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The orbit and θ correspondence for some dual pairs

By

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§0. Introduction

Since the fundamental paper of A. Weil [W] on the construction of the θ representation for the two-fold covering of Sp_n (symplectic group), there have been several approaches to extend Weil's basic construction to a more general context. The issue at hand is which of the many important properties of the θ representation are to be generalized and in which direction the generalization will go. For instance one possibility is to find analogues of θ for higher metaplectic coverings of GL_n [K-P]. Here the point is to find an "automorphic module" which comes from the residual spectrum and to determine certain uniqueness properties about certain types of Fourier coefficients of elements in this module. Such Fourier coefficients are related to Dirichlet series associated to higher order Gauss sums. However in these cases the size of the Fourier coefficients is not (except in low dimensional cases) "minimal". Thus another possible direction of generalizing [W] is to determine for a reductive group (other than Sp_n) the automorphic modules which have "smallest" Fourier coefficients. In particular this means that at least one of the local components of the automorphic module has smallest Gelfand-Kirillov dimension. It is in this direction that we are concerned in this paper. We now describe first our general setup and then the basic questions that we examine.

Let G be a semi-simple Lie group defined over some local field. If A and B are two subgroups of G we say that (A, B) is a dual pair if A is the commutant of B and G and B the commutant of A. A unitary irreducible representation π of G is called minimal if it is associated to a coadjoint orbit of minimal dimension. If we restrict π to $A \times B$ we may ask whether we get a Howe type correspondance between suitable subsets of the admissible duals of A and B. For a finite prime v and for the class of unramified representations (those admitting a non zero fixed vector under the maximal compact subgroup) the Howe conjecture takes a more precise form. Specifically if $\mathscr{H}_{A_v}$ and $\mathscr{H}_{B_v}$ are the spherical Hecke algebras of A_v and B_v, we want a homomorphism

$$\psi_v\colon \mathscr{H}_{A_v} \longrightarrow \mathscr{H}_{B_v} \qquad \mathrm{rank}\,(A_v) \geq \mathrm{rank}\,(B_v)$$

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with the property that for all $z_v \in \mathscr{H}_{A_v}$

$$\pi_{sm}(z_v) = \pi_{sm}(\psi_v(z_v))$$

where π_{sm} is the smooth version of π. For the infinite primes the corresponding conjecture is that there exists a homomorphism

$$\psi_\infty: \mathscr{Z}_{A_\infty} \longrightarrow \mathscr{Z}_{B_\infty}$$

between the centers of the enveloping algebras of A_v and B_v so that

$$\pi_{sm}(z_\infty) = \pi_{sm}(\psi_\infty(z_\infty))$$

for all $z_\infty \in \mathscr{Z}_{A_\infty}$. For the symplectic case this was proved, a long time ago, by R. Howe himself [Ho].

It is this very precise form of the Howe type conjecture that we investigate in this paper but only for the infinite primes. We expect that if there is a functorial construction of the collapsing ψ_∞ above then a corresponding functorial construction of ψ_v exists for finite v. The evidence comes from several dual pairs analyzed in [G-R-S] and from [K].

Let us briefly review what seems to be known about our proposed set-up. Assume G to be simple. If G is not of type A_n then there is a unique coadjoint orbit of minimal dimension; it is a nilpotent one. In the A_n case there is also a one parameter familly of semi-simple coadjoint orbits of minimal dimension. We exclude the A_n-case once for all. Consider the minimal nilpotent orbit. In the archimedean case, D. Vogan [V] proved the existence and the unicity of the minimal representation, except in the B_n, $n \geq 4$ situation where no such representation seems to exist. Over a p-adic field, for a group of Chevalley type, a minimal representation has been obtained in the simply-laced case by D. Kazhdan and G. Savin [K-S] and also in the G_2-case by G. Savin [S]. However one should stress that the available models for this representation are not easy to work with. The key technical problem is to be able to determine an effective computable model for π_{sm}, as in the classical θ case for Sp_n.

For dual pairs we restrict ourselves to the case where both groups are semi-simple. There is an obvious notion of dual pair of semi-simple subalgebras of the Lie algebra of G. Over $\boldsymbol{C}$ such pairs have just been classified by H. Rubenthaler [R]. At the present stage we may content ourselves with the split case so we do get a lot of examples. Some of them, notably in the exceptional cases, seem so weird that it is hard to believe that a Howe correspondence will always exists. On the other hand, one finds in [R] large famillies of dual pairs built in a uniform and conceptual way. Also [R] exhibits towers of dual pairs with many see-saws.

Our goal is to present evidence that positive answers exist for the precise Howe conjecture stated above, at least for some dual pairs in the archimedean case. We shall work with "direct sums" situations. Namely consider the extended Dynkin diagram and remove one of the simple roots. In general we are left with two connected components each corresponding to a simple subalgebra. By an old result of Dynkin this is a dual pair. These are very elementary examples. In fact they even do not appear explicitly in [R] where some irreducibility condition is imposed. We call α the root connected to the highest root and δ the simple root connected to α (for the orthogonal case we take α_3). We shall remove either α or δ. Call $\mathfrak{a}(\alpha)\times\mathfrak{b}(\alpha)$ and $\mathfrak{a}(\delta)\times\mathfrak{b}(\delta)$ those two dual pairs; we suppose that the highest root "belongs" to the $\mathfrak{a}$ subalgebra. Then in the α case the subalgebra is of type A_1, in the δ-case of type A_2 or A_3.

To check if we can expect a local correspondence we shall start from a remark of [K-K-S]. As a particular case of their work the authors point out that, in the symplectic case, by projecting the minimal coadjoint orbit onto the dual of $\mathfrak{a}\times\mathfrak{b}$ one obtains a correspondence between coadjoint orbits of $\mathfrak{a}$ and $\mathfrak{b}$, at least generically. Thus one can predict the existence of a correspondence of Howe type. The symplectic case was investigated further by J. D. Adams [A]. Our first result is that a such generic correspondence appears in the two above cases.

Next, as explained above, we want to find a map Ψ between the centers of the enveloping algebras of $\mathfrak{a}$ and $\mathfrak{b}$ such that $\Psi(Z)-Z$ belongs to the kernel of the minimal representation. We do get such a Ψ, in a completely explicit way, for our two examples. The key point is that the kernel of π is under control, thanks to a paper of A. Joseph [J].

Let us now describe the organization of the paper. The first § recalls, with a few complements, known facts about a particular class of prehomogeneous vectors spaces, those built from a parabolic subalgebra with a commutative nilpotent radical. This will provide us with some results from invariant theory which play a crucial role in the second part of the paper.

Then in §2 we study the projections of the minimal orbit. In the C_n case it is trivial (but useful) that, for direct sums, the restriction of the oscillator representation is the tensor product of the oscillator representations of A and B which are smaller symplectic groups. This is very atypical. Although we do not treat this case, let us mention that for A_n, starting with a coadjoint orbit of minimal codimension and any direct sum dual pair (so we just imbed in an obvious way $A_p\times A_q$ into A_{p+q+1}), it is a simple exercice to show the existence of a correspondence; generically only minimal representations of A and B will occur. Note also that the orthogonal cases (and direct sums) could be worked out along the same lines, using Witt's theorem. Anyhow we exclude the A_n and C_n case. Using Bruhat's decomposition we first get the result for the α-case. This is theorem 2.4. The δ-case is more elaborate. The key is to exhibit a certain subalgebra of type D_4 and to reduce all the computations to this subalgebra. The final result is given by Theorem 2.10 for the exceptional cases and Theorem 2.11 for the orthogonal groups. Although not included in this paper let us mention that we also tested examples of "tensor product" type and in particular obtained an orbital correspondence for the G_2

$\times A_1$ pair in F_4. On the contrary, it is unclear whether all "tensor product" pairs for the orthogonal groups will give rise to correspondences. Strangely enough, for the problems at hand, the exceptional cases seem to be much more well behaved.

The second part of the paper deals with the collapsing the centers of the enveloping algebras. In §3 we recall Joseph's construction. Then in §4 we study the $\mathfrak{a}(\delta)\times\mathfrak{b}(\delta)$ case. The main idea, taken from [J] is roughly speaking to build the "minimal" Verma module and then to restrict it to the dual pair. In fact we only do the minimum in this direction: following Joseph we start with a very large module and then find the highest weight vectors for $\mathfrak{a}(\delta)\times\mathfrak{b}(\delta)$. This is where the results of §1 come into the picture. We are able to write in a completely explicit way such vectors. It is then a simple matter to obtain the collapsing (Theorems 4.7 and 4.11). This is stated in terms of the Harish-Chandra homomorphisms and is essentially given by a map between the Cartan subalgebras. This map is affine, the linear part being the one predicted by the orbital decomposition. However there is a "tail" whose significance certainly requires further investigation. Its appearance should be meaningful from the point of view of θ-liftings and also from the point of view of the orbit method.

Also it is very tempting to ask whether one can build a Howe type correspondence for Verma modules.

Finally in §5 we do the $\mathfrak{a}(\alpha)\times\mathfrak{b}(\alpha)$ case. The method is the same; the results are stated in Theorems 5.10 and 5.11.

The proofs, in the last two §, involve rather heavy-handed computations. We hope to have given enough details to allow a serious reader to check the results but we did skip some repetitive computations in order to keep the paper within a not to unreasonable length.

In conclusion, and although there is a need to test more examples of the tensor product type (there is a wealth of candidates in [R]), it seems likely that many dual pairs will give rise to Howe's correspondences and θ-liftings. Whether or not these liftings will be non-trivial examples of functoriality or produce new automorphic forms of particular interest remains to be seen. At the very least this line of work should give new insight into the structure of the exceptional groups.

§1. Prehomogeneous vector spaces of parabolic commutative type

In this section we shall recall known facts about a particular class of prehomogeneous vector spaces. Unless otherwise stated the proofs may be found in [M-R-S] and in [B-R]

The base field is always $\boldsymbol{C}$. Let $\mathfrak{m}$ be a simple Lie algebra; fix a Cartan subalgebra $\mathfrak{h}$ and let R be the root system. Choose a system Ψ of simple roots and denote by R^+ (resp. R^-) the set of positive (resp. negative) roots. For each root σ fix a root vector $X_\sigma\in\mathfrak{m}^\sigma$ and assume that $[X_{-\sigma}, X_\sigma]=H_\sigma$ where H_σ

is the usual coroot.

Let δ be a simple root. We assume that, in the decomposition of the highest root as a linear combination of simple roots, δ appears with the coefficient 1. Let $\mathfrak{L}$ be the Levi component of the standard maximal parabolic subalgebra corresponding to δ and let $\mathfrak{n}^+$ be the nilpotent radical. By our assumption on δ the subalgebra $\mathfrak{n}^+$ is commutative. Let K be the unique element of $\mathfrak{h}$ such that $\sigma(K)=0$ for all simple roots except δ for which $\delta(K)=2$.

Let M be the adjoint group of $\mathfrak{m}$ and L the centralizer of K in M. By a theorem of Vinberg, under the action of L, the vector space $\mathfrak{n}^+$ is prehomogeneous which means that there exists a Zariski-open orbit Ω. The elements X of Ω are characterized by the equality $\mathrm{ad}(\mathfrak{L})(X)=\mathfrak{n}^+$.

Put $\theta=\Psi-\{\delta\}$ and let $\langle\theta\rangle$ be the set of roots which are linear combinations of elements of θ. Use the exponents $^+$ or $^-$ to mean positive or negative roots. Then

$$\mathfrak{n}^+=\sum_{\sigma\in R^+-\langle\theta\rangle^+}\mathfrak{m}^\sigma$$

and we define

$$\mathfrak{n}^-=\sum_{\sigma\in R^--\langle\theta\rangle^-}\mathfrak{m}^\sigma .$$

In this context, the prehomogeneous space $(L, \mathfrak{n}^+)$ is regular if and only if there exists an SL(2) $-$ triplet (X, K, Y) with $X\in\mathfrak{n}^+$ and $Y\in\mathfrak{n}^-$. The elements of Ω are precisely the X for which such a triplet exists. For SL(2) $-$ triplets our convention will be:

$$[K, X]=2X\ ,\quad [K, Y]=2Y\ ,\quad [X, Y]=-K\ .$$

For the end of the § assume regularity.

Let $\{\beta_1, \beta_2, \dots, \beta_m\}$ be a maximal set of long (*) roots, strongly orthogonal and contained in $R^+-\langle\theta\rangle^+$. For each one consider the root vector $X_{\beta_i}=X_i\in\mathfrak{m}^{\beta_i}$. Then

$$0,\quad X_1,\quad X_1+X_2,\quad \dots\quad , X_1+\cdots+X_m$$

is a complete set of representatives of the orbits of L in $\mathfrak{n}^+$. In particular the last term of the above list belongs to Ω. There is a canonical choice for the β_i. First take $\beta_1=\delta$; then let R_1 be the set of all roots which are orthogonal to δ. Consider $R_1\cap(R^+-\langle\theta\rangle^+)$. If this set is empty we take $m=1$ and we are done. If not let θ_1 be the set of simple roots orthogonal to $\beta_1=\delta$. Then (see [M-R-S]) there is a unique root $\beta_2\in R_1\cap(R^+-\langle\theta\rangle^+)$ such that $R_1\cap(R^+-\langle\theta\rangle^+)\subset\beta_2+\langle\theta_1\rangle^+$. Furthermore $\theta_1\cup\{\beta_2\}$ is a basis of R_1. Let

$$\mathfrak{h}_1=\sum_{\sigma\in R_1}H_\sigma\ ,\quad \mathfrak{m}_1=\mathfrak{h}_1\bigoplus_{\sigma\in R_1}\mathfrak{m}^\sigma\ .$$

(*) that is to say long in their simple component

The subalgebra $\mathfrak{h}_1$ is a Cartan subalgebra of the semi-simple Lie algebra $\mathfrak{m}_1$, the simple root β_2 is such that the highest root of R_1 relative to the basis $\theta_1 \cup \{\beta_2\}$ contains β_2 exactly once so that we get a situation analogous to the original one (including regularity) and we can pursue the construction. Note that the nilpotent radical $\mathfrak{n}_1^+$ is the direct sum of the root spaces $\mathfrak{m}^\sigma$ where the root σ is positive, contains β_1 and is orthogonal to β_1. So we get a decreasing sequence of "commutative nilpotent radicals"

$$\mathfrak{n}^+ \supset \mathfrak{n}_1^+ \supset \ldots \supset \mathfrak{n}_{m-1}^+ \supset (0)$$

and using root spaces we have, at each step, a well defined projection map. In an obvious manner we define the subalgebras $\mathfrak{n}_i^-$.

Also, at this stage, we may recall that the roots orthogonal to β_1 are in fact strongly orthogonal to β_1 and also that the regularity assumption implies (and is in fact equivalent to) the equality $K = \sum_1^m H_{\beta_i}$.

Let

$$I^+ = \sum_1^m X_{\beta_i}\ , \quad I^- = \sum_1^m X_{-\beta_i}\ .$$

It is not difficult to check that (I^+, K, I^-) is an SL(2)-triplet.

There exists, up to a constant factor, a unique irreducible polynomial function Δ_1 on $\mathfrak{n}^+$ which is relatively invariant under the action of L. We normalize it by $\Delta_1(I^+) = 1$. We may, with the Killing form, identify $\mathfrak{n}^-$ with the dual space of $\mathfrak{n}^+$; then we consider that $\Delta_1 \in \mathbf{S}(\mathfrak{n}^-)$, the symmetric algebra of the dual. It is known that

$$\Delta_1(t_1X_1 + \cdots + t_mX_m) = t_1t_2 \ldots t_m$$

so that Δ_1 is (homogeneous) of degree m. Similarly one defines, for each i a polynomial function Δ_i on $\mathfrak{n}_i$. Using the projection from $\mathfrak{n}^+$ to $\mathfrak{n}_i^+$, one considers the Δ_i, as functions on $\mathfrak{n}^+$; they have the value 1 at I^+ and Δ_i is homogeneous of degree $m - i + 1$.

Let

$$\mathfrak{u}^+ = \bigoplus_{\sigma \in \langle\theta\rangle^+} \mathfrak{m}^\sigma$$

and

$$\mathfrak{u}^- = \bigoplus_{\sigma \in \langle\theta\rangle^-} \mathfrak{m}^\sigma\ .$$

In [$M-R-S$] it is proved that, for the action of $(\mathfrak{h} \cap \mathfrak{m}) \oplus \mathfrak{u}^-$, the vector space $\mathfrak{n}^+$ is still prehomogeneous. The relative invariants are the monomials $\Delta_1^{s_1} \ldots \Delta_m^{s_m}$. The polynomials Δ_i, are algebraically independent and the subalgebra of $\mathfrak{u}^-$ invariant polynomials is exactly $\boldsymbol{C}[\Delta_1, \ldots, \Delta_m]$.

In a dual way we have m polynomial functions Δ_i^* on $\mathfrak{n}^-$ with completely similar properties. In particular they are normalized by the condition $\Delta_i^*(I^-)=1$ and also

$$\Delta_1^*(t_1X_{-\beta_1}+\cdots+t_mX_{-\beta_m})=t_it_{i+1}\ldots t_m \ .$$

Let z_0 be the element of the Weyl group of $\mathfrak{L}$ which carries all the negative roots into positive roots and choose a representative of z_0 in L. Then z_0 is an automorphism of $\mathfrak{m}$ which leaves $\mathfrak{L}$ fixed and exchanges $\mathfrak{u}^+$ and $\mathfrak{u}^-$. But z_0 fixes $\mathfrak{n}^\pm$. It is easy to prove that, under the action of z_0, the roots $\beta_1, \ldots, \beta_m$ are transformed into the roots $\beta_m, \ldots, \beta_1$. It follows that $\mathfrak{n}^+$ is also prehomogeneous under the action of $(\mathfrak{h}\cap\mathfrak{m})\oplus\mathfrak{u}^+$. This time we get polynomials ∇_i on $\mathfrak{n}^+$ which are invariant under $\mathfrak{n}^+$, algebraically independent... We normalize them by the condition $\nabla_i(I^+)=1$; then

$$\nabla_i(t_1X_{\beta_1}+\cdots+t_mX_{\beta_m})=t_1t_2\ldots t_{m-i+1} \ .$$

In a similar way we consider the polynomials ∇_i^* on $\mathfrak{n}^-$, invariant under $\mathfrak{n}^-$, normalized in the obvious way ... The polynomial ∇_i defines a differential operator $\nabla_i(\partial)$ with constant coefficients on $\mathfrak{n}^-$. For example we have, for $X\in\mathfrak{n}^+$ and $Y\in\mathfrak{n}^-$

$$\nabla_i(\partial)e^{B(X,Y)}=\nabla_i(X)e^{B(X,Y)} \ .$$

For

$$s=(s_1, s_2, \ldots, s_m)$$

and $1\leq j\leq m$ put

$$t_j(s)=(s_1-1, s_2, \ldots, s_{m+1-j}, s_{m+2-j}+1, s_{m+3-j}, \ldots, s_m)$$

with the convention

$$t_1(s)=(s_1-1, s_2, \ldots, s_m) \ .$$

Also let k be the dimension of $\mathfrak{n}^+$ and define d through

$$k=m+\frac{d}{2}m(m-1) \ .$$

The constant d is an integer which is tabulated for example in [M-R-S]. Finally let

$$b_j(s)=\left(\frac{-m}{4k}\right)^{m+1-j}\prod_1^{m+1-j}\left(s_1+s_2+\cdots+s_i+(i-1)\frac{d}{2}\right) \ .$$

Theorem 1.0. *If* $(\Delta^*)^s=(\Delta_1^*)^{s_1}\ldots(\Delta_m^*)^{s_m}$, *then*

$$\nabla_j(\partial)(\Delta^*)^s=b_j(s)(\Delta^*)^{t_j(s)} \ .$$

Up to a shift in the notations this is theorem 3.19 of [B-R].

Let

$$w_0 = e^{\mathrm{ad}I^+} e^{\mathrm{ad}I^-} e^{\mathrm{ad}I^+}$$

Then (see [B-R.]) w_0 is an involution of $\mathfrak{m}$. One has $w_0K = -K$ and also $w_0I^+ = I^-$, $w_0I^- = I^+$. In particular, w_0 fixes the Levi component $\mathfrak{L}$. Define

$$\mathfrak{s} = \{T \in \mathfrak{L} | w_0T = T\} \quad , \quad \mathfrak{q} = \{T \in \mathfrak{L} | w_0T = -T\}$$

If S is the isotropy subgroup of I^+ in L then it is also the isotropy subgroup of I^-. The Lie algebra of S is $\mathfrak{s}$ and S is an open subgroup of the commutant of w_0 in M.

Proposition 1.1. *The set of $X \in \mathfrak{n}^+$ such that, for some $s \in S$,*

$$sX \in \oplus \boldsymbol{C}^* X_i$$

contains a non-empty Zariski open subset of $\mathfrak{n}^+$

Remark. If we take $\mathfrak{m}$ of type A_n with n odd and for δ the middle root, this simply means that "almost all" matrices are diagonalizable.

In the context of symmetric spaces the result is well known; for the convenience of the reader we include a proof.

Put $Y_i = X_{-\beta_i}$, $H_i = H_{\beta_i}$ and remark that the m $\mathbf{SL}(2)$ $-$triplets (X_i, H_i, Y_i) commute one with each other. Let

$$\mathfrak{k} = \oplus \boldsymbol{C} H_i$$

One has $\mathfrak{k} \subset \mathfrak{q}$.

If σ is a root such that $\sigma(H_i) = 0$ for $i = 1, \ldots, m$, then σ is strongly orthogonal to β_1 so that $[X_\sigma, X_{\pm\beta_1}] = 0$. Next $\sigma \in R_1$ and as an element of R_1 is strongly orthogonal to β_2 so that $[X_\sigma, X_{\pm\beta_2}] = 0$. Proceeding by induction we conclude that X_σ commutes with the (X_i, H_i, Y_i). From the definition of w_0 this implies that $w_0X_\sigma = X_\sigma$.

If $\sigma \in \langle\theta\rangle$ has a non zero restriction to $\mathfrak{k}$, then the kernel of this restriction is an hyperplane in $\mathfrak{k}$. There is only a finite number of roots so that we can choose an $H \in \mathfrak{k}$ such that $\sigma(H) \neq 0$ for all $\sigma \in \langle\theta\rangle$ such that $\sigma_{|\mathfrak{k}} \neq 0$. We shall further assume that $\beta_i(H) \neq 0$ for $i = 1, \ldots, m$.

The restriction of w_0 to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{L}$ is the product (in any order) of the symmetries with respect to the roots β_i. In particular w_0 is -1 on $\mathfrak{k}$. For any $\sigma \in \langle\theta\rangle$ we thus get

$$[H, X_\sigma + w_0X_\sigma] = \sigma(H)\,(X_\sigma - w_0X_\sigma) \ .$$

If $\sigma_{|\mathfrak{k}} = 0$ then $X_\sigma = w_0X_\sigma$ and if not then, by our choice of H, we have $\sigma(H) \neq 0$. In both cases we see that $X_\sigma - w_0X_\sigma \in [\mathfrak{s}, H]$. Furthermore for $X \in \mathfrak{h}$

$$w_0X = X - \sum_i^m \beta_i(X) H_i$$

so that $X - w_0 X \in \mathfrak{k}$

Now

$$\mathfrak{L} = \mathfrak{h} \oplus_{\sigma \in \langle\theta\rangle} \mathfrak{m}^{\sigma}$$

and $\mathfrak{q}$ is the image of $\mathfrak{L}$ by the linear map $Id + w_0$ so we obtain

$$\mathfrak{q} = \mathfrak{h} + \mathrm{ad}(H)\mathfrak{s} .$$

This implies the proposition. Indeed consider the map φ from $\mathfrak{s} \times \mathfrak{k}$ to $\mathfrak{n}^+$ given by

$$\varphi(U, X) = e^{\mathrm{ad}(U)} e^{\mathrm{ad}(X)} [H, I^+] \ , \quad U \in \mathfrak{s} \ , \quad X \in \mathfrak{k} \ .$$

The differential of φ at the origin is the linear map

$$(u, x) \mapsto \mathrm{ad}(u)[H, I^+] + \mathrm{ad}(x)[H, I^+] \ , \quad u \in \mathfrak{s} \ , \quad x \in \mathfrak{h}$$

and because $[u, I^+] = 0$

$$\mathrm{ad}(u)[H, I^+] = -[\mathrm{ad}(H)u, I^+] \ .$$

Also

$$\mathrm{ad}(x)[H, I^+] = \mathrm{ad}(H)\,\mathrm{ad}(x) I^+$$

and note that, because $\beta_i(H) \neq 0$ for all i,

$$\mathrm{ad}(H)\,\mathrm{ad}(\mathfrak{h}) I^+ = \oplus \boldsymbol{C} X_i = \mathrm{ad}(\mathfrak{k}) I^+ \ .$$

It follows that the image of the differential of φ at the origin is

$$[\mathrm{ad}(H)\mathfrak{s} + \mathfrak{k}, I^+] = [\mathfrak{q}, I^+] = \mathfrak{n}^+ \ .$$

Hence the image of φ contains a non-empty Zariski open subset of $\mathfrak{n}^+$. However

$$e^{\mathrm{ad}(X)} I^+ \in \boldsymbol{C}^* X_i$$

so that the proposition is proved.

Proposition 1.2. (commutative regular case).

1) *The subspace $\mathfrak{k}$ is a Cartan subspace of the symmetric pair* $(\mathfrak{L}, w_0)$.

2) *The restricted root system is of type C_m.*

The first assertion is well known (see [B-R] for example); the second one is due to Rossmann but as we later need some details we reprove it.
We need a standard result of the theory of root systems. Let V be a vector space, say over $\boldsymbol{Q}$, and $R \subset V$ a root system. Let τ be a linear involution of V. Denote by V^+ (resp. V^-) the eigenspace of τ for the eigenvalue $+1$ (resp. -1). Then we have $V = V^+ \oplus V^-$ and also for the dual spaces $V^* = (V^+)^* \oplus (V^-)^*$ Let $R^{\tau} \subset V^+$ be the set of non zero restrictions to $(V^+)^*$ of elements of R.

Lemma 1.3. *Assume that the following condition is satisfied: if $\sigma \in R$ is such that $\sigma+\tau(\sigma) \notin R$ then $\sigma-\tau(\sigma) \notin R$. Then*
a) *R^{τ} is a root system,*
b) *If R is irreducible so is R^{τ}.*

Next we choose a positive chamber in V such that $\sigma>0$ and $\sigma_{|(V^+)^*} \neq 0$ imply $\tau(\sigma)>0$ and we let Ψ be the corresponding system of simple roots. Let R_- be the roots whose restriction to $(V^+)^*$ is 0. Then $R_- \subset V^-$ and is a root system in the vector subspace that it generates. Finally we denote by $W(R)$, $W(R^{\tau})$, $W(R_-)$, the various Weyl groups and by W^{τ} the commutant of τ in W.

Lemma 1.4. *Assume the same condition as in Lemma 1.2. Then*
a) The non zero restrictions to $(V^+)^*$ of the elements of Ψ are a set of simple roots of R^{τ},
b) The following sequence is exact

$$1 \rightarrow W(R_-) \rightarrow W^{\tau} \rightarrow W(R^{\tau}) \rightarrow 1$$

A proof of these two lemmas may be found in [H].

In our situation, let us check the assumption of Lemma 1.3. for $\tau=-w_0$ acting on $V=\mathfrak{h}^*$. We have $(V^+)^*=\mathfrak{k}$. Let σ be a root such that $\sigma-w_0(\sigma) \notin R$ and suppose that $\sigma+w_0(\sigma) \in R$. This implies, using the chain $(\sigma+\boldsymbol{Z}w_0(\sigma)) \cap R$ that $n(\sigma, w_0(\sigma)) \leq -1$. However σ and $w_0(\sigma)$ have same length so that the only possibility is $n(\sigma, w_0(\sigma))=-1$ which, computing $w_0(\sigma)$, gives

$$-1=2-\sum_1^n \sigma(H_i)\beta_i(H_\sigma)$$

that is to say

$$(*) \qquad \sum_1^n \sigma(H_i)\beta_i(H_\sigma)=3 \ .$$

Note that this a sum of positive integers. Furthermore $\sum H_i=K$ implies that $\sum \sigma(H_i)$ is an even integer (in fact ± 2 or 0). This parity property rules out, in $(*)$ the possibility $3=3$ and $1+1+1=3$ so that there exists i, j such that

$$n(\sigma, \beta_i)n(\beta_i, \sigma)=1 \ , \quad n(\sigma, \beta_j)n(\beta_j, \sigma)=2 \ .$$

The first equality implies that σ and β_i, have the same length and the second that σ and β_j have different length but this is impossible because we know that all the β_s have the same length. So we have a contradiction in all cases and we conclude that both Lemmas are valid. We still have to prove that the restricted root system is of type C_n.

Let σ be a root with a non zero restriction to $\mathfrak{k}$; we analyse the integers $\sigma(H_i)$. by [M-R-S page 113] we have

$$\sum |\sigma(H_i)| \leq 2$$

and we know that $\Sigma\sigma(H_i)$ is 0 or ± 2. If it is zero then there exists i such that $\sigma(H_i) = +1$ and j such that $\sigma(H_j) = -1$ all the others being 0. If we call β_s^* the restriction of β_s then the restriction σ^* of σ is $(\beta_i^* - \beta_j^*)/2$. As shown in [B-R, Lemma 2.7] the restricted root is positive if and only if $i<j$. Next assume that $\Sigma\sigma(H_i)$ is 2. If exactly one of the $\sigma(H_i)$ is non zero then it has to be 2 and σ^* is one of the β_i^*. The other case is that there exists i and j such that $\sigma(H_i) = \sigma(H_j) = 1$ the others being 0 and this means that $\sigma^* = (\beta_i^* + \beta_j^*)/2$. The restricted root system is irreducible of rank m. Going through the classification we check that it can only be of type C_m. The β_i^* are long roots. In particular $\beta_1 = \delta$ is a long root and it belongs to the set of simple roots given by Lemma 1.4 so it is the unique long root of this basis. The other roots of this basis are the linear forms

$$\frac{1}{2}(\beta_i^* - \beta_{i+1}^*) \qquad i=1, 2, \dots, m-1 \ .$$

In the general situation of Lemma 1.3

Lemma 1.5. *Two elements x and y of $(V^+)^*$ are conjugate under $W(R^\tau)$ if and only if they are conjugate under W.*

By Lemma 1.4 if x and y are conjugate under $W(R^\tau)$ they are conjugate under $W^\tau \subset W$. To prove the converse we may assume that both x and y belongs to the positive Weyl chamber of R^τ that is to say, by Lemma 1.4, that, for all $\sigma \in \Psi$

$$\sigma(x) \geq 0 \ , \quad \sigma(y) \geq 0 \ .$$

But then x and y belongs to the positive Weyl chamber of R so that if they are conjugate by W they are equal.

Proposition 1.6. Let

$$x = \sum \lambda_i H_i \qquad \text{with} \quad \sum \lambda_i = 0 \ ,$$

$$y = \sum \mu_i H_i \qquad \text{with} \quad \sum \mu_i = 0 \ ,$$

They are conjugate under L if and only if there exists a permutation ε of $\{1, 2, \dots, m\}$ such that, for all i

$$\lambda_i = \mu_{\varepsilon(i)}$$

Let R_0 be the root system of $\mathfrak{L}$ relative to the Cartan subalgebra $\mathfrak{h}$. A basis of R_0 is $\theta = \Psi - \{\delta\}$. The Cartan subalgebra $\mathfrak{h}_0$ of the semi-simple part is the orthogonal of K with respect to the Killing form of $\mathfrak{m}$. Also

$$\mathfrak{k}_0=\mathfrak{k}\cap\mathfrak{h}_0$$

is the subspace of all elements $\sum\lambda_i H_i$ with $\sum\lambda_i=0$. We have $-w_0(K)=K$ so that $-w_0(\mathfrak{h}_0)=\mathfrak{h}_0$ and we get an involution on $\mathfrak{h}_0$ with subspace of fixed points $\mathfrak{k}_0$. We claim that the assumption of Lemma 1.3 is satisfied for R_0. Indeed suppose that $\sigma\in R_0$ is such that $\sigma-w(\sigma)\notin R_0$. Because $w(R_0)=R_0$, this implies that $\sigma-w(\sigma)\notin R$ thus $\sigma+w(\sigma)\notin R\supset R_0$.

By Proposition 1.2 the restricted root system in $\mathfrak{k}$ is of type C_m so that the restriciton to $\mathfrak{k}_0$ is of type A_{m-1} and Proposition 1.6 now follows from Lemma 1.5 and the well known fact that, in a semi-simple Lie algebra, two elements of a Cartan subalgebra are conjugate under the adjoint group if and only if they are conjugate under the Weyl group. (note that L is connected).

§2. Decompositions of the minimal orbit Let $\mathfrak{g}$ be a complex simple Lie algebra; fix a Cartan subalgebra $\mathfrak{h}$. Let Δ be the root system and choose a basis Σ. As usual Δ^+ is the set of positive roots. Let β be the highest root. For each root σ, let H_σ be the coroot and choose a root vector $X_\sigma\in\mathfrak{g}^\sigma$ such that $[X_{-\sigma}, X_\sigma]=H_\sigma$ We consider the extended Dynkin diagram $\mathscr{D}$ of Δ; this means that we add $-\beta$ to the usual Dynkin diagram.
In this section we assume that Δ is not of type A_ℓ so that $-\beta$ is connected to a unique simple root; we call this simple root α.

Let $\eta\in\Sigma$ be a simple root. Let $\Sigma'(\eta)$ be the connected component of $-\beta$ in $\Sigma\cup\{-\beta\}-\{\eta\}$ and $\Sigma''(\eta)$ the union of all the other components. For any subset Ψ of $\Sigma\cup\{-\beta\}-\{\eta\}$ let $\langle\Psi\rangle$ be the set of roots which are linear combinations of elements of Ψ. It is well known that

$$\bigoplus_{\sigma\in\Psi} CH_\sigma \bigoplus_{\sigma\in\langle\Psi\rangle}\mathfrak{g}^\sigma$$

is a semi-simple Lie algebra admitting Ψ as a system of simple roots.

We call $\mathfrak{a}(\eta)$ the simple subalgebra associated to $\Sigma'(\eta)$ and $\mathfrak{b}(\eta)$ the semi-simple algebra associated to $\Sigma''(\eta)$. By a result of Dynkin $\mathfrak{a}(\eta)$ and $\mathfrak{b}(\eta)$ are a dual pair. *We call this type of dual pair the direct sum case (see the classical cases)*. In this section we will show that, in some cases at least, the geometry of the minimal orbits predicts the existence of a correspondence similar to the one given by the oscillator representation. We will choose for η either the root α or a root connected to α.

We need some preparation. Let $H\in\mathfrak{h}$ be defined by $\alpha(H)=1$ and $\sigma(H)=0$ for $\sigma\in\Sigma-\{\alpha\}$. For any root γ and any simple root $\sigma\in\Sigma$ we write $|\gamma|_\sigma$ for the coefficient of σ in the decomposition of γ as a linear combination of simple roots; note that this integer is non positive if $\gamma<0$. Using Bourbaki's tables it is easy to check that $|\beta|_\alpha=2$ and that β is the only root with this property. It follows that the eigenvalues of $\mathrm{ad}(H)$ are $\{-2, -1, 0, +1, +2\}$ and we obtain a graduation

$$\mathfrak{g}=\bigoplus_{-2}^{+2}\mathfrak{g}_i$$

where $\mathfrak{g}_i$ is the eigenspace for the eigenvalue i, Furthermore

$$\mathfrak{g}_2=\boldsymbol{C}X_\beta\ ,\quad \mathfrak{g}_{-2}=\boldsymbol{C}X_{-\beta}\ .$$

Let Δ_i be the set of roots γ such that $|\gamma|_\alpha=i$; thus $\Delta_2=\{\beta\}$. The subalgebra $\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2$ is maximal parabolic with Levi component $\mathfrak{g}_0$. The unipotent radical $\mathfrak{g}_1\oplus\mathfrak{g}_2$ is of Heisenberg type. More precisely:

Lemma 2.1. *a) The coroot H_β is equal to H.*
b) Let $\gamma\in\Delta_1$; then $n(\gamma,\beta)=1$ and $\beta-\gamma$ is a root.
c) For $\gamma=\alpha$ we have $\beta-2\alpha\in\Delta$ if and only if Δ is of type C_ℓ and $\beta-3\alpha$ never belongs to Δ.

Let $\mathfrak{m}=[\mathfrak{g}_0,\mathfrak{g}_0]$ be the semi-simple part of the Levi component $\mathfrak{g}_0$. Then $\mathfrak{g}_0=\boldsymbol{C}H\oplus\mathfrak{m}$. There exists a linear form f on $\mathfrak{g}_0$ such that, for $x\in\mathfrak{g}_0$

$$[X,X_\beta]=f(X)X_\beta\ .$$

Clearly f is 0 on $\mathfrak{m}$ and $f(H)=2$. Now consider $H_\beta=[X_{-\beta},X_\beta]$. It belongs to $\mathfrak{g}_0$ and it commutes with $\mathfrak{m}$ and also with H so it lies in the center of $\mathfrak{g}_0$ hence is a multiple of H. But $[H_\beta,X_\beta]=2X_\beta=[H,X_\beta]$ and finally $H_\beta=H$.

Let us prove b) and c). We have $n(\gamma,\beta)=\gamma(H_\beta)=\gamma(H)$ because $\gamma\in\Delta_1$. The β-chain of roots $\gamma+j\beta$ goes from $\gamma-q\beta$ to $\gamma+p\beta$ and $p-q=-n(\gamma,\beta)$. As β is the heighest root we get $p=0$ and $q=1$. Thus $\gamma-\beta$ is a root and so is $\beta-\gamma$. Next consider the α-chain $\beta+j\alpha$. We still have $p=0$ so that $q=n(\beta,\alpha)$. However

$$n(\beta,\alpha)=n(\gamma,\beta)\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle}=\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle}\ .$$

It is an experimental fact that, except in the C_ℓ case the roots β and α have the same length. In the C_ℓ case $\langle\beta,\beta\rangle=2\langle\alpha,\alpha\rangle$ which gives c).

Now going back to the nilpotent radical $\mathfrak{g}_1\oplus\mathfrak{g}_2$ we define an alternating form $A(X,Y)$ on $\mathfrak{g}_1$ by

$$[X,Y]=A(X,Y)X_\beta\ .$$

Then, for $\gamma,\gamma'\in\Delta_1$ we have $A(X_\gamma,X_{\gamma'})=0$ except if $\gamma+\gamma'=\beta$ and in this last case it is not 0 because $[\mathfrak{g}^\gamma,\mathfrak{g}^{\beta-\gamma}]=\mathfrak{g}^\beta$. Hence A is non degenerate. Of course A depends on the choice of X_β.

Let G be the adjoint group of $\mathfrak{g}$ and G_0 the centralizer of H in G. The one dimensional subspace $\mathfrak{g}_2$ is invariant under G_0. Let χ be the character of G_0 given by

$$gX_\beta=\chi(g)X_\beta\ .$$

The relation $[X_{-\beta},X_\beta]=H$ implies that

$$gX_{-\beta}=\chi^{-1}(g)X_{-\beta} .$$

Let M be the kernel of χ; it is a semi-simple group with Lie algebra $\mathfrak{m}$. Furthermore M commutes with $\mathrm{Exp}(\mathrm{ad}(\mathfrak{g}_{\pm 2}))$

The group G_0 acts on $\mathfrak{g}_1$ and, by a theorem of Vinberg, has a Zariski open orbit: $\mathfrak{g}_1$ is a prehomogeneous vector space. The same is true for $\mathfrak{g}_{-1}$ which, via the Killing form, we view as the dual space of $\mathfrak{g}_1$. In this context the prehomogeneous space is regular if and only if there exists $Y^+\in\mathfrak{g}_1$ and $Y^-\in\mathfrak{g}_{-1}$ such that (Y^-, H, Y^+) is an SL(2)-triplet.

Lemma 2.2. *The prehomogeneous spaces $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are regular if and only if Δ is not of type C_ℓ. Furthermore $(X_{-\alpha}+X_{-(\beta-\alpha)}, H, X_\alpha+X_{(\beta-\alpha)}$ is an $SL(2)$ -triplet.*

Suppose that Δ is not of type C_ℓ. By Lemma 2.1 we know that $2\alpha-\beta\notin\Delta$, hence

$$[X_{-\alpha}+X_{\alpha-\beta}, X_\alpha+X_{\beta-\alpha}]=H_\alpha+H_{\beta-\alpha} .$$

But $s_\alpha(\beta)=\beta-\beta(H_\alpha)\alpha=\beta-\alpha$ so that β and $\beta-\alpha$ have the same length and we saw that α and β have the same length. This implies that $H_\alpha+H_{\beta-\alpha}=H$. The other two relations being trivially satisfied we have an SL(2)-triplet with the required properties. In the C_ℓ case the non regularity is well known [M-R-S]

Until the end of the section assume that Δ is not of type C_ℓ

2.1. The case $\eta=\alpha$. The root system Δ is either of exceptional type or of type B_ℓ with $\ell\geq 3$ or D_ℓ with $\ell\geq 4$. We identify $\mathfrak{g}$ with its dual using the Killing form. The minimal coadjoint orbit is

$$\Omega(\mathfrak{g})=GX_\beta$$

In this subsection we remove the root α. Then $\mathfrak{b}(\alpha)$ is simply the semi-simple part $\mathfrak{m}$ of the Levi subalgebra $\mathfrak{g}_0$ and

$$\mathfrak{a}(\alpha)=\boldsymbol{C}X_{-\beta}\oplus\boldsymbol{C}H\oplus\boldsymbol{C}X_\beta$$

is of type A_1. The subgroup $A(\alpha)$ is defined as the subgroup of G generated by $\mathrm{Exp}(\boldsymbol{C}\mathrm{ad}(X_{\pm\beta}))$ and the subgroup $B(\alpha)=M$ has already been defined. Note that M and $A(\alpha)$ commute. We want to find the "generic" orbits of the adjoint action of $A(\alpha)\times B(\alpha)$ on Ω. By generic we mean that we look only at a non empty Zariski open subset of Ω.

For $0<|i|\leq 2$ we put $G_i=\mathrm{Exp}(\mathrm{ad}(\mathfrak{g}_i))$. Then $G_0G_1G_2$ is a maximal parabolic subgroup of G with unipotent radical G_1G_2. The subgroup G_2 is the center of this radical. Now $G_0=M\mathrm{Exp}(\boldsymbol{C}H)$. The subgroup MG_1G_2 fixes X_β so

$$G_0G_1G_2X_\beta=\boldsymbol{C}^*X_\beta .$$

It then follows from the Bruhat's decomposition relative to the above parabolic subgroup that

$$\Omega' = \boldsymbol{C}^* G_{-2} G_{-1} X_\beta$$

is a non empty Zariski open subset of Ω. In such cases we write

$$\Omega \approx \boldsymbol{C}^* G_{-2} G_{-1} X_\beta$$

the symbol $\approx$ meaning that the right side contains a non empty Zariski open subset of the left side. Furthermore $G_{-2} \subset A(\alpha)$ and, by Lemma 2.2 we can choose $Y^- \in \mathfrak{g}_{-1}$ such that $G_0 Y^-$ is Zariski open in $\mathfrak{g}_{-1}$. But, for $x \in G_0$

$$\operatorname{Exp}(\operatorname{ad}(xY^-)) X_\beta = x \operatorname{Exp}(\operatorname{ad}(Y^-)) x^{-1} X_\beta$$

and $x \in \operatorname{Exp}(\boldsymbol{C}H)M$ while $x^{-1} X_\beta \in \boldsymbol{C}^* X_\beta$. Finally

$$\Omega \approx \boldsymbol{C}^* A(\alpha) M \operatorname{Exp}(\operatorname{ad}(Y^-) X_\beta \ . \tag{2-1}$$

Before we proceed let us make our goal more specific. If $B_\mathfrak{g}$ is the Killing form of $\mathfrak{g}$ and if $Z \in \mathfrak{g}$ then it defines the linear form $Y \mapsto B_\mathfrak{g}(Z, Y)$. For any semi-simple subalgebra $\mathfrak{s}$ with Killing form $B_\mathfrak{s}$ there is a unique element $p_{\mathfrak{g}/s}(Z) \in \mathfrak{s}$ such that, for all $Y \in \mathfrak{s}$ we have

$$B_\mathfrak{g}(Z, Y) = B_\mathfrak{s}(p_{\mathfrak{g}/s}(Z), Y) \ .$$

Note that the projection $p_{\mathfrak{g}/s}$ commutes with the action of $\mathfrak{s}$ and more generally with the action of the normalizer of $\mathfrak{s}$ in G.

Going back to formula (2-1) we put

$$Z = \operatorname{Exp}(\operatorname{ad}(Y^-)) X_\beta$$

and, for $t \neq 0$ project tZ on $\mathfrak{a}(\alpha)$ and $\mathfrak{b}(\alpha) = \mathfrak{m}$. We get

$$p_{\mathfrak{g}/a(\alpha)}(\Omega) \approx \bigcup_{t \in \boldsymbol{C}^*} tA(\alpha) p_{\mathfrak{g}/a(\alpha)}(Z) \ ,$$

$$p_{\mathfrak{g}/m}(\Omega) \approx \bigcup_{t \in \boldsymbol{C}^*} tM p_{\mathfrak{g}/m}(Z) \ .$$

We have a correspondence between the (co)-adjoint orbits:

$$tA(\alpha) p_{\mathfrak{g}/a(\alpha)}(Z) \qquad \text{and} \qquad tM p_{\mathfrak{g}/m}(Z) \ .$$

We want to make it explicit.

Because we ruled out the C_ℓ case, the two roots $-\beta$ and α have the same length. Hence there is a well-defined $\mathfrak{a}_2$ subalgebra of type A_2 having $\{-\beta, \alpha\}$ as a set of simple roots The positive roots of $\mathfrak{a}_2$ are then $\{-\beta, \alpha, -\beta+\alpha\}$. We may assume the roots vectors so chosen that there exists an isomorphism of $\mathfrak{sl}(3)$ onto $\mathfrak{a}_2$ such that:

$$X_{-\beta} \mapsto \begin{pmatrix} 0 & +1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{+\alpha} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_{-\beta+\alpha} \mapsto \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_{+\beta}\mapsto\begin{pmatrix}0&0&0\\-1&0&0\\0&0&0\end{pmatrix},\quad X_{-\alpha}\mapsto\begin{pmatrix}0&0&0\\0&0&0\\0&-1&0\end{pmatrix},\quad X_{+\beta-\alpha}\mapsto\begin{pmatrix}0&0&0\\0&0&0\\-1&0&0\end{pmatrix},$$

$$H_{-\beta}\mapsto\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad H_{+\alpha}\mapsto\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix},\quad H_{-\beta+\alpha}\mapsto\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

We simply identify $\mathfrak{sl}(3)$ and $\mathfrak{a}_2$. Then, by a straightforward computation

$$Z=\operatorname{Exp}(\operatorname{ad}(X_{-\alpha}+X_{-\beta+\alpha}))X_\beta=\begin{pmatrix}1/2&-1/4&-1/2\\-1&1/2&1\\1&-1/2&-1\end{pmatrix}.$$

We project Z on $\mathfrak{a}(\alpha)$. We have $\mathfrak{a}(\alpha)\subset\mathfrak{a}_2$:

$$\mathfrak{a}(\alpha)\mapsto\begin{pmatrix}*&*&0\\ *&*&0\\0&0&0\end{pmatrix}.$$

In this matrix realization, the Killing form of $\mathfrak{a}(\alpha)$ is $4\operatorname{Tr}(XY)$. There exists a constant c such that the restriction to $\mathfrak{a}_2$ of $B_\mathfrak{g}$ is $c\operatorname{Tr}(XY)$. We have, for example

$$c=\frac{1}{2}B_\mathfrak{g}(H,H)\ .$$

So, on $\mathfrak{a}(\alpha)$ the linear form defined by Z is

$$\begin{pmatrix}a&b&0\\c&-a&0\\0&0&0\end{pmatrix}\longrightarrow c\operatorname{Tr}\left[\begin{pmatrix}a&b&0\\c&-a&0\\0&0&0\end{pmatrix}\begin{pmatrix}1/2&-1/4&-1/2\\-1&1/2&1\\1&-1/2&-1\end{pmatrix}\right]$$

which is equal to

$$\frac{1}{2}B_\mathfrak{g}(H,H)\left(-b-\frac{1}{4}c\right)$$

and using the Killing form of $\mathfrak{a}(\alpha)$ we see that this linear form is identified with the following element of $\mathfrak{a}(\alpha)$

$$-\frac{1}{8}B_\mathfrak{g}(H,H)\begin{pmatrix}0&1/4&0\\1&0&0\\0&0&0\end{pmatrix}$$

We are only interested in the coadjoint orbit so we may replace the above element by an inner conjugate in $\mathfrak{a}(\alpha)$ and in particular by

$$\frac{1}{16}B_{\mathfrak{g}}(H, H)\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}=\frac{B_{\mathfrak{g}}(H, H)}{16}H_{\beta} .$$

Now we find the projection of Z on $\mathfrak{m}$. let $Z^{\perp}$ be the orthogonal of Z relative to the Killing form $B_{\mathfrak{g}}$. Because of the orthogonality property of the root subspaces we have

$$\mathfrak{m}=(\mathfrak{m}\cap\mathfrak{a}_2)\oplus(Z^{\perp}\cap\mathfrak{m})$$

Furthermore $\mathfrak{m}\cap\mathfrak{a}_2$ is the one dimensional subspace of $\mathfrak{h}\cap\mathfrak{a}_2$ orthogonal to $H=H_{\beta}$. Finally only the "diagonal" part of Z has a non zero projection and the diagonal part is

$$\begin{pmatrix}1/2 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & -1\end{pmatrix}=-\frac{1}{2}H_{\beta}+H_{\alpha} .$$

But H_{β} is orthogonal to $\mathfrak{m}$ so, taking the Killing forms into account we can conclude that the projection of Z onto $\mathfrak{m}$ is the unique element $U\in\mathfrak{h}\cap\mathfrak{m}$ such that

$$B_{\mathfrak{m}}(U, X)=B_{\mathfrak{g}}(H_{\alpha}, X)\qquad\text{for all}\quad X\in\mathfrak{h}\cap\mathfrak{m} .$$

In fact U has a simple expression in terms of fundamental weights. Let $C_1, \dots, C_r$ be the connected components of $\Sigma-\{\alpha\}$. With obvious notations

$$\mathfrak{m}=\mathfrak{m}_1\oplus\dots\oplus\mathfrak{m}_r\ ,\quad \mathfrak{h}\cap\mathfrak{m}=\mathfrak{h}_1\oplus\dots\oplus\mathfrak{h}_r .$$

For each i, there is a unique simple root $\delta_i\in C_i$ such that $\langle\alpha, \delta_i\rangle\neq 0$. Define the weight $\varpi_i\in\mathfrak{h}_i^*$ by $\varpi_i(H_{\delta_i})=1$ and $\varpi_i(H_{\gamma})=0$ for $\gamma\in C_i-\{\delta_i\}$. Then, for $X\in\mathfrak{h}_i$,

$$B_{\mathfrak{g}}(H_{\alpha}, X)=\varpi_i(X)B_{\mathfrak{g}}(H_{\alpha}, H_{\delta_i}) .$$

The roots α and β have the same length so $B_{\mathfrak{g}}(H_{\alpha}, H_{\alpha})=B_{\mathfrak{g}}(H, H)$, hence

$$B_{\mathfrak{g}}(H_{\alpha}, H_{\delta_i})=\frac{1}{2}B_{\mathfrak{g}}(H, H)n(\alpha,\delta_i) .$$

On $\mathfrak{a}(\alpha)$ the unique fundamental weight ϖ is the linear form on $\mathfrak{h}\cap\mathfrak{a}(\alpha)=\boldsymbol{C}H$:

$$xH\mapsto\frac{1}{8}B_{\alpha}(H, x) .$$

Thus the projection of Z onto $\mathfrak{a}(\alpha)$ is conjugate to

$$\frac{1}{2}B_{\mathfrak{g}}(H, H)\varpi .$$

Replacing t by $2t/B_{\mathfrak{g}}(H, H)$ we get that generically, the projection from the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ to the dual $\mathfrak{a}(\alpha)^*\times\mathfrak{m}^*$ of $\mathfrak{a}(\alpha)\times\mathfrak{m}$ induces a correspondence between the coadjoint orbits of

$$t\varpi \quad \text{and} \quad \sum tn(\alpha, \delta_i)\varpi_i \ .$$

There is one point to take care of. The parameter t is different from 0. The coadjoint orbits of $t\varpi$ and $t'\varpi$ coincide if and only if $t = \pm t'$. We claim that the same is true for the coadjoint orbits of $t \sum\varpi_i$ and $t'\sum\varpi_i$. We have to prove that, for each i the coadjoint orbits of $t\varpi_i$ and $t'\varpi_i$ are the same if and only if $t = \pm t'$. By invariance of the Killing form by the coadjoint action we get that if $t\varpi_i$ and $t'\varpi_i$ are conjugate then they have the same length, hence $t = \pm t'$. Finally we have to prove that ϖ_i, and $-\varpi_i$ are conjugate under some element of the Weyl group of $\mathfrak{m}_i$.

Lemma 2.3. *For each i, the maximal parabolic subalgebra of $\mathfrak{m}_i$, associated to the simple root δ_i, has a commutative nilpotent radical which is regular in the sense of* §1.

There is an a priori proof in [R] but it is perhaps more instructive to proceed case by case. Indeed there is a list of commutative prehomogeneous vector spaces regular and of parabolic commutative type in [M-R-S page 98]. We check the various cases, using Bourbaki's notations for root systems. By hypothesis $\mathfrak{g}$ is not of type A_ℓ or C_ℓ.

• $\mathfrak{g}$ is of type B_ℓ. Then $\alpha = \alpha_2$ and, for $\ell \geq 3$ we have two delta roots: $\delta_1 = \alpha_1$ and $\delta_2 = \alpha_3$. The simple component $\mathfrak{m}_1$ is of type A_1; its unique standard maximal parabolic subalgebra is of commutative regular type. The simple component $\mathfrak{m}_2$ is of type $B_{\ell-2}$ and δ_2 is the "first root" and this corresponds to a regular commutative case. For $\ell = 2$ there is just one δ root, α_1 and $\mathfrak{m}$ is of type A_1.

• $\mathfrak{g}$ is of type D_ℓ with $\ell \geq 5$. The situation is completely similar to the preceeding case. For $\ell = 4$ we have three delta roots each corresponding to a component of $\mathfrak{m}$ of type A_1.

• $\mathfrak{g}$ is of type E_6. Then $\alpha = \alpha_2$ and the unique delta root is α_4. The subalgebra $\mathfrak{m}$ is of type A_5. The simple root δ is the middle one so we do get a commutative regular case.

• $\mathfrak{g}$ is of type E_7. Then $\alpha = \alpha_1$ and the unique delta root is α_3. The subalgebra $\mathfrak{m}$ is of type D_6 and we have the case $D_{6,2}$ of [M-R-S].

• $\mathfrak{g}$ is of type E_8. Then $\alpha = \alpha_8$ and the unique delta root is α_7. The subalgebra $\mathfrak{m}$ is of type E_7 where the last root is singled out and this is the only regular commutative case of exceptional type.

• $\mathfrak{g}$ is of type F_4. Then $\alpha = \alpha_1$ and the unique delta root is α_2. The subalgebra $\mathfrak{m}$ is of type C_3 with the third and last root singled out and this again is of commutative regular type.

• $\mathfrak{g}$ is of type G_2. Then $\alpha = \alpha_2$ and $\delta = \alpha_1$ so that $\mathfrak{m}$ is of type A_1.

Our assertion follows. Indeed fix i. Consider as in §1, the element $K_i \in \mathfrak{h}_i$ defined by $\delta_i(K_i) = 2$ and $\gamma(K_i) = 0$ for $\gamma \in C_i - \{\delta_i\}$. Then the weight

ϖ_i is proportional to the linear form $X \mapsto B_{\mathfrak{m}_i}(K_i, X)$. Define w_0 as in §1; we have $w_0(K_i) = -K_i$, hence $w_0(\varpi_i) = -\varpi_i$. Summarizing:

Theorem 2.4. (Δ **of type** B_ℓ **with** $\ell \geq 3$ **or** D_ℓ **with** $\ell \geq 4$ **or exceptional**).

In $\mathfrak{a}(\alpha)$ *the coadjoint orbits of* $t\varpi$ *and* $t'\varpi$ *coincide if and only if* $t = \pm t'$. *In* $\mathfrak{m} = \mathfrak{b}(\alpha)$ *the coadjoint orbits of* $t\sum n(\alpha, \delta_i)\varpi$ *and* $t'\sum n(\alpha, \delta_i)\varpi_i$ *coincide if and only if* $t = \pm t'$.

Generically, the projection from the dual $\mathfrak{g}^*$ *of* $\mathfrak{g}$ *to the dual* $\mathfrak{a}(\alpha)^* \times \mathfrak{m}^*$ *of* $\mathfrak{a}(\alpha) \times \mathfrak{m}$ *induces a correspondence between the coadjoint orbits of*

$$t\varpi \quad \text{and} \quad \sum tn(\alpha, \delta_i)\varpi_i$$

2.2. A computation in D_4. Notations are specific to this subsection. We let $\mathfrak{g}$ be a simple algebra of type D_4. We fix a Cartan subalgebra $\mathfrak{h}$ and a basis of simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where α_2 is the middle root. Let $\mathfrak{t}$ be the intersection of the kernels of the two roots α_1 and α_2. The centralizer $\mathfrak{t}$ in $\mathfrak{g}$ is a Levi subalgebra of a standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. We call $\mathfrak{a}$ the semi-simple part of this Levi subalgebra which is thus equal to $\mathfrak{a} \oplus \mathfrak{t}$. Note that $\mathfrak{a}$ is of type A_2. Let S be the adjoint group of $\mathfrak{g}$ and P the standard parabolic subgroup with Lie algebra $\mathfrak{p}$. Let L be the Levi subgroup of P and N its unipotent radical Finally N^- is the unipotent group opposed to N and its Lie algebra is $\mathfrak{n}^-$. We wish to study the projection onto $\mathfrak{a} \oplus \mathfrak{t}$ of the minimal (co)adjoint orbit Ω of $\mathfrak{g}$.

Let β be the highest root and $X_\beta \in \mathfrak{g}^\beta$. The Bruhat decomposition relative to P gives

$$\Omega \approx LN^- X_\beta$$

The nilpotent radical $\mathfrak{n}^-$ is the direct sum of the 3 subspaces:

$$\mathfrak{n}^-_{1,0} = \mathfrak{g}^{-\alpha_3} \oplus \mathfrak{g}^{-\alpha_3-\alpha_2} \oplus \mathfrak{g}^{-\alpha_3-\alpha_2-\alpha_1}$$

$$\mathfrak{n}^-_{0,1} = \mathfrak{g}^{-\alpha_4} \oplus \mathfrak{g}^{-\alpha_4-\alpha_2} \oplus \mathfrak{g}^{-\alpha_4-\alpha_2-\alpha_1}$$

$$\mathfrak{n}^-_{1,1} = \mathfrak{g}^{-\alpha_3-\alpha_4-\alpha_2} \oplus \mathfrak{g}^{-\alpha_3-\alpha_4-\alpha_2-\alpha_1} \oplus \mathfrak{g}^{-\alpha_3-\alpha_4-\alpha_2-\alpha_1}$$

The first two are abelian and their bracket is equal to the third one which is the center of this nilpotent Lie algebra. This shows that, with obvious notations

$$N^- = N^-_{1,1} N^-_{1,0} N^-_{0,1}$$

so that

$$\Omega \approx L N^-_{1,1} N^-_{1,0} N^-_{0,1} X_\beta$$

Our first remark is that both $\mathfrak{g}^{-\alpha_3}$ and $\mathfrak{g}^{-\alpha_4}$ commute with X_β; using the above facts on the structure of $\mathfrak{n}^-$ we conclude that

$$\Omega \approx LN_{1,1}^{-}\mathrm{Exp}\,(\mathfrak{g}^{-\alpha_3-\alpha_2}\oplus\mathfrak{g}^{-\alpha_3-\alpha_2-\alpha_1})\,\mathrm{Exp}\,(\mathfrak{g}^{-\alpha_4-\alpha_2}\oplus\mathfrak{g}^{-\alpha_4-\alpha_2-\alpha_1})\,X_\beta$$

Next we use the stabilizer of $\boldsymbol{C}X_\beta$ in L; it is a maximal parabolic subgroup of L. The Levi subgroup contains a subgroup of type $GL(2)$ with simple root α_1 which acts irreducibly on each of the two planes

$$\mathfrak{g}^{-\alpha_3-\alpha_2}\oplus\mathfrak{g}^{-\alpha_3-\alpha_2-\alpha_1}\ ,$$
$$\mathfrak{g}^{-\alpha_4-\alpha_2}\oplus\mathfrak{g}^{-\alpha_4-\alpha_2-\alpha_1}\ .$$

Choosing root vectors we see that $(X_{-\alpha_1-\alpha_2-\alpha_3}, X_{-\alpha_2-\alpha_4})$ has a Zariski open orbit, in the product of the two planes. Furthermore this $GL(2)$ group normalizes $N_{1,1}^{-}$. It follows that:

$$\Omega\approx \boldsymbol{C}^*LN_{1,1}^{-}\mathrm{Exp}\,(X_{-\alpha_1-\alpha_2-\alpha_3})\,\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4})\,X_\beta\ .$$

Now we consider the two dimensional abelian subgroup of L

$$\mathrm{Exp}\,(\mathfrak{g}^{\alpha_1+\alpha_2}\oplus\mathfrak{g}^{\alpha_2})\ .$$

It fixes X_β. By an easy computation and denoting by $c_1, c_2\ldots$ non zero constants (they are structural constants of D_4) we prove that, for x, y, t complex,

$$\begin{aligned}&\mathrm{Exp}\,(xX_{\alpha_2}+yX_{\alpha_1+\alpha_2})\,\mathrm{Exp}\,(tX_{-\beta})\,\mathrm{Exp}\,(X_{-\alpha_1-\alpha_2-\alpha_3})\,\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4})=\\ &\mathrm{Exp}\,(txc_1X_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}+tyc_2X_{-\alpha_2-\alpha_3-\alpha_4}+tX_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4})\times\\ &\mathrm{Exp}\,(X_{-\alpha_1-\alpha_2-\alpha_3}+yc_3X_{-\alpha_3})\,\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4}+xc_4X_{-\alpha_4})\,X_\beta\ .\end{aligned}$$

Now

$$\begin{aligned}&\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4}+xc_4X_{-\alpha_4})\,X_\beta=\\ &\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4})\,\mathrm{Exp}\,(xc_4X_{-\alpha_4})\,X_\beta=\\ &\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4})\,X_\beta\ .\end{aligned}$$

Next

$$\begin{aligned}&\mathrm{Exp}\,(X_{-\alpha_1-\alpha_2-\alpha_3}+yc_3X_{-\alpha_3})\,\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4})\,X_\beta=\\ &\mathrm{Exp}\,(X_{-\alpha_1-\alpha_2-\alpha_3})\,\mathrm{Exp}\,(ycX_{-\alpha_3})\,\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4})\,X_\beta=\\ &\mathrm{Exp}\,(X_{-\alpha_1-\alpha_2-\alpha_3})\,\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4}+yc_5X_{-\alpha_2-\alpha_3-\alpha_4})\,\mathrm{Exp}\,(yc_3X_{-\alpha_3})\,X_\beta=\\ &\mathrm{Exp}\,(X_{-\alpha_1-\alpha_2-\alpha_3})\,\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4})\,\mathrm{Exp}\,(yc_5X_{-\alpha_2-\alpha_3-\alpha_4})\,X_\beta\ .\end{aligned}$$

However $X_{-\alpha_2-\alpha_3-\alpha_4}$ belongs to the center $\mathfrak{n}_{1,1}^{-}$ so that we finally get

$$\begin{aligned}&\mathrm{Exp}\,(xX_{\alpha_2}+yX_{\alpha_1+\alpha_2})\,\mathrm{Exp}\,(tX_{-\beta})\,\mathrm{Exp}\,(X_{-\alpha_1-\alpha_2-\alpha_3})\,\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4})\,X_\beta=\\ &\mathrm{Exp}\,(txc_1X_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}+y\,(tc_2+c_5)\,X_{-\alpha_2-\alpha_3-\alpha_4}+tX_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4})\times\\ &\mathrm{Exp}\,(X_{-\alpha_1-\alpha_2-\alpha_3})\,\mathrm{Exp}\,(X_{-\alpha_2-\alpha_4})\,X_\beta\ .\end{aligned}$$

But the set of all elements

$$txc_1X_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}+y\,(tc_2+c_5)\,X_{-\alpha_2-\alpha_3-\alpha_4}+tX_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}$$

is Zariski dense in $\mathfrak{n}_{1,1}^{-}$ so we conclude that

$$\Omega \approx \boldsymbol{C}^* L \operatorname{Exp}(\boldsymbol{C} X_{-\beta}) \operatorname{Exp}(X_{-\alpha_1-\alpha_2-\alpha_3}) \operatorname{Exp}(X_{-\alpha_2-\alpha_4}) X_\beta \ .$$

Consider the subalgebra of type A_2 built with the two simple roots $\alpha_2+\alpha_4$ and $\alpha_1+\alpha_2+\alpha_3$ (beware this is not the same as $\mathfrak{a}$!); the sum of these two roots is β. Choosing suitably the roots vectors we assume that there exists an isomorphism of $\mathfrak{sl}(3)$ on this subalgebra such that

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto X_{-\alpha_1-\alpha_2-\alpha_3} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \mapsto X_{-\alpha_2-\alpha_4}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \mapsto X_{-\beta} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto X_\beta$$

Then computing in SL(3) we have

$$\operatorname{Exp}(tX_{-\beta}) \operatorname{Exp}(X_{-\alpha_1-\alpha_2-\alpha_3}) \operatorname{Exp}(X_{-\alpha_2-\alpha_4}) X_\beta = \begin{pmatrix} 1+t & 1 & 1 \\ -1-t & -1 & -1 \\ -t(1+t) & -t & -t \end{pmatrix} .$$

Call $\eta(t)$ this matrix viewed as an element of $\mathfrak{g}$. We still have to use dilations. For $u \in \boldsymbol{C}^*$, put $\xi(u, t) = u\eta(t/u)$.

Proposition 2.5. *The minimal coadjoint orbit Ω of $\mathfrak{g}$ has the following generic decomposition*

$$\Omega \approx \bigcup_{u \in \boldsymbol{C}^*, t \in \boldsymbol{C}} L\zeta(u, t) \ .$$

We still have to check that this is a disjoint union. If $\xi(u, t)$ and $\xi(u', t')$ are conjugate under L so are their projections on the Lie algebra $\mathfrak{a} \oplus \mathfrak{t}$ of L. Only the diagonal part has a non zero projection. In terms of coroots, this diagonal part is

$$uH_{\alpha_1+\alpha_2+\alpha_3} + tH_\beta$$

Put

$$2u + t = \lambda_1 \ , \quad -u - 2t = \lambda_2 \ .$$

Then

$$uH_{\alpha_1+\alpha_2+\alpha_3} + tH_\beta = \lambda_1 H_{\alpha_1} - \lambda_2 H_{\alpha_2} + \lambda_1 (H_{\alpha_3} - H_{\alpha_4}) - \lambda_2 (H_{\alpha_1} + 2H_{\alpha_2} + H_{\alpha_3} + 2H_{\alpha_4}) \ .$$

On the right side of this equation, the first two terms belong to $\mathfrak{a}$ while the two last are in $\mathfrak{t}$ and are thus invariant under L. This proves our assertion about the non conjugacy of the $\xi(u, t)$.

Remark. Note that the dual pair is not $(\mathfrak{a}, \mathfrak{t})$ but $(\mathfrak{a}\oplus\mathfrak{t}, \mathfrak{t})$ and that for this dual pair we do get (generically) a one to one correspondence between coadjoint orbits of $\mathfrak{t}$ and a two parameters familly of coadjoint orbits of $\mathfrak{a}\oplus\mathfrak{t}$; see [K].

2.3. The case $\eta=\delta$. The root system is still of type B_ℓ, D_ℓ or exceptional. In the exceptional case, we let δ be the unique simple root which is connected to α. In the orthogonal cases we let $\delta=\alpha_3$ (Bourbaki's notations). From the extended Dynkin diagram $\sum \cup \{-\beta\}$ we remove δ. Then $\mathfrak{a}(\delta)$ is the simple subalgebra built on the connected component $\sum'(\delta)$ of $-\beta$ while $\mathfrak{b}(\delta)$ is the semi-simple subalgebra built on the union of the other connected components. For example in the orthogonal cases, the subalgebra $\mathfrak{a}(\delta)$ is of type A_3 while $\mathfrak{b}(\delta)$ is of orthogonal type. Three cases are peculiar and excluded: the G_2 and B_3 cases where $\mathfrak{b}(\delta)=(0)$ and the D_4 case where $\mathfrak{a}(\delta)=\mathfrak{g}$ and $\mathfrak{b}(\delta)=0$. Our goal is a theorem similar to Theorem 2.4. We first need to fix subgroups $A(\delta)$ and $B(\delta)$ with Lie algebras $\mathfrak{a}(\delta)$ and $\mathfrak{b}(\delta)$ respectively.

For $A(\delta)$ we choose the connected subgroup of the adjoint group G with Lie algebra $\mathfrak{a}(\delta)$ and for $B(\delta)$ we take the commutant in G of $\mathfrak{a}(\delta)$. Then $A(\delta)$ and $B(\delta)$ commute and $B(\delta)$ is exactly the commutant of $A(\delta)$ in G.

We now suppose that $\mathfrak{g}$ is of exceptional type; the orthogonal case is slightly different and we shall deal with it later.

Thus δ is the unique simple root connected to α which in turn is the unique simple root connected to $-\beta$ in the extended Dynkin diagram. By definition the subalgebra $\mathfrak{a}(\delta)$ admits $\{-\beta, \alpha\}$ as a set of simple roots. Hence the coroots $H_\beta=H$ and H_α are a basis of a Cartan subalgebra $\mathfrak{h}\cap\mathfrak{a}(\delta)$ of $\mathfrak{a}(\delta)$. We claim that

$$\mathfrak{h}\cap\mathfrak{a}(\delta)=\bigcap_{\gamma\in\Sigma-\{\alpha,\delta\}}\operatorname{Ker}(\gamma)$$

Indeed both spaces have dimension 2 and the two roots β and α are orthogonal to $\sum-\{\alpha, \delta\}$ so that $\mathfrak{h}\cap\mathfrak{a}(\delta)$ is included in the intersection of the kernels. Let $L(\delta)$ be the centralizer of $\mathfrak{h}\cap\mathfrak{a}(\delta)$ in G. It is a reductive connected subgroup, a Levi component of a parabolic subgroup of G. The Lie algebra $\mathfrak{l}(\delta)$ of $L(\delta)$ is

$$\mathfrak{l}(\delta)=(\mathfrak{h}\cap\mathfrak{a}(\delta))\oplus\mathfrak{b}(\delta) .$$

The Lie algebra $\mathfrak{l}(\delta)$ hence also the group $L(\delta)$, normalize $\mathfrak{a}(\delta)$.

Lemma 2.6 ($\mathfrak{g}$ **of type** E_6, E_7, E_8, F_4,). *$B(\delta)$ is the commutant of $\mathfrak{a}(\delta)$ in $L(\delta)$ and if T is the connected subgroup with Lie algebra $\mathfrak{h}\cap\mathfrak{a}(\delta)$, then $L(\delta)=TB(\delta)$.*

By definition $B(\delta)$ is the commutant of $\mathfrak{a}(\delta)$ hence is contained in the commutant $L(\delta)$ of $\mathfrak{h}\cap\mathfrak{a}(\delta)$. If $g\in L(\delta)$ then g acts trivially on the Cartan subalgebra $\mathfrak{h} \cap \mathfrak{a}(\delta)$ of $\mathfrak{a}(\delta)$. Hence there exists an element $t \in T$, the Cartan subgroup, such that g and t coincide on $\mathfrak{a}(\delta)$. We conclude that $gt^{-1} \in B(\delta)$.

Suppose that $\mathfrak{g}$ is not of type G_2. We are going to construct a subalgebra of type D_4 and eventually reduce the problem to this subalgebra. Following Tits [T], a set $\mathfrak{R}$ of roots is called a P-system if the elements of $\mathfrak{R}$ are linearly independent and if $\sigma_1, \sigma_2 \in \mathfrak{R}$ implies that $\sigma_1 - \sigma_2$ is not a root. For such a P-system $\mathfrak{R}$ let $\Delta(\mathfrak{R})$ be the set of all roots which are linear combination with integer coefficients of elements of $\mathfrak{R}$. Then $\Delta(\mathfrak{R})$ is a root system (in the vector space $\mathfrak{h}(\mathfrak{R})$ generated by $\mathfrak{R}$) and $\mathfrak{R}$ is a basis of this system. The subalgebra

$$\mathfrak{h}(\mathfrak{R}) \bigoplus_{\sigma \in \Delta(\mathfrak{R})} \mathfrak{g}^{\sigma}$$

is semi-simple with the obvious root data...

For example any proper subset of the extended Dynkin diagram is a P-system. Given such a subset A we can then add its highest root to get a new extended Dynkin diagram and then take a proper subset and so on...

In particular consider $\Sigma - \{\alpha\}$, a basis for the root system of $\mathfrak{m} = \mathfrak{b}(\alpha)$ and let τ be the highest root relative to this basis. Although easy to prove the following result is basic for this work.

Proposition 2.7 ($\mathfrak{g}$ **of exceptional type different from** G_2). *The subset $\{-\beta, \alpha, \delta, \tau\}$ is a P-system of type D_4.*

We have to prove that the difference of two elements of this subset is not a root. Because β is the highest root of the original system $-\beta-\alpha$, $-\beta-\delta$, $-\beta-\tau$ are not roots. Also $\alpha-\delta$, the difference of two simple roots is not a root and τ being a linear combination with positive integer coefficients of elements of $\Sigma - \{\alpha\}$ the difference $\tau-\alpha$ cannot be a root. Finally τ is the highest root of the root system based on $\Sigma - \{\alpha\}$. If this system is not of type A_ℓ then τ is orthogonal to all the simple roots except one and this one is "contained" twice in τ so that it cannot be δ; in the A_ℓ case τ is orthogonal to all roots except the first and the last one. We excluded G_2 so that this case occurs only for E_6 and we have $\ell = 5$. Because δ is the middle root, it is orthogonal to τ. This imply [M-R-S. lemme 2.1] that τ is strongly orthogonal to δ hence $\tau-\delta$ is not a root.

We must also check that the 4 roots are linearly independent. The root β is orthogonal to δ and to τ and we just saw that τ and δ are also orthogonal. Hence the linear space generated by our 4 roots is of dimension 3 or 4. Now the scalar product of α and δ is strictly negative, the root τ is the sum of δ and of roots orthogonal to α so that the scalar product of α and τ is also strictly negative and finally because $\beta-\alpha$ is a root, the scalar product of α and $-\beta$ is negative, strictly because α and β are not orthogonal. Next we claim that the 4 roots have the same length; only the F_4 case has to be checked but for F_4, the roots β, $\alpha = \alpha_1$, $\delta = \alpha_2$ are long roots and so is $\tau = \alpha_2 + 2\alpha_3 +$

$2\alpha_4$. Because α is not proportionnal to $-\beta$ this implies that the angle between α and $-\beta$ is $2\pi/3$ and this remains valid for τ and δ. The only possible linear relation is thus $\alpha=-\frac{1}{2}(-\beta+\tau+\delta)$ or $\beta=2\alpha+\tau+\delta$ and this is impossible because it is known [R] that $\beta-2\alpha-3\delta$ is a linear combination of simple roots other than α and δ and that we saw that $\tau-\delta$ is also such a linear combination. This tells us that we have a P-system and the above computation also shows that it is of type D_4.

Here again a case by case proof is faster. For example consider the E_6 case. Then $\alpha=\alpha_2$ and $\delta=\alpha_4$. We have $\tau=\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6$ but it can also be obtained as follows. From the extended Dynkin diagram remove α_1 and α_6 getting a P-system of type D_5. By a trivial computation based on the explicit decomposition of β as a sum of simple roots, we check that τ is also the highest root of this D_5 P-system. Then from the extended Dynkin diagram of D_5 remove α_3 and α_5. This gives the required P-system of type D_4. The other three cases can be treated in a similar fashion.

Going back to the general case (exceptional but not G_2...), call $\mathfrak{s}$ the simple algebra of type D_4 built with the above P-system. Note that $\mathfrak{a}(\delta)\subset\mathfrak{s}$ is the subalgebra of type A_2 associated to the P-system $\{-\beta, \alpha\}$. The intersection of $\mathfrak{b}(\delta)$ with $\mathfrak{s}$ is of course the commutator of $\mathfrak{a}(\delta)$ in $\mathfrak{s}$ and is reduced to the two dimensional space $\mathfrak{h}\cap\mathfrak{s}\cap\mathfrak{b}(\delta)$ which can also be described as the intersection of the kernels of the restriction to $\mathfrak{h}\cap\mathfrak{s}$ of the two linear forms $-\beta$ and α. Note that $\mathfrak{a}(\delta)\oplus(\mathfrak{b}(\delta)\cap\mathfrak{s})$ is a parabolic subalgebra of $\mathfrak{s}$.

Consider in $\mathfrak{m}$ the maximal parabolic subgroup built with the simple root δ. Call $\mathfrak{n}^+$ the unipotent radical. It is a commutative subalgebra. In particular let $\beta_1, \beta_2, \ldots, \beta_m$ be the maximal set of roots as defined in §1. Recall that we have $\mathfrak{g}^{\beta_i}\subset\mathfrak{n}^+$, that the β_i are strongly orthogonal and that

$$H_{\beta_1}+\cdots+H_{\beta_m}=K .$$

Another key point is the following one.

Proposition 2.8. *The integer m is equal to* 3 *and we have*

$$\beta_1=\delta\ ,\quad \beta_2=\varepsilon\ ,\quad \beta_3=\tau\ .$$

Let us first check that $H=2H_\alpha+K$. We prove that for $\sigma\in\Sigma$ we have $\sigma(H)=2\sigma(H_\alpha)+\sigma(K)$. If $\sigma\neq\alpha, \delta$, then $\sigma(H)=\sigma(H_\alpha)=0$ and also $\sigma(K)=0$ by definition of K. If $\sigma=\delta$, then $\delta(H)=0$ and $\delta(K)=2$. But neither $\delta-\alpha$ nor $\delta+2\alpha$ are roots while $\alpha+\delta$ is a root which implies that $\delta(H_\alpha)=n(\delta, \alpha)=-1$. Finally instead of $\sigma=\alpha$ we can prove the equality for $\sigma=\beta$. We have $\beta(H)=2$ and $\beta(K)=0$ and arguing as above $\beta(H_\alpha)=n(\beta, \alpha)=1$.

The roots β_i have the same length and $\beta_1=\delta$. Furthermore β, δ, τ, α also have the same length and finally recall that $H=H_\beta$. The equality $H=2H_\alpha+K$ which may be written as

$$H_\beta=2H_\alpha+\sum H_{\beta_i}$$

is thus equivalent to

$$\beta=2\alpha+\sum\beta_i\ .$$

Comparing the coefficients of δ we get $m=3$. We have $\beta_3=\tau$, the highest root of $\mathfrak{m}$. This is buried in [M-R-S.] (in the regular case the last of the β_i is always the highest root). Now $\beta_1=\delta$ so that $\beta_2=\beta-2\alpha-\beta_1-\beta_3=\varepsilon$. Note that $-\varepsilon$ is the highest root of $\mathfrak{s}$. Of course, once more, in the four cases at hand, a case by case verification of the Proposition is a trivial matter. Put

$$\mathfrak{d}^+=\mathfrak{g}^\delta\oplus\mathfrak{g}^\varepsilon\oplus\mathfrak{g}^\tau\ .$$

The subspace $\mathfrak{d}^+$ is contained in $\mathfrak{n}^+$. We let $\mathfrak{n}^-$ be the opposed nilpotent radical and we define $\mathfrak{d}^-$ in an obvious manner. Remark that we are in position to apply Proposition 1.1 and also that $\mathfrak{d}^\pm$ are both contained in $\mathfrak{s}$.

We can now turn our attention back to the minimal orbit Ω; to avoid confusion put $\Omega_{\mathfrak{g}}=\Omega$ and let $\Omega_{\mathfrak{s}}$ be the minimal (co)adjoint orbit of $\mathfrak{s}$.

Proposition 2.9 ($\mathfrak{g}$ **of type** F_4, E_6, E_7, **or** E_8)**.** *One has*

$$\Omega_{\mathfrak{g}}\approx B(\delta)\,\Omega_{\mathfrak{s}}\ .$$

The connected reductive group G_0 is the centralizer of H in G and M is the centralizer of X_β in G_0. Then (§2.1) implies that

$$\Omega_{\mathfrak{g}}\approx \boldsymbol{C}^*A(\alpha)\,G_0Z$$

By definition L is the Levi component of a standard maximal parabolic subgroup of G_0. Let $\mathfrak{n}^+$ be the nilpotent radical of the Lie algebra of this parabolic subgroup and N^+ the unipotent radical. Define as usual $\mathfrak{n}^-$ and N^-. By Bruhat's decomposition

$$G_0\approx LN^-N^+$$

so that

$$\Omega_{\mathfrak{g}}\approx \boldsymbol{C}^*A(\alpha)\,LN^-N^+Z\ .$$

Recall that

$$Z=\operatorname{Exp}(X_{-\alpha}+X_{-\beta+\alpha})\,X_\beta\ ,$$

note that $L=TB(\delta)$ normalizes $\mathfrak{a}(\delta)$ and that $LX_\beta=\boldsymbol{C}^*X_\beta$. The action of L on $\mathfrak{n}^\pm$ is prehomogeneous (commutative type) and

$$I^+=X_\delta+X_\varepsilon+X_\tau$$

is a generic element so

$$\Omega_{\mathfrak{g}} \approx \boldsymbol{C}^* A(\alpha) L N^- \operatorname{Exp}(I^+) A(\delta) X_\beta .$$

By Proposition 1.1 we further get that

$$\Omega_{\mathfrak{g}} \approx \boldsymbol{C}^* A(\alpha) L \operatorname{Exp}(\mathfrak{d}^-) \operatorname{Exp}(I^+) A(\delta) X_\beta .$$

However

$$\boldsymbol{C}^* A(\alpha) T \operatorname{Exp}(\mathfrak{d}^-) \operatorname{Exp}(I^+) A(\delta) X_\beta \subset \Omega_{\mathfrak{s}} .$$

so

$$\Omega_{\mathfrak{g}} \approx B(\delta) \Omega_{\mathfrak{s}} .$$

We use subsection 2.2. Being careful with the numbering of the roots we see that

$$\xi(u, t) = u \operatorname{Exp}((t/u) X_\varepsilon) \operatorname{Exp}(X_{\beta-\alpha-\delta}) \operatorname{Exp}(X_{-\alpha-\tau}) X_{-\varepsilon} .$$

By Proposition 2.5 and 2.9 we have

$$\Omega_{\mathfrak{g}} \approx \bigcup_{u \in \boldsymbol{C}^*, t \in \boldsymbol{C}} A(\delta) B(\delta) \xi(u, t) .$$

To study the projections of $\xi(u, t)$ on $\mathfrak{a}(\delta)$ and $\mathfrak{b}(\delta)$ we may compute inside $\mathfrak{s}$. Define λ_1 and λ_2 as in subsection 2.2. We can replace $\xi(u, t)$ by

$$u H_{\beta-\alpha-\delta} + t H_{-\varepsilon} = (\lambda_1 H_{-\beta} - \lambda_2 H_\alpha) + (\lambda_1 (H_\delta - H_\tau) - \lambda_2 (H_{-\beta} + 2H_\alpha + H_\delta + 2H_\tau)) .$$

We have

$$\lambda_1 H_{-\beta} - \lambda_2 H_\alpha \in \mathfrak{a}(\delta)$$

and this element is orthogonal to $\mathfrak{b}(\delta)$. Also

$$\lambda_1 (H_\delta - H_\tau) - \lambda_2 (H_{-\beta} + 2H_\alpha + H_\delta + 2H_\tau) \in \mathfrak{m}$$

and is orthogonal to $\mathfrak{a}(\delta)$.
Then the linear form on $\mathfrak{a}(\delta)$ defined by $\xi(u, t)$ is

$$X \mapsto B_{\mathfrak{g}}(\lambda_1 H_{-\beta} - \lambda_2 H_\alpha, X) .$$

Let

$$B_{\mathfrak{g}|\mathfrak{a}(\delta)} = c B_{\mathfrak{a}(\delta)}$$

so that

$$c = \frac{B_{\mathfrak{g}}(H, H)}{B_{\mathfrak{a}(\delta)}(H, H)} = \frac{1}{12} B_{\mathfrak{g}}(H. H) .$$

We get

$$p_{\mathfrak{g}/\mathfrak{a}(\delta)} \xi(u, t) = c(\lambda_1 H_{-\beta} - \lambda_2 H_\alpha) .$$

However $\{-\beta, \alpha\}$ is a system of simple roots of $\mathfrak{a}(\delta)$. If we identify $\mathfrak{a}(\delta)$ and $\mathfrak{sl}(3)$ accordingly we see that the projection is

$$\frac{B_{\mathfrak{g}}(H,H)}{12}\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_3 & 0\\ 0 & 0 & \lambda_2\end{pmatrix}$$

where $\lambda_3=-\lambda_1-\lambda_2$.

Next consider the projection on $\mathfrak{b}(\delta)$. We replace $\xi(u,t)$ by

$$\lambda_1(H_\delta-H_\tau)-\lambda_2(H_{-\beta}+2H_\alpha+H_\delta+2H_\tau)\in\mathfrak{m}\ .$$

The roots δ and τ have the same length so $B_{\mathfrak{g}}(K, H_\delta-H_\tau)$ is a multiple of $(\tau-\delta)(K)$. However $\tau-\delta$ is a sum of roots σ such that $\sigma(K)=0$. This shows that $H_\delta-H_\tau\in\mathfrak{b}(\delta)\cap\mathfrak{h}$. In a similar fashion β, α, δ, τ, ε having the same length we find that $H_{-\beta}+2H_\alpha+H_\delta+2H_\tau\in\mathfrak{b}(\delta)\cap\mathfrak{h}$ and also, using the relation $\beta=2\alpha+\delta+\varepsilon+\tau$ that $H_{-\beta}+2H_\alpha+H_\delta+2H_\tau=H_\tau-H_\varepsilon$. Then we consider the linear form on $\mathfrak{b}(\delta)$ defined by the Killing form of $\mathfrak{g}$ and

$$\lambda_1H_\delta+\lambda_2H_\varepsilon+\lambda_3H_\tau$$

where $\lambda_3=-\lambda_1-\lambda_2$. Because $\lambda_1H_\delta+\lambda_2H_\varepsilon+\lambda_3H_\tau$ belongs to $\mathfrak{b}(\delta)$ the only remaining point is to compare the Killing forms of $\mathfrak{g}$ and of $\mathfrak{b}(\delta)$.

As in §1 we put $k=\dim(\mathfrak{n}^+)$ and define the integer d by $k=m+m(m-1)d/2$ that is to say $k=3(1+d)$. By Lemma 3-3 of [R-S] we know that the restriction of the Killing form $B_{\mathfrak{m}}$ of $\mathfrak{m}$ to $\mathfrak{b}(\delta)$ is $cB_{\mathfrak{b}(\delta)}$ with

$$c=\frac{4}{3}\frac{1+d}{d}=\frac{4}{3}\frac{k}{k-3}$$

Furthermore we have $H=K+2H_\alpha$ and H has the same length and is orthogonal to H_α so

$$B_{\mathfrak{g}}(K,K)=3B_{\mathfrak{g}}(H,H)\ .$$

Also $B_{\mathfrak{m}}(K,K)=8k$. In the cases at hand if $\mathfrak{g}$ is simple so is $\mathfrak{m}$ so we get

$$B_{\mathfrak{g}|\mathfrak{m}}=\frac{3}{8k}B_{\mathfrak{g}}(H,H)B_{\mathfrak{m}}$$

and altogether

$$B_{\mathfrak{g}|\mathfrak{b}(\delta)}=\frac{B_{\mathfrak{g}}(H,H)}{2(k-3)}B_{\mathfrak{b}(\delta)}\ .$$

The projection on $\mathfrak{b}(\delta)$ is

$$\frac{B_{\mathfrak{g}}(H,H)}{2(k-3)}(\lambda_1H_\delta+\lambda_2H_\varepsilon+\lambda_3H_\tau)\ .$$

By Proposition 1.6 and Lemma 2.6 two such elements are conjugate if and only if the two sets of coefficients λ_i, are identical up to a permutation. Changing slightly the normalization of the λ_i we get the following result

Theorem 2.10 ($\mathfrak{g}$ **of type** F_4, E_6, E_7, E_8)**.** *Generically the projection from the dual* $\mathfrak{g}^*$ *of* $\mathfrak{g}$ *onto the dual* $\mathfrak{a}(\delta)^* \times \mathfrak{b}(\delta)^*$ *of* $\mathfrak{a}(\delta) \times \mathfrak{b}(\delta)$ *induces a correspondence between the coadjoint orbits of*

$$\lambda_1 H_{-\beta} - \lambda_2 H_\alpha \qquad in \quad \mathfrak{a}(\delta)^*$$

and

$$\frac{6}{k-3}(\lambda_1 H_\delta + \lambda_2 H_\varepsilon - (\lambda_1+\lambda_2) H_\tau) \qquad in \quad \mathfrak{b}(\delta)^*$$

where $k=3(1+d)$ *is given by*

$$k=\begin{cases} 6 & \text{if } \mathfrak{g} \text{ is of type } F_4 \\ 9 & \text{if } \mathfrak{g} \text{ is of type } E_6 \\ 15 & \text{if } \mathfrak{g} \text{ is of type } E_7 \\ 27 & \text{if } \mathfrak{g} \text{ is of type } E_8 \ . \end{cases}$$

If $\lambda_3 = -\lambda_1 - \lambda_2$, *then, in both situations, the coadjoint orbits corresponding to* $(\lambda_1, \lambda_2, \lambda_3)$ *and* $(\lambda_1', \lambda_2', \lambda_3')$ *coincide if and only if the* λ_i' *are equal to the* λ_i, *up to a permutation. Thus we get a well defined and one to one map between the semi-simple coadjoint orbits.*

We now investigate the orthogonal case. Assume that $\mathfrak{g}$ is of type B_ℓ with $\ell \geq 4$ or D_ℓ with $\ell \geq 5$. With Bourbaki's notations we have $\alpha = \alpha_2$ and there are two simple roots connected to α, namely α_1 and α_3. We take $\delta = \alpha_3$. Then $\mathfrak{a}(\delta)$ is of type A_3 and $\mathfrak{b}(\delta)$ of type $B_{\ell-3}$ or $D_{\ell-3}$ (as usual we consider that $B_1 = A_1$, $B_2 = C_2$, $D_2 = A_1 \times A_1$ and $D_3 = A_3$). The subgroups $A(\delta)$ and $B(\delta)$ have already been defined at the beginning of the section.

We now build a subalgebra of type D_4. Recall that $\mathfrak{m} = \mathfrak{a}(\alpha)$. In this case it has two simple components, $\mathfrak{m}_1$ of type A_1, with simple roots α_1 and $\mathfrak{m}_2$ of type $B_{\ell-2}$ or $D_{\ell-2}$ a system of simple roots being $\alpha_3, \alpha_4, \ldots, \alpha_\ell$. In this second component $\delta = \alpha_3$ corresponds to a maximal parabolic subalgebra with a commutative nilpotent radical $\mathfrak{n}_2^+$. We call k_2 the dimension of $\mathfrak{n}_2^+$. With the notations of §1 we have $m_2 = 2$ and the canonical choice for the β_i is $\beta_1 = \delta = \alpha_3$ and $\beta_2 = \tau$ the highest root of $\mathfrak{m}_2$. Also there is an integer d_2 attached to the situation and we have:

$$k_2 = 2 + d_2 \ .$$

We put $\alpha_1 = \varepsilon$ and, as before, define the diagonal

$$\mathfrak{d}^+ = \mathfrak{g}^\delta \oplus \mathfrak{g}^\varepsilon \oplus \mathfrak{g}^\tau \ .$$

We claim that Proposition 2.7 remains valid: the subset $\{-\beta, \alpha, \delta, \tau\}$ is a P-system of type D_4. Indeed $\{-\beta, \alpha_1, \alpha_2, \alpha_3\}$ is clearly a P-system of type D_4. The highest root is

$$-\beta + 2\alpha_2 + \alpha_1 + \alpha_3 = -\tau \ .$$

Adding $-(-\tau)$ and removing α_1 gives a new system of type D_4 (in fact another basis of the same D_4). Note that the relation

$$\beta = 2\alpha + \delta + \varepsilon + \tau$$

remains valid. We call $\mathfrak{s}$ the subalgebra of type D_4 built on the above P-system. With some trivial modifications in the proof, Proposition 2.9 remains true in this case so we are again reduced to compute the projections of $\xi(u, t)$. Choosing suitably the root vectors we get

$$\begin{aligned}\xi(u, t) = &+uH_{-\beta+\alpha+\delta} + tH_{-\varepsilon} \\ &+uX_{-\beta+\alpha+\delta} + uX_{-\varepsilon} - uX_{\alpha+\tau} \\ &-(u+t)X_{\beta-\alpha-\delta} - t(u+t)X_{\varepsilon} - tX_{-\alpha-\tau}\ .\end{aligned}$$

Let

$$\lambda_1 = -t - \frac{u}{2}\ ,\quad \lambda_2 = -\frac{u}{2}\ .$$

Then

$$\xi(u, t) = [\lambda_1 H_{\varepsilon} - \lambda_2 H_{-\beta} + (\lambda_2^2 - \lambda_1^2) X_{\varepsilon} + X_{-\varepsilon}] + [\lambda_2 (H_{\tau} - H_{\delta})] + \ldots$$

where the dots represent a term orthogonal to both $\mathfrak{a}(\delta)$ and $\mathfrak{b}(\delta)$ while the first (resp. the second) term belongs to $\mathfrak{a}(\delta)$ (resp. to $\mathfrak{b}(\delta)$) and is orthogonal to $\mathfrak{b}(\delta)$ (resp. to $\mathfrak{a}(\delta)$). If we identify $\mathfrak{a}(\delta)$ with $\mathfrak{sl}(4, \boldsymbol{C})$, using as simple roots $\{-\beta, \alpha, \varepsilon\}$, then, taking the Killing forms into account, the projection of $\xi(u, t)$ on $\mathfrak{a}(\delta)$ is

$$\frac{B_{\mathfrak{g}}(H, H)}{16}\begin{pmatrix} -\lambda_2 & 0 & 0 & 0 \\ 0 & +\lambda_2 & 0 & 0 \\ 0 & 0 & +\lambda_1 & \lambda_2^2 - \lambda_1^2 \\ 0 & 0 & 1 & -\lambda_1 \end{pmatrix}.$$

If λ_2 is different from 0, then this projection is conjugate to the diagonal element

$$\frac{B_{\mathfrak{g}}(H, H)}{16}\begin{pmatrix} +\lambda_2 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & +\lambda_2 & 0 \\ 0 & 0 & 0 & -\lambda_2 \end{pmatrix}.$$

To compute the projection onto $\mathfrak{b}(\delta)$ we need to evaluate the quotient $B_{\mathfrak{g}}/B_{\mathfrak{b}(\delta)}$. By Lemma 3.3 of [R-S]

$$B_{\mathfrak{m}_2}/B_{\mathfrak{b}(\delta)} = \frac{2+d_2}{d_2}\ .$$

Let K_2 be the unique element of $\mathfrak{h} \cap \mathfrak{m}_2$ such that $\delta(K_2) = 2$ and $\sigma(K_2) = 0$ for σ

a simple root different from α and δ. Then $\beta(K_2)=0$ and

$$\beta=\varepsilon+2\alpha+2\delta+\dots$$

implies $\alpha(K_2)=-2$. Also Δ_1 consists of the 4 roots $\{\alpha, \alpha+\varepsilon, \beta-\alpha, \beta-\alpha-\varepsilon\}$, and of the roots of type $\alpha+\delta+\dots$ and of roots of type $\beta-\alpha-\delta-\dots$ where the... stands for a combination of simple roots belonging to $\Sigma-\{\varepsilon, \alpha, \delta\}$. In particular the dimension of $\mathfrak{g}_1$ is $4+2k_2$. It follows that

$$B_{\mathfrak{g}}(K_2, K_2)=8k_2+32=8(k_2+4)=8(6+d_2)$$

and

$$B_{\mathfrak{m}_2}(K_2, K_2)=8k_2=8(2+d_2)\ .$$

Thus

$$B_{\mathfrak{g}}/B_{\mathfrak{m}_2}=\frac{6+d_2}{2+d_2}\ .$$

Finally we note that

$$B_{\mathfrak{g}}(H, H)=8+2(4+2k_2)=4(4+k_2)=4(6+d_2)$$

so

$$B_{\mathfrak{g}}/B_{\mathfrak{b}(\delta)}=\frac{6+d_2}{d_2}=\frac{B_{\mathfrak{g}}(H, H)}{16}\frac{4}{d_2}\ .$$

The projection onto $\mathfrak{b}(\delta)$ is

$$\frac{4}{d_2}(\lambda_2(H_\tau-H_\delta))\ .$$

By Proposition 1.6

$$\lambda_2(H_\tau-H_\delta) \qquad \text{and} \qquad \lambda_2'(H_\tau-H_\delta)$$

are conjugate if and only if $\lambda_2=\pm\lambda_2'$. The final statement is the following theorem:

Theorem 2.11 ($\mathfrak{g}$ **of type** B_ℓ, $\ell\geq 4$, **or** D_ℓ, $\ell\geq 5$). *Generically the projection from the dual* $\mathfrak{g}^*$ *of* $\mathfrak{g}$ *onto the dual* $\mathfrak{a}(\delta)^*\times\mathfrak{b}(\delta)^*$ *of* $\mathfrak{a}(\delta)\times\mathfrak{b}(\delta)$ *induces a correspondence between the coadjoint orbits of*

$$\lambda_2(H_{-\beta}+H_\varepsilon) \qquad \textit{in} \quad \mathfrak{a}(\delta)^*$$

and

$$\frac{4}{d_2}(\lambda_2(H_\tau-H_\delta)) \qquad \textit{in} \quad \mathfrak{b}(\delta)^*\ ,$$

where $d_2=2\ell-7$ *for* $\mathfrak{g}$ *of type* B_ℓ *and* $d_2=2\ell-8$ *for* $\mathfrak{g}$ *of type* D_ℓ.

In both situations, the coadjoint orbits corresponding to λ_2 *and* λ_2' *coincide if and only if* $\lambda_2=\pm\lambda_2'$. *Thus we get a well defined and one to one map between semi-simple coadjoint orbits.*

§3. The Φ map

3.0. Joseph's construction. We keep the notations of the beginning of §2; in particular $\mathfrak{g}$ is not of type A_ℓ but, for the present time at least the case C_ℓ is not excluded. We consider the Heisenberg subalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and also the subalgebra

$$\mathfrak{r} = \boldsymbol{C}H \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

which is the image of ad (X_β) and may be identified with the tangent space to the minimal orbit Ω at the point X_β. Following Joseph we put $E = X_\beta$ and introduce the localization.

$$\mathcal{A} = \mathcal{U}(\mathfrak{r})_E$$

in E of the enveloping algebra $\mathcal{U}(\mathfrak{r})$ of $\mathfrak{r}$.

Theorem (A. Joseph [J]). *There exists a unique algebra homomorphism Φ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ into $\mathcal{A}$ which is the identity on $\mathfrak{r}$.*

The kernel of this map is the unique completely prime two-sided ideal whose characteristic variety is the closure of the minimal orbit J. It is called Joseph's ideal. As said in the introduction, our goal is to check that, at least in some cases, given a dual pair $\mathfrak{a} \times \mathfrak{b}$ in $\mathfrak{g}$ the images in $\mathcal{U}(\mathfrak{g})/J$ of the centers of the enveloping algebras of $\mathfrak{a}$ and $\mathfrak{b}$ coincide.

We shall need the explicit construction of Φ given in [J]. Roots vectors X_γ are fixed once for all; although it is not crucial let us assume that we have a Chevalley basis. Define the coefficients $N_{\gamma,\sigma}$ as usual by

$$[X_\gamma, X_\sigma] = N_{\gamma,\sigma} X_{\gamma+\sigma} \ .$$

If $\gamma \in \Delta_1$ so does $\beta - \gamma$; put

$$F_\gamma = \frac{1}{N_{\gamma,\beta-\gamma}} X_{\beta-\gamma} \ .$$

so that $[X_\gamma, F_\gamma] = X_\beta$. Also define $F_\beta = -H/2$; then again $[X_\beta, F_\beta] = X_\beta$.
Next define

$$D\colon \mathfrak{g} \otimes S(\boldsymbol{C}H \oplus \mathfrak{g}_1) \longrightarrow \mathfrak{g} \otimes S(\boldsymbol{C}H \oplus \mathfrak{g}_1)$$

by

$$D(X \otimes T) = \sum_{\Delta_1 \cup \Delta_2} \mathrm{ad}(X_\gamma) X \otimes F_\gamma T \ .$$

Note that D is a nilpotent operator $(D^5 = 0)$ so that e^D is well-defined.
Also let

$$u\colon \mathfrak{g} \otimes S(\boldsymbol{C}H \oplus \mathfrak{g}_1) \longrightarrow S(\boldsymbol{C}H \oplus \mathfrak{g}_1)$$

be the contraction map

$$u(Y\otimes T)=\frac{B(Y,X_{-\beta})}{B(X_\beta,X_{-\beta})}T\ .$$

For $X\in\mathfrak{g}$ put

$$\Psi(X)=u\circ e^D(X\otimes 1)$$

Finally Joseph defines a twisted symmetrization map

$$\sigma\colon S(\boldsymbol{C}H\oplus\mathfrak{g}_1)\longrightarrow\mathscr{A}$$

by

$$\sigma(A_1A_2\ldots A_r)=E\frac{1}{r!}\sum_{\tau\in\mathfrak{S}_r}(E^{-1}A_{\tau(1)}\ldots E^{-1}A_{\tau(r)})\ .$$

Then, for $X\in\mathfrak{g}$

$$\Phi(X)=\sigma(\Psi(X))+c(\mathfrak{g})E^{-1}\frac{B(X,X_\beta)}{B(X_{-\beta},X_\beta)}\ ,$$

where $c(\mathfrak{g})$ is some rational number. Note that our normalization for $c(\mathfrak{g})$ is not the same as Joseph. Also Joseph does not compute the exact value of this constant. As a by product of our computation we will get the explicit value.

$$\Phi([X,Y])=\Phi(X)\Phi(Y)-\Phi(Y)\Phi(X)\qquad\text{for}\quad X,Y\in\mathfrak{g}\ .$$

Furthermore $\Phi(X)=X$ for $X\in\mathfrak{r}$. Note that $[X_{-\beta},\mathfrak{g}_1]=\mathfrak{g}_{-1}$ so that to gain some control on Φ it is sufficient to compute $\Phi_{|\mathfrak{m}}$ and $\Phi(X_{-\beta})$.

Lemma 3.1. *If $X\in\mathfrak{m}$, then*

$$\Phi(X)=\frac{1}{4}\sum_{\gamma_1,\gamma_2\in\Delta_1}\frac{B(\operatorname{ad}(X_{\gamma_1})\operatorname{ad}(X_{\gamma_2})X,X_{-\beta})}{B(X_\beta,X_{-\beta})}(F_{\gamma_1}F_{\gamma_2}+F_{\gamma_2}F_{\gamma_1})E^{-1}\ .$$

In particular $\Phi(X)\in\mathcal{U}(\mathfrak{g}_1\oplus\mathfrak{g}_2)_E$.

We use the explicit formula for Φ. We have to apply e^D and then keep only the terms with a non-zero component along $\mathfrak{g}_2$. Because $X\in\mathfrak{m}$ we have $[X_\beta,X]=0$ and so the only terms which matter are the ones coming from D^2 and a couple of roots in Δ_1. The lemma follow modulo a trivial computation. Note that

$$B(\operatorname{ad}(X_{\gamma_1})\operatorname{ad}(X_{\gamma_2})X,X_{-\beta})$$

is symmetric in γ_1 and γ_2 (in fact the adjoint action of $\mathfrak{m}$ in $\mathfrak{g}_1$ is an imbedding into the symplectic Lie algebra for the symplectic form on $\mathfrak{g}_1$ given by the bracket and $\Phi_{|\mathfrak{m}}$ is essentially given by this embedding).
More generally, for any element of $Z\in\mathcal{U}(\mathfrak{m})$ we see that $\Phi(Z)\in\mathcal{U}(\mathfrak{g}_1\oplus\mathfrak{g}_2)_E$.

3.1. Polarization. Now following again [J], consider the Heisenberg subalgebra $\mathfrak{g}_1\oplus\mathfrak{g}_2$. Suppose that we split Δ_1 as a union of two disjoint subsets Γ_1 and Γ_2 such that $\gamma\in\Gamma_1$ implies that $\beta-\gamma\in\Gamma_2$. Let

$$V_1=\oplus\mathfrak{g}^\gamma \qquad \text{for} \quad \gamma\in\Gamma_1$$

and put

$$\mathcal{S}=\bigoplus_{r\in Q} S(V_1)E^r\ .$$

We define a representation π of $\mathfrak{r}$ in $\mathcal{S}$ by

$$\pi(X_\gamma)=\text{multiplication by } X_\gamma \quad \text{if} \quad \gamma\in\Gamma_1\ .$$
$$\pi(F_\gamma)=-E\frac{\partial}{\partial X_\gamma} \quad \text{if} \quad \gamma\in\Gamma_1\ .$$
$$\pi(E)=\text{multiplication by } E\ ,$$
$$\pi(H)X_{\gamma_1}\dots X_{\gamma_j}E^r=(j+2r)X_{\gamma_1}\dots X_{\gamma_j}E^r\ .$$

The representation π extends to a representation of the algebra $\mathcal{A}$. Composing with Φ we get a representation of $\mathfrak{g}$.

Lemma 3.2. *Let* $X\in\mathfrak{m}$ *such that* $\mathrm{ad}(X)V_i\subset V_i$. *Then*

$$\pi(\Phi(X))=\mathrm{ad}(X)+\frac{1}{2}\mathrm{Tr}(\mathrm{ad}(X)_{V_1})Id\ .$$

In the above formula $\mathrm{ad}(X)$ is extended to $S(V_1)$ as a derivation and then to $\mathcal{S}$ in the obvious way. Note that $\mathrm{ad}(X)V_i\subset V_i$. We start from lemma 3.1. With the notations of this lemma, if $\gamma_2\in\Gamma_2$ then $\mathrm{ad}(X_{\gamma_2})X\in V_2$ so that to get a non zero term we must take $\gamma_1\in\Gamma_1$ and conversely. Because $B(\mathrm{ad}(X_{\gamma_1})\mathrm{ad}(X_{\gamma_2})X, X_{-\beta})$ is symmetric with respect to γ_1 and γ_2 we thus have

$$\Phi(X)=\frac{1}{2}\sum_{\gamma_1\in\Gamma_1,\gamma_2\in\Gamma_2}\frac{B(\mathrm{ad}(X_{\gamma_1})\mathrm{ad}(X_{\gamma_2})X, X_{-\beta})}{B(X_{-\beta}, X_\beta)}(F_{\gamma_1}F_{\gamma_2}+F_{\gamma_2}F_{\gamma_1})E^{-1}\ .$$

Hence

$$\pi(\Phi(X))=$$
$$-\frac{1}{2}\sum_{\gamma_1\in\Gamma_1,\gamma_2\in\Gamma_2}\frac{B(\mathrm{ad}(X_{\gamma_1})\mathrm{ad}(X_{\gamma_2})X, X_{-\beta})}{B(X_{-\beta}, X_\beta)}\frac{1}{N_{\gamma_2,\beta-\gamma_2}}\left(X_{\beta-\gamma_2}\frac{\partial}{\partial X_{\gamma_1}}+\frac{\partial}{\partial X_{\gamma_1}}X_{\beta-\gamma_2}\right)\ .$$

This is equal to A_1+A_2 with

$$A_1=-\frac{1}{2}\sum_{\gamma_1\in\Gamma_1}\frac{B(\mathrm{ad}(X_{\gamma_1})\mathrm{ad}(X_{\beta-\gamma_1})X, X_{-\beta})}{B(X_{-\beta}, X_\beta)}\frac{1}{N_{\beta-\gamma_1,\gamma_1}}\ ,$$

$$A_2=-\sum_{\gamma_1\in\Gamma_1,\gamma_2\in\Gamma_2}\frac{B(\mathrm{ad}(X_{\gamma_1})\mathrm{ad}(X_{\gamma_2})X, X_{-\beta})}{B(X_{-\beta}, X_\beta)}\frac{1}{N_{\gamma_2,\beta-\gamma_2}}X_{\beta-\gamma_2}\frac{\partial}{\partial X_{\gamma_1}}\ .$$

For $X=X_\mu$ we have $A_1=0$ and also the restriction of $\mathrm{ad}(X_\mu)$ to V_1 is a nilpotent operator, of trace 0. If $X\in\mathfrak{h}\cap\mathfrak{m}$ then

$$A_1=-\frac{1}{2}\sum_{\gamma_1\in\Gamma_1}(\beta-\gamma_1)(X)Id=\frac{1}{2}\sum_{\gamma_1\in\Gamma_1}\gamma_1(X)Id\ .$$

Now consider A_2. Fix $\gamma \in \Gamma_1$ and let us compute $A_2 X_\gamma$,

$$A_2 X_\gamma = -\sum_{\gamma_2 \in \Gamma_2} \frac{B(\mathrm{ad}(X_\gamma)\,\mathrm{ad}(X_{\gamma_2})X, X_{-\beta})}{B(X_{-\beta}, X_\beta)} \frac{1}{N_{\gamma_2, \beta-\gamma_2}} X_{\beta-\gamma_2} .$$

This is also equal to

$$A_2 X_\gamma = -\sum_{\gamma_2 \in \Gamma_2} \frac{B(\mathrm{ad}(X)X_\gamma, X_{-\beta+\gamma_2})}{B(X_{-\beta}, X_\beta)} \frac{N_{\gamma_2,-\beta}}{N_{\gamma_2, \beta-\gamma_2}} X_{\beta-\gamma_2} .$$

However (see [B-2, page 83])

$$-\frac{N_{\gamma_2,-\beta}}{N_{\gamma_2,\beta-\gamma_2}} = \frac{B(X_\beta, X_{-\beta})}{B(X_{\beta-\gamma_2}, X_{\beta-\gamma_2})}$$

so

$$A_2 X_\gamma = \sum_{\gamma_2 \in \Gamma_2} \frac{B(\mathrm{ad}(X)X_\gamma, X_{-\beta+\gamma_2})}{B(X_{-\beta}, X_\beta)} X_{\beta-\gamma_2} = \mathrm{ad}(X)X_\gamma .$$

Also $A_2 E = 0$ so that the two derivations A_2 and ad (X) concide on a set of generators of $\mathcal{S}$ which proves that they are equal.

§4. The explicit collapsing of the centers: the $\mathfrak{a}(\delta) \times \mathfrak{b}(\delta)$ case

The algebra $\mathfrak{g}$ is simple, of rank at least 3 and, for the present time, assumed to be of type E_6, E_7, E_8, or F_4. The cases A_n and C_n have long been excluded, and in fact are best dealt with separately. The cases B_ℓ and D_ℓ will be taken care of at the end of the §.

We have a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the set of roots Δ and a set of simple roots Σ. The highest root β is orthogonal to all simple roots except one of them called α. In turn α is orthogonal to all simple roots except one which is denoted δ. If we remove one root from the extended Dynkin diagram we get a dual pair. We consider the pair $\mathfrak{a}(\delta) \times \mathfrak{b}(\delta)$ obtained by removing δ. We wish to compute explicitly the collapsing of the centers of the enveloping algebras under Joseph Φ map. This will achieved using a polarization slightly different from the one chosen by Joseph and computing sufficiently many highest weight vectors.

4.0. The polarization. To simplify put, as before $\mathfrak{m} = \mathfrak{a}(\alpha)$. It is a simple algebra admitting $\Sigma - \{\alpha\}$ as a set of simple roots. The root δ corresponds to a maximal parabolic subalgebra of $\mathfrak{m}$ whose standard nilpotent radical $\mathfrak{n}^+$ is commutative; also the action of the Levi subalgebra on this radical is irreducible, prehomogeneous and regular. Let $\mathfrak{n}^-$ be the negative nilpotent radical. The semi-simple part of the Levi is the subalgebra $\mathfrak{b}(\delta)$. The center is one dimensional and generated by the element $K \in \mathfrak{h} \cap \mathfrak{m}$ such that $\sigma(K) = 0$ for σ a simple root of $\mathfrak{m}$ distinct from δ and $\delta(K) = 2$.

Let $H = H_\beta$ the usual coroot. The derivation ad (H) defines a graduation of $\mathfrak{g}$:

$$\mathfrak{g}=\oplus\mathfrak{g}_i \qquad \Delta=\cup\Delta_i \qquad -2\leq i\leq +2 .$$

In particular $\Delta_{\pm 2}=\pm\beta$. We choose root vectors and define the alternating form A on $\mathfrak{g}_1$ by $[X, Y]=A(X, Y)X_\beta$; it is non-degenerate.

For any root σ and any simple root $\eta\in\Sigma$ the coordinate of σ relative to η and the basis Σ is called $|\sigma|_\eta$; it is an integer, possibly negative. We know that $|\beta|_\delta=3$. Hence if $\gamma\in\Delta_1$ then $|\gamma|_\delta$ is equal to 0, 1, 2 or 3. let

$$C_i=\{\gamma\in\Delta_1 \,\big|\, |\gamma|_\delta=i\}$$

and

$$W_i=\oplus\mathfrak{g}^\gamma \qquad \text{for} \quad \gamma\in C_i .$$

The involution $\gamma\mapsto\beta-\gamma$ of Δ_1 sends C_i onto C_{3-i}. If we remove δ from Σ then $\{\alpha\}$ is a connected component so $C_1=\{\alpha\}$ and consequently $C_3=\{\beta-\alpha\}$. Let D^+ be the set of roots μ such that $X_\mu\in\mathfrak{n}^+$ and $D^-=-D^+$. A root μ belongs to D^+ if and only if $|\mu|_\alpha=0$ and $|\mu|_\delta=1$.

Lemma 4.1. *The map $\mu\mapsto\alpha+\mu$ is a bijection of D^+ onto C_1 and*

$$\mathrm{ad}(X_\alpha):\ \mathfrak{n}^+\longrightarrow W_1$$

is an isomorphism of irreducible $\mathfrak{b}(\delta)$ —modules.

The map $\nu\mapsto\beta-\alpha+\nu$ is a bijection of D^- onto C_2 and

$$\mathrm{ad}(X_{\beta-\alpha}):\ \mathfrak{n}^-\longrightarrow W_2$$

is an isomorphism of irreducible $\mathfrak{b}(\delta)$ —modules.

Indeed if $\mu\in D^+$ then $\langle\alpha, \mu\rangle=\langle\alpha, \delta\rangle<0$ so that $\alpha+\mu$ is a root which, by definition of C_1 belongs to C_1. Conversely if $\gamma\in C_1$ then $\langle\gamma, \alpha\rangle=\langle\alpha+\delta, \alpha\rangle=\langle\alpha, \alpha\rangle(1+n(\delta, \alpha)/2)$. But we know (see for example Proposition 2.7) that $n(\delta, \alpha)=-1$ so that $\langle\gamma, \alpha\rangle>0$ and it follows that $\gamma-\alpha$ is a root which of course belongs to D^+. The subalgebra $\mathfrak{b}(\delta)$ normalizes each W_i and also operates irreducibly onto $\mathfrak{n}^+$. But X_α commutes with $\mathfrak{b}(\delta)$ so the restriction of $\mathrm{ad}(X_\alpha)$ to $\mathfrak{n}^+$ is trivially an isomorphism of $\mathfrak{b}(\delta)$ —modules. The second part of the Lemma follows from the involution of Δ_1.

Lemma 4.2. *If $X\in\mathfrak{n}^+$, then* $\mathrm{ad}(X_{-\alpha})\,\mathrm{ad}(X_\alpha)X=-X$ *and also if $Y\in\mathfrak{n}^-$ then* $\mathrm{ad}(X_{\alpha-\beta})\,\mathrm{ad}(X_{\beta-\alpha})Y=-Y$.

Then for $X\in\mathfrak{n}^+$ and $Y\in\mathfrak{n}^-$

$$cB_{\mathfrak{m}}(X, Y)=A(\mathrm{ad}(X_\alpha)X, \mathrm{ad}(X_{\beta-\alpha})Y)$$

where c is a constant given by

$$c=-\frac{1}{2}\|\delta\|^2_{\mathfrak{m}}\, N_{\alpha,\beta-\alpha} .$$

If $\mu\in D^+$, then $-\alpha+\mu$ is not a root so $\mathrm{ad}(X_{-\alpha})\mathfrak{n}^+=0$. Thus

$$\mathrm{ad}(X_{-\alpha})\,\mathrm{ad}(X_{\alpha})X=[H_{\alpha},H]\ .$$

While proving Proposition 2.8 we saw that $H=2H_{\alpha}+K$. Now $[H,X]=0$ and $[K,X]=X$ and our first assertion is proved. The proof of the second one is similar. The alternating form A on $\mathfrak{g}_1$ is invariant under $\mathfrak{m}$. In particular it defines an $\mathfrak{b}(\delta)$ invariant pairing between W_1 and W_2. Such a pairing is unique, up to a constant factor. Also $B_{\mathfrak{m}}$, the Killing form of $\mathfrak{m}$ defines a $\mathfrak{b}(\delta)$ invariant pairing between $\mathfrak{n}^+$ and $\mathfrak{n}^-$. The existence of c is then clear. To compute the constant c we choose $X=X_{\delta}$ and $Y=X_{-\delta}$. Then we consider

$$[\mathrm{ad}(X_{\alpha})X_{\delta},\ \mathrm{ad}(X_{\beta-\alpha})X_{-\delta}]\ .$$

The roots β and δ are orthogonal and $\beta+\delta$ is not a root so they are strongly orthogonal and $[X_{\alpha},X_{\beta-\alpha-\delta}]=0$ so that the above expression may be rewritten as

$$\mathrm{ad}(X_{\alpha})\,[X_{\delta},\ \mathrm{ad}(X_{\beta-\alpha})X_{-\delta}]\ .$$

In the same way $\beta-\alpha+\delta$ is not a root so we get

$$\mathrm{ad}(X_{\alpha})\,\mathrm{ad}(X_{\beta-\alpha})\,[X_{\delta},X_{-\delta}]=(\beta-\alpha)(H_{\delta})N_{\alpha,\beta-\alpha}X_{\beta}\ .$$

Finally $\beta(H_{\delta})=0$ and $\alpha(H_{\delta})=-1$ so

$$A\,(\mathrm{ad}(X_{\alpha})X_{\delta},\ \mathrm{ad}(X_{\beta-\alpha})X_{-\delta})=N_{\alpha,\beta-\alpha}\ .$$

On the other hand

$$B_{\mathfrak{m}}(X_{\delta},X_{-\delta})=-\frac{2}{\langle\delta,\delta\rangle_{\mathfrak{m}}}$$

so that

$$c=-\frac{1}{2}\|\delta\|_{\mathfrak{m}}^2N_{\alpha,\beta-\alpha}\ .$$

Let Δ^* be an element of $S(\mathfrak{n}^+)$. We may consider it as a polynomial function on $\mathfrak{n}^-$. We then define a polynomial function R on W_2 by

$$R(\mathrm{ad}(X_{\beta-\alpha})Y)=\Delta^*(Y)\ .$$

Note that if we extend $\mathrm{ad}(X_{\alpha})$ into an algebra isomorphism of $S(\mathfrak{n}^+)$ onto $S(W_1)$ then, if Δ^* is homogeneous of degree t,

$$\mathrm{ad}(X_{\alpha})\Delta^*=c^tR\ .$$

Similarly if ∇ is an element of $S(\mathfrak{n}^-)$ viewed as a polynomial function on $\mathfrak{n}^+$ then we define D, a polynomial function on W_1 by

$$D(\mathrm{ad}(X_{\alpha})X)=\nabla(X)$$

and, assuming ∇ to be homogeneous of degree t, we may also write this as

$$\mathrm{ad}(X_{\beta-\alpha})\,\nabla=c^tD\ .$$

Now D defines a differential operator $D(\partial)$ on $S(W_1)$ and ∇ defines a differential operator on $S(\mathfrak{n}^+)$. If ∇ is homogeneous of degree t then

$$(D(\partial)R)(\mathrm{ad}(X_{\beta-\alpha})Y)=c^{-t}(\nabla(\partial)\Delta^*)(Y) .$$

We proved (Proposition 2.7) that $\{-\beta, \alpha, \delta, \tau\}$ is a P-system of type D_4 and furthermore that $-\varepsilon$ is the highest root. Call $\mathfrak{S}$ this subalgebra of type D_4. We now assume the root vectors choosen in such a way that, for the roots belonging to $\mathfrak{S}$, the values of the N... are the ones obtained with the standard model of D_4 (as in [B-2]). In particular this implies that

$$N_{\alpha,\delta}=-N_{\alpha,\varepsilon}=N_{\alpha,\tau}=N_{\beta-\alpha,-\delta}=N_{\beta-\alpha,-\varepsilon}=N_{\beta-\alpha,-\tau}=1 .$$

Thus

$$\mathrm{ad}(X_\alpha)(t_1X_\delta+t_2X_\varepsilon+t_3X_\tau)=t_1X_{\alpha+\delta}-t_2X_{\alpha+\varepsilon}+t_3X_{\alpha+\tau}$$

and

$$\mathrm{ad}(X_{\beta-\alpha})(t_1X_{-\delta}+t_2X_{-\varepsilon}+t_3X_{-\tau})=t_1X_{\beta-\alpha-\delta}+t_2X_{\beta-\alpha-\varepsilon}+t_3X_{\beta-\alpha-\tau} .$$

We apply to our situation Theorem 1.0. We have $m=3$ (Proposition 2.8) and

$$\beta_1=\delta,\ \beta_2=\varepsilon=\beta_3=\tau .$$

The subalgebra of $\mathfrak{u}^+$ invariants in $S(\mathfrak{n}^+)$ is $\boldsymbol{C}[\Delta_1^*, \Delta_2^*, \Delta_3^*]$. The polynomials Δ_i^* are homogeneous of degree $4-i$ and are normalized by

$$\Delta_1^*(t_1X_{-\delta}+t_2X_{-\varepsilon}+t_3X_{-\tau})=t_1t_2t_3 ,$$
$$\Delta_2^*(t_1X_{-\delta}+t_2X_{-\varepsilon}+t_3X_{-\tau})=t_2t_3 ,$$
$$\Delta_3^*(t_1X_{-\delta}+t_2X_{-\varepsilon}+t_3X_{-\tau})=t_3 .$$

Using the above convention we get the corresponding polynomials functions R_i on W_2. In particular

$$R_1(t_1X_{\beta-\alpha-\delta}+t_2X_{\beta-\alpha-\varepsilon}+t_3X_{\beta-\alpha-\tau})=t_1t_2t_3$$

and similar relations for R_2 and R_3. Also we have 3 basic $\mathfrak{u}^+$ invariants ∇_i in $S(\mathfrak{n}^-)$, hence the D_i... Theorem 1.0 is then equivalent to

$$D_j(\partial)R^s=c^{-4+j}b_j(s)R^{t_j(s)} .$$

Fortunately there is a simplification. Indeed using the definition of the Killing form we have $\|K\|_{\mathfrak{m}}^2=8k$ with $k=\dim(\mathfrak{n}^+)$. But also $K=H_\delta+H_\varepsilon+H_\tau$ so that the 3 roots being strongly orthogonal and of the same length $\|K\|_{\mathfrak{m}}^2=3\|H_\delta\|_{\mathfrak{m}}^2=12/\|K\|_{\mathfrak{m}}^2$. Hence $\frac{3}{2}=k\|\delta\|_{\mathfrak{m}}^2$. We check that $N_{\alpha,\beta-\alpha}=1$ and it follows that

$$c^{-1}\frac{3}{4k}=-1$$

so

$$(c)^{-4+j}b_j(s)=\prod_1^{4-j}\left(s_1+\cdots+s_i+(i-1)\frac{d}{2}\right) .$$

Call $c_j(s)$ this last expression:

$$D_j(\partial)R^s=c_j(s)R^{t_j(s)} .$$

We now define the polarization that we are going to use:

$$\Gamma_1=C_0\cup C_1 , \quad \Gamma_2=C_2\cup C_3 ,$$

and

$$V_1=W_0\oplus W_1 , \quad \mathcal{S}=\bigoplus_{r\in Q} S(V_1)E^r ,$$

and also

$$V_2=W_2\oplus W_3 .$$

The formulas for π are the same as in §3.

4.1. The highest weight vectors for $\mathfrak{a}(\delta)$. It follows from Lemma 3.2 that the subalgebra of $\mathfrak{u}^+$ invariants vectors in $\mathcal{S}$ is $\oplus_{q\in Q}\boldsymbol{C}[R_1, R_2, R_3, X_\alpha]E^q$. Note also that the monomials

$$R^sX_\alpha^pE^q=R_1^{s_1}R_2^{s_2}R_3^{s_3}X_\alpha^pE^q$$

are weight vectors. In particular they are highest weight vectors for $\mathfrak{b}(\delta)$. This subspace is invariant under $\mathfrak{a}(\delta)$ and we wish to describe explicitly the action.

Proposition 4.3.

$$\begin{aligned}\Phi(X_{-\alpha}) &= -\frac{1}{4}(2H+1)E^{-1}F_\alpha+\frac{1}{2}E^{-1}F_\alpha X_\alpha E^{-1}F_\alpha \\ &\quad +\frac{1}{2}E^{-1}F_\alpha\sum_{C_1}X_\gamma E^{-1}F_\gamma-\frac{k}{4}E^{-1}F_\alpha-D_1E^{-2} ,\end{aligned}$$

$$\pi(\Phi(X_{-\alpha}))=\left(\sum_{C_1}X_\gamma\frac{\partial}{\partial X_\gamma}+X_\alpha\frac{\partial}{\partial X_\alpha}+E\frac{\partial}{\partial E}+\frac{k+3}{4}\right)\frac{\partial}{\partial X_\alpha}+D_1(\partial)E ,$$

$$\begin{aligned}\Phi(X_{-\beta+\alpha}) &= +\frac{1}{4}(2H+1)X_\alpha E^{-1}+\frac{1}{2}X_\alpha E^{-1}F_\alpha X_\alpha E^{-1} \\ &\quad +\frac{1}{2}X_\alpha E^{-1}\sum_{C_1}X_\gamma E^{-1}F_\gamma-\frac{k}{4}X_\alpha E^{-1}+R_1E^{-2} ,\end{aligned}$$

$$\pi(\Phi(X_{-\beta+\alpha}))=\left(E\frac{\partial}{\partial E}+\frac{1-k}{4}\right)X_\alpha E^{-1}+R_1E^{-2} .$$

We have to go back to the definition of Φ. Let us start with $X_{-\alpha}$. We consider the operator D of §3.0. We apply D several times but look only for the component of X_β. We have contributions from D^2 and D^3.

For D^2 the only possibilities are

$$\mathrm{ad}(X_\beta)\,\mathrm{ad}(X_\alpha)X_{-\alpha} \qquad \text{and} \qquad \mathrm{ad}(X_\alpha)\,\mathrm{ad}(X_\beta)X_{-\alpha}$$

which gives, before symmetrization

$$\frac{1}{2!}\left(2\frac{B(\mathrm{ad}(X_\alpha)\,\mathrm{ad}(X_\beta)X_{-\alpha}, X_{-\beta})}{B(X_\beta, X_{-\beta})}F_\alpha F_\beta\right) .$$

We know that $\beta(H_\alpha)=1$ so that after symmetrization we get

$$-\frac{1}{4}(2H+1)F_\alpha E^{-1} .$$

For D^3 we must pick up 3 roots $\gamma_i \in \Delta_1$ such that $\gamma_1+\gamma_2+\gamma_3=\beta+\alpha$. This gives 3 different cases:

a) $\beta-\alpha, \alpha, \alpha$,

b) $\alpha, \gamma, \beta-\gamma$, with $\gamma \in C_1$,

c) 3 roots in C_1.

In case a) the 3 roots are $\alpha, \alpha, \beta-\alpha$ so that there is 3 ways to order them:

$$\begin{aligned} &\mathrm{ad}(X_\alpha)\,\mathrm{ad}(X_\alpha)\,\mathrm{ad}(X_{\beta-\alpha})X_{-\alpha} , \\ &\mathrm{ad}(X_\alpha)\,\mathrm{ad}(X_{\beta-\alpha})\,\mathrm{ad}(X_\alpha)X_{-\alpha} , \\ &\mathrm{ad}(X_{\beta-\alpha})\,\mathrm{ad}(X_\alpha)\,\mathrm{ad}(X_\alpha)X_{-\alpha} . \end{aligned}$$

As $\beta-2\alpha$ is not a root, the first line is 0. The sum of the last 2 is

$$(\beta-2\alpha)\,(H_\alpha)N_{\alpha,\beta-\alpha}X_\beta .$$

So, before symmetrization we have

$$-\frac{1}{3!}(3F_\alpha^2 F_{\beta-\alpha}N_{\alpha,\beta-\alpha})$$

and, after,

$$\frac{1}{6}[F_\alpha F_\alpha X_\alpha+F_\alpha X_\alpha F_\alpha+X_\alpha F_\alpha F_\alpha]E^{-2}=\frac{1}{2}F_\alpha X_\alpha F_\alpha E^{-2} .$$

In case b) fix a root $\gamma \in C_1$. There are 6 ways to order the 3 roots $\gamma, \alpha, \beta-\gamma$ but because X_α commutes with the two other roots vectors, what we get is

$$3(\mathrm{ad}(X_\gamma)\,\mathrm{ad}(X_{\beta-\gamma})+\mathrm{ad}(X_{\beta-\gamma})\,\mathrm{ad}(X_\gamma))\,\mathrm{ad}(X_\alpha)X_{-\alpha}$$

which is equal to

$$-3N_{\gamma,\beta-\gamma}X_\beta .$$

So, before symmetrization we have

$$\frac{1}{2}F_\alpha F_\gamma F_{\beta-\gamma}$$

and after symmetrization and summation over γ

$$\frac{1}{2}E^{-1}F_{\alpha}\sum_{\gamma\in C_1}X_{\gamma}E^{-1}F_{\gamma}-\frac{k}{4}E^{-1}F_{\alpha}$$

(recall that $k=\#C_1=\dim(\mathfrak{n}^+)$).

Finally in case c) we note that the 3 roots vectors X_{γ_i} commute one with each other and the same is true for the F_{γ_i}. So, before symmetrization, we get

$$\frac{1}{6}\sum\frac{B(\operatorname{ad}(X_{\gamma_1})\operatorname{ad}(X_{\gamma_2})\operatorname{ad}(X_{\gamma_3})X_{-\alpha},X_{\beta})}{B(X_{\beta},X_{-\beta})}F_{\gamma_1}F_{\gamma_2}F_{\gamma_3}.$$

The summation is over C_1^3 but we may limit ourselves to the cases where $\gamma_1+\gamma_2+\gamma_3=\beta+\alpha$. If we call C the above expression, then after symmetrization we have simply CE^{-2}.

Putting everything together gives

$$\Phi(X_{-\alpha})=-\frac{1}{4}(2H+1)E^{-1}F_{\alpha}+\frac{1}{2}E^{-1}F_{\alpha}X_{\alpha}E^{-1}F_{\alpha}$$
$$+\frac{1}{2}E^{-1}F_{\alpha}\sum_{C_1}X_{\gamma}E^{-1}F_{\gamma}-\frac{k}{4}E^{-1}+CE^{-2}\ .$$

Clearly

$$\pi(H)=X_{\alpha}\frac{\partial}{\partial X_{\alpha}}+2E\frac{\partial}{\partial E}+\sum_{C_1}X_{\gamma}\frac{\partial}{\partial X_{\gamma}}$$

so that, by an easy computation

$$\pi(\Phi(X_{-\alpha}))=\Big(\sum_{C_1}X_{\gamma}\frac{\partial}{\partial X_{\gamma}}+X_{\alpha}\frac{\partial}{\partial X_{\alpha}}+E\frac{\partial}{\partial E}+\frac{k+3}{4}\Big)\frac{\partial}{\partial X_{\alpha}}+\pi(C)E^{-2}\ .$$

But $X_{-\alpha}$ commutes with $\mathfrak{b}(\delta)$. Using Lemma 3.2, we see that, for $X\in\mathfrak{b}(\delta)$ we have $\pi(\Phi(X))=\operatorname{ad}(X)$ so that we conclude that $\pi(C)$ commutes with $\operatorname{ad}(X)$. However $C\in S(W_2)$ and $\pi(C)$ is the corresponding differential operator. Hence C is invariant under $\operatorname{ad}(\mathfrak{b}(\delta))$. This implies that $C\in\mathbf{C}[D_1]$ but C is homogeneous of degree 3 so it is a multiple of D_1. To find the constant we view C as a polynomial function on W_1 and evaluate

$$C(X_{\alpha+\delta}-X_{\alpha+\varepsilon}+X_{\alpha+\tau})\ .$$

Take $\gamma_1,\gamma_2,\gamma_3\in C_1$ such that $\gamma_1+\gamma_2+\gamma_3=\beta+\alpha$. Then

$$(F_{\gamma_1}F_{\gamma_2}F_{\gamma_3})(X_{\alpha+\delta}-X_{\alpha+\varepsilon}+X_{\alpha+\tau})\neq 0$$

if and only if each of the γ_i belongs to $\{\alpha+\delta,\alpha+\varepsilon,\alpha+\tau\}$. But there is only one way to write $\beta-2\alpha$ as a linear combination of δ,ε,τ and that is $\beta-2\alpha=\delta+\varepsilon+\tau$. So up to permutation the γ_i must be the roots $\alpha+\delta,\alpha+\varepsilon,\alpha+\tau$. However the roots vectors X_{γ_i} commute so we find

$$C(X_{\alpha+\delta}-X_{\alpha+\varepsilon}+X_{\alpha+\tau})=-N_{\alpha+\tau,-\alpha}N_{\alpha+\varepsilon,\tau}N_{\alpha+\delta,\alpha+\tau+\varepsilon}$$

and a little computation in D_4 tells us that

$$C(X_{\alpha+\delta}-X_{\alpha+\varepsilon}+X_{\alpha+\tau})=-1 \ .$$

But D_1 has been normalized by

$$D_1(\mathrm{ad}(X_\alpha)\ (X_\delta+X_\varepsilon+X_\tau))=1$$

which is equivalent to

$$D_1(N_{\alpha+\delta}-X_{\alpha+\varepsilon}+X_{\alpha+\tau})=1$$

so $C=-D_1$. Finally we note that $\pi(D_1)=-D_1(\partial)E^3$ and this is the end of the proof of the first part of the Proposition. The second part being entirely similar we omit the details.

In $\mathfrak{a}(\delta)$ we choose as simple roots $\beta-\alpha$ and $-\beta$. A vector in $\mathscr{S}$ is a highest weight if it a weight and if it belongs to the kernels of $\pi(\Phi(X_{\beta-\alpha}))$ and $\pi(\Phi(X_{-\beta}))$. Then it also belongs to the kernel of $\pi(\Phi(X_{-\alpha}))$.

Proposition 4.4. *The monomial $R_1^{s_1}R_2^{s_2}R_3^{s_3}X_\alpha^pE^q$ is a highest weight of $\mathfrak{a}(\delta)$ if and only if $s_1=p=0$ and q is one of the two numbers*

$$q_1=-s_2+\frac{k+3}{12}\ ,\quad q_2=-s_2-s_3-\frac{k-9}{12}\ .$$

First $\pi(\Phi(X_{\beta-\alpha}))$ is up to a constant factor $\partial/\partial X_\alpha$ so we must take $p=0$. Next, using $\pi(\Phi(X_{-\alpha}))$ we obtain

$$D_1(\partial)R_1^{s_1}R_2^{s_2}R_3^{s_3}E^q=0$$

which means that $c_1(s)=0$. However this is true only for $s_1=0$. Finally we note that $[X_{-\alpha}, X_{-\beta+\alpha}]=X_{-\beta}$ so that our last condition is

$$\pi(\Phi(X_{-\alpha}))\pi(\Phi(X_{-\beta+\alpha}))R_2^{s_2}R_3^{s_3}E^q=0\ .$$

Using Proposition 4.3 we get

$$\pi(\Phi(X_{-\alpha}))\pi(\Phi(X_{-\beta+\alpha}))R_2^{s_2}R_3^{s_3}E^q=$$

$$\left(q-\frac{k+3}{4}\right)\left(2s_2+s_3+q-1+\frac{k+3}{4}\right)R_2^{s_2}R_3^{s_3}E^{q-1}$$

$$+\left(1+s_2+\frac{d}{2}\right)(1+s_2+s_3+d)R_2^{s_2}R_3^{s_3}E^{q-1}\ .$$

Hence the condition:

$$\left(q-\frac{k+3}{4}\right)\left(2s_2+s_3+q-1+\frac{k+3}{4}\right)+\left(1+s_2+\frac{d}{2}\right)(1+s_2+s_3+d)=0\ .$$

But $d=(k-3)/3$ and the above condition turns out to be

$$q^2-(1-2s_2-s_3)q+\left(\frac{k+3}{12}-s_2\right)\left(-\frac{k-9}{12}-s_2-s_3\right)=0 \ .$$

The two roots of this quadratic equation are precisely q_1 and q_2.

We still have to check that the monomials we consider are weight vectors for $\mathfrak{h}$. More generally consider the action of $\mathfrak{h}$ on

$$R^sE^q=R_1^{s_1}R_2^{s_2}R_3^{s_3}E^q \ .$$

We have

$$\pi(\Phi(H))R^sE^q=(3s_1+2s_2+s_3+2q)R^sE^q \ .$$

Recall that $\beta(K)=0$, $\delta(K)=2$ and $\sigma(K)=0$ for σ a simple root different from α and δ. But $\beta=2\alpha+3\delta+\cdots$ so that $0=2\alpha(K)+6$ and $\alpha(K)=-3$. It follows that, for any $\gamma\in C_1$, $\gamma=\alpha+\delta+\cdots$ we have $\gamma(K)=-1$. Lemma 3.2 then implies readily that

$$\pi(\Phi(K))R^sE^q=-\frac{1}{2}(k+3)-(3s_1+2s_2+s_3) \ .$$

If $X\in\mathfrak{b}(\delta)\cap\mathfrak{h}$, by Lemma 3.2 we know that $\pi(\Phi(X))=\mathrm{ad}(X)$. Going back to $S(\mathfrak{n}^+)$ it is enough to evaluate $\mathrm{ad}(X)(\Delta^*)^s$. But we know [M-R-S] that the (Δ_i^*) are eigenvectors and the eigenvalues are computed by restriction to the diagonal:

$$\mathrm{ad}(X)(\Delta^*)^s=(s_1(\delta+\varepsilon+\tau)+s_2(\varepsilon+\tau)+s_3(\tau))(X)(\Delta^*)^s$$

and we have the same eigenvalue for R^s.

There is one more technical point we have to take care about. Let

$$M_i(s_2,s_3)=\pi(\Phi(\mathcal{U}(\mathfrak{a}(\delta)\times\mathfrak{b}(\delta)))R_2^{s_2}R_3^{s_3}E^{q_i}$$

and

$$M_i=\sum_{s_2,s_3}M_i(s_2,s_3) \ .$$

Lemma 4.5. *Fix $i=1$ or 2. Let $X\in\mathcal{U}(\mathfrak{g})$. If $\pi(\Phi(X))M_i=(0)$, then X belongs to the Joseph ideal.*

We first prove that $M_i\supset S(V_1)E^{q_i}$. The subspace M_i is stable under multiplication by X_α and E. Also

$$\pi(\Phi(X_{-\beta+\alpha}X_\beta^2))=R_1+\left(E\frac{\partial}{\partial E}+\frac{1-k}{4}\right)X_\alpha E$$

so that

$$R_1R_2^{s_2}R_3^{s_3}E^{q_i}+\left(q+1+\frac{1-k}{4}\right)X_\alpha R_2^{s_2}R_3^{s_3}E^{q_i+1}\in M_i \ .$$

It follows that

$$R_1 R_2^{s_2} R_3^{s_3} E^{q_i} \in M_i$$

and, by an obvious induction that, for any integer $s_1 \geq 0$,

$$R_1^{s_1} R_2^{s_2} R_3^{s_3} E^{q_i} \in M_i$$

Next we recall ([M-R-S]) that as a $\mathfrak{b}(\delta) \oplus \boldsymbol{C}K$ module the algebra $S(\mathfrak{n}^+)$ is a multiplicity free direct sum of irreducible sub-modules and that the highest weight vector are the monomials $(\Delta^*)^s$. This implies that, under the adjoint action of $\mathfrak{b}(\delta) \oplus \boldsymbol{C}K$ the module $S(W_1)$ has the same properties, the highest weight vector being the R^s. It follows that those monomials generate $S(W_1)$ as a $\mathfrak{b}(\delta) \oplus \boldsymbol{C}K$ module. But the restriction of $\pi \circ \Phi$ to $\mathfrak{b}(\delta) \oplus \boldsymbol{C}K$ is just a twisting of the adjoint action by a central character. This proves that $S \supset S(W_1)E^{q_i}$. Finally we can mutiply by any power of X_α so $S \supset S(W_0 \oplus W_1)E^{q_i}$.

Let $X \in \mathcal{U}(\mathfrak{g})$. Then $\pi(\Phi(X))$ can be decomposed as a finite sum:

$$\pi(\Phi(X)) = \sum T_{\ell j} E^{r_\ell} \left(E \frac{\partial}{\partial E}\right)^j$$

where the j are non negative integers, the r_ℓ integers and the $T_{\ell j}$ differential operators with polynomial coefficients with respect to the variables X_γ for $\gamma \in C_0 \cup C_1$. We also assume that $(r_\ell, j) = (r_{\ell'}, j)$ implies that $\ell = \ell'$. If $\pi(\Phi(X))M_i = 0$ then for any $U \in S(V_1)$ and any positive integer n we have $UE^{q_i+n} \in M_i$ so

$$\sum T_{\ell j}(U)\,(q_i+n)^j E^{q_i+\ell+n} = 0 \ .$$

It follows that, for any $n \geq 0$ and a fixed ℓ

$$\sum_j T_{\ell j}(U)\,(q_i+n)^j = 0 \ .$$

By a standard argument this means that $T_{\ell j}(U) = 0$ and U being arbitrary $T_{\ell j} = 0$. Thus $\pi(\Phi(X)) = 0$ and because π is one to one we conclude that $\Phi(X) = 0$ which by definition means that $X \in J$.

Fix s_2 and s_3. Let us go back to the weight of $R_2^{s_2} R_3^{s_3} E^{q_i}$. Consider first $\mathfrak{a}(\delta)$. The weight is given by

$$H_{-\beta} \mapsto -2s_2 - s_3 - 2q_i,\ H_\alpha \mapsto 2s_2 + s_3 + q_i + \frac{k+3}{4} \ .$$

We worked with $\{\beta - \alpha, -\beta\}$ as system of simple roots; the corresponding half sum of the positive roots is $-\alpha$. We add $-\alpha$ to the above weight and obtain a linear form on $\mathfrak{a}(\delta) \cap \mathfrak{h}$ which we call $\lambda_i(\mathfrak{a}(\delta), s_2, s_3)$. Now let $j_{\mathfrak{a}(\delta)}$ be the Harish-Chandra isomorphism of the canter $\mathcal{Z}(\mathcal{U}(\mathfrak{a}(\delta)))$ of the enveloping algebra of $\mathfrak{a}(\delta)$ onto the subalgebra of Weyl group invariants in $S(\mathfrak{a}(\delta) \cap \mathfrak{h})$. Then, for Z in this center

$$\pi(\Phi(Z))R_2^{s_2}R_3^{s_3}E^{q_i}=j_{\mathfrak{a}(\delta)}(Z)(\lambda_i(\mathfrak{a}(\delta),s_2,s_3))R_2^{s_2}R_3^{s_3}E^{q_i}$$

We identify $\mathfrak{a}(\delta)$ with $\mathfrak{sl}(3)$ reverting to our original set of simple roots: $\{-\beta,\alpha\}$. Take $q=q_1$ and put $q_3=-q_1-q_2$ so that $q_3=2s_2+s_3-1$. Then $\lambda_1(\mathfrak{a}(\delta),s_2,s_3)$ is given by

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \longrightarrow \left(-q_1-\frac{1}{3}q_3+\frac{k-1}{12}\right)t_1+\left(-q_2-\frac{1}{3}q_3+\frac{k-1}{12}\right)t_2 + \left(-\frac{1}{3}q_3-\frac{k-1}{6}\right)t_3 .$$

Choosing q_2 instead of q_1 gives the same formulas after permutation of q_1 and q_2. The two linear forms are conjugate under the Weyl group so that the choice is irrelevant. Finally in order to compare with the result of §2 we note that the linear form $\lambda_1(\ldots)$ is defined, via the Killing form of $\mathfrak{a}(\delta)$ by the following matrix

$$Y(s_2,s_3)=\frac{1}{6}\begin{pmatrix} -q_1-\frac{1}{3}q_3+\frac{k-1}{12} & 0 & 0 \\ 0 & -q_2-\frac{1}{3}q_3+\frac{k-1}{12} & 0 \\ 0 & 0 & -\frac{1}{3}q_3-\frac{k-1}{6} \end{pmatrix}.$$

Now we look at $\mathfrak{b}(\delta)$. The weight of $R_2^{s_2}R_3^{s_3}E^{q_i}$ is the restriction to $\mathfrak{b}(\delta)\cap\mathfrak{h}$ of the linear form

$$s_2(\varepsilon+\tau)+s_3\tau .$$

We know that the roots δ, ε, τ have the same length and that $H_\delta+H_\varepsilon+H_\tau=K$. But $\|K\|_{\mathfrak{m}}^2=8k$ so $\|H_\delta\|_{\mathfrak{m}}^2=\|H_\varepsilon\|_{\mathfrak{m}}^2=\|H_\tau\|_{\mathfrak{m}}^2=8k/3$. Then the weight is the linear form

$$X\longrightarrow\frac{3}{4k}B_{\mathfrak{m}}((s_2+s_3)H_\tau+s_2H_\varepsilon,X) .$$

We saw thet $t_1H_\delta+t_2H_\varepsilon+t_3H_\tau$ belongs to $\mathfrak{b}(\delta)\cap\mathfrak{h}$ if and only if $t_1+t_2+t_3=0$ and is orthogonal to $\mathfrak{b}(\delta)\cap\mathfrak{h}$ if $t_1=t_2=t_3$. Furthermore, by lemma 3.3 of [R-S] we have, on $\mathfrak{b}(\delta)$

$$\frac{B_{\mathfrak{m}}}{B_{\mathfrak{b}(\delta)}}=\frac{4}{3}\,\frac{1+d}{d}=\frac{4}{3}\,\frac{k}{k-3} .$$

Hence the weight is the linear form on $\mathfrak{b}(\delta)\cap\mathfrak{h}$ defined, using the Killing form of $\mathfrak{b}(\delta)$, by the element

$$\frac{1}{k-3}\left(\left(s_2-\frac{2s_2+s_3}{3}\right)H_\varepsilon+\left(s_2+s_3-\frac{2s_2+s_3}{3}\right)H_\tau-\frac{2s_2+s_3}{3}H_\delta\right) .$$

We substitue for s_2 and s_3 their expression in terms of q_1 and q_2:

$$\frac{1}{k-3}\left[\left(-\frac{1}{3}q_3-\frac{k-1}{6}+\frac{k-3}{6}\right)H_\delta+\left(-q_1-\frac{1}{3}q_3+\frac{k-1}{12}\right)H_\varepsilon +\left(-q_2-\frac{1}{3}q_3+\frac{k-1}{12}-\frac{k-3}{6}\right)H_\tau\right] .$$

We still must add the half sum $\rho_{\mathfrak{b}(\delta)}$ of the positive roots. Finally to (s_2, s_3) corresponds in $\mathfrak{b}(\delta)$ the coadjoint orbit of

$$X(s_2, s_3)=\frac{1}{k-3}\left[\left(-\frac{1}{3}q_3-\frac{k-1}{6}\right)H_\delta +\left(-q_1-\frac{1}{3}q_3+\frac{k-1}{12}\right)H_\varepsilon+\left(-q_2-\frac{1}{3}q_3+\frac{k-1}{12}\right)H_\tau\right] +\left[\frac{1}{6}(H_\delta-H_\tau)+2\frac{H_{\rho_{\mathfrak{b}(\delta)}}}{\|H_{\rho_{\mathfrak{b}(\delta)}}\|^2_{\mathfrak{b}(\delta)}}\right]$$

let $j_{\mathfrak{b}(\delta)}$ be the Harish-Chandra isomorphism of the center $\mathscr{Z}(\mathscr{U}(\mathfrak{b}(\delta)))$ of the universal enveloping algebra of $\mathfrak{b}(\delta)$ onto the subalgebra of Weyl group invariants in $S(\mathfrak{b}(\delta)\cap\mathfrak{h})$. Then, for such a Z we have

$$\pi(\Phi(Z))R_2^{s_2}R_3^{s_3}E^{q_i}=j_{\mathfrak{b}(\delta)}(Z)(X(s_2, s_3))$$

For each choice of (s_2, s_3) we have obtained an element $Y(s_2, s_3)$ of $\mathfrak{a}(\delta)\cap\mathfrak{h}$ and an element $X(s_2, s_3)$ of $\mathfrak{b}(\delta)\cap\mathfrak{h}$.

We define a map ψ

$$\psi\colon \mathfrak{a}(\delta)\cap\mathfrak{h}\longrightarrow\mathfrak{b}(\delta)\cap\mathfrak{h}$$

by

$$\psi(x_1H_{-\beta}-x_2H_\alpha)=\frac{6}{k-3}\left(x_2H_\delta+x_1H_\varepsilon+x_3H_\tau\right)+\left(H_\delta-H_\tau+12\frac{H_\rho}{\|H_\rho\|^2}\right)$$

where $x_1+x_2+x_3=0$ and where ρ is the half sum of the positive roots of $\mathfrak{b}(\delta)$ and the norm is relative to the Killing form of $\mathfrak{b}(\delta)$. This is an affine, one to one, first degree map form $\mathfrak{a}(\delta)\cap\mathfrak{h}$ into $\mathfrak{b}(\delta)\cap\mathfrak{h}$. we have

$$\psi(Y(s_2, s_3))=X(s_2, s_3) .$$

Note that the linear part of ψ agrees exactly with the map of §2 (Theorem 2.10); however there is a "tail".

Lemma 4.6. *If p is a permutation of $\{1, 2, 3\}$, then*

$$\psi(x_1H_{-\beta}-x_2H_\alpha) \quad \text{and} \quad \psi(x_{p(1)}H_{-\beta}-x_{p(2)}H_\alpha)$$

are conjugate under the Weyl group of $\mathfrak{b}(\delta)$.

If the tail was 0, the situation would be the same as in §2 and we would just need to apply Proposition 1.6. We proceed case by case following as usual Bourbaki's notations for root systems. We denote by $e_1, e_2 \ldots$ the dual basis of the ε_i basis and by $\langle\, ,\rangle$ the scalar product such that the basis (e_i) is orthonormal. Suppose that $\mathfrak{g}$ is of type F_4. Then $\mathfrak{m}$ is of type C_3. We use the notations of C_3. The various roots are given as follows:

$$\delta=\alpha_3=2\varepsilon_3\ ,\quad \varepsilon=\alpha_3+2\alpha_2=2\varepsilon_2\ ,\quad \tau=\alpha_1+2\alpha_2+2\alpha_3=2\varepsilon_1$$

and half the sum of the positive roots of $\mathfrak{b}(\delta)$ is

$$\rho=\alpha_1+\alpha_2=\varepsilon_1-\varepsilon_3\ .$$

Then

$$H_\delta=e_3\ ,\quad H_\tau=e_1\ ,\quad H_\rho=e_1-e_3\ .$$

Also using the Killing form of $\mathfrak{b}(\delta)$ we have

$$\|H_\rho\|^2=6\langle e_1-e_3, e_1-e_3\rangle=12$$

and, *in this F_4 case the tail is 0.*

Suppose that $\mathfrak{g}$ is of type E_6. Then $\mathfrak{m}$ is of type A_5 and we use the notations for A_5. Then

$$\delta=\alpha_3=\varepsilon_3-\varepsilon_4\ ,\quad \varepsilon=\alpha_2+\alpha_3+\alpha_4=\varepsilon_2-\varepsilon_5\ ,\quad \tau=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\varepsilon_1-\varepsilon_6$$

while

$$\rho=\alpha_1+\alpha_2+\alpha_4+\alpha_5=\varepsilon_1-\varepsilon_3+\varepsilon_4-\varepsilon_6.$$

Then

$$H_\delta=e_3-e_4\ ,\quad H_\tau=e_1-e_6\ ,\quad H_\rho=\frac{1}{2}(e_1-e_3+e_4-e_6)\ .$$

Also

$$\|H_\rho\|^2=\frac{6}{4}\langle e_1-e_3+e_4-e_6, e_1-e_3+e_4-e_6\rangle=6$$

and so *in the E_6 case the tail is 0.*

Suppose that $\mathfrak{g}$ is of type E_7 so that $\mathfrak{m}$ is of type D_6 and let use the notations for D_6. Then

$$\delta=\alpha_6=\varepsilon_5+\varepsilon_6\ ,\quad \varepsilon=\varepsilon_3+\varepsilon_4=\alpha_3+2\alpha_4+\alpha_5+\alpha_6\ ,$$
$$\tau=\varepsilon_1+\varepsilon_2=\alpha_1+2(\alpha_2+\alpha_3+\alpha_4)+\alpha_5+\alpha_6$$

while

$$\rho=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=\frac{1}{2}(5(\varepsilon_1-\varepsilon_6)+3(\varepsilon_2-\varepsilon_5)+(\varepsilon_3-\varepsilon_4))\ .$$

Then

$$H_\delta=e_5+e_6\ ,\quad H_\tau=e_1+e_2\ ,\quad H_\rho=\frac{2}{35}(5(e_1-e_6)+3(e_2-e_5)+(e_3-e_4))\ .$$

Also

$$\|H_\rho\|^2=12\langle H_\rho, H_\rho\rangle=\frac{96}{35}$$

so that, *for the E_7 case the tail is*

$$e_5+e_6-e_1-e_2+\frac{1}{4}(5(e_1-e_6)+3(e_2-e_5)+(e_3-e_4))$$

or

$$\frac{1}{4}(e_1-e_2+e_3-e_4+e_5-e_6) \ .$$

We identify $\mathfrak{b}(\delta)$ to SL(6). Then we have to prove that if in the matrix

$$\frac{1}{2}\begin{pmatrix} x_3 &&&&& \\ & x_3 &&&& \\ && x_2 &&& \\ &&& x_2 && \\ &&&& x_1 & \\ &&&&& x_1 \end{pmatrix}+\begin{pmatrix} +1 &&&&& \\ & -1 &&&& \\ && +1 &&& \\ &&& -1 && \\ &&&& +1 & \\ &&&&& -1 \end{pmatrix}$$

we make a permutation of the x_i then the new matrix is conjugate to the original one by some element of the Weyl group of A_5; this is obvious. *Note that in this case the tail is not 0. However it is orthogonal to H_δ, H_τ and H_ε.*

Last (but not least...) suppose that $\mathfrak{g}$ is of type E_8. Then $\mathfrak{m}$ is of type E_7 and $\mathfrak{b}(\delta)$ of type E_6. We use the E_n notations. Then

$$\delta=\varepsilon_6-\varepsilon_5=\alpha_7 \ , \quad \varepsilon=\varepsilon_5+\varepsilon_6=\alpha_2+\alpha_3+2(\alpha_4+\alpha_5+\alpha_6)+\alpha_7 \ ,$$
$$\tau=\varepsilon_8-\varepsilon_7=2(\alpha_1+\alpha_2)+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$$

while

$$\rho=\varepsilon_2+2\varepsilon_3+3\varepsilon_4+4\varepsilon_5+4(\varepsilon_8-\varepsilon_7-\varepsilon_6) \ .$$

Then

$$H_\delta=e_6-e_5 \ , \quad H_\tau=e_8-e_7 \ , \quad H_\rho=\frac{1}{39}(e_2+2e_3+3e_4+4(e_5+e_8-e_7-e_6)) \ .$$

Also

$$\|H_\rho\|^2=\frac{48}{39}$$

so that, for E_8, the tail turns out to be

$$e_2+2e_3+3e_4$$

and we are considering

$$\frac{1}{4}(x_3(e_6-e_5)+x_2(e_5+e_6)+x_1(e_8-e_7))+(e_2+2e_3+3e_4) .$$

We have to prove that any permutation of the x_i may be realized by the Weyl group of E_6. Consider the product of the symetries with respect to the two roots $\varepsilon_5+\varepsilon_1$ and $\varepsilon_5-\varepsilon_1$. This changes e_5 into $-e_5$ and e_1 into $-e_1$ the other basis vectors being invariant. So H_ε and H_δ are exchanged, H_τ and the tail are invariant. We have realized the permutation of x_2 and x_3. Next put

$$\alpha'=\frac{1}{2}(\varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_5) , \quad \alpha''=\frac{1}{2}(-\varepsilon_4+\varepsilon_3+\varepsilon_2+\varepsilon_1) ,$$

and

$$H'=\frac{1}{2}(e_8-e_7-e_6-e_5) , \quad H''=\frac{1}{2}(-e_4+e_3+e_2+e_1) ,$$

Then $\sigma^+=\alpha'+\alpha''$ and $\sigma^-=\alpha'-\alpha''$ are roots of E_6 and

$$H_{\sigma^+}=H'+H'' , \quad H_{\sigma^-}=H'-H'' .$$

Then the product of the symetries with respect to σ^+ and σ^- is given by

$$x \longrightarrow x -2\alpha'(x)H'-2\alpha''(x)H'' .$$

In particular

$$\begin{aligned} e_6-e_5 &\mapsto e_6-e_5 \\ e_6+e_5 &\mapsto e_8-e_7 \\ e_8-e_7 &\mapsto e_6+e_5 \\ e_2+2e_3+3e_4 &\mapsto e_2+2e_3+3e_4 . \end{aligned}$$

This gives the permutation of x_1 and x_2. This is enough to generate the group of permutations of the 3 variables.

Remark: in all cases the tail is orthogonal to H_δ, H_ε and H_τ. This is a particular case of Lemma 3.9 of [R-S]. One could perhaps avoid the preceeding case by case verification by expanding the arguments of the proof of this Lemma.

Let us go back to the ψ map. Let $P\in S(\mathfrak{b}(\delta)\cap\mathfrak{h})$; it is a polynomial function on the dual space but, using the Killing form of $\mathfrak{b}(\delta)$ we consider it as a polynomial function on $\mathfrak{b}(\delta)\cap\mathfrak{h}$. Then $\Xi(P)$ defined by $\Xi(P)=P\circ\psi$ is a polynomial function on $\mathfrak{a}(\delta)\cap\mathfrak{h}$ which, using the Killing form of $\mathfrak{a}(\delta)$ we consider as an element of $S(\mathfrak{a}(\delta)\cap\mathfrak{h})$.

Theorem 4.7 ($\mathfrak{g}$ **of type** F_4, E_6, E_7 **or** E_8). *The map* Ξ *is an algebra homomorphism and it maps the Weyl group invariants in* $S(\mathfrak{b}(\delta)\cap\mathfrak{h}$ *into the Weyl group invariants in* $S(\mathfrak{a}(\delta)\cap\mathfrak{h})$.

Let

$$\Theta\colon \mathfrak{Z}(\mathcal{U}(\mathfrak{b}(\delta))) \longrightarrow \mathfrak{Z}(\mathcal{U}(\mathfrak{a}(\delta)))$$

be defined by

$$\Theta = j_{\mathfrak{a}(\delta)}^{-1} \circ \Xi \circ j_{\mathfrak{b}(\delta)}$$

Then Θ is an algebra homomorphism and, for any $Z \in \mathscr{Z}(\mathcal{U}(\mathfrak{b}(\delta)))$

$$Z - \Theta(Z) \in J\ .$$

By Lemma 4.6 the Ξ map carries invariant polynomials into invariant polynomials. Also ϕ has been defined in such a way that for any (s_2, s_3) we have

$$\pi(\Phi(Z - \Theta(Z))) R_2^{s_2} R_3^{s_3} E^{q_i} = 0$$

But $\pi(\Phi(Z - \Theta(Z)))$ is a scalar operator on $M_i(s_2, s_3)$ hence on M_i. So Lemma 4.5 implies that $Z - \Theta(Z)$ belongs to the Joseph ideal J.

4.2. The case of the orthogonal groups. In this subsection we assume that $\mathfrak{g}$ is of type B_ℓ with $\ell \geq 4$ or D_ℓ with $\ell \geq 5$. We prove an analog of Theorem 4.7, following the same line of proof. Using as usual the notations of Bourbaki's table, we have $\alpha = \alpha_2$. There are two simple roots connected to α, namely α_1 and α_3. We put $\delta = \alpha_3$. Then $\mathfrak{a}(\delta)$ is the subalgebra of type A_3 admitting $\{-\beta, \alpha, \alpha_1\}$ as a set of simple roots while $\mathfrak{b}(\delta)$ is the subalgebra of type B_{n-3} or D_{n-3} admitting $\{\alpha_4, \dots, \alpha_n\}$ as a set of simple roots.

The highest root is $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots$ so that $|\beta|_\delta + |\beta|_{\alpha_1} = 3$. We call C_i the set of roots γ such that

$$|\gamma|_\alpha = 1 \qquad \text{and} \qquad |\gamma|_\delta + |\gamma|_{\alpha_1} = i\ .$$

Then $C_0 = \{\alpha\}$ and $C_3 = \{\beta - \alpha\}$ as before. In C_1 we find $\alpha_1 + \alpha_2$ and also all the roots $\alpha_2 + \alpha_3 + \cdots$ while $C_2 = \beta - C_1$ contains $\beta - \alpha_1 - \alpha_2 = \alpha_2 + 2\alpha_3 + \cdots$ and all the roots $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$. We put

$$\Gamma_1 = C_0 \cup C_1\ , \quad \Gamma_2 = C_2 \cup C_3\ ,$$

and define π as before.

The subalgebra $\mathfrak{m}$ has two simple components, $\mathfrak{m}_1$ based on α_1 and $\mathfrak{m}_2$ based on $\{\alpha_3, \dots, \alpha_n\}$. We put $\mathfrak{n}_1^\pm = \boldsymbol{C} X_{\pm\alpha_1}$; the simple root $\delta = \alpha_3$ of $\mathfrak{m}_2$ defines a maximal parabolic subalgebra with commutative nilpotent radical $\mathfrak{n}_2^+$ and we let $\mathfrak{n}_2^-$ be the "negative" nilpotent radical. Define $K_1 = H_{\alpha_1}$ and K_2 by $\beta(K_2) = 0$, $\delta(K_2) = 2$, $\alpha_j(K_2) = 0$ for $j \geq 4$. Then $\mathfrak{n}_2^\pm$ are prehomogeneous (regular) relative to the adjoint action of $\mathfrak{b}(\delta)$. Also

$$H = 2H_\alpha + K_1 + K_2\ .$$

Let $D_1^\pm = \{\pm\alpha_1\}$ and $D_2^\pm$ be the set of roots σ such that $X_\sigma \in \mathfrak{n}_2^\pm$. If we define $D^\pm = D_1^\pm \cup D_2^\pm$ them Lemma. 4.1 remains valid. So does Lemma 4.2 except that we have two constants c_1 and c_2, one for each simple component of $\mathfrak{m}$:

$$c_i = -\frac{1}{2} \|\delta_i\|_{\mathfrak{m}_i}^2 N_{\alpha, \beta - \alpha}$$

with the notation $\delta_1=\alpha_1$, $\delta_2=\alpha_3=\delta$.

We still have a subalgebra $\mathfrak{s}$ of type D_4 with simple roots $\{-\beta, \alpha, \delta, \tau\}$ where τ is the highest root of $\mathfrak{m}_2$. Now $-\alpha_1=-\beta+2\alpha+\delta+\tau$ is the highest root of $\mathfrak{s}$ so that α_1 plays the role of ε. We choose root vectors for $\mathfrak{s}$, as before and examine the situation from the point of view of Theorem 1.0. Let $\mathfrak{u}^{\pm}$ be the sum of the root spaces relative to the positive (or negative) roots of $\mathfrak{b}(\delta)$. The subalgebra of $\mathfrak{u}^+$ invariants in $S(\mathfrak{n}^+)$ is $\boldsymbol{C}[\Delta_2^*, \Delta_3^*]$. Here we call Δ_2^* the relative invariant of $\mathfrak{b}(\delta)$; it is homogeneous of degree 2, normalized by

$$\Delta_2^*(t_1X_{-\delta}+t_3X_{-\tau})=t_1t_3 \ .$$

The polynomial Δ_3^* is of degree 1: it it is a multiple of X_τ normalized by

$$\Delta_3^*(t_1X_{-\delta}+t_3X_{-\tau})=t_3 \ .$$

Also define Δ_1^* to be the constant multiple of X_{α_1} such that $\Delta_1^*(X_{-\alpha_1})=1$. We make a slight change of notation by putting

$$W_1=\oplus\mathfrak{g}^\gamma \qquad \text{for} \quad |\gamma|_\alpha=|\gamma|_\delta=1$$

so that

$$V_1=W_1\oplus\boldsymbol{C}X_\alpha\oplus\boldsymbol{C}X_{\alpha+\alpha_1}$$

and also

$$W_2=\oplus\mathfrak{g}^\gamma \qquad \text{for} \quad |\gamma|_\alpha=1=|\gamma|_\delta=|\gamma|_{\alpha_1}=1$$

so that

$$V_2=W_2\oplus\boldsymbol{C}X_{\beta-\alpha}\oplus\boldsymbol{C}X_{\beta-\alpha_1-\alpha}.$$

Then $\mathrm{ad}(X_\alpha)$ maps $\mathfrak{n}_2^+$ onto W_1 and $\mathrm{ad}(X_{\beta-\alpha})$ maps $\mathfrak{n}_2^-$ onto W_2. Let R_i be defined by

$$R_i(\mathrm{ad}(X_{\beta-\alpha})Y)=\Delta_i^*(Y)$$

with $Y\in\mathfrak{n}_2^+$ if $i=2$ or 3 and $Y\in\mathfrak{n}_1^+$ if $i=1$. By our choice of normalization:

$$R_2(t_1X_{\beta-\alpha-\delta}+t_3X_{\beta-\alpha-\tau})=t_1t_3$$
$$R_3(t_1X_{\beta-\alpha-\delta}+t_3X_{\beta-\alpha-\tau})=t_3$$
$$R_1(t_2X_{\beta-\alpha-\alpha_1})t_2 \ .$$

In the same way we have two basic invariants ∇_2 and ∇_3 in $S(\mathfrak{n}_2^-)$ corresponding to $D_2, D_3\in S(W_2)$. They satisfy

$$D_2(t_1X_{\alpha+\delta}+t_3X_{\alpha+\tau})=t_1t_3$$
$$D_3(t_1X_{\alpha+\delta}+t_3X_{\alpha+\tau})=t_1$$

and we define ∇_1 as the unique element of $S(\mathfrak{n}_1^-)$ such that $\nabla_1(X_{\alpha_1})=1$. Then D_1 defined by $D_1\circ\mathrm{ad}(X_\alpha)=\nabla_1$ is the unique element of $\boldsymbol{C}X_{\beta-\alpha-\alpha_1}$ such that $D_1(X_{\alpha+\alpha_1}) = -1$. Remember that on $\mathfrak{m}$ we use the Killing form of $\mathfrak{m}$ while on $\mathfrak{g}_1$ we use the alternating form given by the bracket. Thus the condition on D_1 means that $[X_{\alpha+\alpha_1}, D_1] = -X_\beta$. We apply Theorem 1.0. First to $\mathfrak{m}_2$. For this subalgebra the integer $d = d_2$ has the value $2n-8$ if $\mathfrak{g}$ is of type D_n and $2n-7$ if $\mathfrak{g}$ is of type B_n. In both cases the dimension of $\mathfrak{n}_2^+$ is $k_2 = d_2 + 2$ and as we have two invariants the integer $m = m_2$ is 2. Being careful with the indices we find

$$D_2(\partial) R_1^{s_1} R_2^{s_2} R_3^{s_3} = s_2\left(s_2 + s_3 + \frac{d_2}{2}\right) R_1^{s_1} R_2^{s_2-1} R_3^{s_3} ,$$

$$D_3(\partial) R_1^{s_1} R_2^{s_2} R_3^{s_3} = s_2 R_1^{s_1} R_2^{s_2-1} R_3^{s_3+1} .$$

For the simple component $\mathfrak{m}_1$ we check that

$$N_{\alpha+\alpha_1, \beta-\alpha-\alpha_1} = -1$$

and this implies that

$$D_1 = X_{\beta-\alpha-\alpha_1} , \quad R_1 = -X_{\alpha+\alpha_1}$$

so that $[R_1, D_1] = X_\beta$ and

$$D_1(\partial) R_1^{s_1} R_2^{s_2} R_3^{s_3} = s_1 R_1^{s_1-1} R_2^{s_2} R_3^{s_3} .$$

We have to choose a system of positive roots for $\mathfrak{a}(\delta)$. As simple roots we take

$$\{-\beta, \beta-\alpha-\alpha_1, \alpha_1\}$$

Put $k = 1 + k_2$; proposition 4.3 is replaced by

Proposition 4.8 ($\mathfrak{g}$ **of type** B_ℓ, $\ell \geq 4$ **or** D_ℓ, $\ell \geq 5$)**.**

$$\Phi(X_{-\alpha}) = -\frac{1}{4}(2H+1)E^{-1}F_\alpha + \frac{1}{2}E^{-1}F_\alpha X_\alpha E^{-1}F_\alpha + \frac{1}{2}E^{-1}F_\alpha \sum_{C_1} X_\gamma E^{-1}F_\gamma - \frac{k}{4}E^{-1}F_\alpha - D_1 D_2 E^{-2}$$

$$\pi(\Phi(X_{-\alpha})) = \left(\sum_{C_1} X_\gamma \frac{\partial}{\partial X_\gamma} + X_\alpha \frac{\partial}{\partial X_\alpha} + E\frac{\partial}{\partial E} + \frac{k+3}{4}\right)\frac{\partial}{\partial X_\alpha} + D_1(\partial) D_2(\partial) E$$

$$\Phi(X_{-\beta+\alpha}) = +\frac{1}{4}(2H+1)X_\alpha E^{-1} + \frac{1}{2}X_\alpha E^{-1}F_\alpha X_\alpha E^{-1} + \frac{1}{2}X_\alpha E^{-1}\sum_{C_1} X_\gamma E^{-1}F_\gamma - \frac{k}{4}X_\alpha E^{-1} + R_1 R_2 E^{-2}$$

$$\pi(\Phi(X_{-\beta+\alpha})) = \left(E\frac{\partial}{\partial E} + \frac{1-k}{4}\right)X_\alpha E^{-1} + R_1 R_2 E^{-2}$$

$$\Phi(X_{-\alpha_1}) = X_\alpha F_{\alpha+\alpha_1} E^{-1} - R_2 E^{-1}$$

$$\pi(\Phi(X_{-\alpha_1}))=-X_\alpha\frac{\partial}{\partial X_{\alpha+\alpha_1}}-R_2E^{-1}$$

$$\Phi(X_{\alpha_1})=-X_{\alpha+\alpha_1}F_\alpha E^{-1}-D_2E^{-1}$$

$$\pi(\Phi(X_{\alpha_1}))=X_{\alpha+\alpha_1}\frac{\partial}{\partial X_\alpha}-D_2(\partial)E\ .$$

The proof is easy and uses the same type of argument that the proof of Proposition 4.3. We omit the details.

The highest weight vectors for $\mathfrak{b}(\delta)$ (using the original set of positive roots) are the weight vectors which are linear combination of the monomials

$$R_1^{s_1}R_2^{s_2}R_3^{s_3}X_\alpha^pE^q$$

with p, s_1, s_2, s_3 positive integers and q a rational number.

Proposition 4.9. *The monomial $R_1^{s_1}R_2^{s_2}R_3^{s_3}X_\alpha^pE^q$ is a highest weight vector for $\mathfrak{a}(\delta)$ if and only if $s_1=s_2=p=0$ and q is one of the two numbers*

$$q_1=\frac{k_2}{4}\ ,\quad q_2=1-s_3-\frac{k_2}{4}\ .$$

Note that $k_2=\dim(\mathfrak{n}_2^+)$ is given by

$$k_2=d_2+2=\begin{cases}2\ell-5 \text{ if } \mathfrak{g} \text{ is of type } B_\ell\\ 2\ell-6 \text{ if } \mathfrak{g} \text{ is of type } D_\ell\end{cases}$$

For our choice of simple roots, $\beta-\alpha$ is positive. The operator $\pi(\Phi(X_{\beta-\alpha})$ is essentially the derivation with respect to X_α so we must have $p=0$. Similarly the positivity of the root $\beta-\alpha-\alpha_1$ implies $s_1=0$. Then

$$\pi(\Phi(X_{\alpha_1}))R_2^{s_2}R_3^{s_3}E^q=s_2\Big(s_2+s_3+\frac{1}{2}d_2\Big)R_2^{s_2-1}R_3^{s_3}E^{q+1}$$

has to be 0 which gives $s_2=0$. Finally we must consider $-\beta=-\beta+(-\alpha_1)$. By an easy computation we get

$$\pi(\Phi(X_{-\beta}))R_3^{s_3}E^q=\Big(\Big(q-\frac{k_2}{4}\Big)\Big(q-1+s_3+\frac{k_2}{4}\Big)\Big)R_3^{s_3}E^{q-1}\ .$$

This is 0 if and only if $q=q_1$ or q_2. We just have to prove that the corresponding monomial is a weight. This is an immediate consequence of Lemma 3.2. on $\mathfrak{h}\cap\mathfrak{m}$ the weight is

$$q\beta+s_3(\alpha+\tau)+\frac{1}{2}\Big[(k_2+2)\alpha+\alpha_1+\sum_{\sigma\in D_2^+}\sigma\Big]\ .$$

Let

$$M_i(s_3)=\pi(\Phi(\mathcal{U}(\mathfrak{a}(\delta)\times\mathfrak{b}(\delta)))R_3^{s_3}E^{q_i}$$

and

$$M_i = \sum_{s_3} M_i(s_3) \quad .$$

Lemma 4.10. *Fix* $i=1$ *or* 2. *Let* $X \in \mathcal{U}(\mathfrak{g})$. *If* $\pi(\Phi(X))M_i = (0)$, *then* X *belongs to the Joseph ideal.*

The proof is similar to the proof of Lemma 4.5 and we omit the details.

Then, for $q = q_1$ or q_2, we compute the restriction to $\mathfrak{h} \cap \mathfrak{a}(\delta)$ and to $\mathfrak{h} \cap \mathfrak{b}(\delta)$ of the weight of $R_3^{s_3} E^{q_i}$.

For $\mathfrak{a}(\delta)$ this restriction satisfies:

$$\begin{aligned} H_{-\beta} &\mapsto -2q_i - s_3 \\ H_\alpha &\mapsto q_i + s_3 + \frac{1}{4}k_2 + 1 \\ H_\varepsilon &\mapsto -s_3 - \frac{1}{2}k_2 \end{aligned}$$

(recall that $\alpha = \alpha_2$ and $\varepsilon = \alpha_1 \ldots$).

$R_3^{s_3} E^{q_i}$ is a highest weight for $\mathfrak{a}(\delta)$ provided we choose as simple roots

$$\{-\beta, \beta - \alpha - \varepsilon, \varepsilon\} \quad .$$

Then half the sum of the positive roots is the linear form given by

$$H_{-\beta} \mapsto 1 \ , \quad H_\alpha \mapsto -3 \ , \quad H_\varepsilon \mapsto 1 \ .$$

Adding this form we obtain

$$\begin{aligned} H_{-\beta} &\mapsto -2q_i - s_3 + 1 \\ H_\alpha &\mapsto q_i + s_3 + \frac{1}{4}k_2 - 2 \\ H_\varepsilon &\mapsto -s_3 - \frac{1}{2}k_2 + 1 \ . \end{aligned}$$

Call this linear form $\lambda_i(\mathfrak{a}(\delta), s_3)$. Identify $\mathfrak{a}(\delta)$ with $\mathfrak{sl}(4)$ using

$$\{-\beta, \alpha, \varepsilon\}$$

as a system of simple roots. The Killing form $B_{\mathfrak{a}(\delta)}$ is given by

$$B_{\mathfrak{a}(\delta)}(X, Y) = 8\mathrm{Tr}(XY) \quad .$$

Identify $\mathfrak{a}(\delta)$ with its dual using the above Killing form. By an easy computation we get

$$\lambda_i(\mathfrak{a}(\delta), s_3) = \frac{1}{16}\begin{pmatrix} -s_3 + \frac{1}{2}k_2 - 2q_i & 0 & 0 & 0 \\ 0 & s_3 + \frac{1}{2}k_2 + 2q_i - 2 & 0 & 0 \\ 0 & 0 & -s_3 + 2 & 0 \\ 0 & 0 & 0 & s_3 + k_2 \end{pmatrix}.$$

Also is is trivially checked that $\lambda_1(\mathfrak{a}(\delta), s_3)$ is conjugate under the Weyl group to $\lambda_2(\mathfrak{a}(\delta), -s_3+2-k_2)$. Let $j_{\mathfrak{a}(\delta)}$ be the Harish-Chandra isomorphism of the center $\mathfrak{Z}(\mathcal{U}(\mathfrak{a}(\delta))$ onto the subalgebra of Weyl group invariant in the symmetric algebra of $\mathfrak{h}\cap\mathfrak{a}(\delta)$. Then for $Z\in\mathfrak{Z}(\mathcal{U}(\mathfrak{a}(\delta))$

$$\pi(\Phi(Z))R_3^{s_3}E^{q_i}=\lambda_i(\mathfrak{a}(\delta), s_3)R_3^{s_3}E^{q_i}$$

Next we look at the restriction to $\mathfrak{b}(\delta)\cap\mathfrak{h}$. It is equal to the restriction of $s_3\tau$ and in particular is the same for q_1 and q_2. We add $\rho_{\mathfrak{b}(\delta)}$, half the sum of the positive roots. Then identifying $\mathfrak{b}(\delta)$ with its dual as usual and after an easy computation we see that, in both B_ℓ and D_ℓ cases, the linear form is

$$\lambda(\mathfrak{b}(\delta), s_3)=\frac{s_3}{16}\frac{4}{d_2}(H_\tau-H_\delta)+\frac{2}{\|H_{\rho_{\mathfrak{b}(\delta)}}\|^2}H_{\rho_{\mathfrak{b}(\delta)}} .$$

Also it is easy to check that $\lambda(\mathfrak{b}(\delta), s_3)$ is conjugate under the Weyl group of $\mathfrak{b}(\delta)$ to $\lambda(\mathfrak{b}(\delta), -s_3+2-k_2)$. With obvious notations we thus get

$$\pi(\Phi(Z))R_3^{s_3}E^{q_i}=\lambda(\mathfrak{b}(\delta), s_3)R_3^{s_3}E^{q_i} .$$

Our final result is

Theorem 4.11 ($\mathfrak{g}$ **of type** B_ℓ **with** $\ell\geq 4$ **or** D_ℓ **with** $\ell\geq 5$). *Let* $Z\in\mathfrak{Z}(\mathcal{U}(\mathfrak{a}(\delta))$ *and* $Z'\in\mathfrak{Z}(\mathcal{U}(\mathfrak{b}(\delta))$. *Then*

$$\pi(\Phi(Z-Z'))=0$$

if and only if, for all $s_3\in\boldsymbol{N}$

$$j_{\mathfrak{a}(\delta)}(Z)(\lambda_1(\mathfrak{a}(\delta), s_3)=j_{\mathfrak{b}(\delta)}(Z')(\lambda(\mathfrak{b}(\delta), s_3)$$

Note that we could replace in the above equality $\lambda_1(\ldots)$ by $\lambda_2(\ldots)$ the two conditions being equivalent. The proof is the same as the proof of Theorem 4.7.

§5. The explicit collapsing of the centers: the $\mathfrak{a}(\alpha)\times\mathfrak{b}(\alpha)$ case

5.1. *Preliminary computations.* For the present time we assume that $\mathfrak{g}$ is simple but not of type A_ℓ. By definition $\mathfrak{b}(\alpha)=\mathfrak{m}$ and

$$\mathfrak{a}(\alpha)=\boldsymbol{C}X_{-\beta}\oplus\boldsymbol{C}H\oplus\boldsymbol{C}X_\beta$$

For any element of $Z\in\mathcal{U}(\mathfrak{m})$ it follows from Proposition 3.1 that $\Phi(Z)\in\mathcal{U}(\mathfrak{g}_1\oplus\mathfrak{g}_2)_E$.

Let

$$P(X)=B(\mathrm{ad}(X)^4X_\beta, X_\beta)\qquad\text{for}\quad X\in\mathfrak{g}_{-1}$$

Then $P\in S(\mathfrak{g}_1)$; let $Q\in\mathcal{U}(\mathfrak{g}_1\oplus\mathfrak{g}_2)$ be the (usual) symmetrization of P. The algebra $\mathfrak{m}$ operates in $S(\mathfrak{g}_1\oplus\mathfrak{g}_2)$ and $\mathcal{U}(\mathfrak{g}_1\oplus\mathfrak{g}_2)$ by the adjoint action

Lemma 5.1. *a) The subalgebra $S(\mathfrak{g}_1 \oplus \mathfrak{g}_2)^{\mathfrak{m}}$ of $\mathfrak{m}$-invariants is*

$$S(\mathfrak{g}_1 \oplus \mathfrak{g}_2)^{\mathfrak{m}} = \begin{cases} \boldsymbol{C}[E] \text{ if } \mathfrak{g} \text{ is of type } C_\ell \\ \boldsymbol{C}[E, P] \text{ if } \mathfrak{g} \text{ is not of type } C_\ell . \end{cases}$$

b) The subalgebra $\mathcal{U}(\mathfrak{g}_1 \oplus \mathfrak{g}_2)^{\mathfrak{m}}$ of $\mathfrak{m}$-invariants is

$$\mathcal{U}(\mathfrak{g}_1 \oplus \mathfrak{g}_2)^{\mathfrak{m}} = \begin{cases} \boldsymbol{C}[E] \text{ if } \mathfrak{g} \text{ is of type } C_\ell \\ \boldsymbol{C}[E, Q] \text{ if } \mathfrak{g} \text{ is not of type } C_\ell . \end{cases}$$

The usual symmetrization operator from $S(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ to $\mathcal{U}(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ commutes with the action of $\mathfrak{m}$ so that it is enough to prove a). Now $E = X_\beta$ and $\mathfrak{m}$ commute so we only have to find the invariants of $\mathfrak{m}$ in $S(\mathfrak{g}_1)$. If $R \in S(\mathfrak{g}_1)$ is invariant under $\mathfrak{m}$ then each homogeneous component of R is invariant under $\mathfrak{m}$. However H operates by dilation in $\mathfrak{g}_1$ so that these homogeneous components are relatively invariants under the action of $\mathfrak{g}_0 = \boldsymbol{C}H \oplus \mathfrak{m}$. But $\mathfrak{g}_1$ is a prehomegeneous $\mathfrak{g}_0$-module irreducible and regular except if $\mathfrak{g}$ is of type C_ℓ. In this last case the only relative invariants of $\mathfrak{g}_0$ are the constants and we are done. In the other cases there exist an irreducible relative invariant P_1 such that the relative invariants are the monomials cP_1^n; in particular P_1 is homogeneous. We claim that, up to a constant factor $P = P_1$. Note that this implies the Lemma.

First P is non zero. Indeed we proved (Lemma 2.2) that $(X_{-\alpha} + X_{-\beta+\alpha}, 2H, X_\alpha + X_{\beta-\alpha})$ is a T.D.S. For the adjoint action of this T.D.S. the vector X_β is of weight 4. Hence $\mathrm{ad}(X_{-\alpha} + X_{-\beta+\alpha})^4 X_\beta$ is non zero mutiple of $X_{-\beta}$. This proves that $P(X_{-\alpha} + X_{-\beta+\alpha}) \neq 0$. If P is not irreducible then the irreducible invariant P_1 has to be of degree 2 (there is clearly no degree one invariant). Define on $\mathfrak{g}_{-1}$ an alternating form by

$$[X, Y] = \omega_-(X, Y) X_{-\beta} .$$

This form is invariant under $[\mathfrak{g}_0, \mathfrak{g}_0]$ and non-degenerate. But P_1 is a non zero quadratic form invariant under $[\mathfrak{g}_0, \mathfrak{g}_0]$. Because of the irreducibility of $\mathfrak{g}_{-1}$ as a $[\mathfrak{g}_0, \mathfrak{g}_0]$ module this is impossible. Thus P_1 is of degree 4 and proportional to P.

If $X \in \mathfrak{m}$ and $Y \in \mathfrak{r}$ then

$$\Phi(\mathrm{ad}(X) Y) = \mathrm{ad}(X) Y = \Phi(XY - YX) = \Phi(X) Y - YH(X)$$

and this relation remains true for $Y \in \mathcal{U}(\mathfrak{r})_E$. Hence if we start with Z an element of the center $\mathcal{Z}(\mathfrak{m})$ of the enveloping algebra of $\mathfrak{m}$ we get that $\Phi(Z)$ commutes with the adjoint action of $\mathfrak{m}$. By the last Lemma this implies that $\Phi(Z)$ belongs to the subalgebra $\boldsymbol{C}[Q, E]$ in general, to $\boldsymbol{C}[E]$ in the C_ℓ situation. Furthermore

$$HE - EH = 2H , \quad HQ - QH = 4Q ,$$

and Z commutes with H thus

$$\Phi(Z)\in \boldsymbol{C}[QE^{-2}]$$

if $\mathfrak{g}$ is not of type C_ℓ. In the C_ℓ case $\Phi(Z)$ has to be a constant. In the C_ℓ case let us put $P=Q=0$

Lemma 5.2. *There exists two constants c_0 and d_0 such that*

$$\Phi(X_{-\beta})=c_0QE^{-3}+\left(\frac{1}{4}H^2+\frac{1}{2}H+d_0\right)E^{-1} .$$

In fact $[X_{-\beta}, \mathfrak{m}]=(0)$ so

$$\Phi(X_{-\beta})\in \mathscr{A}^{\mathfrak{m}}=\bigoplus_{t\geq 0}\mathcal{U}(\mathfrak{g}_1\oplus\mathfrak{g}_2)^{\mathfrak{m}}_{E}H^t$$

Furthermore $H\Phi(X_{-\beta})-\Phi(X_{-\beta})H=-2\Phi(X_{-\beta})$ thus

$$\Phi(X_{-\beta})=\sum_{\text{finite}} c_{u,v}Q^uH^vE^{-1-2u} \qquad u, v\geq 0$$

The explicit expression for Φ tells us that

$$\Phi(X_{-\beta})E^3\in \mathcal{U}(\mathfrak{r})^{\mathfrak{m}}$$

so that

$$\Phi(X_{-\beta})\in \boldsymbol{C}[H]E^{-1}\oplus \boldsymbol{C}[H]QE^{-3} .$$

let u and v be the polynomials such that

$$\Phi(X_{-\beta})=u(H)E^{-1}+v(H)QE^{-3} .$$

Writing that $[X_{-\beta}, X_\beta]=H$ we get

$$H=(u(H)E^{-1}+v(H)QE^{-3})E-E(u(H)E^{-1}+v(H)QE^{-3}) .$$

But

$$Eu(H)=u(H-2)E$$

so that the above condition may be rewritten

$$u(H)+v(H)QE^{-2}-u(H-2)-v(H-2)QE^{-3}=H$$

which implies that

$$v(H)=v(H-2) \quad , \quad u(H)-u(H-2)=H .$$

Because u and v and polynomials this is possible if and only if $v=c_0$, a constant and

$$u(H)=\frac{1}{4}H^2+\frac{1}{2}H+d_0$$

for some constant d_0.

We will need a more explicit formula. Let

$$s_\beta = e^{\mathrm{ad}X_\beta} e^{\mathrm{ad}X_{-\beta}} e^{\mathrm{ad}X_\beta}$$

Lemma 5.3. *If $\gamma \in \Delta_1$, then*

$$s_\beta(X_{-\gamma}) = N_{\beta,-\gamma} X_{\beta-\gamma} .$$

Indeed

$$e^{\mathrm{ad}X_\beta} X_{-\gamma} = X_{-\gamma} + [X_\beta, X_{-\gamma}] = X_{-\gamma} + N_{\beta,-\gamma} X_{\beta-\gamma} .$$

Next

$$\begin{aligned} e^{\mathrm{ad}X_\beta} e^{\mathrm{ad}X_{-\beta}} X_{-\gamma} &= X_{-\gamma} + N_{\beta,-\gamma} X_{\beta-\gamma} + N_{\beta,-\gamma}[X_{-\beta}, X_{\beta-\gamma}] \\ &= (1 + N_{\beta,-\gamma} N_{-\beta,\beta-\gamma}) X_{-\gamma} + N_{\beta,-\gamma} X_{\beta-\gamma} . \end{aligned}$$

Finally

$$s_\beta X_{-\gamma} = (1 + N_{\beta,-\gamma} N_{-\beta,\beta-\gamma}) X_{-\gamma} + N_{\beta,-\gamma} X_{\beta-\gamma} + (1 + N_{\beta,-\gamma} N_{-\beta,\beta-\gamma}) N_{\beta,-\gamma} X_{\beta-\gamma} .$$

To prove the lemma we thus have to check that

$$1 + N_{\beta,-\gamma} N_{-\beta,\beta-\gamma} = 0 .$$

Let p, q be the positive integers such that

$$(-\gamma + \mathbf{Z}\beta) \cap \Delta = \{-\gamma - q\beta, -\gamma - (q-1)\beta, \dots, -\gamma + (p-1)\beta, -\gamma + p\beta\} .$$

It is known that

$$N_{\beta,-\gamma} N_{-\beta,\beta-\gamma} = -p(q+1) .$$

But $-\gamma-\beta \notin \Delta$ so that $q=0$ and $-\gamma+2\beta \notin \Delta$ so that $p=1$ and we are done.

Lemma 5.4. *Let $\gamma \in \Delta_1$ and put*

$$r = \sup\{j \mid \beta - j\gamma \in \Delta\} .$$

Then

$$N_{\beta,-\beta+\gamma} N_{\gamma,\beta-\gamma} = -r$$

and

$$r\|\gamma\|^2 = \|\beta\|^2 .$$

Let p and q be the two positive integers such that $\beta + j(-\beta+\gamma)$ is a root if and only if $-q \le j \le p$. Then

$$N_{-\beta+\gamma,\beta} N_{\beta-\gamma,\gamma} = -p(q+1) .$$

However

$$N_{-\beta+\gamma,\beta} = -N_{\beta,-\beta+\gamma} , \quad N_{\beta-\gamma,\beta} = -N_{\gamma,\beta-\gamma}$$

so we have to prove that

$$r=p(q+1)\ .$$

Now $\beta-(-\beta+\gamma)=2\beta-\gamma\notin\Delta$ so that $q=0$ and

$$p(q+1)=p=\sup\{j\,|\beta+j(-\beta+\gamma)\in\Delta\}\ .$$

Furthermore

$$s_\beta(\beta+j(-\beta+\gamma)=-\beta+j\gamma=-(\beta-j\gamma)$$

and, by definition of r we get $r=p$.

Using the same sequence of roots we get

$$\frac{\|\gamma\|^2}{\|\beta\|^2}=\frac{q+1}{p}=\frac{1}{r}\ .$$

Next we go back to P. By definition

$$P(X)=B(\mathrm{ad}(X)^4X_\beta,X_\beta)\ .$$

The subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are put in duality by the Killing form B.

Lemma 5.5. *As an element of $S[\mathfrak{g}_1]$ the polynomial function P is equal to*

$$P=\frac{\|\beta\|^8}{16}\sum_{\Delta_1^4}B(\mathrm{ad}(X_{\gamma_1})\,\mathrm{ad}(X_{\gamma_2})\,\mathrm{ad}(X_{\gamma_3})\,\mathrm{ad}(X_{\gamma_4})X_{-\beta},X_{-\beta})F_{\gamma_1}F_{\gamma_2}F_{\gamma_3}F_{\gamma_4}.$$

Put

$$X=\sum_{\gamma\in\Delta_1}x_\gamma X_{-\gamma}\qquad x_\gamma=\frac{B(X,X_\gamma)}{B(X_{-\gamma},X_{-\gamma})}$$

so that

$$P(X)=\sum_{\Delta_1^4}x_{\gamma_1}x_{\gamma_2}x_{\gamma_3}x_{\gamma_4}B(\mathrm{ad}(X_{-\gamma_1})\,\mathrm{ad}(X_{-\gamma_2})\,\mathrm{ad}(X_{-\gamma_3})\,\mathrm{ad}(X_{-\gamma_4})X_\beta,X_\beta)$$

or, equivalently

$$P=\sum_{\Delta_1^4}\frac{X_{\gamma_1}}{B(X_{\gamma_1},X_{-\gamma_1})}\,\frac{X_{\gamma_2}}{B(X_{\gamma_2},X_{-\gamma_2})}\,\frac{X_{\gamma_3}}{B(X_{\gamma_3},X_{-\gamma_3})}\,\frac{X_{\gamma_4}}{B(X_{\gamma_4},X_{-\gamma_4})}$$

$$B(\mathrm{ad}(X_{-\gamma_1})\,\mathrm{ad}(X_{-\gamma_2})\,\mathrm{ad}(X_{-\gamma_3})\,\mathrm{ad}(X_{-\gamma_4})X_\beta,X_\beta)\ .$$

We use the invariance of the Killing form under s_β and Lemma 5.3 to get

$$P=\sum_{\Delta_1^4}\frac{X_{\gamma_1}}{B(X_{\gamma_1},X_{-\gamma_1})}N_{\beta,-\gamma_1}\frac{X_{\gamma_2}}{B(X_{\gamma_2},X_{-\gamma_2})}N_{\beta,-\gamma_2}\frac{X_{\gamma_3}}{B(X_{\gamma_3},X_{-\gamma_3})}N_{\beta,-\gamma_3}\frac{X_{\gamma_4}}{B(X_{\gamma_4},X_{-\gamma_4})}N_{\beta,-\gamma_4}$$

$$B(\mathrm{ad}(X_{\beta-\gamma_1})\,\mathrm{ad}(X_{\beta-\gamma_2})\,\mathrm{ad}(X_{\beta-\gamma_3})\,\mathrm{ad}(X_{\beta-\gamma_4})X_{-\beta},X_{-\beta}).$$

Then we note that $\gamma\mapsto\beta-\gamma$ is an involution of Δ_1 and that

$$B(X_{\beta-\gamma}, X_{-\beta+\gamma}) = B(X_\gamma, X_{-\gamma}) = -\frac{2}{\|\gamma\|^2}$$

so that, using Lemma 5.4

$$\begin{aligned} N_{\beta,-\beta+\gamma}\frac{X_{\beta-\gamma}}{B(X_{\beta-\gamma}, X_{-\beta+\gamma})} &= -\frac{\|\gamma\|^2}{2}N_{\beta,-\beta+\gamma}X_{\beta-\gamma} \\ &= -\frac{\|\gamma\|^2}{2}N_{\gamma,\beta-\gamma}N_{\beta,-\beta+\gamma}F_\gamma \\ &= +\frac{\|\beta\|^2}{2}F_\gamma \end{aligned}$$

In the last expression for P we change all the γ_i into $\beta-\gamma_i$ and apply the above transformation. This gives the required result.

Finally we compute again $\Phi(X_{-\beta})$, this time using directly the definition of Φ. Consider $\Psi(X_{-\beta})$. We apply e^D and the contraction operator which means that we only keep the coefficient of X_β. A typical element arising from $e^D(X_{-\beta})$ is, up to the factorials of the exponential series,

$$\mathrm{ad}X_{\delta_1}\ldots\mathrm{ad}X_{\delta_r}X_{-\beta}$$

with $\delta_i \in \Delta_1 \cup \Delta_2$. Obviously we have to consider 3 cases:

a) $r=2$ and $\delta_1=\delta_2=\beta$,

b) $r=3$ and one of the 3 roots is β the others belonging to Δ_1,

c) $r=4$ and the 4 roots belong to Δ_1.

Case a) occurs while computing $D^2(X_{-\beta})$. Its contribution to $\Psi(X_{-\beta})$. is

$$\frac{1}{2!}\frac{B(\mathrm{ad}(X_\beta)\,\mathrm{ad}(X_\beta)X_{-\beta}, X_{-\beta})}{B(X_\beta, X_{-\beta})}\frac{H^2}{4}$$

which is equal to

$$\frac{1}{4}H^2 .$$

Case b) occurs while computing $D^3(X_{-\beta})$. The 3 roots δ_i are γ, $\beta-\gamma$, β where $\gamma \in \Delta_1$ (others choices would give 0 after contraction). However the order of those 3 roots is relevant so that we get a sum of 6 terms. Now X_β commutes with X_γ and $X_{\beta-\gamma}$ so we have to evaluate

$$3(\mathrm{ad}(X_\gamma)\,\mathrm{ad}(X_{\beta-\gamma})\,\mathrm{ad}(X_\beta) + \mathrm{ad}(X_{\beta-\gamma})\,\mathrm{ad}(X_\gamma)\,\mathrm{ad}(X_\beta))X_{-\beta}$$

However $[X_\beta, X_{-\beta}] = -H$ so that

$$\mathrm{ad}(X_\gamma)\,\mathrm{ad}(X_{\beta-\gamma})\,\mathrm{ad}(X_\beta)X_{-\beta} = \mathrm{ad}(X_\gamma)([X_{\beta-\gamma}, -H]) = [X_\gamma, X_{\beta-\gamma}]$$

and as the second part gives $[X_{\beta-\gamma}, X_\gamma]$ the contribution turns out to be 0.

Finally case c) occurs while computing $D^4(X_{-\beta})$. The contribution is

$$\frac{1}{4!}\frac{1}{B(X_\beta, X_{-\beta})}\sum_{\Delta_1^4} B(\mathrm{ad}(X_{\gamma_1})\,\mathrm{ad}(X_{\gamma_2})\,\mathrm{ad}(X_{\gamma_3})\,\mathrm{ad}(X_{\gamma_4})X_{-\beta}, X_{-\beta})F_{\gamma_1}F_{\gamma_2}F_{\gamma_3}F_{\gamma_4}$$

which is equal to

$$-\frac{P}{3\|\beta\|^6} .$$

Hence

$$\Psi(X_{-\beta}) = \frac{1}{4}H^2 - \frac{1}{3\|\beta\|^6}P .$$

Applying the symmetrization operator σ we obtain

$$\Phi(X_{-\beta}) = \frac{1}{4}HE^{-1}H - \frac{1}{3\|\beta\|^6}QE^{-3} + c(\mathfrak{g})E^{-1} .$$

Also

$$HE^{-1}H = H(H+2)E^{-1}$$

so that the following proposition has been completely proved.

Proposition 5.6. *We have*

$$\Phi(X_{-\beta}) = \left[\frac{1}{4}H(H+2) + c(\mathfrak{g})\right]E^{-1} - \frac{1}{3\|\beta\|^6}QE^{-3} .$$

Note that $c(\mathfrak{g})$ is computed at the end of the paper. Comparing with Lemma 5.2 we see that

$$c_0 = -\frac{1}{3\|\beta\|^6} , \quad d_0 = c(\mathfrak{g}) .$$

The subalgebra $\mathfrak{a}(\alpha)$ is of type A_1. We define its Casimir element using the Killing form of $\mathfrak{a}(\alpha)$. So it is equal to

$$\omega(\mathfrak{a}(\alpha)) = \frac{1}{8}(H^2 - 2H - 4X_\beta X_{-\beta}) .$$

Then using Proposition 5.6:

Proposition 5.7.

$$\Phi(\omega(\mathfrak{a}(\alpha))) = -\frac{1}{2}c(\mathfrak{g}) + \frac{1}{6\|\beta\|^6}QE^{-2} .$$

Now we go back to the representation π. Suppose that $\mathfrak{g}$ is not of type C_ℓ and put

$$\rho_{\Gamma_1} = \frac{1}{2}\sum_{\gamma\in\Gamma_1}\gamma .$$

Proposition 5.8. *If r is any rational number, then*

$$\pi(Q)E^r = c_Q(\mathfrak{g})E^{r+2}$$

where

$$c_Q(\mathfrak{g})=3\|\beta\|^4\left(\|\rho_{\Gamma_1}\|^2-\frac{1}{8}\#\Gamma_1(1+\#\Gamma_1)\langle\beta,\alpha\rangle\right).$$

By definition Q is the usual symmetrization of

$$P=\frac{\|\beta\|^8}{16}\sum_{\Delta_1^4}B(\mathrm{ad}(X_{\gamma_1})\,\mathrm{ad}(X_{\gamma_2})\,\mathrm{ad}(X_{\gamma_3})\,\mathrm{ad}(X_{\gamma_4})X_{-\beta},X_{-\beta})F_{\gamma_1}F_{\gamma_2}F_{\gamma_3}F_{\gamma_4}.$$

We may assume that

$$\gamma_1+\gamma_2+\gamma_3+\gamma_4=2\beta$$

otherwise the coefficient $B(\dots,X_{-\beta})$ is 0. If $\gamma\in\Gamma_1$ then $\pi(F_\gamma)$ is $-E\partial/\partial X_\gamma$ and if $\gamma\in\Gamma_2$ then $\pi(F_\gamma$ is essentially multiplication by $X_{\beta-\gamma}$. Computing $\pi(Q)E^r$ gives a sum of terms of type

$$\pi(F_{\gamma_1})\,\pi(F_{\gamma_2})\,\pi(F_{\gamma_3})\,\pi(F_{\gamma_4})E^r.$$

Now at least one of the 4 roots $\gamma_i\in\Gamma_1$, for example $\gamma_1\in\Gamma_1$; in order not to get 0, then before taking derivative with respect to X_{γ_1} we must first multiply by X_{γ_1}. This means that one of the 3 remaining roots, say γ_2 has to be $\beta-\gamma_1$. But then $\gamma_3+\gamma_4=\beta$ so that we can argue in the same way once more. This means that we may limit ourselves to the situation where, up to a permutation the 4 roots are

$$\{\sigma_1,\beta-\sigma_1,\sigma_2,\beta-\sigma_2\}$$

with σ_1 and σ_2 in Γ_1.

In the above expression for P the summation is over all elements of Δ_1^4. To such an element $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ we associate the subset of Δ_1 consisting of the distinct elements among the γ_i. In other words, assuming for example the γ_i to be distinct we carefully distinguish between $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)\in\Delta_1^4$ and $\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}\subset\Delta_1$.

Consider first the case where $\sigma_1\neq\sigma_2$. Then the subset $\{\sigma_1,\beta-\sigma_1,\sigma_2,\beta-\sigma_2\}$ of Δ_1 corresponds to 24 differents elements of Δ_1^4. However we note that X_{σ_1} commutes with X_{σ_2} and with $X_{\beta-\sigma_2}$ and so does $X_{\beta-\sigma_1}$. Hence

$$\sum B(\mathrm{ad}(X_{\gamma_1})\,\mathrm{ad}(X_{\gamma_2})\,\mathrm{ad}(X_{\gamma_3})\,\mathrm{ad}(X_{\gamma_4})X_{-\beta},X_{-\beta})$$

where the sum is over the 24 elements of Δ_1^4 such that

$$\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}=\{\sigma_1,\sigma_2,\beta-\sigma_1,\beta-\sigma_2\}$$

is in fact a sum of 4 terms each taken 6 times. One of the 4 terms is

$$B(\mathrm{ad}(X_{\beta-\sigma_1})\,\mathrm{ad}(X_{\sigma_1})\,\mathrm{ad}(X_{\sigma_2})\,\mathrm{ad}(X_{\beta-\sigma_2})X_{-\beta},X_{-\beta})$$

which is equal to

$$B(\mathrm{ad}(X_{\sigma_2})\,\mathrm{ad}(X_{\beta-\sigma_2})X_{-\beta},\ \mathrm{ad}(X_{\beta-\sigma_1})\,\mathrm{ad}(X_{\sigma_1})X_{-\beta})$$

or to

$$N_{\beta-\sigma_1,-\beta}N_{\beta-\sigma_2,-\beta}B(H_{\sigma_2}, H_{\sigma_1}) = N_{\beta-\sigma_1,-\beta}N_{\beta-\sigma_2,-\beta}\frac{4\langle\sigma_1, \sigma_2\rangle}{\langle\sigma_1, \sigma_1\rangle\langle\sigma_2, \sigma_2\rangle} .$$

Now if u, v and $u+v$ are roots then

$$N_{-u,u+v}\langle H_v, H_v\rangle = -N_{-u,-v}\langle H_{u+v}, H_{u+v}\rangle .$$

If we take

$$u=-\beta+\sigma_i , \quad v=-\sigma_2 , \quad u+v=-\beta$$

we obtain

$$N_{\beta-\sigma_i,-\beta}\langle H_{\sigma_i}, H_{\sigma_i}\rangle = -N_{\beta-\sigma_i,\sigma_i}\langle H_\beta, H_\beta\rangle$$

and our expression may be written

$$\frac{4}{\|\beta\|^4}N_{\beta-\sigma_1,\sigma_1}N_{\beta-\sigma_2,\sigma_2}\langle\sigma_1, \sigma_2\rangle .$$

The other 3 terms are obtained by changing σ_1 and/or σ_2 into $\beta-\sigma_1$ and/or $\beta-\sigma_2$. Adding the 4 of them, mutiplying by 6 and taking into account the coefficient $\|\beta\|^8/16$ in front of P we end up with

$$3\|\beta\|^4(2\langle\sigma_1, \sigma_2\rangle - \langle\beta, \alpha\rangle)$$

(note also that $\langle\beta, \sigma_i\rangle = \langle\beta, \alpha\rangle = \langle\beta, \beta\rangle/2$).

Next suppose that $\sigma_1=\sigma_2=\sigma$. This time we have a priori 6 terms but it is easy to reduce this partial sum to

$$3B(\mathrm{ad}(X_\sigma)\,\mathrm{ad}(X_{\beta-\sigma})\,\mathrm{ad}(X_\sigma)\,\mathrm{ad}(X_{\beta-\sigma})X_{-\beta}, X_{-\beta})$$
$$+3B(\mathrm{ad}(X_{\beta-\sigma})\,\mathrm{ad}(X_\sigma)\,\mathrm{ad}(X_{\beta-\sigma})\,\mathrm{ad}(X_\sigma)X_{-\beta}, X_{-\beta})$$

Arguing as before we get

$$\frac{3}{2}\|\beta\|^4N^2_{\beta-\sigma,\sigma}(\langle\sigma, \sigma\rangle - \langle\beta, \alpha\rangle) .$$

Finally in order to compute $\pi(Q)E^r$ we may replace Q by the usual symmetrization of

$$3\|\beta\|^4\sum_{\{\sigma_1,\sigma_2\}}(2\langle\sigma_1, \sigma_2\rangle - \langle\beta, \alpha\rangle)F_{\sigma_1}X_{\sigma_1}F_{\sigma_2}X_{\sigma_2}$$
$$+\frac{3}{2}\|\beta\|^4\sum_{\Gamma_1}(\langle\sigma, \sigma\rangle - \langle\beta, \alpha\rangle)F_\sigma X_\sigma F_\sigma X_\sigma .$$

The end of the computation is straightforward and we omit the details.

5.2. Collapsing of the centers. We exclude the C_ℓ case. If $Z \in \mathfrak{Z}(\mathfrak{a}(\alpha))$, the center of the enveloping algebra of $\mathfrak{a}(\alpha)$, then $\Phi(Z)$ commutes with $\mathfrak{m}$ hence belongs to $\boldsymbol{C}[QE^{-2}]$. This center is isomorphic to $\boldsymbol{C}[\omega(\mathfrak{a}(\alpha))]$ so Proposition 5.7 implies that Φ is an isomorphism of $\mathfrak{Z}(\mathfrak{a}(\alpha))$ onto $\boldsymbol{C}[QE^{-2}]$. Now if Z belongs to the center $\mathfrak{Z}(\mathfrak{m})$ of the universal enveloping algebra of $\mathfrak{m}$, then $\Phi(Z)$ commutes with $\mathfrak{m}$ and thus belongs to $\boldsymbol{C}[QE^{-2}]$. We have an homomorphism of $\mathfrak{Z}(\mathfrak{m})$ into $\boldsymbol{C}[QE^{-2}]$. Combining with the inverse of the above isomorphism this gives an homomorphism

$$\Theta\colon \mathfrak{Z}(\mathfrak{m}) \longrightarrow \mathfrak{Z}(\mathfrak{a}(\alpha))$$

with the property that for any Z the element $Z-\Theta(Z)$ belongs to Joseph's ideal. Our goal is to compute Θ explicitly. This will be done by finding sufficiently many highest weight vectors for $\mathfrak{a}(\alpha)\times\mathfrak{m}$. It will be convenient to distinguish several cases. The computations being very similar to the computations of §4 we shall skip most details

First assume that $\mathfrak{g}$ *is of type* G_2.
As usual we follow the notations of Bourbaki's tables. hence the two simple roots are denoted α_1 and α_2 and the highest root $\beta=3\alpha_1+2\alpha_2$ is connected to α_2. We have $\alpha=\alpha_2$ and $\delta=\alpha_1$. Also

$$\Gamma_1=\{\alpha,\alpha+\delta\}\quad,\quad \Gamma_2=\{\alpha+2\delta,\alpha+3\delta\}\quad.$$

Lemma 5.9 ($\mathfrak{g}$ **of type** G_2).

$$\pi(\Phi(X_{-\alpha})=\left(X_{\alpha+\delta}\frac{\partial}{\partial X_{\alpha+\delta}}+X_\alpha\frac{\partial}{\partial X_\alpha}+E\frac{\partial}{\partial E}+1\right)\frac{\partial}{\partial X_\alpha}-\frac{\partial^3}{\partial X^3_{\alpha+\delta}}E\ ,$$

$$\pi(\Phi(X_{-\beta+\alpha})=E\frac{\partial}{\partial E}X_\alpha E^{-1}-\frac{1}{27}X^3_{\alpha+\delta}E^{-2}\ .$$

The roots vectors are choosen so that the structural constants are given by the table in [G-S]. The formulas follow from a direct computation starting with the definition of Φ and π.

The subalgebra $\mathfrak{m}$ is of type A_1 with simple root $\alpha_2=\delta$. We have $\delta=(\alpha+\delta)+(-\alpha)$. Using the above Lemma we obtain

$$\pi(\Phi([X_{\alpha+\delta},X_{-\alpha}])X^p_\alpha X_{\alpha+\delta}E^q=-pX^{p-1}_\alpha X^{r+1}_{\alpha+\delta}E^q+3r(r-1)X^p_\alpha X^{r-2}_{\alpha+\delta}E^{q+1}\ .$$

The right hand side is 0 if and only if $p=0$ and $r=0$ or 1. This leaves us with

$$E^q\ ,\quad X_{\alpha+\delta}E^q\ .$$

Next we take $X_{-\beta}$ as the unique positive root of $\mathfrak{a}(\alpha)$. We compute the action of $X_{-\beta}$ using the equality $-\beta=(-\beta+\alpha)+(-\alpha)$. We find that $X_{\alpha+\delta}E^q$ is a highest weight for $\mathfrak{a}(\alpha)$ if and only if $q=\pm1/3$ and that E^q is a highest weight vector for $q=1/3$ and $q=2/3$. Thus we have four highest weights vectors for

$\mathfrak{a}(\alpha)\times\mathfrak{b}(\alpha)$.

Relative to $\mathfrak{a}(\alpha)$ the weight of $X^r_{\alpha+\delta}E^q$ is given by

$$H_{-\beta}\mapsto -r-2q$$

and if we add $\rho_{\mathfrak{a}(\alpha)}=-\beta/2$ we get the linear form

$$H\mapsto r+2q-1\ .$$

If $\varpi_{\mathfrak{a}(\alpha)}=\beta/2$ then this linear form is

$$(r+2q-1)\varpi_{\mathfrak{a}(\alpha)}$$

Relative to $\mathfrak{m}$ the weight of $X^r_{\alpha+\delta}E^q$ is given by

$$H_\delta\mapsto(\alpha+\delta)(H_\delta)+\rho_{\Gamma_1}(H_\delta)=-r-2\ .$$

Adding $\rho_{\mathfrak{m}}$ and defining $\varpi_{\mathfrak{m}}=\delta/2$ we have the linear form

$$(-\mathrm{r}-1)\varpi_{\mathfrak{m}}\ .$$

Let

$$\psi(t\varpi_{\mathfrak{a}(\alpha)})=3t\varpi_{\mathfrak{m}}\ .$$

For each of the 4 possible choices of (r, q) we have

$$\psi((r+2q-1)\varpi_{\mathfrak{a}(\alpha)})=\pm(-r-1)\varpi_{\mathfrak{m}}$$

Let $j_{\mathfrak{a}(\alpha)}$ and $j_{\mathfrak{m}}$ be the Harish-Chandra isomorphism of the centers $\mathfrak{Z}(\mathcal{U}(\mathfrak{a}(\alpha)))$ and $\mathfrak{Z}(\mathcal{U}(\mathfrak{m}))$ onto the algebra of Weyl group invariants. It follows from the above remarks that for any $Z\in\mathfrak{Z}(\mathcal{U}(\mathfrak{m}))$

$$\pi(\Phi(j^{-1}_{\mathfrak{a}(\alpha)}\circ{}^t\psi\circ j_{\mathfrak{m}}(Z)-Z))X^r_{\alpha+\delta}E^q=0$$

in the 4 cases.

Let us apply this to the Casimir elements $\omega(\mathfrak{a}(\alpha))$ and $\omega(\mathfrak{m})$. Proposition 5.7 and the definition of Φ imply that there exist two constants a and b such that

$$\Phi(\omega(\mathfrak{m}))=a\Phi(\omega(\mathfrak{a}(\alpha)))+b\ .$$

However

$$j_{\mathfrak{a}(\alpha)}(\omega(\mathfrak{a}(\alpha)))=\frac{1}{8}H^2-1\ ,$$

$$j_{\mathfrak{m}}(\omega(\mathfrak{m}))=\frac{1}{8}(H_\delta^2-1)\ .$$

Thus, for the 4 values of (r, q)

$$a((r-1+2q)^2-1)+8b=(r+1)^2-1\ .$$

This gives $a=9$ and $b=1$ so

$$\Phi(\omega(\mathfrak{m}))-9\Phi(\omega(\mathfrak{a}(\alpha)))=1 .$$

Also this computation shows that $\Theta(\omega(\mathfrak{m}))$ which, by definition, is $a\omega(\mathfrak{a}(\alpha))+b$ is also equal to $j_{\mathfrak{a}(\alpha)}^{-1}\circ\psi\circ j_{\mathfrak{m}}(\omega(\mathfrak{m}))$

As we are dealing with an $A_1\times A_1$ situation this is enough to prove the following theorem:

Theorem 5.10 (The G_2 case). *The homomorphism Θ of $\mathfrak{Z}(\mathcal{U}(\mathfrak{m}))$ onto $\mathfrak{Z}(\mathcal{U}(\mathfrak{a}(\alpha)))$ is given by*

$$\Theta=j_{\mathfrak{a}(\alpha)}^{-1}\circ{}^t\psi\circ j_{\mathfrak{m}}$$

In this case Θ is in fact an isomorphism.

Suppose now that $\mathfrak{g}$ *is of type* F_4, E_6, E_7 *or* E_8.

Going back to §4, we start with the highest weight vectors for $\mathfrak{a}(\delta)\times\mathfrak{b}(\delta)$

$$R_2^{s_2}R_3^{s_3}E^{q_i}$$

where

$$q_1=-s_2+\frac{k+3}{12}\ ,\quad q_2=-s_2-s_3-\frac{k-9}{12}\ .$$

We note that such a vector is also an highest weight vector for $\mathfrak{a}(\alpha)\times\mathfrak{m}$ provided it belongs to the kernel of $\pi(\Phi(X_\delta))$. However we have $\delta=(\alpha+\delta)+(-\alpha)$ so that we have an explicit formula for this operator (Proposition 4.3). First we check that

$$\pi(\Phi(X_{-\alpha}))R_2^{s_2}R_3^{s_3}E^{q_i}=0.$$

We are than left with the condition

$$D_1(\partial)X_{\alpha+\delta}R_2^{s_2}R_3^{s_3}E^{q_i}=0.$$

We claim that

$$D_1(\partial)X_{\alpha+\delta}X_{\alpha+\tau}^{s_3}=0.$$

Indeed

$$D_1(\delta)=\sum c_{\gamma_1,\gamma_2,\gamma_3}\frac{\partial^3}{\partial X_{\alpha+\gamma_1}\partial X_{\alpha+\gamma_2}\partial X_{\alpha+\gamma_3}}$$

where the $C\dots$ are some constants and where the sum is to be taken over all triple roots of in D^+ such that $\gamma_1+\gamma_2+\gamma_3=\delta+\varepsilon+\tau$ (cf. the definition of $D_1(\partial)$ in §1). If we apply this differential operator to $X_{\alpha+\delta}X_{\alpha+\tau}^{s_3}$ to obtain a non zero term we must take, up to permutation

$$\gamma_1=\delta,\ \gamma_2=\gamma_3=\tau$$

but $\delta+2\tau\neq\delta+\varepsilon\neq\tau$ so that this monomial does not apear.

However R_3 is a constant multiple of $X_{\alpha+\tau}$ so the above remark proves that

$$R_3^{s_3}E^{q_i}$$

are highest weight vectors for $\mathfrak{a}(\alpha)\times\mathfrak{m}$.

Let us compute the corresponding weights. For $\mathfrak{a}(\alpha)$ it is simply given by

$$H_{-\beta}\mapsto -2q_i-s_3.$$

As before we add $-\beta/2$ which gives

$$H=H_\beta\mapsto s_3+2q_i-1=\pm(q_1-q_2)\ .$$

The sign is $+$ for $q=q_1$ and $-$ for $q=q_2$. It is the linear form

$$\pm(q_1-q_2)\varpi_{\mathfrak{a}(\alpha)}\ .$$

Next we look at the restriction to $\mathfrak{m}\cap\mathfrak{h}$. The weight is the restriction of

$$s_3(\alpha+\tau)+\rho_{\Gamma_1}\ .$$

and we add $\rho_{\mathfrak{m}}$. Denote by $\varpi_{\mathfrak{m}}$ the fundamental weight of $\mathfrak{m}$ corresponding to the simple root δ. Then the restriction of α is $n(\alpha,\delta)\varpi_{\mathfrak{m}}=-\varpi_{\mathfrak{m}}$. Now $\mathfrak{m}$ operates via the adjoint action into V_1 and $2\rho_{\Gamma_1}$ is the restriction to $\mathfrak{m}\cap\mathfrak{h}$ of the trace. This trace is 0 on the semi-simple part $\mathfrak{b}(\delta)$. This means that the restriction of ρ_{Γ_1} is a multiple of $\varpi_{\mathfrak{m}}$. We saw that $K=H_\delta+H_\varepsilon+H_\tau$ a sum of 3 orthogonal vectors of same length $8k/3$ with respect to the Killing form $B_{\mathfrak{m}}$ of $\mathfrak{m}$ so that

$$B_{\mathfrak{m}}(K,H_\delta)=B_{\mathfrak{m}}(H_\delta,H_\delta)=\frac{1}{3}B_{\mathfrak{m}}(K,K)=\frac{8k}{3}\ .$$

Because $\mathfrak{b}(\delta)\cap\mathfrak{h}$ is the orthogonal of K in $\mathfrak{h}\cap\mathfrak{m}$ this implies that

$$H_\delta-\frac{1}{3}K\in\mathfrak{b}(\delta)\cap\mathfrak{h}\ .$$

So

$$\rho_{\Gamma_1}(H_\delta)=\frac{1}{3}\rho_{\Gamma_1}(K)\ .$$

Also $\alpha(K)=3\alpha(H_\delta)=3n(\alpha,\delta)=-3$ and, for $\gamma\in D^+$ we have $\gamma(K)=2$. Hence

$$2\rho_{\Gamma_1}(K)=\alpha(K)+\sum_{D^+}(\alpha+\gamma)(K)=-3(k+1)+2k=-k-3$$

which means that ρ_{Γ_1} restricted to $\mathfrak{h}\cap\mathfrak{m}$ is equal to

$$-\frac{k+3}{6}\varpi_{\mathfrak{m}}$$

hence, on $\mathfrak{h}\cap\mathfrak{m}$ the weight is, after adding $\rho_{\mathfrak{m}}$ equal to

$$s_3(-\varpi_{\mathfrak{m}}+\tau)-\frac{k+3}{6}\varpi_{\mathfrak{m}}+\rho_{\mathfrak{m}} \ .$$

For our purpose there is no harm applying some element in the Weyl group. We use the symmetry s_τ with respect to the highest root τ. Then it is clear that $s_\tau(\varpi_{\mathfrak{m}})=(\varpi_{\mathfrak{m}}-\tau)$ and we check case by case that $s_\tau(\rho_{\mathfrak{m}})=\rho_{\mathfrak{m}}-(2d+1)\tau$. Recalling that $k=3(d+1)$ we finally obtain the linear form

$$-(q_1-q_2)\varpi_{\mathfrak{m}}-\varpi_{\mathfrak{m}}-\frac{3}{2}d\tau+\rho_{\mathfrak{m}} \ .$$

Also note that

$$q_1-q_2=s_3+\frac{1}{2}d\in\boldsymbol{N}+\frac{1}{2}d \ .$$

Let $X\in\mathfrak{Z}(\mathfrak{U}(\mathfrak{a}(\alpha)))$. Then

$$\pi(\Phi(X))R_3^{s_3}E^{q_i}=j_{\mathfrak{a}(\alpha)}(X)(\pm(q_1-q_2)\varpi_{\mathfrak{a}(\alpha)}) \ .$$

If this is 0 for all s_3 then $j_{\mathfrak{a}(\alpha)}(X)=0$ hence $X=0$.

It follows that, given $Z\in\mathfrak{Z}(\mathfrak{U}(\mathfrak{m}))$ there is at most one $X\in\mathfrak{Z}(\mathfrak{U}(\mathfrak{a}(\alpha)))$ such that, for all s_3

$$\pi(\Phi(Z))R_3^{s_3}E^{q_i}=\pi(\Phi(X))R_3^{s_3}E^{q_i} \ .$$

Recall that we proved the existence of an algebra homomorphism Θ of $\mathfrak{Z}(\mathfrak{U}(\mathfrak{m}))$ into $\mathfrak{Z}(\mathfrak{U}(\mathfrak{a}(\alpha)))$ such that $X=\Theta(Z)$ has the above property.

Thus for all $t\in\mathbf{N}+d/2$ and all $Z\in\mathfrak{Z}(\mathfrak{U}(\mathfrak{m}))$

$$j_{\mathfrak{a}(\alpha)}(\Theta(Z))(\pm t)=j_{\mathfrak{m}}(Z)\left(-t\varpi_{\mathfrak{m}}-\varpi_{\mathfrak{m}}-\frac{3}{2}d\tau+\rho_{\mathfrak{m}}\right)$$

Because both sides are polynomials in t the equality remains valid for any real t. Also the left hand side is an even polynomial so the right hand side is also even. This shows that for any symmetric invariant ξ in $S(\mathfrak{h}\cap\mathfrak{m})$

$$\xi\left(-t\varpi_{\mathfrak{m}}-\varpi_{\mathfrak{m}}-\frac{3}{2}d\tau+\rho_{\mathfrak{m}}\right)=+t\varpi_{\mathfrak{m}}-\varpi_{\mathfrak{m}}-\frac{3}{2}d\tau+\rho_{\mathfrak{m}} \ .$$

We conclude that

$$-t\varpi_{\mathfrak{m}}-\varpi_{\mathfrak{m}}-\frac{3}{2}d\tau+\rho_{\mathfrak{m}}$$

and

$$+t\varpi_{\mathfrak{m}}-\varpi_{\mathfrak{m}}-\frac{3}{2}d+\rho_{\mathfrak{m}}$$

are conjugate under the Weyl group of $\mathfrak{m}$.

Put

$$\psi(t\varpi_{\mathfrak{a}(\alpha)}) = -t\varpi_{\mathfrak{m}} - \varpi_{\mathfrak{m}} - \frac{3}{2}d\tau + \rho_{\mathfrak{m}}$$

so that *the tail is*

$$-t\varpi_{\mathfrak{m}} - \frac{3}{2}d\tau + \rho_{\mathfrak{m}} .$$

Then ${}^t\psi$ is an affine map from $\mathfrak{h} \cap \mathfrak{m}$ onto $\mathfrak{h} \cap \mathfrak{a}(\alpha)$. We extend it to the symmetric algebra. It carries symmetric polynomials onto symmetric polynomials so that

$$\Theta_1 = j_{\mathfrak{a}(\alpha)}^{-1} \circ {}^t\psi \circ j_{\mathfrak{m}}$$

is a well defined algebra homomorphism from $\mathcal{Z}(\mathcal{U}(\mathfrak{m}))$ onto $\mathcal{Z}(\mathcal{U}(\mathfrak{a}(\alpha)))$. By definition

$$\pi(\Phi(Z) - \Phi(\Theta_1(Z)))R_3^{s_3}E^{q_i} = 0$$

so $\Theta_1 = \Theta$:

Theorem 5.11 (**Cases** F_4, E_6, E_7, E_8). *The homomorphism Θ is given by*

$$\Theta = j_{\mathfrak{a}(\alpha)}^{-1} \circ {}^t\psi \circ j_{\mathfrak{m}} .$$

It is easy to finish the computation for the Casimir operators, We give the end result:

$$\frac{\Phi(\omega(\mathfrak{m}))}{\|\varpi_{\mathfrak{m}}\|_{\mathfrak{m}}^2} - \frac{\Phi(\omega(\mathfrak{a}(\alpha)))}{\|\varpi_{\mathfrak{a}(\alpha)}\|_{\mathfrak{a}(\alpha)}^2} = -d(d+2) .$$

Suppose that $\mathfrak{g}$ *is of type* B_ℓ *with* $\ell \geq 4$ *or* D_ℓ *with* $\ell \geq 5$.
We start with the set of highest weight vectors of the $\mathfrak{a}(\delta) \times \mathfrak{b}(\delta)$ case:

$$R_3^{s_3}E^{q_i} \qquad q_1 = \frac{k_2}{4}, \quad q_2 = 1 - s_3 - \frac{k_2}{4} .$$

Such a vector will be a highest vector for $\mathfrak{a}(\alpha) \times \mathfrak{m}$ provided it belongs to the kernel of $\pi(\Phi(X_\delta))$ (recall that $\delta = \alpha_3$ with the usual notations for the root systems of type D_ℓ or B_ℓ). Now $\delta = (\alpha + \delta) + (-\alpha)$ so that using Proposition 4.8 we can can compute the action of X_δ. It turns out that all the above vectors are highest weight vectors for $\mathfrak{a}(\alpha) \times \mathfrak{m}$. The explicit computation of the weight is trivial and leads to the *following definition of* ψ:

$$\psi(t\varpi_{\mathfrak{a}(\alpha)}) = \left(t\varpi_{\mathfrak{m}_1}, t\varpi_{\mathfrak{m}_2} + \left(\frac{1}{2}d_2 - 1\right)\varpi_{\mathfrak{m}_2} - d_2\tau + \rho_{\mathfrak{m}_2}\right)$$

Then Theorem 5.11 is valid. For the Casimir operators we find:

$$\Phi(\varpi(\mathfrak{a}(\alpha))) = \Phi(\varpi(\mathfrak{m}_1))$$

$$\frac{\Phi(\omega(\mathfrak{m}_2))}{\|\varpi_{\mathfrak{m}_2}\|^2_{\mathfrak{m}_2}}-\frac{\Phi(\omega(\mathfrak{a}(\alpha)))}{\|\varpi_{\mathfrak{a}(\alpha)}\|^2_{\mathfrak{a}(\alpha)}}=-\frac{d_2(d_2+4)}{4}\ .$$

Suppose that $\mathfrak{g}$ *is of type* B_3.
We first have prove the analog of Proposition 4.3. We use the Chevalley basis given in [B-2]. In particular $\alpha=\alpha_2$. Proposition 4.3 remains valid provided we make the following substitutions. In the formula for $\pi(\Phi(X_{-\alpha})$ replace

$$+D_1(\partial)E \qquad \text{by} \quad +\frac{\partial}{\partial X_{\alpha+\alpha_1}}\frac{\partial^2}{\partial X^2_{\alpha+\alpha_3}}E\ .$$

In the formula for $\pi(\Phi(X_{-\beta+\alpha}))$ replace

$$+R_1E^{-2} \qquad \text{by} \quad +\frac{1}{4}X_{\alpha+\alpha_1}X^2_{\alpha+\alpha_3}E^{-2}\ .$$

Also $k=2$. Then an easy computation shows that the following vectors are highest vectors for $\mathfrak{m}$

$$X^s_{\alpha+\alpha_1}E^q\ ,\quad X_{\alpha+\alpha_3}E^q\ ,$$

with $q\in\boldsymbol{Q}$ and s a positive integer. Then expliciting the action of $X_{-\beta}$ we get the following highest vectors for the dual pair $\mathfrak{a}(\alpha)\times\mathfrak{m}$

$$X^s_{\alpha+\alpha_1}E^{q_i} \qquad \text{with} \quad q_1=\frac{3}{4},\ q_2=\frac{1}{4}-s$$

and also

$$X_{\alpha+\alpha_3}E^{\pm 1/4}\ .$$

Now $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2$ with α_1 the simple root of $\mathfrak{m}_1$ and α_3 the simple root of $\mathfrak{m}_2$.
The map ψ *is*

$$\psi(t\varpi_{\mathfrak{a}(\alpha)})=(t\varpi_{\mathfrak{m}_1},\ 2t\varpi_{\mathfrak{m}_2})$$

and for the Casimir operators we get

$$\Phi(\varpi(\mathfrak{a}(\alpha)))=\Phi(\varpi(\mathfrak{m}_1))$$

$$\frac{\Phi(\omega(\mathfrak{m}_2))}{4\|\varpi_{\mathfrak{m}_2}\|^2_{\mathfrak{m}_2}}-\frac{\Phi(\omega(\mathfrak{a}(\alpha)))}{\|\varpi_{\mathfrak{a}(\alpha)}\|^2_{\mathfrak{a}(\alpha)}}=\frac{3}{4}\ .$$

Finally suppose that $\mathfrak{g}$ *is of type* D_4.
We first have prove the analog of Proposition 4.3. We use the Chevalley basis given in [B-2]. In particular $\alpha=\alpha_2$. Proposition 4.3 remains valid provided we make the following substitutions. In the formula for $\pi(\Phi(X_{-\alpha}))$ replace

$$+D_1(\partial)E \qquad \text{by} \quad -\frac{\partial}{\partial X_{\alpha+\alpha_1}}\frac{\partial}{\partial X_{\alpha+\alpha_3}}\frac{\partial}{\partial X_{\alpha+\alpha_4}}E\ .$$

In the formula for $\pi(\Phi(X_{-\beta+\alpha}))$ replace

$$+R_1E^{-2} \qquad \text{by} \quad -X_{\alpha+\alpha_1}X^2_{\alpha+\alpha_3}X_{\alpha+\alpha_4}E^{-2} \ .$$

Also $k=3$. Now $\mathfrak{m}$ has 3 components, $\mathfrak{m}_1$ with simple root α_1, $\mathfrak{m}_3$ with simple root α_3 and $\mathfrak{m}_4$ with simple root α_4. The following vectors are highest vectors for $\mathfrak{m}$

$$X^s_{\alpha+\alpha_1}E^q \ , \quad X^s_{\alpha+\alpha_3}E^q \ , \quad X^s_{\alpha+\alpha_4}E^q$$

and they are also highest vectors for $\mathfrak{a}(\alpha)$ if

$$q=\frac{1}{2} \qquad \text{or} \qquad q=\frac{1}{2}-s \ .$$

So this time we put

$$\psi(t\varpi_{\mathfrak{a}(\alpha)})=(t\varpi_{\mathfrak{m}_1}, t\varpi_{\mathfrak{m}_3}, t\varpi_{\mathfrak{m}_4})$$

and we get

$$\Phi(\omega(\mathfrak{a}(\alpha)))=\Phi(\omega(\mathfrak{m}_1))=\Phi(\omega(\mathfrak{m}_3))=\Phi(\omega(\mathfrak{m}_4))$$

Note that in all cases there is at least one q such that E^q is a highest weight vector for $\mathfrak{a}(\alpha)\times\mathfrak{m}$ and in particular satisfies $\pi(\Phi(X_{-\beta}))E^q=0$. Then combining Proposition 5.6 and Proposition 5.8 we see that

$$c(\mathfrak{g})=q(1-q)+\frac{\langle\rho_{r_1}, \rho_{r_1}\rangle}{\langle\beta, \beta\rangle}-\frac{(k+1)(k+2)}{16} \ .$$

The C_n case is excluded. However the computation of $c(\mathfrak{g})$ can also be done in this case. Proposition 5.6 remains valid if we take $Q=0$ and $\pi(\Phi(X_{-\beta}))E^{1/4}=0$. This implies that $c(\mathfrak{g})=3/16$. The constant appearing in [J] is $-1/2c(\mathfrak{g})$.

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