepsilon}(\Gamma)$ for $0 < \varepsilon < \frac{1}{2}$ uniformly in t. This implies that $Mu(t) = 0$ on Γ for $t \in [0, T]$. Now we consider the case where $m = 1$. Since $H^1_{**}(\Omega) = H^1(\Omega)$, we have

$$M = \sup_k \max_{0 \le t \le T} \|\tilde{P}u^k(t)\|_1 < \infty$$

by Proposition 3.3. We fix t arbitrarily and note that the imbedding $H^1(\Omega) \subsetneq H^{1-\varepsilon}(\Omega)$ is compact for $0 < \varepsilon < 1$. Therefore there is a subsequence of $\{\tilde{P}u^k(t)\}$ which converges in $H^{1-\varepsilon}(\Omega)$. But by Lemma 3.4 the sequence $\{\tilde{P}u^k(t)\}$ converges weakly in $L^2(\Omega)$ to $\tilde{P}u(t)$. It follows that the sequence $\{\tilde{P}u^k(t)\}$ converges in $H^{1-\varepsilon}(\Omega)$ to $\tilde{P}u(t)$ without choosing a subsequence. Let $\tilde{M}$ be a smooth extension of M. Then

$$\tilde{M}\tilde{P}u^k(t) \to \tilde{M}\tilde{P}u(t) \qquad \text{in } H^{1-\varepsilon}(\Omega),$$

for $t \in [0, T]$, if $0 < \varepsilon < 1$. Hence

$$\tilde{M}\tilde{P}u^k(t)|_\Gamma \to \tilde{M}\tilde{P}u(t)|_\Gamma \qquad \text{in } H^{\frac{1}{2}-\varepsilon}(\Gamma),$$

for $t \in [0, T]$, if $0 < \varepsilon < \frac{1}{2}$, that is,

$$Mu^k(t)|_\Gamma \to Mu(t)|_\Gamma \qquad \text{in } H^{\frac{1}{2}-\varepsilon}(\Gamma).$$

Notice that $MP = M$ on Γ. Since $Mu^k(t) = 0$ on Γ for $k \ge 1$ and $t \in [0, T]$, we see that $Mu(t) = 0$ on Γ for $t \in [0, T]$. Thus u satisfies the boundary condition (0.2).

Finally we check the initial condition (0.3). By our assumption, $u^k(0) = f_k$ converges to f in $H^m(\Omega)$. On the other hand, $u^k(0)$ converges to $u(0)$ weakly in $L^2(\Omega)$ by Lemma 3.4. Then we have $u(0) = f$ by the uniqueness of the limit. Hence u satisfies the initial condition (0.3). Notice that actually $u \in C([0, T]; L^2(\Omega))$, which follows from the fact that $\partial_t u \in C_w([0, T]; L^2(\Omega))$. This completes the proof of Proposition 3.5.

Lemma 3.6. *Let* $m \ge 2$. *Let* $u^k \in X^{m+1}([0, T]; \Omega)$ *be the solution of the initial boundary value problem* (3.34), (3.35), (3.36) *obtained in Proposition* 3.3. *Then the whole sequence* $\{u^k\}$ *is a Cauchy sequence in* $C([0, T]; L^2(\Omega))$.

Proof. Since each of the u^k's satisfies (3.34), we have

$$L(v_k; v, k)u^k - L(v_l; v, l)u^l = F''_k - F''_l. \tag{3.48}$$

We rewrite this equation as

$$
\begin{aligned}
(3.49) \qquad & A_0(v_k)\partial_t w^{k,l} + \sum_{j=1}^{n} A_j(v_k)\partial_j w^{k,l} \\
& + \frac{1}{C_k^2}\sum_{j=1}^{n} v_j\partial_j w^{k,l} + B(v_k)w^{k,l} \\
& = J_{k,l}.
\end{aligned}
$$

Here

$$
\begin{aligned}
w^{k,l} &= u^k - u^l, \\
J_{k,l} &= -(A_0(v_k) - A_0(v_l))\partial_t u^l - \sum_{j=1}^{n} (A_j(v_k) - A_j(v_l))\partial_j u^l \\
&\quad - \left(\frac{1}{C_k^2} - \frac{1}{C_l^2}\right)\sum_{j=1}^{n} v_j\partial_j u^l - (B(v_k) - B(v_l))u^l + (F_k - F_l) \\
&\quad + \frac{1}{C_k^2}\sum_{j=1}^{n} v_j\partial_j U_k - \frac{1}{C_l^2}\sum_{j=1}^{n} v_j\partial_j U_l.
\end{aligned}
$$

For simplicity, we write in what follows v, w, and J instead of v_k, $w^{k,l}$, and $J_{k,l}$, respectively. We take the inner product of (3.49) with w and integrate it over Ω. Then we estimate each term by a standard method. The nonnegativity of the boundary condition for (3.49) is used when we deal with the integrals on the boundary. Since A_0 is positive definite, we obtain finally

$$
(3.50) \qquad \|w(t)\| \le C\|w(0)\| + CM(v_k)\int_0^t \|w(\tau)\|\,d\tau + C\int_0^t \|J(\tau)\|\,d\tau.
$$

where

$$
\begin{aligned}
M(v_k) &\equiv \max_{0\le t\le T}\left(1 + \|\operatorname{Div}\vec{A}(v_k(t))\|_{[\frac{n}{2}]+1} + \|B(v_k(t))\|_{[\frac{n}{2}]+1}\right), \\
\operatorname{Div}\vec{A} &= \partial_t A_0 + \sum_{j=1}^{n}\partial_j A_j.
\end{aligned}
$$

Since $v_k \to v$ in $X_*^{\mu}([0, T]; \Omega)$ with $\mu = \max\left(m, 2\left[\frac{n}{2}\right] + 6\right)$, $M(v_k)$ is uniformly bounded in k, say, by M. Then, by Gronwall's inequality, we get

$$
(3.51) \qquad \|w(t)\| \le C\left(\|w(0)\|e^{CMT} + \int_0^T \|J(\tau)\|\,d\tau e^{CMT}\right).
$$

It is easy to see that

$$
\|w(0)\| = \|u^k(0) - u^l(0)\| = \|f_k - f_l\|.
$$

$J = J_{k,l}$ is estimated as follows.

$$\int_0^T \| J_{k,l}(\tau) \| \, d\tau \leq CKT ||| v_k - v_l |||_{X_*^{2[\frac{n}{2}]+2}([0,T];\Omega)} ||| u^l |||_{X_*^2([0,T];\Omega)}$$

$$+ CT\left(\frac{1}{C_k^2} + \frac{1}{C_l^2} \right) ||| u^l |||_{X_*^2([0,T];\Omega)}$$

$$+ \int_0^T \| F_k(\tau) - F_l(\tau) \| \, d\tau + CT\left(\frac{1}{C_k} + \frac{1}{C_l} \right).$$

Here K is a constant depending on $||| v_k |||_{X_*^{2[\frac{n}{2}]+2}([0,T];\Omega)}$ and $||| v_l |||_{X_*^{2[\frac{n}{2}]+2}([0,T];\Omega)}$. Since $v_k \to v$ in $X_*^\mu([0, T]; \Omega)$ as we noted above, we see that K is uniformly bounded in k, l. On the other hand, $||| u^l |||_{X_*^2([0,T];\Omega)}$ is uniformly bounded in l. Then, by the properties of f_k, v_k, and F_k stated in Lemma 3.1, we conclude from (3.51) that $\| u^k(t) - u^l(t) \| \to 0$ uniformly in t as $k, l \to \infty$. Therefore the sequence u^k is a Cauchy sequence in $C^0([0, T]; L^2(\Omega))$. The proof of Lemma 3.6 is complete.

Lemma 3.7. *Let X and Y be Hilbert spaces such that $X \hookrightarrow Y$. Assume that X is dense in Y. Let $I = [a, b]$ and let the sequence $\{u_k\}$ in $C_w(I; X)$, $k \geq 1$, have the following properties:*

i) *The supremum norm of u_k in $C_w(I; X)$ is bounded by a constant not depending on k, that is,*

$$M = \sup_k \sup_{0 \leq t \leq T} \| u_k(t) \|_X < +\infty.$$

ii) *There is a $u \in C_w(I; Y)$ such that the sequence $\{u_k(t)\}$ converges to $u(t)$ as $k \to \infty$ in the weak topology of Y uniformly in $t \in [0, T]$.*

Then the limit u lies in $C_w(I; X)$.

Proof. Let $t \in I$. Since $\{u_k(t)\}$ is weakly sequentially compact, there is a subsequence $\{u_{k_j}(t)\}$ such that $u_{k_j}(t)$ converges as $j \to \infty$ weakly in X. This implies that there is at least one accumulation point of $\{u_k(t)\}$ in X endowed with the weak topology. On the other hand, by condition ii), there exists at most one accumulation point of $\{u_k(t)\}$ in the weak topology of X. Hence the whole sequence $\{u_k(t)\}$ converges weakly to some limit in X for any $t \in [0, T]$. This limit must coincide with $u(t)$ stated in condition ii). Next we show that $\{u_k(t)\}$ converges to u in the weak topology of X uniformly in t. Let X^* and Y^* be the adjoint spaces of X and Y, respectively. We have $Y^* \hookrightarrow X^*$ and furthermore Y^* is dense in X^*. Let $f \in T^*$. Then $(u_k(t), f)$ is a continuous function of t by condition i). It follows from condition ii) that $(u_k(t), f) \to (u(t), f)$ as $k \to \infty$ uniformly in $t \in I$. Hence $(u(t), f)$ is a continuous function of t. Now let $g \in X^*$. We have

$$|(u_k(t), g) - (u(t), g)| \tag{3.52}$$
$$\leq |(u_k(t), g - f)| + |(u_k(t), f) - (u(t), f)| + |(u(t), f - g)|.$$

Then the first term on the right hand side of (3.52) is bounded by

$$\sup_k \sup_{0\le t\le T} \|u_k(t)\|_X \|g-f\|_{X^*} < M\varepsilon,$$

if we choose $f\in Y^*$ such that $\|g-f\|_{X^*}<\varepsilon$. The middle term of the right hand side of (3.52) becomes smaller than ε by taking k sufficiently large for fixed f. The last term of the right hand side of (3.52) is estimated by

$$\sup_{0\le t\le T} \|u(t)\|_X \|f-g\|_{X^*} < M\varepsilon,$$

since $\|u(t)\|_X \le \varliminf_{k\to\infty} \|u_k(t)\|_X \le M$. This completes the proof of Lemma 3.7.

Lemma 3.8. *Let X and Y be Hilbert spaces and let $I=[a,b]$. Let $T(t)\in\mathscr{L}(X,Y)$ for $t\in I$ and let $T(t)$ be a continuous function of t in the norm of $\mathscr{L}(X,Y)$. We define Tf for $f\in C_w(I;X)$ by*

$$(Tf)(t)=T(t)f(t), \qquad t\in I.$$

Then we have $Tf\in C_w(I;Y)$. The mapping $f\mapsto Tf$ is a continuous linear operator from $C_w(I;X)$ into $C_w(I;Y)$.

Proof. We denote by X^* and Y^* the adjoint spaces of X and Y, respectively. Since $T(t)\in\mathscr{L}(X,Y)$, we have $T(t)^*\in\mathscr{L}(Y^*,X^*)$. $T(t)^*$, $t\in I$, is a continuous function of t in the norm of $\mathscr{L}(Y^*,X^*)$. Let $\phi\in Y^*(t)$. Then

$$(T(t)f(t),\phi)=(f(t),T(t)^*\phi).$$

The right hand side is a continuous function of t. The last assertion of the lemma is easily seen. This completes the proof of Lemma 3.8.

Proposition 3.9. *The initial boundary value problem* (0.1), (0.2), (0.3) *has a unique solution u in $Y_*^m([0,T];\Omega)$.*

Proof. We recall that $\{u^k\}$ is a subsequence of the solution of (3.34), (3.35), (3.36) such that, for any $0\le j\le m$, $\{\partial_t^j u^k\}$ converges in the weak* topology of $L^\infty(0,T;H_*^{m-j}(\Omega))$. Let u be the limit of $\{u^k\}$ as $k\to\infty$. Then we have $\partial_t^j u^k\to\partial_t^j u$ as $k\to\infty$ for $1\le j\le m$. We shall show that $\partial_t^j u\in C_w([0,T];H_*^{m-j}(\Omega)))$ for $0\le j\le m$. First we consider the case where $j=0$. Since $u^k\in C([0,T];H_*^m(\Omega))$, $k\ge 1$, we have a fortiori $u^k\in C_w([0,T];H_*^m(\Omega))$, $k\ge 1$. Moreover,

$$\sup_k \max_{0\le t\le T} \|u^k(t)\|_{m,*}<\infty.$$

On the other hand, u^k converges to u in $C([0,T];L^2(\Omega))$ by Lemma 3.6 and hence in $C_w([0,T];L^2(\Omega))$. Combining these observations and applying Lemma 3.7 with $X=H_*^m(\Omega)$ and $Y=L^2(\Omega)$ to $\{u^k\}$, we conclude that $u\in C_w([0,T];H_*^m(\Omega))$. Next we prove the general case by the induction on j. Let us recall that u satisfies (0.1) which is rewritten as

$$\partial_t u = Gu + \tilde{F},$$

where

$$G = -\sum_{j=1}^{n} \tilde{A}_j \partial_j - \tilde{B},$$

$$\tilde{A}_j = A_0^{-1} A_j,\ 1 \le j \le n,\ \tilde{B} = A_0^{-1} B,\ \tilde{F} = A_0^{-1} F.$$

We set

$$G_i = -\sum_{j=1}^{n} \tilde{A}_j^{(i)} \partial_j - \tilde{B}^{(i)}, \qquad i \ge 1,$$

$$\tilde{A}_j^{(i)} = \partial_t^i \tilde{A}_j, \quad \tilde{B}^{(i)} = \partial_t^i \tilde{B}.$$

Then, by Leibnitz's rule, we have

$$\partial_t^l u = G(t)\partial_t^{l-1} u + \sum_{i=0}^{l-2} \binom{l-1}{i} G_{l-1-i}(t)\partial_t^i u + \partial_t^{l-1}\tilde{F}. \tag{3.53}$$

We assume that $\partial_t^i u \in C_w([0, T]; H_*^{m-i}(\Omega))$ for $0 \le i \le l-1$ and want to show that $\partial_t^l u \in C_w([0, T]; H_*^{m-l}(\Omega))$. Since $F \in W_*^m(0, T; \Omega)$, we have $\tilde{F} \in W_*^m(0, T; \Omega)$ also. This implies that $\partial_t^j \tilde{F} \in C([0, T]; H_*^{m-1-j}(\Omega))$, $0 \le j \le m-1$. In particular $\partial_t^{l-1}\tilde{F} \in C([0, T]; H_*^{m-l}(\Omega))$. Therefore, the last term on the right hand side of (3.53) is a member of $C_w([0, T]; H_*^{m-l}(\Omega))$. We note that $\partial_j \in \mathscr{L}(H_*^{m-i}(\Omega), H_*^{m-i-2}(\Omega))$ for $1 \le j \le n$ where $0 \le i \le l-2$. Then we have $\partial_j(\partial_t^i u) \in C_w([0, T]; H_*^{m-i-2}(\Omega))$. It can be shown by using Lemma B.1 iii) that $\tilde{A}_j^{(l-1-i)}(t)$ is an operator of $\mathscr{L}(H_*^{m-i-2}(\Omega), H_*^{m-l}(\Omega))$ for each $t \in I$ and that it is a continuous function of t in the norm of $\mathscr{L}(H_*^{m-i-2}(\Omega), H_*^{m-l}(\Omega))$ for $0 \le i \le l-2$ and $1 \le j \le n$. The same is true for $\tilde{B}^{(l-1-i)}(t)$. Hence by applying Lemma 3.8 we see that

$$G_{l-1-i}(t)\partial_t^i u = -\sum_{j=1}^{n} \tilde{A}_j^{(l-1-i)} \partial_j(\partial_t^i u) - \tilde{B}^{(l-1-i)}(t)\partial_t^i u$$

is a member of $C_w([0, T]; H_*^{m-l}(\Omega))$ provided that $0 \le i \le l-2$. Thus the middle term on the right hand side of (3.53) is a member of $C_w([0, T]; H_*^{m-l}(\Omega))$. Finally we consider the first term on the right hand side of (3.53). To this end, we write

$$G(t) = A_0(t)^{-1}(A_\nu(t)\partial_\nu + \Lambda(t) + B(t)),$$

where

$$A_\nu(t) = \sum_{j=1}^{n} \nu_j A_j(t), \quad \partial_\nu = \sum_{j=1}^{n} \nu_j \partial_j,$$

$$\Lambda(t) = \sum_{j=1}^{n} (A_j(t) - \nu_j A_\nu(t))\partial_j,$$

and where $\nu=(\nu_1,\dots,\nu_n)$ is a smooth extension of the outward unit normal to Γ. Let us look at

$$A_0^{-1}(t)(A(t)+B(t))\partial_t^{l-1}u.$$

We have $\partial_t^{l-1}u\in C_w([0,T];H_*^{m-l+1}(\Omega))$ by the assumption of the induction. We observe that $A(t)$ is a tangential vector field, although its coefficient matrices are not in $C^\infty(\bar{\Omega})$. $A_0(t)^{-1}A(t)$ is an operator of $\mathscr{L}(H_*^{m-l+1}(\Omega),H_*^{m-l}(\Omega))$ for each $t\in I$ and continuous in t in the norm of $\mathscr{L}(H_*^{m-l+1}(\Omega),H_*^{m-l}(\Omega))$. This can be shown by using Lemma B.1 iii). Therefore, by Lemma 3.8, we see that $A_0(t)^{-1}A(t)\partial_t^{l-1}u\in C_w([0,T];H_*^{m-l}(\Omega))$. The same argument holds for $A_0(t)^{-1}B(t)\partial_t^{l-1}u$. Finally we treat $A_0(t)^{-1}A_\nu(t)\partial_\nu(\partial_t^{l-1}u)$ that remains. Let $\tilde{P}=\tilde{P}(x)$, $x\in\Omega$, be a smooth extension of $P=P(x)$, $x\in\Gamma$, where $P(x)$ is the orthogonal projection onto $\mathscr{N}(x)^\perp$. Recall that, by Proposition 3.3, $\tilde{P}\partial_t^{l-1}u^k\in C([0,T];H_{**}^{m-l+1}(\Omega))$ and that

$$M=\sup_k\max_{0\le t\le T}\|\tilde{P}(\partial_t^{l-1}u^k(t))\|_{m-l+1,**}<\infty.$$

Since $1\le l\le m$ and $H_{**}^1(\Omega)=H^1(\Omega)$, the imbedding $H_{**}^{m-l+1}\hookrightarrow L^2(\Omega)$ is compact. Hence, for each $t\in[0,T]$, $\{\tilde{P}\partial_t^{l-1}u^k(t)\}$ has a subsequence that converges in $L^2(\Omega)$. This subsequence also converges weakly in $H_{**}^{m-l+1}(\Omega)$ by the uniform estimate in this space. Actually, by the uniqueness of the limit of $\{\tilde{P}\partial_t^{l-1}u^k(t)\}$, we need not employ the subsequence. The whole sequence $\{\tilde{P}\partial_t^{l-1}u^k(t)\}$ converges weakly in $H_{**}^{m-l+1}(\Omega)$ for each $t\in[0,T]$. The limit $\tilde{P}\partial_t^{l-1}u(t)$ satisfies the estimate

$$\sup_{0\le t\le T}\|\tilde{P}(\partial_t^{l-1}u(t))\|_{m-l+1,**}\le M.$$

Since $\tilde{P}\partial_t^{l-1}u\in C_w([0,T];H_*^{m-l+1}(\Omega))$, we get $\tilde{P}\partial_t^{l-1}u\in C_w([0,T];H_{**}^{m-l+1}(\Omega))$. Then it follows that $\partial_\nu\tilde{P}(\partial_t^{l-1}u)\in C_w([0,T];H_*^{m-l}(\Omega))$. Note that if $m\ge1$ we have $\partial_\nu f\in H_*^{m-1}(\Omega)$ for any $f\in H_{**}^m(\Omega)$. Now $A_0(t)^{-1}A_\nu(t)$ is an operator of $\mathscr{L}(H_*^{m-l}(\Omega))$ for each $t\in I$. Moreover, it is a continuous function of t in the norm of $\mathscr{L}(H_*^{m-l}(\Omega))$. Hence we have $\tilde{A}_\nu\partial_\nu\tilde{P}\partial_t^{l-1}u\in C_w([0,T];H_*^{m-l}(\Omega))$, where we set $\tilde{A}_\nu(t)=A_0^{-1}(t)A_\nu(t)$. Let us write

$$\begin{aligned}\tilde{A}_\nu(t)\partial_\nu(\partial_t^{l-1}u)&=\tilde{A}_\nu(t)\partial_\nu(\tilde{P}\partial_t^{l-1}u)\\&\quad-\tilde{A}_\nu(t)(\partial_\nu\tilde{P})(\partial_t^{l-1}u)+\tilde{A}_\nu(t)(1-\tilde{P})\partial_\nu(\partial_t^{l-1}u)\end{aligned}\tag{3.54}$$

The second term on the right hand side is a member of $C_w([0,T];H_*^{m-l+1}(\Omega))$ by Lemma 3.8. To discuss the last term on the right hand side, we note that $\tilde{A}_\nu(1-P)\partial_\nu$ is a tangential vector field because $\tilde{A}_\nu(1-\tilde{P})$ vanishes on Γ. Hence this term can be dealt with in the same way as we treated $A_0(t)^{-1}A_0(t)(\partial_t^{l-1}u)$. The first term on the right hand side of (3.54) was discussed above. Consequently, we get $\tilde{A}_\nu\partial_\nu(\partial_t^{l-1}u)\in C_w([0,T];H_*^{m-l}(\Omega))$. Summing up these observations, we conclude that the first term on the right hand side of (3.53) is a member of $C_w([0,T];H^{m-l}(\Omega))$. Therefore all the terms on the right hand side of (3.53) lie in $C_w([0,T];H^{m-l}(\Omega))$. This implies that $\partial_t^l u\in C_w([0,T];H^{m-l}(\Omega))$.

We prove the uniqueness of the solution of the initial boundary value problem (0.1), (0.2), (0.3) in $Y_*^m([0, T]; \Omega)$. For simplicity we assume that $m = 1$. If $u \in Y_*^1([0, T]; \Omega)$, then we have

$$(u(t), \phi) - (u(s), \phi) = \int_s^t \left(\frac{\partial}{\partial \tau} u(\tau), \phi \right) d\tau$$

for $0 \leq s \leq t \leq T$ and $\phi \in L^2(\Omega)$. This implies that $u = u(t)$ is strongly continuous and in addition weakly differentiable in $L^2(\Omega)$ in $t \in [0, T]$. Hence the following equality holds for $t \in [0, T]$.

$$\frac{\partial}{\partial t}(u, A_0(t)u) = (\partial_t u, A_0(t)u) + (u, (\partial_t A_0(t))u) + (u, A_0(t)(\partial_t u)).$$

This enables us to obtain the $L^2(\Omega)$-estimate of the solution $u \in Y_*^1([0, T]; \Omega)$ by the standard energy method. Then, it is clear that if u_1 and u_2 belong to $Y_*^1([0, T]; \Omega)$ and satisfy (0.1), (0.2), (0.3), they must coincide with each other. This proves the uniqueness assertion. The proof of Proposition 3.9 is now complete.

Remark. As a consequence of the above proposition, it turns out that the whole sequence $\{u^k\}$ converges in the weak* topology of $L^\infty(0, T; H_*^m(\Omega))$ to u without passing to a subsequence. Also, the whole sequence $\{\partial_t^j u^k\}$ converges in the weak* topology of $L^\infty(0, T; H_*^{m-j}(\Omega))$ to $\partial_t^j u$ for $1 \leq j \leq m$.

Proposition 3.10. *The solution u obtained in Proposition* 3.9 *lies in* $X_*^m([0, T]; \Omega)$.

Proof. One of the ingredients of our proof is the use of Rauch's mollifier introduced in [20]. Except for this point, the argument is analogous to that of Majda [13], where the Cauchy problem is studied. The detailed proof will be given in a forthcoming paper [24].

§ 4. Proof of Lemma 3.1 A

We follow the line of the proof of Lemma 3.3 in [21]. However, we must argue more carefully, for lack of regularity of the coefficient matrices of the equation. Besides this, the boundary matrix is singular in our case. Therefore we can employ the proof of Lemma 3.3 in [21] only after suitable modifications. The f_p's mentioned in § 2 are defined inductively by

(4.1)

$$\begin{cases} f_0 = f, \\ f_p = \sum_{i=0}^{p-1} \binom{p-1}{i} G_i(0) f_{p-1-i} + \partial_t^{p-1}(A_0(v)^{-1}(A_0(v)^{-1}F)(0), \quad p \geq 1, \text{ in } \Omega. \end{cases}$$

Here

$$G_0(t) = -\sum_{j=1}^{n} A_0(v)^{-1} A_j(v)\partial_j - A_0(v)^{-1} B(v),$$

$$\begin{aligned} G_i(t) &= -\sum_{j=1}^{n} \partial_t^i (A_0(v)^{-1} A_j(v))\partial_j - \partial_t^i (A_0(v)^{-1} B(v)) \\ &= [\partial_t, G_{i-1}(t)], \qquad i \geq 1. \end{aligned}$$

We observe that f_p, $p \geq 0$, defined by (4.1) can be written as

$$f_p = B_p f + E_p F, \quad p \geq 0, \text{ in } \Omega. \tag{4.2}$$

Here B_p and E_p are defined respectively by

$$\begin{cases} B_0 f = f, \\ B_p f = \displaystyle\sum_{i_1+\cdots+i_q+q=p} C(p; q; i_1,\dots,i_q) G_{i_1}(0)\cdots G_{i_q}(0) f, \qquad p \geq 1, \end{cases} \tag{4.3}$$

and

$$\begin{cases} E_0 F = 0, \\ E_1 F = (A_0(v)^{-1} F)(0), \\ E_p F \\ \quad = \displaystyle\sum_{\eta=0}^{p-2} \sum_{i_1+\cdots+i_q+q=p-1-\eta} C(p; q; i_1,\dots,i_q) G_{i_1}(0)\cdots G_{i_q}(0) \times \\ \quad \times \partial_t^{\eta}(A_0(v)^{-1} F)(0) + \partial_t^{p-1}(A_0(v)^{-1} F)(0), \qquad p \geq 2, \end{cases} \tag{4.4}$$

with

$$\begin{aligned} &C(p; q; i_1,\dots,i_q) \\ &= \binom{p-1}{i_1}\binom{p-2-i_1}{i_2}\cdots\binom{p-q-(i_1+\cdots+i_{q-1})}{i_q}. \end{aligned}$$

The summation on the right hand side of (4.3) is taken over all $1 \leq q \leq p$ and the q-tuples $(i_1,\dots,i_q)$ such that $i_1 + \cdots + i_q + q = p$. The summation on the right hand side of (4.4) is analogous to this. In order to get a concrete expression for the product of the first order differential operators $G_{i_1}(0),\dots,G_{i_q}(0)$ appearing in (4.3) and (4.4), we set for $i \geq 0$

$$\begin{aligned} A_j^{(i)} &= \partial_t^i (A_0(v)^{-1} A_j(v))(0), \qquad 1 \leq j \leq n. \\ A_{n+1}^{(i)} &= \partial_t^i (A_0(v)^{-1} B(v))(0). \end{aligned}$$

When ∂_{n+1} appears in the following, it should always be replaced by the identity operator. Let $S(q)$ be the set of $q \times q$ upper triangular matrices σ whose entries are either 0 or 1 and whose rows contain at most one entry which equals 1. Let $1 \leq j_1,\dots,j_q \leq n+1$. We define $S(q; j_1,\dots,j_q)$ to be the set of $\sigma \in S(q)$ such that, if $j_k = n+1$ then each entry of the k-th row is zero, and if $j_k \neq n+1$ then the

k-th row contains one entry which equals 1. Then, we get

$$G_{i_1}(0)\cdots G_{i_q}(0)f = \sum_{1\le j_1,\dots,j_q\le n+1} (A_{j_1}^{(i_1)}\partial_{j_1})(A_{j_2}^{(i_2)}\partial_{j_2})\cdots(A_{j_q}^{(i_q)}\partial_{j_q})f$$

$$= \sum_{1\le j_1,\dots,j_q\le n+1}\ \sum_{\sigma\in S(q;j_1,\dots,j_q)} A_{j_1}^{(i_1)}(\partial_{j_1}^{\sigma(1,2)}A_{j_2}^{(i_2)})(\partial_{j_1}^{\sigma(1,3)}\partial_{j_2}^{\sigma(2,3)}A_{j_3}^{(i_3)})\cdots$$
$$\times(\partial_{j_1}^{\sigma(1,q)}\cdots\partial_{j_{q-1}}^{\sigma(q-1,q)}A_{j_q}^{(i_q)})\partial_{j_1}^{\sigma(1,1)}\cdots\partial_{j_q}^{\sigma(q,q)}f,$$

where $\sigma(m, k)$, $1\le m, k\le q$, stands for the (m, k)-entry of σ.

Let $\sigma\in S(q; j_1,\dots,j_q)$ and let $1\le i\le n$. We define $\varphi_i(\sigma)$ to be the number of k for which $\sigma(k, k) = 1$ and, in addition, $j_k = i$. We write $\varphi(\sigma) = (\varphi_1(\sigma),\dots, \varphi_n(\sigma))$. Now let $l = (l_1,\dots,l_n)$. We set

$$A(p; q; i_1,\dots,i_q; l) = \sum_{1\le j_1,\dots,j_q\le n+1}\ \sum_{\substack{\sigma\in S(q;j_1,\dots,j_q)\\ \varphi(\sigma)=l}} A_{j_1}^{(i_1)}(\partial_{j_1}^{\sigma(1,2)}A_{j_2}^{(i_2)})$$
$$\times(\partial_{j_1}^{\sigma(1,3)}\partial_{j_2}^{\sigma(2,3)}A_{j_3}^{(i_3)})\cdots(\partial_{j_1}^{\sigma(1,q)}\cdots\partial_{j_{q-1}}^{\sigma(q-1,q)}A_{j_q}^{(i_q)}).$$

Then, B_pf and E_pF are written as follows.

$$B_pf = \sum_{|l|\le p} A(p, l)\partial_x^l f, \qquad p\ge 0, \tag{4.5}$$

$$E_pF = \sum_{\eta=0}^{p-1}\ \sum_{|l|\le p-1-\eta} A(p-1-\eta, l)\partial_x^l\partial_t^\eta(A_0(v)^{-1}F)(0), \qquad p\ge 1. \tag{4.6}$$

Here

$$A(p, l) = \sum_{i_1+\cdots+i_q+q=p} C(p; q; i_1,\dots,i_q)A(p; q; i_1,\dots,i_q; l), \qquad p\ge 1.$$

We set $A(0, 0) = I$ for convenience.

Lemma 4.1. *Let* $v\in X_*^\mu([0, T]; \Omega)$ *and let* $\partial_t^i v(0)\in H^{2\mu+2-i}(\Omega)$, $0\le i\le\mu$, *where* $\mu = \max\left(m, 2\left[\frac{n}{2}\right]+6\right)$. *Let* $\nu(p, |l|) = 2\mu+2-p+\max(|l|, 1)$. *Then*

$$A(p, l)\in H^{\nu(p,|l|)}(\Omega), \qquad 0\le|l|\le p\le\mu+1. \tag{4.7}$$

Proof. We have by Leibniz's rule

$$A_j^{(i)} = (\partial_t^i(A_0(v)^{-1}A_j(v)))(0) = \sum_{s+r=i}\binom{i}{s}(\partial_t^s A_0(v)^{-1})(0)(\partial_t^r A_j(v))(0),$$
$$0\le i\le\mu,\ 1\le j\le n+1.$$

By exploiting $\partial_t^i v(0)\in H^{2\mu+2-i}(\Omega)$, $0\le i\le\mu$, we can use Lemma C.5 and Lemma C.3 to obtain

$$\partial_t^s A_0(v)^{-1}(0)\in H^{2\mu+2-s}(\Omega), \qquad 0\le s\le i, \tag{4.8}$$

and

(4.9) $$\partial_t^r A_j(v)(0) \in H^{2\mu+2-r}(\Omega), \qquad 0 \le r \le i,\ 1 \le j \le n+1.$$

This yields by Lemma C.1 i)

(4.10) $$A_j^{(i)} \in H^{2\mu+2-i}(\Omega), \qquad 0 \le i \le \mu,\ 1 \le j \le n+1.$$

It follows that

$$\partial_{j_1}^{\sigma(1,k)} \cdots \partial_{j_{k-1}}^{\sigma(k-1,k)} A_{j_k}^{(i_k)} \in H^{2\mu+2-(i_k+\sigma(1,k)+\cdots+\sigma(k-1,k))}(\Omega), \qquad 2 \le k \le q.$$

Applying Lemma C.1 i) repeatedly to each term of $A(p; q; i_1,\dots,i_q; l)$, we have finally

$$A(p; q; i_1,\dots,i_q; l) \in H^{2\mu+2-(i_1+\cdots+i_q+\Sigma_{k=2}^q \Sigma_{m=1}^{k-1} \sigma(m,k))}(\Omega).$$

We define $\rho = \rho(j_1,\dots,j_q)$ to be the number of k such that $j_k = n+1$. Let $\sigma \in S(q; j_1,\dots,j_q)$. Then we have

$$\sum_{k=1}^{q} \sigma(k, k) + \sum_{\mu=2}^{q} \sum_{m=1}^{k-1} \sigma(m, k) = q - \rho.$$

Since $\operatorname{tr}\sigma = |\varphi(\sigma)|$, this implies

$$\sum_{k=2}^{q} \sum_{m=1}^{k-1} \sigma(m, k) = q - \rho - |\varphi(\sigma)|.$$

Therefore

$$A(p; q; i_1,\dots,i_q; l) \in H^{2\mu+2-(i_1+\cdots+i_q-\rho-|l|)}(\Omega).$$

We observe that if $|\varphi(\sigma)| = 0$, then $\sigma(q, q) = 0$, that is, each entry of the q-th row of σ is zero. Thus, $|\varphi(\sigma)| = 0$ implies that $\rho \ge 1$. Consequently, we have $1 \le \rho + |\varphi(\sigma)|$. Hence

$$\rho + |l| \ge \max(|l|, 1).$$

We see therefore

$$A(p; q; i_1,\dots,i_q) \in H^{2\mu+2-(i_1+\cdots+i_q+q-\max(|l|,1))}(\Omega).$$

Since $i_1 + \cdots + i_q + q = p$, we obtain

$$A(p, l) \in H^{\nu(p,|l|)}(\Omega).$$

The proof of Lemma 4.1 is now complete.

Corollary 4.2. *Let v be as in Lemma 4.1. Let B_p and E_p be the differential operators defined by (4.5) and (4.6), respectively. Then*

(4.11) $$B_p \in \mathscr{L}(H^s(\Omega), H^{s-p}(\Omega)), \qquad 1 \le p \le \mu+1,\ p \le s \le 2\mu+3,$$

(4.12) $$E_p \in \mathscr{L}(V_*^s(0, T; \Omega), H^{s-p}(\Omega)), \qquad 1 \le p \le \mu+1,\ p \le s \le 2\mu+3.$$

Proof. First we prove (4.11). Let $0 \le |l| \le p \le \mu + 1$. Then, by Lemma 4.1, $A(p, l)$ is a member of $H^{\nu(p,|l|)}(\Omega)$. Since $p \le s \le 2\mu + 3$ and $|l| \le p$, we have

$$\min\left\{\nu(p, |l|),\ s - |l|,\ \nu(p, |l|) + (s - |l|) - \left[\frac{n}{2}\right] - 1\right\} \ge s - p.$$

Hence, by Lemma C.1 i),

$$\| B_p f \|_{s-p} \le C \sum_{|l| \le p} \| A(p, l) \|_{\nu(p,|l|)} \| \partial_x^l f \|_{s-|l|} \le C \| f \|_s \tag{4.13}$$

for $f \in H^s(\Omega)$, $p \le s \le 2\mu + 3$. This proves (4.11).

Next we show (4.12). Let $F \in V_*^s(0, T; \Omega)$ where $p \le s \le 2\mu + 3$. We recall that $A(p - 1 - \eta, l)$ is a member of $H^{\nu(p-1-\eta,|l|)}(\Omega)$, $0 \le \eta \le p - 1$, $0 \le |l| \le p - 1 - \eta \le \mu + 1$, by Lemma 4.1. Since $p \le s \le 2\mu + 3$ and $|l| \le p - 1 - \eta$, we have

$$\min\left\{\nu(p-1-\eta, |l|),\ s-1-\eta-|l|,\ \nu(p-1-\eta, |l|)+(s-1-\eta-|l|)-\left[\frac{n}{2}\right]-1\right\}$$
$$\ge s - p.$$

By using Lemma C.1 i) and Leibniz's rule, it is seen that

$$\begin{aligned} \| E_p F \|_{s-p} &\le \sum_{\eta=0}^{p-1} \sum_{|l| \le p-1-\eta} \| A(p - 1 - \eta, l) \partial_x^l \partial_t^\eta (A_0(v)^{-1} F)(0) \|_{s-p} \\ &\le C \sum_{\eta=0}^{p-1} \sum_{|l| \le p-1-\eta} \| A(p - 1 - \eta, l) \|_{\nu(p-1-\eta,|l|)} \\ &\quad \times \| \partial_x^l \partial_t^\eta (A_0(v)^{-1} F)(0) \|_{s-1-\eta-|l|} \\ &\le C \sum_{\eta=0}^{p-1} \| \partial_t^\eta (A_0(v)^{-1} F)(0) \|_{s-1-\eta} \\ &\le C \sum_{\eta=0}^{p-1} \sum_{\xi+\zeta=\eta} \| \partial_t^\xi A_0(v)^{-1}(0) \partial_t^\zeta F(0) \|_{s-1-\eta}. \end{aligned} \tag{4.14}$$

We see that

$$\min\left\{2\mu+2-\xi,\ s-1-\zeta,\ (2\mu+2-\xi)+(s-1-\zeta)-\left[\frac{n}{2}\right]-1\right\} \ge s-1-\eta.$$

Then we use Lemma C.1 i) to obtain

$$\begin{aligned} &\sum_{\eta=0}^{p-1} \sum_{\xi+\zeta=\eta} \| \partial_t^\xi A_0(v)^{-1}(0) \partial_t^\zeta F(0) \|_{s-1-\eta} \\ &\le C \sum_{\eta=0}^{p-1} \sum_{\xi+\zeta=\eta} \| \partial_t^\xi A_0(v)^{-1}(0) \|_{2\mu+2-\xi} \| \partial_t^\zeta F(0) \|_{s-1-\zeta} \end{aligned} \tag{4.15}$$

$$\le C \sum_{\eta=0}^{p-1} \| \partial_t^\eta F(0) \|_{s-1-\eta}$$

$$\le C ||| F |||_{V_*^s(0,T;\Omega)}.$$

Hence, it follows from (4.14) and (4.15) that

$$\| E_p F \|_{s-p} \le C ||| F |||_{V_*^s(0,T;\Omega)}.$$

This proves (4.12).

Remark. Corollary 4.2 can be proved directly by using the formulae (4.3) and (4.4). Nevertheless, it is worth while presenting the above proof that is not the shortest, because it serves as preliminaries to our proof of Lemma 4.4.

Corollary 4.3. *Let* $f \in H^m(\Omega)$ *and let* $F \in V_*^m(0, T; \Omega)$, *where* $m \ge 1$. *Let* $v \in X_*^\mu([0, T]; \Omega)$ *and let* $\partial_t^i v(0) \in H^{\mu-1-i}(\Omega)$, $0 \le i \le \mu - 1$, *where* $\mu = \max\left(m, 2\left[\frac{n}{2}\right] + 6\right)$. *Then we have*

$$\| B_p f \|_{m-p} \le C(K_{\mu-1}) \| f \|_m, \tag{4.16}$$

$$\| E_p F \|_{m-p} \le C(K_{\mu-1}) ||| F(0) |||_{m-1}, \tag{4.17}$$

for $0 \le p \le m$. *Here* $K_{\mu-1}$ *is a constant such that* $||| v(0) |||_{\mu-1} \le K_{\mu-1}$ *and* $C(\cdot)$ *depends increasingly on its argument.*

Proof. Let

$$\kappa(p, |l|) = \mu - 1 - p + \max(|l|, 1).$$

Then, as a particular consequence of Lemma 4.1, we have

$$A(p, l) \in H^{\kappa(p,|l|)}(\Omega), \qquad 0 \le |l| \le p \le m.$$

Replacing $v(p, |l|)$ by $\kappa(p, |l|)$ in the proof of Lemma 4.1 and verifying that the use of Lemma C.1 i) is still valid, we see that

$$\| A(p, l) \|_{\kappa(p,|l|)} \le C(K_{\mu-1}), \qquad 0 \le |l| \le p \le m.$$

Then we retrace the proof of Corollary 4.2, replacing again $v(p, |l|)$ by $\kappa(p, |l|)$. Lemma C.1 i) is also applicable to this case. This should be checked whenever we use the lemma. Except for this point, the proof is similar to that of Corollary 4.2. We obtain finally the estimates (4.16) and (4.17).

In what follows, we make use of the surfaces parallel to Γ. We mark off a segment of constant length δ, directed inward (resp. outward) to Γ, along the normals at every point of Γ. For a sufficiently small δ, the locus of the end points of these segments forms a closed surface, which does not cut itself, and which lies inside (resp. outside) Γ and has a smoothly varying tangent plane. Let Γ_δ denote this surface. For every point $\bar{x}$ on Γ there is a corresponding definite point x on Γ_δ, which lies on the normal to Γ at $\bar{x}$. Conversely, for every point x on Γ_δ there is a corresponding definite point $\bar{x}$ on Γ. The normal to Γ at $\bar{x}$ is also normal to Γ_δ at x. To every point x in a neighborhood of Γ we associate the outward unit normal to Γ at the corresponding point $\bar{x} = \bar{x}(x)$ on Γ, which is the nearest point to x on Γ. This is equivalent to saying that to every point x near Γ we associate the outward unit normal to Γ_δ at this point, where δ is the distance from x to Γ. Thus we obtain an extension of the outward unit normal originally defined only on Γ. Explicitly this is given by $\nu(\bar{x}(x))$, but we continue to denote this extension by the same ν. The vector field $\partial_\nu = \sum_{i=1}^n \nu_i \partial_i$ is defined on a neighborhood of Γ in $\mathbf{R}^n$, say G, in this sense.

Lemma 4.4. *Let $x \in G \cap \bar{\Omega}$ and let f be a smooth function defined on $G \cap \bar{\Omega}$. Then the differential operator B_p can be written in the form*

$$B_p f = (A_0(v(0))^{-1} A_\nu(v(0)))^p \partial_\nu^p f + \sum_{i=0}^{p-1} C_{p,p-i} \partial_\nu^i f, \qquad 1 \le p \le \mu + 1, \tag{4.18}$$

where $C_{p,p-i}$ is a differential operator of order at most $p - i$ involving only the differentiation in the direction tangential to the surface which is parallel to Γ and on which x lies. Moreover, we have

$$C_{p,p-i} \in \mathscr{L}(H^{s-i}(G \cap \Omega), H^{s-p}(G \cap \Omega)), \qquad 1 \le p \le \mu + 1,\ p \le s \le 2\mu + 3. \tag{4.19}$$

Proof. We study B_p by using a partition of unity and changes of systems of local coordinates. Let U be a neighborhood of some point on the boundary Γ and let Φ be the diffeomorphism from U to $B_1(0)$ defined in § 5 after the proof of Lemma 5.2, where $B_1(0)$ is the open ball of radius 1 centered at the origin. We regard B_p as an operator acting on the space of smooth functions with supports contained in U. Since B_p is the sum of the terms like constant times $G_{i_1}(0) \cdots G_{i_q}(0)$, we study each $G_i(0)$ in the local coordinates. We denote by Φ_* the transformation of linear differential operators induced by Φ. Let

$$\Phi_*(G_i(0)) = \sum_{j=1}^n \hat{A}_j^{(i)} D_j + \hat{B}^{(i)},$$

where $D_j = \partial/\partial y_j$, $1 \le j \le n$. Then

$$\hat{A}_j^{(i)} = \sum_{k=1}^n \varphi_{j,k} A_k^{(i)}|_{x=\Psi(y)},$$

$$\hat{B}^{(i)} = B^{(i)}|_{x=\Psi(y)},$$

where $\varphi_{j,k} = \partial \Phi_j / \partial x_k$ and $\Psi = \Phi^{-1}$. It is obvious that

$$\Phi_*(G_{i_1}(0) \cdots G_{i_q}(0)) = \Phi_*(G_{i_1}(0)) \cdots \Phi_*(G_{i_q}(0)).$$

Hence each term of $\Phi_*(B_p)$ has the form

$$\text{const.}\,(\sum_{j=1}^{n} \hat{A}_j^{(i_1)} D_j + \hat{B}^{(i_1)}) \cdots (\sum_{j=1}^{n} \hat{A}_j^{(i_q)} D_j + \hat{B}^{(i_q)}).$$

It follows that

$$\Phi_*(B_p) = (\hat{A}_1^{(0)})^p D_1^p + \sum_{i=0}^{p-1} (\text{const.} \sum_{|l'| \le p-i} \hat{A}(p, (i, l')) D_{y'}^{l'}) D_1^i$$

where $\hat{A}(p, (i, l'))$ is analogous to $A(p, (i, l'))$ and $y' = (y_2, \dots, y_n)$, $l' = (l_2, \dots, l_n)$. We set

$$\text{(4.20)} \qquad C_{p,p-i} = \Psi_*(\text{const.} \sum_{|l'| \le p-i} \hat{A}(p, (i, l')) D_{y'}^{l'}).$$

Then $C_{p,p-i}$ is a differential operator of order at most $p - i$ having the property described in the statement of Lemma 4.4. We prove that $C_{p,p-i} \in \mathscr{L}(H^{s-i}(U \cap \Omega), H^{s-p}(U \cap \Omega))$, $1 \le p \le \mu + 1$, $p \le s \le 2\mu + 3$. Let $f \in H^{s-i}(U \cap \Omega)$, $1 \le s \le 2\mu + 3$. It is clear that $f(\Psi(\cdot)) \in H^{s-i}(B_1^+(0))$ with $B_1^+(0) = B_1(0) \cap \mathbf{R}_+^n$. We note that $\hat{A}(p, (i, l')) \in H^{\nu(p,i+|l'|)}(B_1^+(0))$ by Lemma 4.1. Since

$$\Phi_*(C_{p,p-i}) f(\Psi(y)) = \text{const.} \sum_{|l'| \le p-i} \hat{A}(p, (i, l')) D_{y'}^{l'} f(\Psi(y)),$$

it can be shown by an argument similar to that given in the proof of Corollary 4.2 that

$$\Phi_*(C_{p,p-i}) \in \mathscr{L}(H^{s-i}(B_1^+(0)), H^{s-p}(B_1^+(0))),$$
$$1 \le p \le \mu + 1,\ p \le s \le 2\mu + 3.$$

We note that so far the operators $C_{p,p-i}$, $0 \le i \le p - 1$, are defined only locally. Let f be a function defined on $G \cap \bar{\Omega}$. We choose a partition of unity subordinate to a suitable finite open covering of Γ. Let $f^{(j)} = \chi_j f$, $j = 1, \dots, N$, with χ_j cut off functions. For each $f^{(j)}$, $C_{p,p-i} f^{(j)}$ is defined by the argument as above. We set

$$\text{(4.21)} \qquad C_{p,p-i} f = \sum_{j=1}^{N} C_{p,p-i} f^{(j)}.$$

Thus $C_{p,p-i}$ is defined as an operator acting on $H^{s-i}(G \cap \Omega)$. It can be shown that the operators $C_{p,p-i}$, $0 \le i \le p - 1$, are determined uniquely by B_p. Hence, the proof of (4.19) is complete.

We define a new inner product in $\mathbf{C}^l$ by

$$\text{(4.22)} \qquad \langle u, w \rangle_0 = (A_0(v(0)) u, w) \quad \text{for } u, w \in \mathbf{C}^l.$$

Then $A_0(v(0))^{-1} A_\nu(v(0))$ becomes a selfadjoint operator, that is,

$$\langle A_0(v(0))^{-1} A_\nu(v(0)) u, w \rangle_0 = \langle u, A_0(v(0))^{-1} A_\nu(v(0)) w \rangle_0 \quad \text{for } u, w \in \mathbf{C}^l.$$

We set

$$L(x) = (A_0(v(0))^{-1} A_\nu(v(0)))(x) \qquad \text{for } x \in G \cap \bar{\Omega}.$$

Let $\bar{x}$ be an arbitrary point lying on Γ. Let $C(\bar{x})$ be a closed rectifiable Jordan curve with positive direction enclosing all the non-zero eigenvalues of $L(\bar{x})$. Define $T(\bar{x})$ by

$$T(\bar{x}) = \frac{1}{2\pi i} \int_{C(\bar{x})} \frac{1}{\lambda} (\lambda - L(\bar{x}))^{-1} d\lambda.$$

Since $\bar{x}$ is an arbitrary point on Γ, we obtain a complex matrix-valued function $T(\cdot)$ on Γ. For any $\bar{x}$, there is a suitable neighborhood of $\bar{x}$ in $\mathbf{R}^n$, say $U(\bar{x})$, such that we have

$$T(x) = \frac{1}{2\pi i} \int_{C(\bar{x})} \frac{1}{\lambda} (\lambda - L(x))^{-1} d\lambda \qquad \text{for } x \in U(\bar{x}) \cap \bar{\Omega}. \tag{4.23}$$

Notice that the eigenvalues of $L(x)$ depend continuously on x because $L(x)$ is a continuous function of x. This enables us to choose one and the same path $C(\bar{x})$ for all $x \in U(\bar{x})$. We may regard $T(x)$ as a matrix-valued function defined on $G \cap \bar{\Omega}$. We define $T_p(x)$, $p \geq 1$, by

$$T_p(x) = \frac{1}{2\pi i} \int_{C(\bar{x})} \frac{1}{\lambda^p} (\lambda - L(x))^{-1} d\lambda \qquad \text{for } x \in U(\bar{x}) \cap \bar{\Omega}. \tag{4.24}$$

$T_p(x)$ is also a complex matrix-valued function on $G \cap \bar{\Omega}$. Then $T_1(x) = T(x)$. We use Lemma C.6 with $r = 2\mu + 1$, $A(\lambda, x) = \lambda - L(x)$, and $\varphi(\lambda) = \dfrac{1}{\lambda^p}$. Then it turns out that

$$T_p(\cdot) \in H^{2\mu+2}(G \cap \Omega; B(\mathbf{C}^l)), \qquad p \geq 1. \tag{4.25}$$

We set

$$L_p(x) = ((A_0(v(0))^{-1} A_\nu(v(0)))(x))^p = L(x)^p, \qquad p \geq 1.$$

Then $L_1(x) = L(x)$. We have $T_p(x) L_p(x) = L_p(x) T_p(x) = P(x)$, $p \geq 1$, $x \in G \cap \bar{\Omega}$, where

$$P(x) = \frac{1}{2\pi i} \int_{C(\bar{x})} (\lambda - L(x))^{-1} d\lambda \qquad \text{for } x \in U(\bar{x}) \cap \bar{\Omega}.$$

Actually, $P(x)$ is the sum of eigenprojections corresponding to the eigenvalues of $L(x)$, which do not belong to the zero-group. Hence $P(x)$ is a projection operator acting on $\mathbf{C}^l$. We call $T_p(x)$ the pseudo-inverse of $L_p(x)$. We shall show that $L_p(\cdot)$ belongs to a Sobolev space on $G \cap \bar{\Omega}$. Since $A_0(v(0))^{-1} \in H^{2\mu+2}(\Omega)$ by (4.8) and $A_\nu(v(0)) \in H^{2\mu+2}(\Omega)$ by (4.9), an application of Lemma C.1 i) yields

$$L_p(\cdot) \in H^{2\mu+2}(\Omega; B(\mathbf{C}^l)), \qquad p \geq 1.$$

We extend $M(x)$ to a C^∞-function defined on $\bar{\Omega}$ which is denoted by the same $M(x)$. We may assume without loss of generality that $M(x)$ is a selfadjoint operator acting on $\mathbf{C}^l$ equipped with the new inner product introduced above. In fact, if this is not the case, we may replace $M(x)$ by $M^{(*)}(x)M(x)$, where $M^{(*)}(x) = (A_0(v(0))^{-1}M^*A_0(v(0)))(x)$. Then $\langle M(x)u, w\rangle_0 = \langle u, M^{(*)}(x)w\rangle_0$ for $u, w \in \mathbf{C}^l$, that is, $M^{(*)}(x)$ is the adjoint operator of $M(x)$. Note that $M(x)u = 0$ if and only if $M^{(*)}(x)M(x)u = 0$. We set

$$Q(x) = \frac{1}{2\pi i}\int_{C(\bar{x})} (\lambda - M(x))^{-1}d\lambda \qquad \text{for } x \in U(\bar{x}) \cap \bar{\Omega} \tag{4.26}$$

where $C(\bar{x})$ is, like the one used in (4.23), a path with positive direction enclosing all the nonzero eigenvalues of $M(x)$ where x lies on Γ. $Q(x)$ can also be regarded as a matrix-valued function defined in $G \cap \bar{\Omega}$. We see that $Q(x)$ is the orthogonal projection onto the direct sum of the eigenspaces of $M(x)$, such that the corresponding eigenvalues do not belong to the zero-group. Let

$$K(x) = \frac{1}{2\pi i}\int_{C(\bar{x})} \frac{1}{\lambda}(\lambda - M(x))^{-1}d\lambda \qquad \text{for } x \in U(\bar{x}) \cap \bar{\Omega}. \tag{4.27}$$

Then $K(x)$ is what we call the pseudo-inverse of $M(x)$. We have $K(x)M(x) = M(x)K(x) = Q(x)$. By using Lemma C.6, we obtain

$$Q(\cdot) \in H^{2\mu+2}(G \cap \Omega; B(\mathbf{C}^l)).$$

Combining this with (4.25), we get

$$T_pQ(\cdot) \in H^{2\mu+2}(G \cap \Omega; B(\mathbf{C}^l)), \qquad p \geq 1.$$

Hence, denoting by T_pQ the multiplication operator defined by $T_pQ(\cdot)$, we conclude that

$$T_pQ \in \mathscr{L}(H^s(G \cap \Omega)), \qquad 0 \leq s \leq 2\mu + 2. \tag{4.28}$$

Proof of Lemma 3.1A. Following the line of the proof of Lemma 3.3 in [21] with suitable modifications, we construct f_k and F_k. By Lemma B.3 with $r = s = 2m + 3$, it is seen that there exists a sequence $\{F_k\}$ in $C^{2m+3}([0, T]; H^{2m+3}(\Omega))$ such that $F_k \to F$ in $V_*^m(0, T; \Omega)$. We choose a sequence $\{g_k\}$ in $H^{2m+3}(\Omega)$ with $g_k \to f$ in $H^m(\Omega)$. Then, we write the desired sequence $\{f_k\}$ as $f_k = g_k - h_k$ where $h_k \in H^{m+2}(\Omega)$ must be so chosen that $h_k \to 0$ in $H^m(\Omega)$ and

$$MB_ph_k = M(B_pg_k + E_pF_k) \qquad \text{on } \Gamma,\ 0 \leq p \leq m. \tag{4.29}$$

The construction of h_k is as follows. By Lemma 4.4, the equation (4.29) is written as

$$\begin{cases} Mh_k = Mg_k, & \\ M(A_0(v(0))^{-1}A_\nu(v(0)))^p\partial_\nu^p h_k + M\sum_{i=0}^{p-1} C_{p,p-i}\partial_\nu^i h_k & \\ = M(B_pg_k + E_pF_k), & 1 \leq p \leq m, \end{cases} \quad \text{on } \Gamma. \tag{4.30}$$

Then it suffices to solve

$$
(4.31)\qquad \begin{cases} h_k = Qg_k, \\ (A_0(v(0))^{-1}A_\nu(v(0)))^p \partial_\nu^p h_k \\ \quad = Q((B_p g_k + E_p F_k) - \sum_{i=p}^{p-1} C_{p,p-i}\partial_\nu^i h_k), \qquad 1 \le p \le m, \end{cases} \qquad \text{on } \Gamma.
$$

Note that $MQ = QM = M$, because M is supposed to be a selfadjoint operator acting on $\mathbf{C}^l$ with the inner product defined by (4.22). To solve (4.31), it suffices in turn to solve

$$
(4.32)\qquad \begin{cases} h_k = Qg_k, \\ \partial_\nu^p h_k = T_p Q((B_p g_k + E_p F_k) - \sum_{i=0}^{p-1} C_{p,p-i}\partial_\nu^i h_k), \\ \qquad\qquad 1 \le p \le m, \end{cases} \qquad \text{on } \Gamma.
$$

Recall that we set $L_p = (A_0(v(0))^{-1}A_\nu(v(0)))^p$ and that $T_p L_p = L_p T_p = P$ for $p \ge 1$. Here P and Q are orthogonal projections onto $(\operatorname{Ker} L)^\perp$ and $(\operatorname{Ker} M)^\perp$, respectively, for $x \in \Gamma$. By the maximal nonnegativity, we have $\operatorname{Ker} A_\nu \subset \operatorname{Ker} M$ on Γ. Hence $\operatorname{Ker} L \subset \operatorname{Ker} M$ on Γ. It follows that $(\operatorname{Ker} L)^\perp \supset (\operatorname{Ker} M)^\perp$ on Γ. This implies that $PQ = Q$ on Γ. Hence (4.31) follows from (4.32). The equation (4.32) reduces to

$$
(4.33)\qquad \partial_\nu^p h_k = b_{p,k} \qquad \text{on } \Gamma,\ 0 \le p \le m,
$$

where

$$
\begin{aligned}
b_{0,k} &= Qg_k, \\
b_{p,k} &= T_p Q(B_p g_k + E_p F_k) - T_p Q \sum_{i=0}^{p-1} C_{p,p-i} b_{i,k}, \qquad 1 \le p \le m.
\end{aligned}
$$

Let $\mathscr{A}_p$, $0 \le p \le m$, denote the operator defined by

$$
\begin{aligned}
\mathscr{A}_0(f, F) &= Qf, \\
\mathscr{A}_p(f, F) &= T_p Q(B_p f + E_p F), \qquad 1 \le p \le m.
\end{aligned}
$$

Then, by Corollary 4.2 and (4.28), we have

$$
(4.34)\qquad \mathscr{A}_p \in \mathscr{L}(H^s(G\cap\Omega) \times V_*^s(0, T; G\cap\Omega), H^{s-p}(G\cap\Omega)), \\ \qquad\qquad 1 \le p \le m,\ p \le s \le 2\mu + 3.
$$

Let $\mathscr{B}_0 = \mathscr{A}_0$. Define the operators $\mathscr{B}_p$, $1 \le p \le m$, inductively by

$$
(4.35)\qquad \mathscr{B}_p = \mathscr{A}_p - T_p Q \sum_{i=0}^{p-1} C_{p,p-i}\mathscr{B}_i,
$$

where

(4.36) $$\mathscr{B}_p \in \mathscr{L}(H^s(G\cap\Omega)\times V_*^s(0, T; G\cap\Omega), H^{s-p}(G\cap\Omega)),$$
$$1\le p\le m,\ p\le s\le 2\mu+3.$$

It follows from (4.19), (4.28), and (4.34) that the operators $\mathscr{B}_p$, $1\le p\le m$, are well defined. Setting $s=2m+3$ in (4.36), we have

(4.37) $$b_{p,k}=\mathscr{B}_p(g_k, F_k)\in H^{2m+3-p}(G\cap\Omega), \qquad 0\le p\le m.$$

Let $a_{p,k}=\mathscr{A}_p(g_k, F_k)$ and let $a_p=\mathscr{A}_p(f, F)$. Noting that $g_k\to f$ in $H^m(G\cap\Omega)$ and $F_k\to F$ in $V_*^m(0, T; G\cap\Omega)$, and then using (4.34) with $s=m$, we have

$$a_{p,k}-a_p=\mathscr{A}_p(g_k-f, F_k-F)\to 0 \quad\text{in}\quad H^{m-p}(G\cap\Omega),\ 0\le p\le m-1.$$

Let $b_p=\mathscr{B}_p(f, F)$. We have also

$$b_{p,k}\to b_p \quad\text{in}\quad H^{m-p}(G\cap\Omega) \quad\text{as}\quad k\to\infty,\ 0\le p\le m-1.$$

Hence

$$\gamma(a_{p,k})\to\gamma(a_p) \quad\text{in}\quad H^{m-p-\frac{1}{2}}(\Gamma) \quad\text{as}\quad k\to\infty,\ 0\le p\le m-1,$$

and

$$\gamma(b_{p,k})\to\gamma(b_p) \quad\text{in}\quad H^{m-p-\frac{1}{2}}(\Gamma) \quad\text{as}\quad k\to\infty,\ 0\le p\le m-1.$$

Here γ denotes the trace operator on Γ. Since $Mf_p=0$ on Γ, $0\le p\le m-1$, and $Q=KM$, we have $\gamma(a_p)=0$, $0\le p\le m-1$. By induction on p, this shows that $\gamma(b_p)=0$, $0\le p\le m-1$. Note that, by (4.35),

$$b_0=a_0,$$
$$b_p=a_p-T_pQ\sum_{i=1}^{p-1}C_{p,p-i}b_i, \qquad 1\le p\le m.$$

This proves that

(4.38) $$\gamma(b_{p,k})\to 0 \quad\text{in}\quad H^{m-p-\frac{1}{2}}(\Gamma) \quad\text{as}\quad k\to\infty,\ 0\le p\le m-1.$$

Recalling (4.33), we define a sequence $\{y_k\}$ in $H^{2m+3}(\Omega)$ by

$$y_k=R_{2m+3,m}(\gamma(b_{0,k}),\dots,\gamma(b_{m-1,k})).$$

Here $R_{2m+3,m}$ is the operator described in Lemma C.2 ii) with p, q replaced by $2m+3$ amd m, respectivly. Then it follows that

(4.39) $$y_k\to 0 \quad\text{in}\quad H^m(\Omega) \quad\text{and}\quad \gamma(\partial_\nu^p y_k)=\gamma(b_{p,k}), \qquad 0\le p\le m-1.$$

We write $h_k=y_k+z_k$ where $z_k\in H^{m+2}(\Omega)$ must be so chosen that

(4.40) $$\begin{cases} z_k\to 0 \text{ in } H^m(\Omega),\\ \partial_\nu^p z_k=0 \text{ on } \Gamma,\ 0\le p\le m-1,\\ \partial_\nu^m z_k=b_{m,k}-\partial_\nu^m y_k=b_{m,k}\equiv w_k \text{ on } \Gamma.\end{cases}$$

Let C_k be a constant such that $\|w_k\|_{H^m(\Gamma)} \leq C_k$. Without loss of generality we may assume that $C_k \to \infty$. To solve the set of equations (4.40) for z_k we reduce our problem to the case where $\Omega = \mathbf{R}^n_+$. The construction of such a sequence of functions for this case is given in [21]. We state it here for the sake of completeness. Let $\psi_k(r) = m!r^m\phi(C_k^4 r)$, where $\phi \in C_0^\infty(\mathbf{R})$ with $\phi(r) = 1$ for r near 0. Then $\psi_k^{(i)}(0) = 0$ for $0 \leq i \leq m-1$ and $\psi_k^{(m)}(0) = 1$. Also, $\|\psi_k\|_{H^m(0,\infty)} \leq$ const. C_k^{-2}. Then the desired sequence $\{z_k\}$ is given by

$$(4.41) \qquad z_k = \psi_k(x_1)w_k(x_2,\dots,x_n).$$

Since $\psi_k \in C_0^\infty(\mathbf{R})$ and $w_k \in H^{m+2}(\mathbf{R}^{n-1})$, we have $z_k \in H^{m+2}(\mathbf{R}^n_+)$. In addition,

$$\partial_1^p z_k|_{x_1=0} = (\partial_1^p \psi_k)w_k|_{x_1=0} = 0, \qquad 0 \leq p \leq m-1,$$

$$\partial_1^m z_k|_{x_1=0} = (\partial_1^m \psi_k)w_k|_{x_1=0} = w_k,$$

and

$$\|z_k\|_{H^m(\mathbf{R}^n_+)} \leq \|\psi_k\|_{H^m(0,\infty)} \cdot \|w_k\|_{H^m(\mathbf{R}^{n-1})} \leq C_k^{-1} \to 0 \qquad \text{as } k \to \infty.$$

Thus, (4.40) is proved. We see that

$$h_k \in H^{m+2}(\Omega), \; h_k \to 0 \quad \text{in} \quad H^m(\Omega).$$

Since

$$\partial_1^p h_k = b_{p,k} \qquad \text{on } \Gamma, \; 0 \leq p \leq m,$$

we find that $\{h_k\}$ is the desired sequence. This completes the proof of Lemma 3.1A.

§5. Reduction to the problem in the half space

In this section, we reduce the initial boundary value problem (3.34), (3.35), (3.36) to the one in the half space. This is a preliminary for the proof of the estimate (3.37) which is uniform in k. For simplicity, we write ε in place of $1/C(k)^2$. We write also $v_\varepsilon, f_\varepsilon, F_\varepsilon, U_\varepsilon$ instead of v_k, f_k, F_k, U_k in the following. Needless to say, ε is small enough. Then the problem (3.34), (3.35), (3.36) is written as

$$(5.1) \qquad A_0(v_\varepsilon)\partial_t u + \sum_{j=1}^n A_j(v_\varepsilon)\partial_j u + B(v_\varepsilon)u + \varepsilon \sum_{j=1}^n v_j \partial_j u = F_\varepsilon + \varepsilon \sum_{j=1}^n v_j \partial_j U_\varepsilon \quad \text{in } [0, T] \times \Omega,$$

$$(5.2) \qquad M(x)u = 0 \qquad \text{on } [0, T] \times \Gamma,$$

$$(5.3) \qquad u(0, x) = f_\varepsilon(x) \qquad \text{for } x \in \Omega.$$

First, we prove the following lemma.

Lemma 5.1. *Assume that conditions* i)-viii) *of Theorem* 2.1 *hold. Then, for any* $\bar{x} \in \Gamma$, *there exists a neighborhood* U *of* $\bar{x}$ *and an* $l \times l$ *unitary matrix valued function* $T(x) \in C^\infty(U \cap \bar{\Omega})$ *having the following properties: Let* $x \in U \cap \Gamma$ *and let* $\mathscr{M}(x) = \operatorname{Ker} M(x)$. *Then,* $u \in \mathscr{M}(x)$ *is equivalent to* $T(x)u \in \tilde{\mathscr{M}}$. *Also,* $u \in \mathscr{N}(x)$ *is equivalent to* $T(x)u \in \tilde{\mathscr{N}}$. *Here* $\tilde{\mathscr{M}}$ *and* $\tilde{\mathscr{N}}$ *are subspaces of* $\mathbf{C}^l$ *independent of* x *such that* $\tilde{\mathscr{M}} \supset \tilde{\mathscr{N}}$.

Proof. Since $\mathscr{N}(x)$ is a smoothly varying subspace by condition vi), we can choose an orthonormal basis $\{e_i(x)\}_{i=l_1+1}^{l}$ of $\mathscr{N}(x)$ which depends smoothly on x in a neighborhood of $\bar{x}$, say, $U \cap \Gamma$. Let $l_2 = l - \dim \mathscr{M}(x)$. Then, by condition ii), l_2 is constant on Γ. The maximal nonnegativity implies that $\mathscr{N}(x) \subset \mathscr{M}(x)$ on Γ. Hence $0 \le l_2 \le l_1$. $\mathscr{M}(x)$ varies smoothly with x, so we can choose an orthonormal basis $\{e_i(x)\}_{i=l_2+1}^{l_1}$ of $\mathscr{M}(x) \cap \mathscr{N}(x)^\perp$ which is also a smooth function of x on $U \cap \Gamma$. Finally, let $\{e_i(x)\}_{i=1}^{l_2}$ be a smoothly varying orthonormal basis of $\mathscr{M}(x)^\perp$ defined on $U \cap \Gamma$. Note that $\mathscr{M}(x)^\perp \subset \mathscr{N}(x)^\perp$. The collection $\{e_i(x)\}_{i=1}^{l}$ is an orthonormal basis of $\mathbf{C}^l$ which belongs to C^∞-class on $U \cap \Gamma$. Define $T(x)$ by

$$T(x) = {}^t(e_1(x)^*, \dots, e_l(x)^*), \qquad x \in U \cap \Gamma.$$

Then $T(x)$ is a unitary matrix valued C^∞-function on $U \cap \Gamma$. Let $v = T(x)u$. Let $\tilde{\mathscr{M}} = \{v \in \mathbf{C}^l \mid v_1 = \cdots = v_{l_2} = 0\}$ and let $\tilde{\mathscr{N}} = \{v \in \mathbf{C}^l \mid v_1 = \cdots = v_{l_1} = 0\}$. Then, $u \in \mathscr{M}(x)$ is equivalent to $v \in \tilde{\mathscr{M}}$. Also, $u \in \mathscr{N}(x)$ is equivalent to $v \in \tilde{\mathscr{N}}$. We take an arbitrary C^∞-extention of $\{e_i(x)\}_{i=1}^{l}$, which we denote again by the same $\{e_i(x)\}_{i=1}^{l}$. We orthogonalize this basis in the descending order of the suffixes of $e_i(x)$, starting from $e_l(x)$, by the method of Schmidt. Let us denote the resulting orthonormal basis by $\{\hat{e}_i(x)\}_{i=1}^{l}$. Observe that $e_i(x) = \hat{e}_i(x)$, $x \in U \cap \Gamma$. Define $T(x)$ by

$$T(x) = {}^t(\hat{e}_1(x)^*, \dots, \hat{e}_l(x)^*), \qquad x \in U \cap \bar{\Omega}.$$

Then $T(x)$ has the desired properties.

Lemma 5.2. *Let* $v \in X^{[\frac{n}{2}]+2}([0, T]; \Omega)$ *and let* v *take values in* $\mathbf{R}^l$. *Let* $M_{[\frac{n}{2}]+2}$ *be a constant such that* $|||v|||_{X^{[\frac{n}{2}]+2}([0,T];\Omega)} \le M_{[\frac{n}{2}]+2}$. *Assume that conditions* i), ii), iv)-viii) *of Theorem* 2.1 *hold. Then, for any* $\bar{x} \in \Gamma$, *there exists a neighborhood* U *of* $\bar{x}$ *that depends only on* $M_{[\frac{n}{2}]+2}$ *having the following properties: Let* $l_1 = l - \dim \mathscr{N}(x)$. (*By condition* viii), l_1 *is constant on* Γ. *Note that, by condition* vii), $0 < l_1 < l$.) *Let* $T(x)$ *be the unitary matrix valued function defined on a neighborhood of* $\bar{x}$ *which was constructed in the proof of the preceding lemma. Let us write* $T(x)A_\nu(v)T(x)^*$ *in the form of a block matrix, namely, let*

$$T(x)A_\nu(v)T(x)^* = \begin{pmatrix} \hat{A}_\nu^{I\,I} & \hat{A}_\nu^{I\,II} \\ \hat{A}_\nu^{II\,I} & \hat{A}_\nu^{II\,II} \end{pmatrix} \qquad \text{on } [0, T] \times (U \cap \bar{\Omega}).$$

Here $\hat{A}_\nu^{I\,I}$ *and* $\hat{A}_\nu^{II\,II}$ are $l_1 \times l_1$ *and* $(l - l_1) \times (l - l_1)$ *submatrices, respectively. Accordingly,* $\hat{A}_\nu^{I\,II}$ *is an* $l_1 \times (l - l_1)$ *submatrix and* $\hat{A}_\nu^{II\,I} = (\hat{A}_\nu^{I\,II})^*$. *Then* $\hat{A}_\nu^{I\,I}$ *is* *invertible on* $[0, T] \times (U \cap \bar{\Omega})$ *and satisfies*

$$|(\hat{A}_\nu^{II})^{-1}| \le C(M_{[\frac{n}{2}]+2}) \quad \text{on } [0, T] \times (U \cap \bar{\Omega}). \tag{5.4}$$

Here $C(M_{[\frac{n}{2}]+2})$ *is a constant depending only on* $M_{[\frac{n}{2}]+2}$. *Furthermore,* $\hat{A}_\nu^{I\,II} = 0$, $\hat{A}_\nu^{II\,I} = 0$, $\hat{A}_\nu^{II\,II} = 0$, *on* $[0, T] \times (U \cap \Gamma)$.

Proof. Let $T(x)$ be the unitary matrix-valued function given in the proof of Lemma 5.1. Assume that $T(x)$ is defined on a neighborhood U of $\bar{x} \in \Gamma$. We write $T(x)A_\nu(v)T(x)^*$ in the form of a block matrix, namely,

$$T(x)A_\nu(v)T(x)^* = \begin{pmatrix} \hat{A}_\nu^{I\,I} & \hat{A}_\nu^{I\,II} \\ \hat{A}_\nu^{II\,I} & \hat{A}_\nu^{II\,II} \end{pmatrix} \quad \text{on } [0, T] \times (U \cap \bar{\Omega}),$$

where $\hat{A}_\nu^{II\,I} = (\hat{A}_\nu^{I\,II})^*$. Then it follows from condition vi) that $\hat{A}_\nu^{I\,II} = 0$, $\hat{A}_\nu^{II\,I} = 0$, $\hat{A}_\nu^{II\,II} = 0$ on $[0, T] \times (U \cap \Gamma)$. Since rank A_ν = rank $\hat{A}_\nu^{I\,I} = l_1$, $\hat{A}_\nu^{I\,I}$ is invertible on $[0, T] \times (U \cap \Gamma)$. Let K be the set of $v \in X^{[\frac{n}{2}]+2}([0, T]; \Omega)$ that takes values in $\mathbf{R}^l$ and satisfies condition iv) and the estimate

$$|||\, v \,|||_{X^{[\frac{n}{2}]+2}([0,T];\Omega)} \le M_{[\frac{n}{2}]+2}.$$

Note that $X^{[\frac{n}{2}]+2}([0, T]; \Omega) \hookrightarrow C^1([0, T]; H^{[\frac{n}{2}]+1}(\Omega))$ is a continuous imbedding. On the other hand, the imbedding of the latter space into $C([0, T]; C(\bar{\Omega})) = C([0, T] \times \bar{\Omega})$ is compact. Therefore, K is a precompact set in $C([0, T] \times \bar{\Omega})$. We denote by $\bar{K}$ the closure of K in this space. Any function belonging to $\bar{K}$ takes values in $\mathbf{R}^l$, satisfies condition iv), and its norm in this space is bounded by $C_0 M_{[\frac{n}{2}]+2}$ where C_0 is the norm of the continuous imbedding $X^{[\frac{n}{2}]+2}([0, T]; \Omega) \hookrightarrow C([0, T] \times \bar{\Omega})$. The map $(t, x, v) \mapsto |\det A_{v(x)}^{I\,I}(x, v(t, x))|$ is continuous from $[0, T] \times (\bar{U} \cap \Gamma) \times \bar{K}$ into $\mathbf{R}$ and the value of this map is always positive. Hence there exists a constant $d(M_{[\frac{n}{2}]+2})$ depending only on $M_{[\frac{n}{2}]+2}$ such that

$$\inf_{t,x,v} |\det \hat{A}_\nu^{I\,I}(x, v(t, x))| \ge d(M_{[\frac{n}{2}]+2}) > 0,$$

where the infimum is taken over $(t, x, v) \in [0, T] \times (\bar{U} \cap \Gamma) \times \bar{K}$. It follows that

$$\sup_{t,x,v} |(\hat{A}_\nu^{I\,I}(x, v(t, x))^{-1}| \le C_1(M_{[\frac{n}{2}]+2}).$$

Here the supremum is taken over $(t, x, v) \in [0, T] \times (\bar{U} \cap \Gamma) \times \bar{K}$, because the cofactor matrix of $\hat{A}_\nu^{I\,I}$ can be estimated by a constant depending only on $M_{[\frac{n}{2}]+2}$. Next, let $x \in U \cap \bar{\Omega}$. Let us write

$$\hat{A}_\nu^{I\,I}(x, v(t, x)) = \hat{A}_\nu^{I\,I}(x'(x), v(t, x'(x))) + R,$$

where $x'(x)$ is the point on Γ nearest to x. Then

$$(\hat{A}_\nu^{I\,I}(x, v(t, x)))^{-1}$$

$$= (1 + (\hat{A}_\nu^{II}(x'(x), v(t, x'(x))))^{-1}R)^{-1}(\hat{A}_\nu^{II}(x'(x), v(t, x'(x))))^{-1}.$$

The remainder R can be estimated by δ times a constant depending only on $M_{[\frac{n}{2}]+2}$ if $|x - x'(x)| < \delta$, because $X^{[\frac{n}{2}]+2}([0, T]; \Omega) \subsetneq C([0, T]; C^1(\Omega))$ is a continuous imbedding. Namely

$$|R| \leq \delta C_2(M_{[\frac{n}{2}]+2}).$$

We choose δ so that

$$0 < \delta < \frac{1}{2C_1(M_{[\frac{n}{2}]+2})C_2(M_{[\frac{n}{2}]+2})}.$$

Then we have

$$|(\hat{A}_\nu^{II}(x, v(t, x)))^{-1}| \leq 2C_1(M_{[\frac{n}{2}]+2})$$

for $(t, x, v) \in [0, T] \times (U \cap V) \times K$, where $V = \{x \in \Omega \,|\, |x - x'(x)| < \delta\}$. We denote $U \cap V$ still by U. This completes the proof of Lemma 5.2.

Let $\bar{x}$ be an arbitrary fixed point on Γ. We may assume that Γ is represented by $x_1 = \psi(x')$ in a neighborhood W of $\bar{x}$ where $x' = (x_2, \dots, x_n)$. We consider a suitable neighborhood V of the origin in $\mathbf{R}^n$ and define a transformation $\Psi = \Psi(y) = (\Psi_1, \dots, \Psi_n)$ from V into W by

$$\begin{cases} \Psi_1(y) = \psi(\bar{x}' + y') - \nu_1(\psi(\bar{x}' + y'), \bar{x}' + y')y_1, \\ \Psi_j(y) = \bar{x}_j + y_j - \nu_j(\psi(\bar{x}' + y'), \bar{x}' + y')y_1, \qquad 2 \leq j \leq n, \end{cases}$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = (\bar{x}_1, \bar{x}')$, $y = (y_1, y_2, \dots, y_n) = (y_1, y')$, and $\nu(x) = (\nu_1, \dots, \nu_n)$ is the outward unit normal to Γ. Note that $(\psi(x'), x')$ lies on Γ. It is shown that the Jacobian $J(\Psi(y))$ evaluated at $y_1 = 0$ does not vanish. Hence the inverse transformation of Ψ exists which we denote by $\Phi = \Phi(x) = (\Phi_1, \dots, \Phi_n)$. Let U be the image of V by Ψ. Then Φ is a diffeomorphism of class C^∞ from U onto V and $\Phi(U \cap \Gamma) = V \cap \{y \,|\, y_1 = 0\}$. Since Ω is represented by $x_1 > \psi(x')$ in U, we have $\Phi(U \cap \Omega) = V \cap \{y \,|\, y_1 > 0\}$. For any $x \in \bar{\Omega} \cap U$, there exists a unique point $x'(x)$ on Γ which is nearest to x. This is assured by the existence of the inverse transformation Φ. The outward unit normal ν to Γ can be extended to a vector-valued function defined in a neighborhood of Γ by setting $\nu(x'(x))$ for $x \in U$. Then the vector field $\sum_{j=1}^n \nu_j \frac{\partial}{\partial x_j}$ defined on U corresponds to the vector field $-\frac{\partial}{\partial y_1}$ by the transformation Φ. Namely, for any differentiable function g, we have

$$\sum_{j=1}^n \nu_j \frac{\partial}{\partial x_j} g(x) = -\frac{\partial}{\partial y_1} g(\Psi(y)).$$

Now we return to the problem (5.1), (5.2), (5.3). The solution u of this problem depends on the parameter ε, although its dependence is not explicitly written. First we observe that each v_ε satisfies the boundary condition and hence we have $\mathscr{N}(x) = \operatorname{Ker} A_{\nu(x)}(v_\varepsilon(t, x))$ for $(t, x) \in [0, T] \times \Gamma$ and ε small enough. This enables us to use Lemma 5.1 for the problem (5.1), (5.2), (5.3) with arbitrary ε. On the other hand, v_ε converges to v in $X_*^\mu([0, T]; \Omega)$ by iii) of Lemma 3.1. This implies that the norm of v_ε in $X^{[\frac{n}{2}]+2}([0, T]; \Omega)$ is uniformly bounded in ε. Therefore, Lemma 5.2 holds with $v = v_\varepsilon$. In particular, the estimate (5.4) holds on a neighborhood $[0, T] \times (U \cap \bar{\Omega})$ independent of ε with constant $C(M_{[\frac{n}{2}]+2})$ which is also independent of ε. We take an appropriate finite covering $\{\mathscr{U}_i\}_{i=0}^N$ of $\bar{\Omega}$ such that $\mathscr{U}_i \cap \Gamma \neq \phi$, $i = 1, \dots, N$, $\mathscr{U}_0 \subset\subset \Omega$ and each $\mathscr{U}_i$ has the above mentioned properties. We choose a partition of unity $\{\varphi_i\}_{i=0}^N$ subordinate to this covering such that $\sum_{i=0}^N \varphi_i^2 = 1$ and $\varphi_i \geq 0$.

Let $w = w(x)$ be a function defined on $\mathscr{U}_i \cap \Omega$. We denote $w(\Psi^i(y))$ defined on $V \cap \{y \mid y_1 \geq 0\}$ by $\tilde{w} = \tilde{w}(y)$. This convention will be used in the following. For any solution u to the initial boundary value problem (5.1), (5.2), (5.3), let us put $u^i = u^i(t, y) = T_i(y)(\varphi_i u)(t, y)$. Here $T_i(y)$ is the unitary matrix valued function constructed in the proof of Lemma 5.1. Recall that $u \in X^{m+1}([0, T]; \Omega)$ by the existence theorem of solutions for the non-characteristic initial boundary value problem. Then, $\operatorname{supp} u^i \subset V \cap \{y \mid y_1 \geq 0\}$ and $u^i \in X^{m+1}([0, T]; V \cap \{y \mid y_1 \geq 0\})$ is the solution of the following mixed problem in the half space.

$$(5.5) \qquad A_0^i(y, \tilde{v}_\varepsilon)\frac{\partial u^i}{\partial t} + \sum_{j=1}^n A_j^i(y, \tilde{v}_\varepsilon)\frac{\partial u^i}{\partial y_j} + B^i(y, \tilde{v}_\varepsilon)u^i - \varepsilon\frac{\partial u^i}{\partial y_1} = H^i \quad \text{in } [0, T] \times \{y \mid y_1 > 0\},$$

$$(5.6) \qquad M^i u^i = 0 \quad \text{on } [0, T] \times \{y \mid y_1 = 0\},$$

$$(5.7) \qquad u^i(0, y) = f_\varepsilon^i(y) \quad \text{for } y \in \{y \mid y_1 > 0\},$$

where

$$\tilde{v}_\varepsilon = \tilde{v}_\varepsilon(t, y),\ \tilde{\nu} = \tilde{\nu}(y) = (\tilde{\nu}_1, \dots, \tilde{\nu}_n),$$

$$A_0^i(y, \tilde{v}_\varepsilon) = T_i(y)A_0(\tilde{v}_\varepsilon)T_i(y)^*,$$

$$A_1^i(y, \tilde{v}_\varepsilon) = -\,T_i(y)\sum_{l=1}^n \tilde{\nu}_l A_l(\tilde{v}_\varepsilon)T_i(y)^*,$$

$$A_j^i(y, \tilde{v}_\varepsilon) = T_i(y)\sum_{l=1}^n A_l(\tilde{v}_\varepsilon)\left(\widetilde{\frac{\partial \Phi_j^i}{\partial x_l}}\right)(y)T_i(y)^*, \qquad 2 \leq j \leq n,$$

$$B^i(y, \tilde{v}_\varepsilon) = T_i(y)B(\tilde{v}_\varepsilon)T_i(y)^*,$$

$$H^i = H^i(\varepsilon; \tilde{F}_\varepsilon, \tilde{U}_\varepsilon, \tilde{v}_\varepsilon, \tilde{u}) = T_i(y)\widetilde{(\varphi_i F_\varepsilon)}(t, y) + \varepsilon T_i(y)\sum_{j=1}^n \widetilde{\left(\nu_j\varphi_i\frac{\partial U_\varepsilon}{\partial x_j}\right)}(t, y)$$
$$+\, T_i(y)\sum_{j=1}^n A_j(\tilde{v}_\varepsilon)\widetilde{\left(\frac{\partial \varphi_i}{\partial x_j}u\right)}(t, y) + \varepsilon T_i(y)\sum_{j=1}^n \widetilde{\left(\nu_j\frac{\partial \varphi_i}{\partial x_j}u\right)}(t, y)$$

$$+ T_i(y) \sum_{j=2}^{n} \sum_{l=1}^{n} A_l(\tilde{v}_\varepsilon)\widetilde{\left(\frac{\partial \Phi_j^i}{\partial x_l}\right)}(y) T_i(y)^* \frac{\partial T_i(y)}{\partial y_j} \widetilde{(\varphi_i u)}(t, y)$$

$$- T_i(y) \sum_{l=1}^{n} \tilde{v}_l A_l(\tilde{v}_\varepsilon) T_i(y)^* \frac{\partial T_i(y)}{\partial y_1} \widetilde{(\varphi_i u)}(t, y) - \varepsilon \frac{\partial T_i(y)}{\partial y_1} \widetilde{(\varphi_i u)}(t, y)$$

and where

$$f_\varepsilon^i(y) = T_i(y) \widetilde{(\varphi_i f_\varepsilon)}(y).$$

Since $\tilde{v}_\varepsilon \in X^{\mu+1}([0, T]; V \cap \{y \mid y_1 \geq 0\})$, $f_\varepsilon \in H^{m+1}(\Omega)$, and $T_i \in C^\infty(V \cap \{y \mid y_1 \geq 0\})$, it is seen by using Lemma C.3 that

$$A_j^i(y, \tilde{v}_\varepsilon) \in X^{\mu+1}([0, T]; V \cap \{y \mid y_1 \geq 0\}), \qquad 0 \leq j \leq n,$$

$$B^i(y, \tilde{v}_\varepsilon) \in X^{\mu+1}([0, T]; V \cap \{y \mid y_1 \geq 0\}),$$

$$f_\varepsilon^i \in H^{m+1}(V \cap \{y \mid y_1 \geq 0\}).$$

We have also

$$H^i(\varepsilon; \tilde{F}_\varepsilon, \tilde{U}_\varepsilon, \tilde{v}_\varepsilon, \tilde{u}) \in X^m([0, T]; V \cap \{y \mid y_1 \geq 0\}),$$

because $U_\varepsilon \in X^{m+1}([0, T]; \Omega)$ and $F_\varepsilon \in H^{m+1}([0, T] \times \Omega)$. Note that M^i is a constant matrix by virtue of Lemma 5.1 and that the boundary subspace $\operatorname{Ker} M^i$ defined by (5.6) is maximal nonnegative on $[0, T] \times (V \cap \{y \mid y_1 = 0\})$ for $-A_1^i(y, \tilde{v}_\varepsilon) + \varepsilon I$. We write $A_1^i(y, \tilde{v}_\varepsilon)$ in the form of a block matrix, namely,

$$A_1^i(y, \tilde{v}_\varepsilon) = \begin{pmatrix} A_1^{iI\,I}(\varepsilon) & A_1^{iI\,II}(\varepsilon) \\ A_1^{iII\,I}(\varepsilon) & A_1^{iII\,II}(\varepsilon) \end{pmatrix} \quad \text{in } [0, T] \times (V \cap \{y \mid y_1 \geq 0\}),$$

where $A_1^{iI\,I}(\varepsilon)$ and $A_1^{iII\,II}(\varepsilon)$ are $l_1 \times l_1$ and $(l - l_1) \times (l - l_1)$ submatrices, respectively. By Lemma 5.2, $A_1^{iI\,I}(\varepsilon)$ is invertible on $[0, T] \times (V \cap \{y \mid y_1 \geq 0\})$ for any ε and satisfies

$$|(A_1^{iI\,I}(\varepsilon))^{-1}| \leq C(M_{[\frac{n}{2}]+2}). \tag{5.8}$$

In addition, $A_1^{iI\,II}(\varepsilon) = 0$, $A_1^{iII\,I}(\varepsilon) = 0$, $A_1^{iII\,II}(\varepsilon) = 0$ on $[0, T] \times (V \cap \{y \mid y_1 = 0\})$.

Finally we observe that, if u is a solution of the initial boundary value problem (5.1), (5.2), (5.3), then, $u_0 = \varphi_0 u$ is the solution of the following Cauchy problem.

$$A_0(v_\varepsilon)\partial_t u^0 + \sum_{j=1}^{n} A_j(v_\varepsilon)\partial_j u^0 + B(v_\varepsilon)u^0 + \varepsilon \sum_{j=1}^{n} v_j \partial_j u^0 = H^0 \quad \text{in } [0, T] \times \mathscr{U}_0, \tag{5.9}$$

$$u^0(0, x) = f_\varepsilon^0(x) \qquad \text{for } x \in \mathscr{U}_0. \tag{5.10}$$

Here

$$H^0 = H^0(\varepsilon; F_\varepsilon, U_\varepsilon, v_\varepsilon, u) = \varphi_0 F_\varepsilon + \sum_{j=1}^{n} A_j(v_\varepsilon)(\partial_j \varphi_0)u + \varepsilon \sum_{j=1}^{n} v_j(\partial_j \varphi_0)u$$

$$+ \varepsilon \sum_{j=1}^{n} v_j \varphi_0 \partial_j U_\varepsilon,$$

$$f_\varepsilon^0 = \varphi_0 f_\varepsilon.$$

§6. The Proof of the uniform estimates

In this section, we prove the estimates (3.37), (3.38). The existence of $u \in X^{m+1}([0, T]; \Omega)$ that satisfies (3.34), (3.35), (3.36) is assumed here. Let $\Omega = \mathbf{R}_+^n$. For an arbitrary smooth function w defined on $[0, T] \times \overline{\mathbf{R}_+^n}$, we set

$$|||w(t)|||_{m,\tan}^2 = \sum_{|\gamma| \leq m} \|D_\star^\gamma w(t)\|^2,$$

where $D_\star^\gamma = \partial_t^j x_1^{\alpha_1} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$ and $\gamma = (j, \alpha)$. We write also

$$|||w(t)|||_{m,(*)}^2 = \sum_{\substack{|\gamma| + 2k \leq m \\ k \geq 1}} \|D_\star^\gamma \partial_1^k w(t)\|^2.$$

Then

$$|||w(t)|||_{m,\tan}^2 + |||w(t)|||_{m,(*)}^2 = |||w(t)|||_{m,*}^2.$$

Let us write for the moment $A_j^i(y, \tilde{v}_\varepsilon) = A_j^i(\varepsilon)$, $0 \leq j \leq n$, $B^i(y, \tilde{v}_\varepsilon) = B^i(\varepsilon)$, and $H^i(\varepsilon; \tilde{F}_\varepsilon, \tilde{U}_\varepsilon, \tilde{v}_\varepsilon, \tilde{u}) = H^i(\varepsilon)$. We rewrite (5.5) as

$$\text{(6.1)} \quad \begin{pmatrix} A_0^{iI\,I}(\varepsilon) & A_0^{iI\,II}(\varepsilon) \\ A_0^{iII\,I}(\varepsilon) & A_0^{iII\,II}(\varepsilon) \end{pmatrix} \partial_t \begin{pmatrix} u_I^i \\ u_{II}^i \end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix} A_j^{iI\,I}(\varepsilon) & A_j^{iI\,II}(\varepsilon) \\ A_j^{iII\,I}(\varepsilon) & A_j^{iII\,II}(\varepsilon) \end{pmatrix} \partial_j \begin{pmatrix} u_I^i \\ u_{II}^i \end{pmatrix}$$

$$+ \begin{pmatrix} B^{iI\,I}(\varepsilon) & B^{iI\,II}(\varepsilon) \\ B^{iII\,I}(\varepsilon) & B^{iII\,II}(\varepsilon) \end{pmatrix} \begin{pmatrix} u_I^i \\ u_{II}^i \end{pmatrix} - \varepsilon \partial_1 \begin{pmatrix} u_I^i \\ u_{II}^i \end{pmatrix} = \begin{pmatrix} H_I^i(\varepsilon) \\ H_{II}^i(\varepsilon) \end{pmatrix}$$

$$\text{in } [0, T] \times \mathbf{R}_+^n.$$

Here $u^i = {}^t(u_I^i, u_{II}^i) \in \mathbf{C}^{l_1} \times \mathbf{C}^{l-l_1}$, $H^i(\varepsilon) = {}^t(H_I^i(\varepsilon), H_{II}^i(\varepsilon)) \in \mathbf{C}^{l_1} \times \mathbf{C}^{l-l_1}$ with $l_1 = l - \dim \mathcal{N}(x)$. $A_j^{iII}(\varepsilon)$ and $A_j^{iII\,II}(\varepsilon)$ are $l_1 \times l_1$ and $(l - l_1) \times (l - l_1)$ submatrices, etc. We prepare two lemmas that play a crucial rôle in the proof of the estimate (3.37). Let $u \in X^{m+1}([0, T]; \Omega)$ satisfy (3.34), (3.35), (3.36). Let u^i, $0 \leq i \leq N$, be the functions defined in terms of u as in the previous section. We recall that u^i, $1 \leq i \leq N$, comes from the boundary patches, while u^0 corresponds to the patch that does not intersect with the boundary. Each u^i satisfies (5.5), (5.6), (5.7), where $1 \leq i \leq N$. On the other hand, u^0 satisfies (5.9), (5.10).

As we observed earlier, iii) of Lemma 3.1 implies that there exist constants $M_{[\frac{n}{2}]+2}$, M_r^*, $r = \mu - 1$, μ, satisfying

$$|||v_\varepsilon|||_{X^{[\frac{n}{2}]+2}([0,T];\Omega)} \leq M_{[\frac{n}{2}]+2},$$
$$|||v_\varepsilon|||_{X_*^r([0,T];\Omega)} \leq M_r^*, \quad r = \mu - 1, \mu$$

for any ε.

The first lemma of this section proves the estimates for u^i that are uniform in ε.

Lemma 6.1. *There exists a positive constant ε_0 depending only on $M_{[\frac{n}{2}]+2}$ such that the following estimates hold: For $t \in [0, T]$ and $0 < \varepsilon < \varepsilon_0$,*

$$(6.2)\qquad |||u^i(t)|||_{m,\tan} \le C(M_{[\frac{n}{2}]+2})\,|||u^i(0)|||_{m,\tan} + C(M_\mu^*)\int_0^t (|||u_I^i(\tau)|||_{m+1,(*)} + |||u^i(\tau)|||_{m,*} + |||H^i(\varepsilon, \tau)|||_{m,\tan})\,d\tau,$$

$$(6.3)\qquad |||u_I^i(t)|||_{m+1,(*)} \le C(M_{\mu-1}^*)\{|||u^i(t)|||_{m,*} + |||H^i(\varepsilon, t)|||_{m-1,*}\},$$

$$(6.4)\qquad |||u_{II}^i(t)|||_{m,(*)} \le C(M_{[\frac{n}{2}]+2})\,|||u_{II}^i(0)|||_{m,(*)} + C(M_\mu^*)\int_0^t (|||u_I^i(\tau)|||_{m+1,(*)} + |||u^i(\tau)|||_{m,*} + |||H^i(\varepsilon, \tau)|||_{m,(*)})\,d\tau.$$

Here M_μ^* and $M_{\mu-1}^*$ are the constants described above. $C(\cdot)$ takes positive values and is an increasing function of its argument that is independent of ε.

The next lemma gives the estimate for u^0 that is also uniform in ε.

Lemma 6.2. *For $t \in [0, T]$, we have for any ε*

$$(6.5)\qquad |||u^0(t)|||_{m,*} \le C(M_{[\frac{n}{2}]+2})\,|||u^0(0)|||_{m,*} + C(M_\mu^*)\int_0^t (|||u^0(\tau)|||_{m,*} + |||H^0(\varepsilon, \tau)|||_{m,*})\,d\tau.$$

Here $M_{[\frac{n}{2}]+2}$ and M_μ^* are the constants described before the preceding lemma and $C(\cdot)$ is similar to the one appearing in the same lemma.

Assuming for a while that the above two lemmas are valid, we prove the estimates (3.37), (3.38).

Proof of the estimates (3.37), (3.38). From the definition of the norm, we have

$$(6.6)\qquad |||u^i(t)|||_{m,*} \le |||u^i(t)|||_{m,\tan} + |||u_I^i(t)|||_{m,(*)} + |||u_{II}^i(t)|||_{m,(*)}$$

for $1 \le i \le N$. The first and the third terms on the right hand side are estimated by (6.2) and (6.4), respectively. The second term on the right hand side is bounded as follows.

$$|||u_I^i(t)|||_{m,(*)} \le \int_0^t |||u_I^i(\tau)|||_{m+1,(*)}\,d\tau + |||u_I^i(0)|||_{m,(*)}.$$

Combining these estimates we obtain from (6.6)

$$|||u^i(t)|||_{m,*} \le C(M_{[\frac{n}{2}]+2})\,|||u^i(0)|||_{m,*}$$

$$+ C(M_\mu^*) \int_0^t (|||u_I^i(\tau)|||_{m+1,(*)} + |||u^i(\tau)|||_{m,*} + |||H^i(\varepsilon, \tau)|||_{m,*}) d\tau.$$

Then we use (6.3) and get

$$(6.7) \qquad |||u^i(t)|||_{m,*}$$
$$\leq C(M_{[\frac{n}{2}]+2}) |||u^i(0)|||_{m,*} + C(M_\mu^*) \int_0^t (|||u^i(\tau)|||_{m,*} + |||H^i(\varepsilon, \tau)|||_{m,*}) d\tau,$$

for $1 \leq i \leq N$. Summing up (6.7) from $i = 1$ to N and adding the resulting estimate and (6.5), we obtain

$$(6.8) \qquad |||u(t)|||_{m,*}$$
$$\leq C(M_{[\frac{n}{2}]+2}) |||u(0)|||_{m,*} + C(M_\mu^*) \int_0^t (|||u(\tau)|||_{m,*} + \sum_{i=0}^N |||H^i(\varepsilon, \tau)|||_{m,*}) d\tau.$$

Notice that $T_i \in C^\infty(V \cap \{y \mid y_1 \geq 0\})$ does not depend on v and that $v \in C^\infty(\bar{\Omega})$, φ_i, $\Phi_i \in C^\infty(\mathscr{U}_i \cap \bar{\Omega})$. We have for $0 \leq t \leq T$

$$\sum_{i=0}^N |||H^i(\varepsilon, t)|||_{m,*} = |||H^0(\varepsilon, t)|||_{m,*} + \sum_{i=1}^N |||H^i(\varepsilon, t)|||_{m,*}$$
$$\leq |||\varphi_0 F_\varepsilon|||_{m,*} + \sum_{j=1}^n |||A_j(v_\varepsilon)(\partial_j \varphi_0) u|||_{m,*}$$
$$+ \varepsilon \sum_{j=1}^n |||v_j (\partial_j \varphi_0) u|||_{m,*} + \varepsilon \sum_{j=1}^n |||v_j \varphi_0 \partial_j U_\varepsilon|||_{m,*}$$
$$+ \sum_{i=1}^N \Bigg\{ |||T_i(\widetilde{\varphi_i F_\varepsilon})|||_{m,*} + \varepsilon |||T_i \sum_{j=1}^n \left(\widetilde{v_j \varphi_i \frac{\partial U_\varepsilon}{\partial x_j}}\right)|||_{m,*}$$
$$+ |||T_i \sum_{j=1}^n A_j(\tilde{v}_\varepsilon) \left(\widetilde{\frac{\partial \varphi_i}{\partial x_j} u}\right)|||_{m,*} + \varepsilon |||T_i \sum_{j=1}^n \left(\widetilde{v_j \frac{\partial \varphi_i}{\partial x_j} u}\right)|||_{m,*}$$
$$+ |||T_i \sum_{j=2}^n \left(\sum_{l=1}^n A_l(\tilde{v}_\varepsilon) \widetilde{\frac{\partial \Phi_j^i}{\partial x_l}} T_i^*\right) \frac{\partial T_i}{\partial y_j} \widetilde{(\varphi_i u)}|||_{m,*}$$
$$+ |||T_i (\sum_{l=1}^n \tilde{v}_l A_l(\tilde{v}_\varepsilon)) T_i^* \frac{\partial T_i}{\partial y_1} \widetilde{(\varphi_i u)}|||_{m,*} + \varepsilon |||\frac{\partial T_i}{\partial y_1} \widetilde{(\varphi_i u)}|||_{m,*} \Bigg\}$$
$$\leq C\{|||F_\varepsilon(t)|||_{m,*} + \sum_{j=1}^n |||(A_j(v_\varepsilon)u)(t)|||_{m,*} + \varepsilon |||u(t)|||_{m,*} + \varepsilon \sum_{j=1}^n |||\partial_j U_\varepsilon(t)|||_{m,*}\}.$$

Here C is a positive constant independent of ε. By using (A.1), (A.2) in Appendix A, we obtain

$$|||(A_j(v_\varepsilon)u)(t)|||_{m,*} \leq C(M_\mu^*) |||u(t)|||_{m,*}, \qquad 1 \leq j \leq n.$$

Hence,

$$(6.9)\qquad \sum_{i=0}^{N} |||H^i(\varepsilon, t)|||_{m,*} \le C(M_\mu^*)\,|||u(t)|||_{m,*} + C\{|||F_\varepsilon(t)|||_{m,*} + \varepsilon\,|||U_\varepsilon(t)|||_{m+1}\}.$$

Similarly, we obtain

$$(6.10)\qquad \begin{aligned}\sum_{i=0}^{N} |||H^i(\varepsilon, t)|||_{m-1,*} &\le C(M_{\mu-1}^*)\,|||u(t)|||_{m-1,*} + C\{|||F_\varepsilon(t)|||_{m-1,*} + \varepsilon\,|||U_\varepsilon(t)|||_{m}\}\\ &\le C(M_{\mu-1}^*)\,|||u(0)|||_{m-1,*} + C\{|||F_\varepsilon(0)|||_{m-1,*} + \varepsilon\,|||U_\varepsilon(0)|||_{m}\}\\ &\quad + C(M_{\mu-1}^*)\int_0^t |||u(\tau)|||_{m-1,*}d\tau + C\int_0^t \{|||F_\varepsilon(\tau)|||_{m-1,*} + \varepsilon\,|||U_\varepsilon(\tau)|||_{m}\}d\tau.\end{aligned}$$

Substituting (6.9) for (6.8) and using Gronwall's inequality, we get

$$\begin{aligned}|||u(t)|||_{m,*} &\le C(M_{[\frac{n}{2}]+2})\,|||u(0)|||_{m,*}e^{C(M_\mu^*)t} + \varepsilon\,|||U_\varepsilon|||_{X^{m+1}([0,T];\Omega)}e^{C(M_\mu^*)t}\\ &\quad + C(M_\mu^*)\int_0^t e^{C(M_\mu^*)(t-\tau)}\,|||F_\varepsilon(\tau)|||_{m,*}d\tau.\end{aligned}$$

This completes the proof of the estimate (3.37). Finally, combining (6.8), (6.9), (6.10) with (6.3), we have the estimate

$$\begin{aligned}&|||u_I(t)|||_{m+1,(*)}\\ &\le \{C(M_{[\frac{n}{2}]+2})\,|||u(0)|||_{m,*} + C(M_{\mu-1}^*)\,|||u(0)|||_{m-1,*}\}e^{C(M_\mu^*)t}\\ &\quad + C(M_{\mu-1}^*)\{|||F_\varepsilon(0)|||_{m-1,*} + \varepsilon\,|||U_\varepsilon(0)|||_{m}\}e^{C(M_\mu^*)t} + \varepsilon\,|||U_\varepsilon|||_{X^{m+1}([0,T];\Omega)}e^{C(M_\mu^*)t}\\ &\quad + C(M_\mu^*)\int_0^t e^{C(M_\mu^*)(t-\tau)}\,|||F_\varepsilon(\tau)|||_{m,*}d\tau,\end{aligned}$$

from which we derive (3.38) immediately.

Now we prove Lemma 6.1.

Proof of Lemma 6.1 In what follows, we omit the indices i and ε for simplicity. For γ such tha $|\gamma| \le m$, take $D_\star^\gamma$ of (6.1), and take the $\mathbf{C}^l$ inner product of the resulting equation with $D_\star^\gamma u$. Then we integrate it over $\mathbf{R}_+^n$ to obtain

$$(6.11)\qquad \begin{aligned}&\frac{1}{2}\int_{\mathbf{R}_+^n} \partial_t(D_\star^\gamma u\cdot\overline{A_0 D_\star^\gamma u})\,dx + \frac{1}{2}\sum_{j=1}^{n}\int_{\mathbf{R}_+^n} \partial_j(D_\star^\gamma u\cdot\overline{A_j D_\star^\gamma u})\,dx\\ &-\frac{\varepsilon}{2}\int_{\mathbf{R}_+^n} \partial_1(D_\star^\gamma u\cdot\overline{D_\star^\gamma u})\,dx + \varepsilon\,\mathrm{Re}\int_{\mathbf{R}_+^n} D_\star^\gamma u\cdot\overline{\alpha_1 D_\star^{\gamma'}\partial_1 u}\,dx\\ &= \mathrm{Re}\int_{\mathbf{R}_+^n} D_\star^\gamma u\cdot\overline{J_\gamma}\,dx.\end{aligned}$$

The independent variable is denoted by x in place of y here, and

$$J_\gamma = -[D_\star^\gamma, A_0]\partial_t u - \sum_{j=1}^{n} [D_\star^\gamma, A_j]\partial_j u - D_\star^\gamma(Bu)$$

$$+ D_\star^\gamma H + \frac{1}{2}\operatorname{Div}\vec{A} D_\star^\gamma u + \alpha_1 A_1 D_\star^{\gamma'}\partial_1 u \equiv \sum_{i=1}^{6} J_\gamma^{(i)},$$

$$\operatorname{Div}\vec{A} = \partial_t A_0 + \sum_{j=1}^{n}\partial_j A_j,$$

$$\gamma' = (j, \alpha_1 - 1, \alpha_2, \ldots, \alpha_n).$$

$J_\gamma^{(i)}$ is defined to be the i-th term in the expression of J_γ. Note that $\partial_1(x_1^{\alpha_1}\partial_1^{\alpha_1}) = (x_1^{\alpha_1}\partial_1^{\alpha_2})\partial_1 + \alpha_1(x_1^{\alpha_1-1}\partial_1^{\alpha_1-1})\partial_1$. Obviously $D_\star^\gamma = x_1 D_\star^{\gamma'}\partial_1$ and $x_1 \geq 0$ in $\mathbf{R}_+^n$. Hence

$$\text{(6.12)} \qquad \operatorname{Re}\int_{\mathbf{R}_+^n} D_\star^\gamma u \cdot \overline{\alpha_1 D_\star^{\gamma'}\partial_1 u}\, dx = \operatorname{Re}\int_{\mathbf{R}_+^n} x_1 D_\star^{\gamma'}\partial_1 u \cdot \overline{\alpha_1 D_\star^{\gamma'}\partial_1 u}\, dx \geq 0.$$

Since supp u is compact, the integration by parts yields

$$\text{(6.13)} \qquad \frac{1}{2}\sum_{j=1}^{n}\int_{\mathbf{R}_+^n}\partial_j(D_\star^\gamma u \cdot \overline{A_j D_\star^\gamma u})\,dx - \frac{\varepsilon}{2}\int_{\mathbf{R}_+^n}\partial_1(D_\star^\gamma u \cdot \overline{D_\star^\gamma u})\,dx$$

$$= \frac{1}{2}\int_{\mathbf{R}^{n-1}}(D_\star^\gamma u \cdot \overline{(-A_1 + \varepsilon)D_\star^\gamma u})|_{x_1=0}\,dx',$$

where $x' = (x_2, \ldots, x_n)$. We notice that $D_\star^\gamma u$ lies in Ker M because M is constant on the boundary. It follows from Lemma 3.2 ii) that

$$\text{(6.14)} \qquad \int_{\mathbf{R}^{n-1}}(D_\star^\gamma u \cdot \overline{(-A_1 + \varepsilon)D_\star^\gamma u})|_{x_1=0}\,dx' \geq 0.$$

Using (6.12), (6.13), (6.14), we deduce from (6.11) that

$$\frac{1}{2}\int_{\mathbf{R}_+^n}\partial_t(D_\star^\gamma u \cdot \overline{A_0 D_\star^\gamma u})\,dx$$

$$\leq \operatorname{Re}\int_{\mathbf{R}_+^n} D_\star^\gamma u \cdot \overline{J_\gamma}\,dx \leq \|D_\star^\gamma\| \cdot \|J_\gamma\|.$$

Summing these inequalities over all γ with $|\gamma| \leq m$, and taking account of the fact that A_0 is positive definite, we obtain

$$\text{(6.15)} \qquad |||u(t)|||_{m,\tan} \leq C(M_{[\frac{n}{2}]+2})\left\{|||u(0)|||_{m,\tan} + \int_0^t \sum_{|\gamma|\leq m}\|J_\gamma\|\,d\tau\right\}.$$

We note that, if $|||v|||_{X^{[\frac{n}{2}]+2}([0,T];\Omega)} \leq M_{[\frac{n}{2}]+2}$, there exists a positive constant c depending only on $M_{[\frac{n}{2}]+2}$ such that $c^{-1} \leq A_0 \leq c$. This is shown by an argument analogous to the one employed in the proof of Lemma 5.2.

We estimate the integrand of the second term on the right hand side of

(6.15) in the following. By using (A.7) and (A.2) in Appendix A, we have

$$\sum_{|\gamma|\le m} \|J_\gamma^{(1)}\| = \sum_{|\gamma|\le m} \|[D_\star^\gamma, A_0]\partial_t u\| \le C(M_\mu^*) |||u(t)|||_{m,*}. \tag{6.16}$$

Also by using (A.7) and (A.2), we see that

$$\sum_{|\gamma|\le m} \|J_\gamma^{(2)}\| \le \sum_{|\gamma|\le m} \|[D_\star^\gamma, A_1]\partial_1 u\| + \sum_{j=2}^{n} \sum_{|\gamma|\le m} \|[D_\star^\gamma, A_j]\partial_j u\|$$
$$\le \sum_{|\gamma|\le m} \|[D_\star^\gamma, A_1]\partial_1 u\| + C(M_\mu^*) |||u|||_{m,*}.$$

The first term on the right hand side of the last inequality is estimated as follows. We observe that $D_\star^\eta A_1^{I\,II}|_{x_1=0} = 0$ and $D_\star^\eta A_1^{II\,II}|_{x_1=0} = 0$. Then by using (A.8), (A.9), and (A.2), we get

$$\sum_{|\gamma|\le m} \|[D_\star^\gamma, A_1]\partial_1 u\| \tag{6.17}$$
$$\le \sum_{|\gamma|\le m} (\|[D_\star^\gamma, A_1^{I\,I}]\partial_1 u_I\| + \|[D_\star^\gamma, A_1^{I\,II}]\partial_1 u_{II}\|$$
$$+ \|[D_\star^\gamma, A_1^{II\,I}]\partial_1 u_I\| + \|[D_\star^\gamma, A_1^{II\,II}]\partial_1 u_{II}\|)$$
$$\le C(M_\mu^*) |||u_I|||_{m+1,(*)} + C(M_\mu^*) |||u|||_{m,*}.$$

Hence

$$\sum_{|\gamma|\le m} \|J_\gamma^{(2)}\| \le C(M_\mu^*) |||u_I|||_{m+1,(*)} + C(M_\mu^*) |||u|||_{m,*}. \tag{6.18}$$

Similarly we apply (A.1) and (A.2) to obtain

$$\sum_{|\gamma|\le m} \|J_\gamma^{(3)}\| = \sum_{|\gamma|\le m} \|D_\star^\gamma(Bu)\| \le C\, |||Bu|||_{m,\tan} \le C(M_\mu^*) |||u|||_{m,*}. \tag{6.19}$$

It is easy to see that

$$\sum_{|\gamma|\le m} \|J_\gamma^{(4)}\| = \sum_{|\gamma|\le m} \|D_\star^\gamma H\| \le C\, |||H|||_{m,\tan}. \tag{6.20}$$

Utilizing (A.1) and (A.2), we have

$$\sum_{|\gamma|\le m} \|J_\gamma^{(5)}\| = \sum_{|\gamma|\le m} \|\mathrm{Div}\, \vec{A} D_\star^\gamma u\| \le C(M_{\mu-1}^*) |||u|||_{m,*}. \tag{6.21}$$

In view of the fact that $A_1^{I\,II}|_{x_1=0} = 0$ and $A_1^{II\,I}|_{x_1=0} = 0$, we use (A.8), (A.9), and (A.2) to obtain

$$(6.22) \qquad \sum_{|\gamma|\le m} \|J_\gamma^{(6)}\| = \sum_{|\gamma|\le m} \|A_1 \alpha_1 D_\star^{\gamma'} \partial_1 u\|$$

$$\le C \sum_{|\gamma|\le m-1} \|A_1 D_\star^\gamma \partial_1 u\|$$

$$\le \sum_{|\gamma|\le m-1} \sum_{|\eta|=1} (\|A_1^{II} D_\star^\gamma \partial_1 u_I\| + \|A_1^{I\,II} D_\star^\gamma \partial_1 u_{II}\|$$

$$+ \|A_1^{II\,I} D_\star^\gamma \partial_1 u_I\| + \|A_1^{II\,II} D_\star^\gamma \partial_1 u_{II}\|)$$

$$\le C(M^*_{2[\frac{n}{2}]+2}) |||u_I|||_{m+1,(*)} + C(M^*_{2[\frac{n}{2}]+2}) |||u|||_{m,*}.$$

Summing up (6.16), (6.18)–(6.22), we conclude that

$$(6.23) \qquad \sum_{|\gamma|\le m} \|J_\gamma\| \le C(M_\mu^*)(|||u|||_{m,*} + |||u_I|||_{m+1,(*)}) + C\,|||H|||_{m,\tan}.$$

Substituting (6.23) for (6.15) yields (6.2).

We prove (6.3). We see by (6.1) that u_I satisfies

$$(6.24) \qquad (A_1^{II} - \varepsilon I)\partial_1 u_I = -A_0^{II}\partial_t u_I - A_0^{I\,II}\partial_t u_{II} - A_1^{I\,II}\partial_1 u_{II}$$

$$- \sum_{j=2}^n (A_j^{II}\partial_j u_I + A_j^{I\,II}\partial_j u_{II}) - B^{II} u_I - B^{I\,II} u_{II} + H_I$$

$$\text{in } [0, T] \times \mathbf{R}_+^n.$$

For γ, k such that $|\gamma| + 2(k-1) \le m-1$, $k \ge 1$, take $D_\star^\gamma \partial_1^{k-1}$ of (6.24). Then we have

$$(A_1^{II} - \varepsilon I) D_\star^\gamma \partial_1^k u_I = K_{\gamma,k},$$

where

$$K_{\gamma,k} = -[D_\star^\gamma \partial_1^{k-1}, A_1^{II}]\partial_1 u_I - D_\star^\gamma \partial_1^{k-1}(A_1^{I\,II}\partial_1 u_{II}) - D_\star^\gamma \partial_1^{k-1}(A_0^{II}\partial_t u_I)$$

$$- D_\star^\gamma \partial_1^{k-1}(A_0^{I\,II}\partial_t u_{II}) - \sum_{j=2}^n D_\star^\gamma \partial_1^{k-1}(A_j^{II}\partial_j u_I + A_j^{I\,II}\partial_j u_{II})$$

$$- D_\star^\gamma \partial_1^{k-1}(B^{II} u_I + B^{I\,II} u_{II}) + D_\star^\gamma \partial_1^{k-1} H_I \equiv \sum_{i=1}^7 K_{\gamma,k}^{(i)}.$$

We define $K_{\gamma,k}^{(i)}$ to be the i-th term of the expression of $K_{\gamma,k}$. Since A_1^{II} is invertible, $(A_1^{II} - \varepsilon I)^{-1}$ exists for ε small enough. To see this, we write

$$(A_1^{II} - \varepsilon I)^{-1} = (I - \varepsilon (A_1^{II})^{-1})^{-1} (A_1^{II})^{-1}.$$

Then, by (5.8), $\sup_{x,t,v} |(A_1^{II})^{-1}| \le C(M_{[\frac{n}{2}]+2})$. Hence, if $\varepsilon \le (2C(M_{[\frac{n}{2}]+2}))^{-1}$, we have

$$\sup_{x,t,v} |(A_1^{II} - \varepsilon I)^{-1}| \le 2C(M_{[\frac{n}{2}]+2}).$$

We conclude that

(6.25)
$$\sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| D_*^\gamma \partial_1^k u_I \| \leq 2C(M_{[\frac{n}{2}]+2}) \sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| K_{\gamma,k} \|.$$

To estimate the right hand side of the above inequality, one proceeds as follows. Applying (A.8) and (A.2) to our situation, we get

(6.26)
$$\sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| K_{\gamma,k}^{(1)} \|$$
$$= \sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| [D_*^\gamma \partial_1^{k-1}, A_1^{II}] \partial_1 u_I \|$$
$$\leq C(M_{\mu-1}^*) \| u \|_{m,(*)}.$$

We have also

(6.27)
$$\sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| K_{\gamma,k}^{(2)} \| = \sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| D_*^\gamma \partial_1^{k-1} (A_1^{I II} \partial_1 u_{II}) \|$$
$$\leq \sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} (\| A_1^{I II} D_*^\gamma \partial_1^{k-1} \partial_1 u_{II} \| + \| [D_*^\gamma \partial_1^{k-1}, A_1^{I II}] \partial_1 u_{II} \|)$$
$$\leq C(M_{2[\frac{n}{2}]+4}^*) \sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| x_1 D_*^\gamma \partial_1^{k-1} \partial_1 u_{II} \| + C(M_{\mu-1}^*) \| u_{II} \|_{m,*}$$
$$\leq C(M_{\mu-1}^*) \| u_{II} \|_{m,*}.$$

Here we used (A.4) and (A.8), taking account of the fact that $A_1^{I II}|_{x_1=0} = 0$. After that we employed (A.2). By using (A.1) and (A.2), we obtain

(6.28)
$$\sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| K_{\gamma,k}^{(3)} \|$$
$$= \sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| D_*^\gamma \partial_1^{k-1} (A_0^{II} \partial_t u_I) \|$$
$$\leq C(M_{\mu-1}^*) \| \partial_t u_I \|_{m-1,*} \leq C(M_{\mu-1}^*) \| u_I \|_{m,*}.$$

Similar arguments show that

(6.29)
$$\sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| K_{\gamma,k}^{(4)} \|$$
$$= \sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| D_*^\gamma \partial_1^{k-1} (A_0^{I II} \partial_t u_{II}) \|$$
$$\leq C(M_{\mu-1}^*) \| u_{II} \|_{m,*},$$

and that

(6.30)
$$\sum_{\substack{|\gamma|+2(k-1)\leq m-1 \\ k\geq 1}} \| K_{\gamma,k}^{(5)} \|$$

$$= \sum_{\substack{|\gamma|+2(k-1)\le m-1\\ k\ge 1}} \sum_{j=2}^{n} \| D_\star^\gamma \partial_1^{k-1}(A_j^{II}\partial_j u_I + A_j^{I\,II}\partial_j u_{II})\|$$

$$\le C(M_{\mu-1}^*)\, |||u|||_{m,*}.$$

Also it is shown that

$$(6.31) \qquad \sum_{\substack{|\gamma|+2(k-1)\le m-1\\ k\ge 1}} \|K_{\gamma,k}^{(6)}\|$$

$$= \sum_{\substack{|\gamma|+2(k-1)\le m-1\\ k\ge 1}} \|D_\star^\gamma \partial_1^{k-1}(B^{II}u_I + B^{I\,II}u_{II})\|$$

$$\le C(M_{\mu-1}^*)\, |||u|||_{m-1,*}.$$

We see easily that

$$(6.32) \qquad \sum_{\substack{|\gamma|+2(k-1)\le m-1\\ k\ge 1}} \|K_{\gamma,k}^{(7)}\| = \sum_{\substack{|\gamma|+2(k-1)\le m-1\\ k\ge 1}} \|D_\star^\gamma \partial_1^{k-1} H_I\|$$

$$\le C\, |||H_I|||_{m-1,(*)}.$$

Summing up (6.26)–(6.32), we obtain

$$(6.33) \qquad \sum_{\substack{|\gamma|+2(k-1)\le m-1\\ k\ge 1}} \|K_{\gamma,k}\| \le C(M_{\mu-1}^*)\, |||u|||_{m,*} + C\, |||H_I|||_{m-1,*}.$$

Substituting (6.33) for (6.25) leads us to

$$(6.34) \qquad |||u_I|||_{m+1,(*)} \le C(M_{\mu-1}^*)\{\, |||u|||_{m,*} + |||H_I|||_{m-1,*}\}.$$

We prove (6.3). We see by (6.1) that u_{II} satisfies

$$(6.35) \qquad A_0^{II\,II}\partial_t u_{II} + \sum_{j=1}^{n} A_j^{II\,II}\partial_j u_{II} - \varepsilon \partial_1 u_{II}$$

$$= -\,(A_0^{II\,I}\partial_t u_I + \sum_{j=1}^{n} A_j^{II\,I}\partial_j u_I + B^{II\,I}u_I + B^{II\,II}u_{II}) + H_{II}$$

$$\text{in } [0, T] \times \mathbf{R}_+^n.$$

For γ, k such that $|\gamma| + 2k \le m$, $k \ge 1$, take $D_\star^\gamma \partial_1^k$ of (6.35), and take the $\mathbf{C}^l$ inner product of it with $D_\star^\gamma \partial_1^k u_{II}$. Then integrate the resulting equation over $\mathbf{R}_+^n$ to obtain

$$(6.36) \qquad \frac{1}{2}\int_{\mathbf{R}_+^n} \partial_t(D_\star^\gamma \partial_1^k u_{II} \cdot \overline{A_0^{II\,II} D_\star^\gamma \partial_1^k u_{II}})\,dx$$

$$+ \frac{1}{2}\sum_{j=1}^{n}\int_{\mathbf{R}_+^n} \partial_j(D_\star^\gamma \partial_1^k u_{II} \cdot \overline{A_j^{II\,II} D_\star^\gamma \partial_1^k u_{II}})\,dx$$

$$- \frac{\varepsilon}{2}\int_{\mathbf{R}_+^n} \partial_1(D_\star^\gamma \partial_1^k u_{II} \cdot \overline{D_\star^\gamma \partial_1^k u_{II}})\,dx$$

$$+ \varepsilon \operatorname{Re} \int_{\mathbf{R}^n_+} D^\gamma_\star \partial^k_1 u_{II} \cdot \overline{\alpha_1 D^{\gamma'}_\star \partial^{k+1}_1 u_{II}}\, dx$$

$$= \operatorname{Re} \int_{\mathbf{R}^n_+} D^\gamma_\star \partial^k_1 u_{II} \cdot \overline{L_{\gamma,k}}\, dx.$$

Here

$$\begin{aligned} L_{\gamma,k} = & \frac{1}{2} \operatorname{Div} \vec{A}^{II\,II} D^\gamma_\star \partial^k_1 u_{II} + \alpha_1 A^{II\,II}_1 D^{\gamma'}_\star \partial^k_1 u_{II} \\ & - [D^\gamma_\star \partial^k_1, A^{II\,II}_0]\, \partial_t u_{II} - \sum_{j=1}^n [D^\gamma_\star \partial^k_1, A^{II\,II}_j]\, \partial_j u_{II} \\ & - D^\gamma_\star \partial^k_1 (A^{II\,I}_0 \partial_t u_I) - \sum_{j=1}^n D^\gamma_\star \partial^k_1 (A^{II\,I}_j \partial_j u_I) \\ & - D^\gamma_\star \partial^k_1 (B^{II\,I} u_I + B^{II\,II} u_{II}) + D^\gamma_\star \partial^k_1 H_{II} \equiv \sum_{i=1}^8 L^{(i)}_{\gamma,k}, \end{aligned}$$

defining $L^{(i)}_{\gamma,k}$ to be the i-th term of the expression of $L_{\gamma,k}$. We recall that $\operatorname{Div} \vec{A}^{II\,II}$ stands for $\partial_t A^{II\,II}_0 + \sum_{j=1}^n \partial_j A^{II\,II}_j$. As in the proof of (6.12), it is seen that

$$\operatorname{Re} \int_{\mathbf{R}^n_+} D^\gamma_\star \partial^k_1 u_{II} \cdot \overline{\alpha_1 D^{\gamma'}_\star \partial^{k+1}_1 u_{II}}\, dx \geq 0. \tag{6.37}$$

We have also

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^n \int_{\mathbf{R}^n_+} \partial_j (D^\gamma_\star \partial^k_1 u_{II} \cdot \overline{A^{II\,II}_j D^\gamma_\star \partial^k_1 u_{II}})\, dx \\ & - \frac{\varepsilon}{2} \int_{\mathbf{R}^n_+} \partial_1 (D^\gamma_\star \partial^k_1 u_{II} \cdot \overline{D^\gamma_\star \partial^k_1 u_{II}})\, dx \\ & = \frac{1}{2} \int_{\mathbf{R}^{n-1}} (D^\gamma_\star \partial^k_1 u_{II} \cdot \overline{(-A^{II\,II}_1 + \varepsilon) D^\gamma_\star \partial^k_1 u_{II}})|_{x_1=0}\, dx' \\ & \geq 0, \end{aligned} \tag{6.38}$$

because $A^{II\,II}_1|_{x_1=0} = 0$. Making use of (6.37) and (6.38), we obtain from (6.36) that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbf{R}^n_+} \partial_t (D^\gamma_\star \partial^k_1 u_{II} \cdot \overline{A^{II\,II}_0 D^\gamma_\star \partial^k_1 u_{II}})\, dx \\ & \leq \operatorname{Re} \int_{\mathbf{R}^n_+} D^\gamma_\star \partial^k_1 u_{II} \cdot \overline{L_{\gamma,k}}\, dx \\ & \leq \| D^{\gamma,k}_{\star 1} u_{II} \| \cdot \| L_{\gamma,k} \|. \end{aligned}$$

Since $A_0^{II\,II}$ is positive definite, it follows that

$$(6.39)\qquad |||u_{II}(t)|||_{m,(*)} \le C(M_{[\frac{n}{2}]+2})\left\{|||u_{II}(0)|||_{m,(*)} + \int_0^t \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|L_{\gamma,k}\|\,d\tau\right\}.$$

Here $C(M_{[\frac{n}{2}]+2})$ is the constant that appears in (6.15). We estimate the integrand of the second term on the right hand side of (6.39) as follows. By using (A.6) and (A.2), it is seen that

$$(6.40)\qquad \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|L_{\gamma,k}^{(1)}\| = \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|\mathrm{Div}\,\vec{A}^{II\,II} D_*^\gamma \partial_1^k u_{II}\|$$

$$\le \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|\partial_t A_0^{II\,II} D_*^\gamma \partial_1^k u_{II}\| + \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \sum_{j=1}^n \|\partial_j A_j^{II\,II} D_*^\gamma \partial_1^k u_{II}\|$$

$$\le C(M_\mu^*)\,|||u_{II}|||_{m,(*)}.$$

We have also by (A.6) and (A.2)

$$(6.41)\qquad \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|L_{\gamma,k}^{(2)}\| = \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \alpha_1 \|A_1^{II\,II} D_*^{\gamma'} \partial_1^k u_{II}\|$$

$$\le C(M_\mu^*)\,|||u_{II}|||_{m-1,(*)}.$$

By using (A.7) and (A.2), it is shown that

$$(6.42)\qquad \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|L_{\gamma,k}^{(3)}\| = \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|[D_*^\gamma \partial_1^k,\ A_0^{II\,II}]\partial_t u_{II}\|$$

$$\le C(M_\mu^*)\,|||u_{II}|||_{m,*}.$$

Noting that $A^{II\,II}|_{x_1=0} = 0$ and using (A.9), (A.7), and (A.2), we have

$$(6.43)\qquad \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|L_{\gamma,k}^{(4)}\|$$

$$\le \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \left(\|[D_*^\gamma \partial_1^k,\ A_1^{II\,II}]\partial_1 u_{II}\| + \sum_{j=2}^n \|[D_*^\gamma \partial_1^k,\ A_j^{II\,II}]\partial_j u_{II}\|\right)$$

$$\le C(M_\mu^*)\,|||u_{II}|||_{m,*}.$$

It is not hard to see that

$$(6.44)\qquad \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|L_{\gamma,k}^{(5)}\| = \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|D_*^\gamma \partial_1^k (A_0^{II\,I}\partial_t u_I)\|$$

$$\le \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|A_0^{II\,I} D_*^\gamma \partial_1^k \partial_t u_I\| + \sum_{\substack{|\gamma|+2k\le m\\ k\ge 1}} \|[D_*^\gamma \partial_1^k,\ A_0^{II\,I}]\partial_t u_I\|$$

$$\le C(M_\mu^*)\,|||u_I|||_{m+1,(*)}.$$

Here we used (A.6), (A.7), and (A.2). Since $A_1^{II\,I}|_{x_1=0}=0$, we employ similar arguments to the above ones to obtain

$$\sum_{\substack{|\gamma|+2k\le m\\k\ge 1}}\|L_{\gamma,k}^{(6)}\| = \sum_{\substack{|\gamma|+2k\le m\\k\ge 1}}\sum_{j=1}^{n}\|D_\star^\gamma\partial_1^k(A_j^{II\,I}\partial_j u_I)\| \tag{6.45}$$

$$\le \sum_{\substack{|\gamma|+2k\le m\\k\ge 1}}(\|A_1^{II\,I}D_\star^\gamma\partial_1^k\partial_1 u_I\| + \|[D_\star^\gamma\partial_1^k,\ A_1^{II\,I}]\partial_1 u_I\|)$$

$$+ \sum_{\substack{|\gamma|+2k\le m\\k\ge 1}}\sum_{j=2}^{n}(\|A_j^{II\,I}D_\star^\gamma\partial_1^k\partial_j u_I\| + \|[D_\star^\gamma\partial_1^k,\ A_j^{II\,I}]\partial_j u_I\|)$$

$$\le C(M_\mu^*)\,|||u_I|||_{m+1,(*)}.$$

Also we get

$$\sum_{\substack{|\gamma|+2k\le m\\k\ge 1}}\|L_{\gamma,k}^{(7)}\| \tag{6.46}$$

$$= \sum_{\substack{|\gamma|+2k\le m\\k\ge 1}}\|D_\star^\gamma\partial_1^k(B^{II\,I}u_I + B^{II\,II}u_{II})\|$$

$$\le (M_\mu^*)\{|||u_I|||_{m,*} + |||u_{II}|||_{m,*}\}$$

and

$$\sum_{\substack{|\gamma|+2k\le m\\k\ge 1}}\|L_{\gamma,k}^{(8)}\| = \sum_{\substack{|\gamma|+2k\le m\\k\ge 1}}\|D_\star^\gamma\partial_1^k H_{II}\| \tag{6.47}$$

$$\le C\,|||H_{II}|||_{m,(*)}.$$

Summing up (6.40)–(6.47), we see that

$$\sum_{\substack{|\gamma|+2k\le m\\k\ge 1}}\|L_{\gamma,k}\| \le C(M_\mu^*)\{|||u|||_{m,*} + |||u_I|||_{m+1,(*)}\} + C\,|||H_{II}|||_{m,(*)}. \tag{6.48}$$

Substituting (6.48) and (6.39), we get (6.4). The proof of Lemma 6.1 is complete.

Lemma 6.2 is proved by a standard argument employed for the Cauchy problem.

Appendix A

We shall prove here several basic inequalities used in §3 and §6. Let $\Omega\subset\mathbf{R}^n$, $n\ge 2$, be an open bounded set with boundary Γ of C^∞-class. Let $u=u(t,x)$ and $v=v(t,x)$ be functions defined on $[0,T]\times\bar{\Omega}$ taking values in $\mathbf{C}^l$. We denote by $u\cdot v$ the standard inner product in $\mathbf{C}^l$ of u and v.

Lemma A.1. *Let $m \geq 1$ be an integer and let $r = \max\left(m, 2\left[\frac{n}{2}\right] + 3\right)$. If $u \in X_*^m([0, T]; \Omega)$ and $v \in X_*^r([0, T]; \Omega)$, then $u \cdot v \in X_*^m([0, T]; \Omega)$. Moreover, we have*

(A.1) $$|||(u \cdot v)(t)|||_{m,*} \leq C\, |||u(t)|||_{m,*}\, |||v(t)|||_{r,*} \quad \text{for } t \in [0, T],$$

where C is a constant independent of u and v. As a consequence,

$$|||u \cdot v|||_{X_*^m([0,T];\Omega)} \leq C\, |||u|||_{X_*^m([0,T];\Omega)}\, |||v|||_{X_*^r([0,T];\Omega)}.$$

Proof. We suppose that $\Omega = \mathbf{R}_+^n$ and that the support of u is contained in $\{x; |x| < 1\} \cap \overline{\mathbf{R}_+^n}$. The general case can be reduced to this case by localization and flattening of the boundary. For $t \in [0, T]$, we have by Leibniz's rule

$$|||(u \cdot v)(t)|||_{m,*} \leq C \sum_{|\gamma| + 2k \leq m} \sum_{\substack{\rho \leq \gamma \\ p \leq k}} \| D_*^{\gamma-\rho} \partial_1^{k-p} u(t) \cdot D_*^{\rho} \partial_1^{p} v(t) \|.$$

Then, by using Lemma C.1 i), we get

$$|||(u \cdot v)(t)|||_{m,*} \leq C\{K_1^m \cdot K_2^m + K_3^m \cdot K_4^m\},$$

with

$$K_1^m = \sum_{|\gamma| + 2k \leq m} \sum_{(\rho,p) \in I(\gamma,k)} \| D_*^{\gamma-\rho} \partial_1^{k-p} u(t) \|_{[\frac{n}{2}]+1},$$

$$K_2^m = \sum_{|\gamma| + 2k \leq m} \sum_{(\rho,p) \in I(\gamma,k)} \| D_*^{\rho} \partial_1^{p} v(t) \|,$$

$$K_3^m = \sum_{k=0}^{[\frac{n}{2}]} \sum_{|\gamma| + 2k \leq m} \sum_{(\rho,p) \in I(\gamma,k,i)} \| D_*^{\gamma-\rho} \partial_1^{k-p} u(t) \|_{[\frac{n}{2}]-i},$$

$$K_4^m = \sum_{i=0}^{[\frac{n}{2}]} \sum_{|\gamma| + 2k \leq m} \sum_{(\rho,p) \in I(\gamma,k,i)} \| D_*^{\rho} \partial_1^{p} v(t) \|_{i+1}.$$

Here

$$I(\gamma, k) = \left\{(\rho, p); \rho \leq \gamma, p \leq k, |\rho| + 2p \geq 2\left(\left[\frac{n}{2}\right] + 1\right)\right\},$$

$$I(\gamma, k; i) = \left\{(\rho, p); \rho \leq \gamma, p \leq k, 2\left(\left[\frac{n}{2}\right] - i\right) \leq |\rho| + 2p \leq 2\left(\left[\frac{n}{2}\right] - i\right) + 1\right\}.$$

If $(\rho, p) \in I(\gamma, k)$, then $2\left(\left[\frac{n}{2}\right] + 1\right) + |\gamma| - |\rho| + 2k - 2p \leq 2\left(\left[\frac{n}{2}\right] + 1\right) + |\gamma| + 2k \leq m$ and $|\rho| + 2p \leq m \leq r$. Hence

$$K_1^m \leq |||u(t)|||_{m,*}, \qquad K_2^m \leq |||v(t)|||_{r,*}.$$

If $(p, p) \in I(\gamma, k; i)$, then we see also that $2\left(\left[\frac{n}{2}\right] - i\right) + |\gamma| - |\rho| + 2k - 2p \leq$

$2\left(\left[\frac{n}{2}\right]-i\right)+|\gamma|+2k-2\left(\left[\frac{n}{2}\right]-i\right)=|\gamma|+2k\leq m$ and $2(i+1)+|\rho|+2p\leq 2(i+1)+2\left(\left[\frac{n}{2}\right]-i\right)+1=2\left[\frac{n}{2}\right]+3\leq r$. Hence

$$K_3^m \leq |||u(t)|||_{m,*}, \qquad K_4^m \leq |||v(t)|||_{r,*}.$$

Therefore, combining these estimates, we have

$$|||(u\cdot v)|||_{m,*} \leq C\, |||u(t)|||_{m,*}\, |||v(t)|||_{r,*}.$$

It follows that

$$\sup_{0\leq t\leq T} |||(u\cdot v)|||_{m,*} \leq C \sup_{0\leq t\leq T} |||u(t)|||_{m,*} \sup_{0\leq t\leq T} |||v(t)|||_{r,*}.$$

Furthermore, for any $t, t'\in[0, T]$ and $0\leq j\leq m$,

$$\begin{aligned} &\|\partial_t^j(u\cdot v)(t)-\partial_t^j(u\cdot v)(t')\|_{m-j,*} \\ &\leq |||(u\cdot v)(t)-(u\cdot v)(t')|||_{m,*} \\ &\leq C\{|||u(t)|||_{m,*}\,|||v(t)-v(t')|||_{r,*}+|||v(t')|||_{r,*}\,|||u(t)-u(t')|||_{m,*}\}. \end{aligned}$$

Since $u\in X_*^m([0,T];\Omega)$ and $v\in X_*^r([0,T];\Omega)$, this implies that $u\cdot v\in X_*^m([0,T];\Omega)$. This completes the proof of Lemma A.1.

Lemma A.2. *Let $r>n+2$ be an integer. Let $v\in X_*^r([0,T];\Omega)$ take values in $\mathbf{R}^l$. Let $A=A(u)$ be a smooth function of $u\in\mathbf{R}^l$ with values in the space of $l\times l$ complex matrices. Then, $A(v)\in X_*^r([0,T];\Omega)$. Moreover, we have*

(A.2) $$|||A(v)|||_{X_*^r([0,T];\Omega)} \leq C(N_{[\frac{n}{2}]+1})\{1+|||v|||^r_{X_*^r([0,T];\Omega)}\} \qquad \text{for } t\in[0,T],$$

where $N_{[\frac{n}{2}]+1}$ is a constant such that $\sup_{0\leq t\leq T}\|v(t)\|_{[\frac{n}{2}]+1}\leq N_{[\frac{n}{2}]+1}$ and $C(\cdot)$ is increasing as a function of its argument.

Proof. We refer the reader to [17].

Lemma A.3. *Let $r>n+2$ be an integer. Let u and v be in $X_*^r([0,T];\Omega)$ and take the values in $\mathbf{R}^l$. Let $A=A(u)$ be a smooth function of $u\in\mathbf{R}^l$ with values in the space of $l\times l$ complex matrices. Then we have*

(A.3) $$\begin{aligned} &|||A(u)-A(v)|||_{X_*^r([0,T];\Omega)} \\ &\leq C(N_{[\frac{n}{2}]+1})\,|||u-v|||_{X_*^r([0,T];\Omega)}(1+|||u|||^r_{X_*^r([0,T];\Omega)}+|||v|||^r_{X_*^r([0,T];\Omega)}), \end{aligned}$$

where $N_{[\frac{n}{2}]+1}$ is a constant such that

$$\max\left(\sup_{0\leq t\leq T}\|u(t)\|_{[\frac{n}{2}]+1},\ \sup_{0\leq t\leq T}\|v(t)\|_{[\frac{n}{2}]+1}\right)\leq N_{[\frac{n}{2}]+1}$$

and where $C(\cdot)$ depends increasingly on its argument.

In what follows we assume for simplicity that $\Omega = \mathbf{R}^n_+$ and $\operatorname{supp} u \subset \{x \mid |x| < 1\} \cap \overline{\mathbf{R}^n_+}$.

Lemma A.4. *Let $A(\cdot)$ be as in Lemma* A.2. *If v is in $X_*^{2[\frac{n}{2}]+4}([0, T]; \mathbf{R}^n_+)$ and u is in $X^1([0, T]; \mathbf{R}^n_+)$, and if $A(v(t, x)) = 0$ for $(t, x) \in [0, T] \times \partial\mathbf{R}^n_+$, then we have the estimate*

$$\text{(A.4)} \qquad \| A(v(t))\partial_1 u(t) \| \leq C \,|||\, A(v(t)) \,|||_{2[\frac{n}{2}]+4,*} \| x_1 \partial_1 u(t) \| \qquad \text{for } t \in [0, T],$$

where C is a positive constant independent of u and v.

Proof. In view of the fact that $A(v(t, x)) = 0$ on $[0, T] \times \partial\mathbf{R}^n_+$, we see that

$$A(v(t, x)) = \int_0^{x_1} \partial_1 A(v(t, \theta, x'))d\theta.$$

Hence, by using Lemma C.1 ii), it follows that

$$\begin{aligned}
&\| A(v(t))\partial_1 u(t) \| \\
&= \left\| \int_0^{x_1} \partial_1 A(v(t, \theta, x'))d\theta \partial_1 u(t) \right\| \\
&\leq \sup_{x \in \mathbf{R}^n_+} |\partial_1 A(v(t, x))| \, \| x_1 \partial_1 u(t) \| \\
&\leq \| \partial_1 A(v(t)) \|_{[\frac{n}{2}]+1} \| x_1 \partial_1 u(t) \| \\
&\leq |||\, A(v(t)) \,|||_{2[\frac{n}{2}]+4,*} \| x_1 \partial_1 u(t) \|.
\end{aligned}$$

This completes the proof of Lemma A.4.

Lemma A.5. *Let $m \geq 1$ be an integer and let $A(\cdot)$ be as in Lemma A.2. Let q be an integer such that $0 \leq q \leq m$ and let $r = \max\left(m, 2\left[\dfrac{n}{2}\right] + 3 + q\right)$. Assume that u lies in $X_*^{m-q}([0, T]; \mathbf{R}^n_+)$ and v lies in $X_*^r([0, T]; \mathbf{R}^n_+)$. Then, for any ρ, ρ', p, p' such that $|\rho| + |\rho'| + 2(p + p') \leq m$ and $q \leq |\rho| + 2p$, we have the estimate*

$$\text{(A.5)} \qquad \| D_\star^\rho \partial_1^p A(v(t)) D_\star^{\rho'} \partial_1^{p'} u(t) \| \leq C \,|||\, A(v(t)) \,|||_{r,*} \,|||\, u(t) \,|||_{m-q,*} \qquad \text{for } t \in [0, T].$$

In particular, when $p' \geq 1$, we get the estimate

$$\text{(A.6)} \qquad \| D_\star^\rho \partial_1^p A(v(t)) D_\star^{\rho'} \partial_1^{p'} u(t) \| \leq C \,|||\, A(v(t)) \,|||_{r,*} \,|||\, u(t) \,|||_{m-q,(*)} \qquad \text{for } t \in [0, T],$$

where C is a positive constant independent of u and v.

Proof. By using Lemma C.1 i), we get

$$\| D_\star^\rho \partial_1^p A(v(t)) D_\star^{\rho'} \partial_1^{p'} u(t) \|$$

$$\leq \begin{cases} \| D_*^\rho \partial_1^p A(v(t)) \| \, \| D_*^{\rho'} \partial_1^{p'} u(t) \|_{[\frac{n}{2}]+1} & \text{for } (\rho, p) \in I(q), \\ \| D_*^\rho \partial_1^p A(v(t)) \|_{i+1} \, \| D_*^{\rho'} \partial_1^{p'} u(t) \|_{[\frac{n}{2}]-i} & \text{for } (\rho, p) \in I(q; i), \ 0 \leq i \leq \left[\frac{n}{2}\right], \end{cases}$$

where

$$I(q) = \left\{ (\rho, p); \, |\rho| + 2p \geq 2\left(\left[\frac{n}{2}\right] + 1 \right) + q \right\},$$

$$I(q; i) = \left\{ (\rho, p); \, 2\left(\left[\frac{n}{2}\right] - i \right) + q \leq |\rho| + 2p \leq 2\left(\left[\frac{n}{2}\right] - i \right) + q + 1 \right\},$$

$$0 \leq i \leq \left[\frac{n}{2}\right].$$

Let $(\rho, p) \in I(q)$. Then, since $|\rho| + |\rho'| + 2(p + p') \leq m$ by assumption, we have $|\rho| + 2p \leq m \leq r$ and $2\left(\left[\frac{n}{2}\right] + 1 \right) + |\rho'| + 2p' = 2\left(\left[\frac{n}{2}\right] + 1 \right) + m - (|\rho| + 2p)$ $\leq 2\left(\left[\frac{n}{2}\right] + 1 \right) + m - 2\left(\left[\frac{n}{2}\right] + 1 \right) - q = m - q$. Hence,

$$\| D_*^\rho \partial_1^p A(v(t)) \| \, \| D_*^{\rho'} \partial_1^{p'} u(t) \|_{[\frac{n}{2}]+1} \leq ||| A(v(t)) |||_{r,*} \, ||| u(t) |||_{m-q,*}.$$

Let $(\rho, p) \in I(q; i)$. Then, by the same reason as before, we have $2(i + 1) + |\rho| + 2p \leq 2(i + 1) + 2\left(\left[\frac{n}{2}\right] - i \right) + q + 1 = 2\left[\frac{n}{2}\right] + 3 + q \leq r$ and $2\left(\left[\frac{n}{2}\right] - i \right) + |\rho'| + 2p' \leq 2\left(\left[\frac{n}{2}\right] - i \right) + m - (|\rho| + 2p) \leq 2\left(\left[\frac{n}{2}\right] - i \right) + m - 2\left(\left[\frac{n}{2}\right] - i \right) - q = m - q$. Therefore we get

$$\| D_*^\rho \partial_1^p A(v(t)) \|_{i+1} \, \| D_*^{\rho'} \partial_1^{p'} u(t) \|_{[\frac{n}{2}]-i} \leq ||| A(v(t)) |||_{r,*} \, ||| u(t) |||_{m-q,*}$$

where $0 \leq i \leq \left[\frac{n}{2}\right]$. Combining these inequalities, we get (A.5). Recalling the definition of the norm $||| \cdot |||_{m,(*)}$, we obtain (A.6) from this at once. The proof of Lemma A.5 is complete.

Lemma A.6. *Let $m \geq 1$ be an integer. Let $A(\cdot)$ be as in Lemma A.2.*

i) *Let $r = \max\left(m, 2\left[\frac{n}{2}\right] + 4 \right)$ and let ∂_j denote $\partial_2, \ldots, \partial_n$ or ∂_t. If v lies in $X_*^r([0, T]; \mathbf{R}_+^n)$ and u lies in $X_*^m([0, T]; \mathbf{R}_+^n)$, then for any γ, k such that $|\gamma| + 2k \leq m$ and $t \in [0, T]$, we have the estimate*

$$\| [D_*^\gamma \partial_1^k, A(v(t))] \partial_j u(t) \| \leq C \, ||| A(v(t)) |||_{r,*} \, ||| u(t) |||_{m,*}. \tag{A.7}$$

Similarly, if v lies in $X_*^r([0, T]; \mathbf{R}_+^n)$ and u lies in $X_*^{m+1}([0, T]; \mathbf{R}_+^n)$, then, for any γ, k such that $|\gamma| + 2k \leq m$ and $t \in [0, T]$, we have the *estimate*

(A.8) $$\| [D_\star^\gamma \partial_1^k, A(v(t))] \partial_1 u(t) \| \leq C ||| A(v(t)) |||_{r,*} ||| u(t) |||_{m+1,(*)}.$$

ii) *Let* $r = \max\left(m, 2\left[\frac{n}{2}\right] + 5\right)$. *If* v *is in* $X^r([0, T]; \mathbf{R}_+^n)$ *and* u *is in* $X_*^m([0, T]; \mathbf{R}_+^n)$, *and if* $A(v(t, x)) = 0$ *on* $[0, T] \times \partial\mathbf{R}_+^n$, *then, for any* γ, k *such that* $|\gamma| + 2k \leq m$ *and* $t \in [0, T]$, *we have the estimate*

(A.9) $$\| [D_\star^\gamma \partial_1^k, A(v(t))] \partial_1 u(t) \| \leq C ||| A(v(t)) |||_{r,*} ||| u(t) |||_{m,*},$$

where C *is a positive constant independent of* u *and* v.

Proof of the first assertion. By Leibniz's rule,

$$\| [D_\star^\gamma \partial_1^k, A(v(t))] \partial_j u(t) \| \leq C \sum_{\substack{\rho + \rho' = \gamma \\ p + p' = k \\ 1 \leq \rho + 2p}} \| D_\star^\rho \partial_1^p A(v(t)) D_\star^{\rho'} \partial_1^{p'} \partial_j u(t) \|.$$

Noting that $|\rho| + |\rho'| + 2(p + p') = |\gamma| + 2k \leq m$, $1 \leq |\rho| + 2p$, and $r \geq 2\left[\frac{n}{2}\right] + 4$, we apply (A.5) with $q = 1$ to the right hand side of the above inequality to obtain

$$\begin{aligned} \| [D_\star^\gamma \partial_1^k, A(v(t))] \partial_j u(t) \| &\leq C ||| A(v(t)) |||_{r,*} ||| \partial_j u(t) |||_{m-1,*} \\ &\leq C ||| A(v(t)) |||_{r,*} ||| u(t) |||_{m,*}. \end{aligned}$$

This proves (A.7). Similarly, we get

$$\begin{aligned} \| [D_\star^\gamma \partial_1^k, A(v(t))] \partial_1 u(t) \| &\leq C ||| A(v(t)) |||_{r,*} ||| \partial_1 u(t) |||_{m-1,*} \\ &\leq C ||| A(v(t)) |||_{r,*} ||| u(t) |||_{m+1,(*)} \end{aligned}$$

by using again (A.5) with $q = 1$. Hence, (A.8) is proved.

Proof of the second assertion. First we observe that

$$\| [D_\star^\gamma \partial_1^k, A(v(t))] \partial_1 u(t) \| \leq C \{ L_1^{\gamma,k} + L_2^{\gamma,k} \},$$

where

$$L_1^{\gamma,k} = \sum_{\substack{\rho + \rho' = \gamma \\ |\rho| = 1}} \| D_\star^\rho A(v(t)) D_\star^{\rho'} \partial_1^k \partial_1 u(t) \|,$$

$$L_2^{\gamma,k} = \sum_{\substack{\rho + \rho' = \gamma \\ p + p' = k \\ 2 \leq |\rho| + 2p}} \| D_\star^\rho \partial_1^p A(v(t)) D_\star^{\rho'} \partial_1^{p'} \partial_1 u(t) \|.$$

It is easy to see that $D_\star^\rho A(v(t, x)) = 0$ on $[0, T] \times \partial\mathbf{R}_+^n$. We use this with $|\rho| = 1$ and apply Lemma A.4. Then we obtain

$$L_1^{\gamma,k} \leq \sum_{\substack{\rho+\rho'=\gamma \\ |\rho|=1}} |||D_\star^\rho A(v(t))|||_{2[\frac{n}{2}]+4,*} \|D_\star^{\rho'+e_1} \partial_1^k u(t)\|$$
$$\leq |||A(v(t))|||_{2[\frac{n}{2}]+5,*} |||u(t)|||_{m,*}$$
$$\leq |||A(v(t))|||_{r,*} |||u(t)|||_{m,*},$$

where $e_1 = (0, 1, 0, \ldots, 0)$. Note that $r = \max\left(m, 2\left[\frac{n}{2}\right] + 5\right)$ and that $|\rho' + e_1| + 2k \leq |\rho'| + 2k + 1 \leq (m-1) + 1 \leq m$. Since $|\rho| + |\rho'| + 2(p + p') \leq m$ and $2 \leq |\rho| + 2p$, we obtain also by using (A.5) with $q = 2$

$$L_2^{\gamma,k} \leq |||A(v(t))|||_{r,*} |||\partial_1 u(t)|||_{m-2,*}$$
$$\leq |||A(v(t))|||_{r,*} |||u(t)|||_{m,*}.$$

Therefore,

$$\|[D_\star^\gamma \partial_1^k, A(v(t))]\partial_1 u(t)\| \leq C\, |||A(v(t))|||_{r,*} |||u(t)|||_{m,*}.$$

Thus (A.9) is proved.

Appendix B

We state here some basic properties of $H_*^m(\Omega)$ and $X_*^m([0, T]; \Omega)$.

Lemma B.1. *Let $m \geq 1$ be an integer. Then,*

i) *$C^\infty(\bar{\Omega})$ is dense in $H_*^m(\Omega)$.*

ii) *$C^\infty([0, T] \times \bar{\Omega})$ is dense in $X_*^m([0, T]; \Omega)$.*

iii) *Let p and q be nonnegative integers and let $r = \min(p, q, p + q - 2[n/2] - 3) \geq 0$. Then $H_*^p(\Omega) \cdot H_*^q(\Omega) \subsetneq H_*^r(\Omega)$.*

Proof. To prove i), we notice that $H_*^m(\Omega)$ can be regarded as a weighted Sobolev space. Let us set

$$\sigma_\alpha(x) = \sum_{(2\alpha_1 + |\alpha'| - m)_+ \leq k \leq \alpha_1} x_1^{2k}.$$

Then we have by a straightforward computation

$$\|u\|_{m,*}^2 = \sum_{\substack{|\alpha| + 2k \leq m \\ k \geq 0}} \|x_1^{\alpha_1} \partial_1^{\alpha_1 + k} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u\|^2$$
$$= \sum_{|\alpha| \leq m} \int_\Omega |\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u|^2 \sigma_\alpha(x)\, dx,$$

where $\Omega = \mathbf{R}_+^n$. It should be noted that $\sigma_\alpha(x)$ defined above is a finite sum of the powers of the distance from x to the boundary. Then we can apply the argument in the proof of Theorem 7.2 in [10] to our situation with suitable modifications. This shows the density of $C^\infty(\bar{\Omega})$ in $H_*^m(\Omega)$. The proof of (ii) is quite similar to that of Lemma B.3. See [17] for the proof of (iii).

Lemma B.2. *Let $p \geq 2$ be an integer.*

i) *There exists also a bounded linear operator S_p of $H^p_*(\Omega) \to \prod_{j=0}^{[\frac{p}{2}]-1} H^{p-2j-1}(\Gamma)$ such that*

$$S_p u = (u|_\Gamma, \partial_\nu u|_\Gamma, \ldots, \partial_\nu^{[\frac{p}{2}]-1} u|_\Gamma),$$

for any $u \in C^\infty(\bar{\Omega})$. The range of S_p coincides with $\prod_{j=0}^{[\frac{p}{2}]-1} H^{p-2j-1}(\Gamma)$.

There exists also a bounded linear operator R_p of $\prod_{j=0}^{[\frac{p}{2}]-1} H^{p-2j-1}(\Gamma) \to H^p_*(\Omega)$ such that $S_p \cdot R_p = 1$.

ii) *The bounded linear operator R_p stated in* i) *can be so chosen that, if we define $R_{p,q}$ for every q with $\left[\frac{p}{2}\right] > \left[\frac{q}{2}\right] \geq 1$ by*

$$R_{p,q}(h_0, \ldots, h_{[\frac{q}{2}]-1}) = R_p(h_0, \ldots, h_{[\frac{q}{2}]-1}, \underbrace{0, \ldots, 0}_{[\frac{p}{2}]-[\frac{q}{2}] \text{ times}}),$$

then we have

$$\| R_{p,q}(h_0, \ldots, h_{[\frac{q}{2}]-1}) \|_{q,*} \leq C_{p,q} \sum_{j=0}^{[\frac{q}{2}]-1} \| h_j \|_{H^{q-2j-1}(\Gamma)}$$

for any $(h_0, \ldots, h_{[\frac{q}{2}]-1}) \in \prod_{j=0}^{[\frac{q}{2}]-1} H^{p-2j-1}(\Gamma)$. Here $C_{p,q}$ is a positive constant depending on p, q. Namely, for such choice of R_p, $R_{p,q}$ defined above extends to a bounded linear operator of $\prod_{j=0}^{[\frac{q}{2}]-1} H^{q-2j-1}(\Gamma) \to H^q_*(\Omega)$ for any q such that $\left[\frac{p}{2}\right] > \left[\frac{q}{2}\right] \geq 1$.

Proof. The proof of ii) is given in another publication [19].

Lemma B.3. *Let $m \geq 1$. Then $C^r([0, T]; H^s(\Omega))$, that is, the space of r times continuously differentiable functions on $[0, T]$ with values in $H^s(\Omega)$, is dense in $V^m_*(0, T; \Omega)$ for any integers r, s large enough.*

Proof. It is known that there is a sequence of operators $\{J_k\}$ such that

i) $J_k \in \mathscr{L}(H^p_*(\Omega), H^q(\Omega))$, $k \geq 1$, for any integers p, q sucn that $0 \leq p \leq q$ *and* J_k converges strongly to I in $H^p_*(\Omega)$ as $k \to \infty$.

ii) $J_k \in \mathscr{L}(H^p(\Omega), H^q(\Omega))$, $k \geq 1$, for any integers p, q such that $0 \leq p \leq q$ and J_k converges strongly to I in $H^p(\Omega)$ as $k \to \infty$.

The existence of such a sequence of operators is shown, for example, in [10]. Let $v \in V^m_*(0, T; \Omega)$, where $m \geq 1$. We define $\tilde{J}_k \in \mathscr{L}(V^m_*(0, T; \Omega))$ by

$$(\tilde{J}_k v)(t) = J_k v(t), \qquad 0 \leq t \leq T.$$

Let $v_k = \tilde{J}_k v$. Then $v_k \in H^m(0, T; H^s(\Omega))$ for any $s \geq m$. Here $H^m(0, T; H^s(\Omega))$ denotes the space of functions such that $\partial_t^j u \in L^2(0, T; H^s(\Omega))$ for $0 \leq j \leq m$. Hence $\partial_t^j v_k(0) \in H^s(\Omega)$, $0 \leq j \leq m-1$. It is easily seen that v_k converges to v as $k \to \infty$ in $V^m_*(0, T; \Omega)$.

On the other hand, there is a sequence of operators $\{K_j\}$ such that

$K_j \in \mathscr{L}(H^l(0, T), C^r[0, T])$, $j \geq 1$, for any $r \geq l$ and K_j converges strongly to I in $H^l(0, T)$. Such a sequence of operators is constructed by using a variant of Friedrichs' mollifier with respect to the time variable t. Let us denote K_j by $\tilde{K}_j$ when it is regarded as an operator acting in the space of functions of t with values in a space of functions of x. Let $w \in H^m(0, T; H^s(\Omega))$, where $s \geq m$. Then $\tilde{K}_j w \in C^r([0, T]; H^s(\Omega))$, $j \geq 1$, for $r \geq m$ and $s \geq m$ and

$$\tilde{K}_j w \to w \quad \text{in} \quad H^m(0, T; H^s(\Omega)) \quad \text{as} \quad j \to \infty.$$

Hence

$$\partial_t^i \tilde{K}_j w(0) \to \partial_t^i w(0) \quad \text{in} \quad H^s(\Omega) \quad \text{as} \quad j \to \infty$$

for $0 \leq i \leq m - 1$. Let $v \in V_*^m(0, T; \Omega)$. We recall that $\tilde{J}_k v \in H^m(0, T; H^s(\Omega))$, where $s \geq m$. Let $v_{j,k} = \tilde{K}_j v_k = \tilde{K}_j \tilde{J}_k v$, $j, k \geq 1$. Then we can choose a suitable j for each k in such a way that the resulting subsequence $\{v_{j_k,k}\}$ converges to v in $V_*^m(0, T; \Omega)$. It follows from the properties enjoyed by the operators J_k and K_j that $v_{j,k} \in C^r([0, T]; H^s(\Omega))$ for any r and s large enough. This completes the proof of Lemma B.3.

Appendix C

We shall state basic facts concerning the usual Sobolev spaces in the following two lemmas. The results are well known and so the proofs are omitted here.

Lemma C.1. *Let p and q be nonnegative integers and let $\Omega \subset \mathbf{R}^n$ be a bounded domain.*

i) *Let* $r = \min\left(p, q, p + q - \left[\frac{n}{2}\right] - 1\right) \geq 0$. *Then we have a continuous imbedding* $H^p(\Omega) \cdot H^q(\Omega) \hookrightarrow H^r(\Omega)$.

ii) *We have a continuous imbedding* $H^{[\frac{n}{2}]+1+p}(\Omega) \hookrightarrow C^p(\bar{\Omega})$.

Lemma C.2. *Let $p \geq 1$ be an integer.*

i) *There exists a bounded linear operator S_p of* $H^p(\Omega) \to \prod_{j=0}^{p-1} H^{p-j-\frac{1}{2}}(\Gamma)$ *such that*

$$S_p u = (u|_\Gamma, \partial_\nu u|_\Gamma, \ldots, \partial_\nu^{p-1} u|_\Gamma)$$

for any $u \in C^\infty(\bar{\Omega})$. *The range of S_p coincides with* $\prod_{j=0}^{p-1} H^{p-j-\frac{1}{2}}(\Gamma)$. *There exists also a bounded linear operator R_p of* $\prod_{j=0}^{p-1} H^{p-j-\frac{1}{2}}(\Gamma) \to H^p(\Omega)$ *such that* $S_p \cdot R_p = 1$.

ii) *The bounded linear operator R_p stated in* i) *can be so chosen that, if we define $R_{p,q}$ for every q with $1 \leq q < p$ by*

$$R_{p,q}(h_0, \ldots, h_{q-1}) = R_p(h_0, \ldots, h_{q-1}, \underbrace{0, \ldots, 0}_{p-q\,\text{times}}),$$

then we have

$$\| R_{p,q}(h_0,\dots,h_{q-1}) \|_q \le C_{p,q} \sum_{j=0}^{q-1} \| h_j \|_{H^{q-j-\frac{1}{2}}(\Gamma)}$$

for any $(h_0,\dots,h_{q-1}) \in \prod_{j=0}^{q-1} H^{p-j-\frac{1}{2}}(\Gamma)$. *Here* $C_{p,q}$ *is a positive constant depending on* p, q. *Namely, for such choice of* R_p, $R_{p,q}$ *defined above extends to a bounded linear operator of* $\prod_{j=0}^{q-1} H^{q-j-\frac{1}{2}}(\Gamma) \to H^q(\Omega)$ *for any* q *such that* $1 \le q < p$.

Proof. For the proof of ii), we refer the reader to p. 310 of [21].

Lemma C.3. *Let* $r \ge \left[\frac{n}{2}\right] + 1$ *be an integer such that* $0 \le m \le r$. *Let* $v \in X_*^m([0, T]; \Omega)$ *and let, furthermore,* $\partial_t^i v(0) \in H^{r-i}(\Omega)$ *for* $0 \le i \le m$. *Assume that* v *takes values in* $\mathbf{R}^l$ *and that* $A = A(u)$ *is a smooth function of* $u \in \mathbf{R}^l$ *with values in the space of* $l \times l$ *complex matrices. Then,* $\partial_t^i A(v)(0) \in H^{r-i}(\Omega)$, $0 \le i \le m$. *Moreover, we have*

$$\| \partial_t^i A(v)(0) \|_{r-i} \le C(L_{[\frac{n}{2}]+1}) \{1 + (\sum_{j=0}^{i} \| \partial_t^j v(0) \|_{r-j})^r\} \tag{C.1}$$

for $0 \le i \le m$, *where* $L_{[\frac{n}{2}]+1}$ *is a constant such that* $\| v(0) \|_{[\frac{n}{2}]+1} \le L_{[\frac{n}{2}]+1}$ *and* $C(\cdot)$ *is an increasing function of its argument. In particular, if* $v \in H^r(\Omega)$, *then* $A(v) \in H^r(\Omega)$ *and we have*

$$\| A(v) \|_r \le C(R_{[\frac{n}{2}]+1}) \{1 + \| v \|_r^r\},$$

where $R_{[\frac{n}{2}]+1}$ *is a constant such that* $\| v \|_{[\frac{n}{2}]+1} \le R_{[\frac{n}{2}]+1}$ *and* $C(\cdot)$ *is similar to the above mentioned one.*

Lemma C.4. *Let* $r \ge \left[\frac{n}{2}\right] + 1$ *be an integer such that* $0 \le m \le r$. *Let* $u, v \in X_*^m([0, T]; \Omega)$ *and let* $\partial_t^i u(0), \partial_t^i v(0) \in H^{r-i}(\Omega)$ *for* $0 \le i \le m$. *Assume that* u, v *take values in* $\mathbf{R}^l$ *and that* $A = A(\cdot)$ *is a smooth function defined on* $\mathbf{R}^l$ *with values in the space of* $l \times l$ *complex matrices. Then we have*

$$\begin{aligned} &\| \partial_t^i A(u)(0) - \partial_t^i A(v)(0) \|_{r-i} \\ &\le C(L_{[\frac{n}{2}]+1}) \sum_{j=0}^{i} \| \partial_t^j u(0) - \partial_t^j v(0) \|_{r-j} \\ &\quad \times \{1 + (\sum_{j=0}^{i} \| \partial_t^j u(0) \|_{r-j})^r + (\sum_{j=0}^{i} \| \partial_t^j v(0) \|_{r-j})^r\} \end{aligned} \tag{C.2}$$

for $0 \le i \le m$, *where* $L_{[\frac{n}{2}]+1}$ *is a constant such that*

$$\max(\|u(0)\|_{[\frac{n}{2}]+1}, \|v(0)\|_{[\frac{n}{2}]+1}) \le L_{[\frac{n}{2}]+1}$$

and $C(\cdot)$ depends increasingly on its argument.

Lemma C.5. *Let $r \ge \left[\frac{n}{2}\right] + 1$ be an integer and let $0 \le m \le r$. Let $A = A(u)$ be a smooth function of $u \in \mathbf{R}^l$ with values in the space of $l \times l$ complex matrices. Let $K \subset R^l$ be a compact set contained in the set of $u \in \mathbf{R}^l$ such that* $A(u)$ is invertible. *If $v \in X_*^m(0, T]; \Omega)$ takes values in K and $\partial_t^i v(0) \in H^{r-i}(\Omega)$ for $0 \le i \le m$, then $\partial_t^i A(v)^{-1}(0) \in H^{r-i}(\Omega)$, $0 \le i \le m$. Moreover, we have*

$$\|\partial_t^i A(v)^{-1}(0)\|_{r-i} \le C(K)\{1 + (\sum_{j=0}^{i} \|\partial_t^j A(v)(0)\|_{r-j})^r\} \tag{C.3}$$

for $0 \le i \le m$, where $C(K)$ is a positive constant depending on K.

Lemma C.6. *Let $r \ge \left[\frac{n}{2}\right] + 1$ be an integer and let C be a closed rectifiable Jordan curve with positive orientation in $\mathbf{C}$. Let $\mathbf{B}(\mathbf{C}^l)$ be the space of $l \times l$ complex matrices. Let $A(\lambda)$ be a continuous function of λ defined on C with values in $H^r(\Omega; \mathbf{B}(\mathbf{C}^l))$ and let $\varphi(\lambda)$ be a complex valued continuous function of λ on C. Assume that $A(\lambda, x)^{-1}$ exists for all $(\lambda, x) \in C \times \Omega$ and that $\sup_{\lambda, x} |A(\lambda, x)^{-1}| < \infty$. If we set*

$$B = \int_C \varphi(\lambda) A(\lambda)^{-1} d\lambda,$$

then B lies in $H^r(\Omega; \mathbf{B}(\mathbf{C}^l))$.

Proof. It is shown that $A(\lambda)^{-1}$ is a continuous function of λ taking values in $H^r(\Omega; \mathbf{B}(\mathbf{C}^l))$. This is proved by using an argument employed in the proof of Lemma 2.13 in [8] with suitable modifications. The result then follows immediately.

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