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Hochschild and cyclic homology of Q-difference operators

By

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Introduction

Let k be an arbitrary field and $A=k[x, x^{-1}]$. The ring D_1 of formal differential operators on A is the k-algebra generated by x, x^{-1}, ∂ and the Heisenberg relation $\partial x - x\partial = 1$. This algebra was studied in [K], where the author built up an explicit 2-cocycle on the Lie algebra underlying D, which restricts to the 2-cocycle defining the Virasoro algebra. To carry out his plan Kassel computed the Hochschild homology of D by using a complex simpler than the canonical one. In this same work, the q-analogue of D, which is the algebra D_q ($q \in k\setminus \{0\}$) of q-difference operators on A, has been studied. By definition D_q is the algebra generated by x, x^{-1}, ∂ and the relation $\partial x - qx\partial = 1$ (which is the q-analogue of the Heisenberg relation). One of the main results proved in [K] is that the Hochschild homology of D_q is the homology of a complex $R_{*,*}(D_q)$ simpler than the canonical one of Hochschild. By using this complex Kassel obtained some partial results about the Hochschild homology $HH_1(D_q)$.

In this paper following the results of [K] we make a further step in the sense that we can compute the Hochschild and cyclic homologies of D_q (for definitions, basic notions and notations about these theories we remit to [L, Ch1, 2 and 3]). When $q=1$ or $q^n \neq 1$ for all $n \in \mathbf{N}$ these homologies are known (see [W], [K] and [G-G]). So, we can assume that q is a primitive m-th root of unity with $m>1$. The results that we obtain resemble the case when $q^n \neq 1$ for all $n \in \mathbf{N}$. In this sense, the case $q=1$ is a special one.

The paper is divided in two sections. In the first one we compute the Hochschild homology of D_q by means of the study of a natural filtration of $R_{*,*}(D_q)$. The graded complex associated to this filtration gives the Hochschild homology of the k-algebra generated by x, x^{-1}, ∂ and the relation $\partial x = qx\partial$. The first time that this homology (that plays a basic role in our work) has been studied was in [T], and has been fully computed in [G-G]. In the second section we give an explicit formula for the morphisms B_*: $\mathrm{HH}_0(D_q) \to \mathrm{HH}_1(D_q)$ and B_*: $\mathrm{HH}_1(D_q) \to \mathrm{HH}_2(D_q)$ (see [L, 2.1.7.4]). This fact together

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with the study of the Gysin-Connes exact sequence allows us to compute the cyclic homology of D_q.

1. Hochschild homology of D_q

Let k be an arbitrary field and $q \in k \setminus \{0\}$. In this section we compute the Hochschild homology of the k-algebra D_q generated by x, x^{-1}, ∂ and the relation $\partial x - qx\partial = 1$. We use the following standard notations. Given $q \in k \setminus \{0\}$ we put $(n)_q = \frac{q^n - 1}{q-1}$ for all $n \in \mathbf{Z}$ (of course, $(n)_1 = n$), $(0)!_q = 1$, $(n)!_q = \prod_{j=1}^{n-1} (n-j)_q$ for all $n \in \mathbf{N}$ and $\binom{n}{j}_q$ for the Gaussian binomial coefficients defined by

$$\binom{n}{j}_q = \begin{cases} 0 & \text{if } -j \in \mathbf{N},\, n \in \mathbf{Z} \\ 1 & \text{if } j = 0,\, n \in \mathbf{Z} \\ \prod_{i=1}^{j} \frac{(n-i+1)_q}{(i)_q} & \text{if } j \in \mathbf{N},\, n \in \mathbf{Z}\ . \end{cases}$$

From the equality $\partial x - qx\partial = 1$ it follows that

$$\partial^j x^n = \sum_{r+s=j} \binom{j}{r}_q \binom{n}{s}_q (s)!_q\, q^{nr-sr} x^{n-s} \partial^r\ .$$

In particular, if $q \neq 1$ is a root of unity of order m, then $\partial x^m = x^m \partial$ and $\partial^m x = x\partial^m$. In [K] it was proved that the Hochschild homology of D_q is the homology of the bicomplex

$$R_{*,*}(D_q): \quad \begin{array}{ccc} D_q & \xleftarrow{\delta_{1,1}} & D_q \Omega^1_{A/k} \\ \downarrow \phi_0 & & \downarrow \phi_1 \\ D_q & \xleftarrow{\delta_{0,1}} & D_q \Omega^1_{A/k'} \end{array}$$

where

$$\phi_0(x^n \partial^j) = (1-q^n) x^n \partial^{j+1} - (n)_q x^{n-1} \partial^j$$
$$\phi_1(x^n \partial^j dx) = (q^n - q^{-1}) x^n \partial^{j+1} dx + (n)_q x^{n-1} \partial^j dx$$
$$\delta_{0,1}(x^n \partial^j dx) = [x^n \partial^j, x] = (q^j - 1) x^{n+1} \partial^j + (j)_q x^n \partial^{j-1}$$
$$\delta_{1,1}(x^n \partial^j dx) = (x^n \partial^j x - q^{-1} x^{n+1} \partial^j) = (q^j - q^{-1}) x^{n+1} \partial^j + (j)_q x^n \partial^{j-1}\ .$$

It is easy to see that D_q is the Ore's extension $A[\partial, \alpha, \delta]$, where $A = k[x, x^{-1}]$ and α: $A \to A$ is the isomorphism that sends x to qx and δ the α-derivative that sends x to 1. So, the complex $R_{*,*}(D_q)$ is a particular case of the one obtained in Proposition 3.2 of [G-G] for an Ore's extension of a smooth algebra. Using this proposition together with Theorem 1.4 of the same paper, we obtain the quasi-isomorphism γ_*: $Tot(R_{*,*}(D_q)) \to \overline{C}_*(D_q)$, where $\overline{C}_*(D_q)$ is the normalized Hochschild complex, defined by

$$\gamma_0 = id \quad \gamma_1(x^n\partial^j) = x^n\partial^j \otimes \partial \ , \quad \gamma_1(x^n\partial^j dx) = x^n\partial^j \otimes x$$
$$\gamma_2(x^n\partial^j dx) = x^n\partial^j \otimes x \otimes \partial - q^{-1}x^n\partial^j \otimes \partial \otimes x \ .$$

Let us consider the filtration $\widetilde{F}^0_{*,*}(D_q) \subseteq \widetilde{F}^1_{*,*}(D_q) \subseteq \widetilde{F}^2_{*,*}(D_q) \subseteq \cdots \subseteq R_{*,*}(D_q)$ of $R_{*,*}(D_q)$ defined by $\widetilde{F}^r_{i,0}(D_q) = \oplus_{j=0}^{r-i} A.\partial^j$ and $\widetilde{F}^r_{i,1}(D_q) = (\oplus_{j=0}^{r-i} A.\partial^j)\,\Omega^1_{A/k}$ $(i=0, 1)$. The r-th component $\widetilde{G}_{*,*}(D_q) = \widetilde{F}^r_{*,*}(D_q)/\widetilde{F}^{r-1}_{*,*}(D_q)$ of the graded complex associated to this filtration is $A \overset{0}{\leftarrow} \Omega^1_{A/k}$ when $r=0$, and

$$\begin{array}{ccc} A & \overset{\delta^{r-1}_{1,1}}{\longleftarrow} & \Omega^1_{A/k} \\ \downarrow \phi^r_0 & & \downarrow \phi^r_1 \\ A & \overset{\delta^r_{0,1}}{\longleftarrow} & \Omega^1_{A/k'} \end{array}$$

when $r>0$, where

$$\phi^r_0(x^n) = (1-q^n)x^n \ , \quad \phi^r_1(x^n dx) = (q^n - q^{-1})x^n dx \ ,$$
$$\delta^r_{0,1}(x^n dx) = (q^r - 1)x^{n+1} \quad \text{and} \quad \delta^{r-1}_{1,1}(x^n dx) = (q^{r-1} - q^{-1})x^{n+1} \ .$$

When q is not a root of unity the complexes $\widetilde{G}^r_{*,*}(D_q)$ $(r>0)$ are exact. Hence, in this case the Hochschild and cyclic homologies of D_q and A coincide (see [G-G, Example 3.5]). On the other hand the Hochschild and cyclic homologies of D_1 were calculated in [W] when k is a characteristic zero field and in [K] in the general case (in spite of that Kassel works with the field **C** of the complex numbers his method works for arbitrary characteristic). So, we assume that q is a primitive m-th root of unity with $m>1$. To carry out the computation of the Hochschild homology of D_q we will need the following result

1.1. Lemma. *Let r be a multiple of m and $u \in \mathbf{Z}$. We have that* $x^{um}\partial^{r-1} \in \widetilde{F}^r_{1,0}(D_q)$, $x^{um-1}\partial^r dx \in \widetilde{F}^r_{0,1}(D_q)$ *and* $\sum_{s=1}^m \left(\frac{q}{1-q}\right)^{s-1} x^{um-s}\partial^{r-s}dx \in \widetilde{F}^r_{1,1}(D_q)$ *are cycles of* $Tot(R_{*,*}(D_q))$.

Proof. It is immediate that $\phi_0(x^{um}\partial^{r-1}) = 0$ and $\delta_{0,1}(x^{um-1}\partial^r dx) = 0$. In order to prove that $(\delta_{1,1} + \phi_1)\left(\sum_{s=1}^m \left(\frac{q}{1-q}\right)^{s-1} x^{um-s}\partial^{r-s}dx\right) = 0$ it suffices to observe that

$$(\delta_{1,1}+\phi_1)(x^{um-s}\partial^{r-s}dx) = (q^{-s}-q^{-1})x^{um-s+1}\partial^{r-s} + (-s)_q x^{um-s}\partial^{r-s-1}$$
$$+ (q^{-s}-q^{-1})x^{um-s}\partial^{r-s+1}dx + (-s)_q x^{um-s-1}\partial^{r-s}dx \ ,$$

which follows by direct computation.

1.2. Theorem. *Let $m>1$ and q be a primitive m-th root of unity. We have*:

$$HH_0(D_q)=\bigoplus_{i\in\mathbf{Z}}k.x^i \quad\oplus\quad \bigoplus_{u\in\mathbf{Z},v>0}k.x^{um}\partial^{vm}$$

$$HH_1(D_q)=\bigoplus_{i\in\mathbf{Z}}k.(x^i\otimes x)\oplus\bigoplus_{u\in\mathbf{Z},v>0}k.Y_{1,u,v}\oplus\bigoplus_{u\in\mathbf{Z},v>0}k.Y_{2,u,v}$$

$$HH_2(D_q)=\bigoplus_{u\in\mathbf{Z},v>0}k.Y_{3,u,v}$$

$$HH_n(D_q)=0\quad\forall n>2\ ,$$

where

$$Y_{1,u,v}=x^{um}\partial^{vm-1}\otimes\partial\ ,\quad Y_{2,u,v}=x^{um-1}\partial^{vm}\otimes x\quad and$$

$$Y_{3,u,v}=\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1}(x^{um-s}\partial^{vm-s}\otimes x\otimes\partial-q^{-1}x^{um-s}\partial^{vm-s}\otimes\partial\otimes x)\ .$$

Proof. It is clear that $HH_n(D_q)=0$ for all $n>2$. If $r>0$ is not a multiple of m, then $H_n(\widetilde{G}^r_{*,*}(D_q))=0$ for all $n\geq 0$. Hence $H_n(\widetilde{F}^r_{*,*}(D_q))=H_n(\widetilde{F}^{r-1}_{*,*}(D_q))$ for all $n\geq 0$. Now let $r>0$ be a multiple of m. In this case

$$H_0(\widetilde{G}^r_{*,*}(D_q))=\bigoplus_{u\in\mathbf{Z}}k.x^{um}$$

$$H_1(\widetilde{G}^r_{*,*}(D_q))=\bigoplus_{u\in\mathbf{Z}}k.x^{um}\oplus\bigoplus_{u\in\mathbf{Z}}k.x^{um-1}dx$$

$$H_2(\widetilde{G}^r_{*,*}(D_q))=\bigoplus_{u\in\mathbf{Z}}k.x^{um-1}dx\ .$$

Moreover, the canonical maps

$$\pi_1\colon H_1(\widetilde{F}^r_{*,*}(D_q))\to H_1(\widetilde{G}^r_{*,*}(D_q))\quad\text{and}\quad\pi_2\colon H_2(\widetilde{F}^r_{*,*}(D_q))\to H_2(\widetilde{G}^r_{*,*}(D_q))$$

are epimorphisms (this follows easily from Lemma 1.1). Hence, from the long exact sequence of homology associated to $0\to\widetilde{F}^{r-1}_{*,*}(D_q))\to\widetilde{F}^r_{*,*}(D_q)\to\widetilde{G}^r_{*,*}(D_q)\to 0$, we obtain

$$H_0(\widetilde{F}^r_{*,*}(D_q))=H_0(\widetilde{F}^{r-1}_{*,*}(D_q))\oplus\bigoplus_{u\in\mathbf{Z}}k.x^{um}\partial^r$$

$$H_1(\widetilde{F}^r_{*,*}(D_q))=H_1(\widetilde{F}^{r-1}_{*,*}(D_q))\oplus\bigoplus_{u\in\mathbf{Z}}k.x^{um}\partial^{r-1}\oplus\bigoplus_{u\in\mathbf{Z}}k.x^{um-1}\partial^r dx$$

$$H_2(\widetilde{F}^r_{*,*}(D_q))=H_2(\widetilde{F}^{r-1}_{*,*}(D_q))\oplus\bigoplus_{u\in\mathbf{Z}}k.\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1}x^{um-s}\partial^{r-s}dx\ .$$

So,

$$H_0(R_{*,*}(D_q))=\bigoplus_{i\in\mathbf{Z}}k.x^i\oplus\bigoplus_{u\in\mathbf{Z},v>0}k.x^{um}\partial^{vm}$$

$$H_1(R_{*,*}(D_q)) = \bigoplus_{i\in\mathbf{Z}} k.x^i dx \oplus \bigoplus_{u\in\mathbf{Z},v>0} k.x^{um}\partial^{vm-1} \oplus \bigoplus_{u\in\mathbf{Z},v>0} k.x^{um-1}\partial^{vm}dx$$

$$H_2(R_{*,*}(D_q)) = \bigoplus_{u\in\mathbf{Z},v>0} k.\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1} x^{um-s}\partial^{vm-s}dx\ .$$

Now, the proof can be finished applying γ_* to this homologies.

2. Cyclic homology of D_q

In this section we compute the cyclic homology of D_q. As in section 1 we assume that q is a primitive m-th root of unity with $m>1$. In this case we will carry out the promised computation giving an explicit formula for the morphisms B_*: $HH_0(D_q) \to HH_1(D_q)$ and B_*: $HH_1(D_q) \to HH_2(D_q)$ and studying the Gysin-Connes exact sequence associated to D_q. We first establish some results that will be needed later.

Preliminaries

2.1. Notation. Let (C_*, d_*) be a chain complex. Following [M] we will say that two cycles z and z' of C_n are homologous if they define the same element in $H_n(C_*)$. That is to say if $z-z'$ is a boundary.

2.2. Proposition. *Let q be a primitive m-th root of unity with $m>1$, $u\in$ $\mathbf{Z}$ and $v\in\mathbf{N}$. We havh:*

1) *The cycles $u(x^{(u-1)m}\partial^{vm-1}\otimes x^m\otimes\partial - x^{(u-1)m}\otimes\partial^{vm-1}\otimes\partial\otimes x^m)$ and $B(x^{um}\partial^{vm-1}\otimes\partial)$ of $\overline{C}_*(D_q)$ are homolrgous.*

2) *The cycles $v(x^{um-1}\partial^{(v-1)m}\otimes\partial^m\otimes x - x^{um-1}\partial^{(v-1)m}\otimes x\otimes\partial^m)$ and $B(x^{um-1}\partial^{vm}\otimes x)$ of $(\overline{C}_*(D_q)$ are homologous.*

Proof. 1) Let $D=k[x, x^{-1}, y]$ be the k-algebra generated by x, x^{-1}, y and the relation $xy=yx$. It is easy to see that the morphism

$$\psi_2 \colon D\otimes\overline{D}^{\otimes^2} \to \Omega^2_{D/k}\ , \quad \psi_2(d_0\otimes d_1\otimes d_2) = d_0\frac{\partial d_1}{\partial x}\frac{\partial d_2}{\partial y}$$

induces an isomorphism from $HH_2(D)$ onto $\Omega^2_{D/k}$, which is the inverse of the canonical isomorphism ε_2: $\Omega^2_{D/k} \to HH_2(D)$. Let T be the cycle $u(x^{u-1}y^{vm-1}\otimes x \otimes y - x^{u-1}y^{vm-1}\otimes y\otimes x) - B(x^u y^{vm-1}\otimes y)$ of $\overline{C}_*(D)$. As $\psi_2(T)=0$, T is a boundary. Let f: $D\to D_q$ be the morphism of algebras that sends x to x^m and y to ∂. The proof can be finished by observing that $H_2(f)(T) = u(x^{(u-1)m}\partial^{vm-1}\otimes x^m\otimes\partial - x^{(u-1)m}\partial^{vm-1}\otimes\partial\otimes x^m) - B(x^{um}\partial^{vm-1}\otimes\partial)$.
2) is similar.

2.3. Proposition. *Let $m>1$ and q be a primitive m-th root of unity.*

The cycles $\partial^{m-1}\otimes x^m\otimes\partial-\partial^{m-1}\otimes\partial\otimes x$, $x^{m-1}\otimes x\otimes\partial^m-x^{m-1}\otimes\partial^m\otimes x$ *and* $mq\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1}(qx^{m-s}\partial^{m-s}\otimes x\otimes\partial-q^{-1}x^{m-s}\partial^{m-s}\otimes\partial\otimes x)$ *of* $\overline{C}_*(D_q)$ *are homologous.*

To prove this proposition we will need the following lemmas

2.4. Lemma. *The cycles*

$$\sum_{s=1}^{m}\left(\frac{1}{1-q}\right)^{s-1}q^{m-s+1}\partial^{m-s}x^{m-s}dx \quad\text{and}\quad q\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1}x^{m-s}\partial^{m-s}dx$$

of $R_{*,*}(D_q)$ *are equal.*

Proof. Using that $\partial x=qx\partial+1$ we can see that there exist $c_s\in k$ $(2\leq s\leq m)$ such that

$$\sum_{s=1}^{m}\left(\frac{1}{1-q}\right)^{s-1}q^{m-s+1}\partial^{m-s}x^{m-s}dx=qx^{m-1}\partial^{m-1}dx+\sum_{s=2}^{m}c_s x^{m-s}\partial^{m-s}dx\ .$$

On the other hand, in the proof of Theorem 1.2 we showed that

$$Z_2(R_{*,*}(D_q))=H_2(R_{*,*}(D_q))=\bigoplus_{u\in\mathbf{Z},v>0}k.\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1}x^{um-s}\partial^{vm-s}dx\ ,$$

where $Z_2(R_{*,*}(D_q))$ is the submodule of the cycles of degree 2 of $R_{*,*}(D_q)$. Consequently there exist $\lambda_{u,v}$ $(u\in\mathbf{Z},\ v\in\mathbf{N})$, with $\{(u,v):\lambda_{u,v}\neq 0\}$ finite, such that

$$\begin{aligned}&qx^{m-1}\partial^{m-1}dx+\sum_{s=2}^{m}c_s x^{m-s}\partial^{m-s}dx\\&=\sum_{u\in\mathbf{Z},v\in N}\lambda_{u,v}\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1}x^{um-s}\partial^{vm-s}dx\ ,\end{aligned}$$

From this fact we can easily deduce that $\lambda_{u,v}=0$ if $(u,v)\neq(1.1)$, $\lambda_{1,1}=q$ and $c_s=q\left(\frac{q}{1-q}\right)^{s-1}$ for all $s\geq 1$.

2.5. Lemma. *Let* B_q *be the algebra generated by* x, ∂ *and the relation* $\partial x-qx\partial=1$. *The cycles* $mq\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1}(x^{m-s}\partial^{m-s}\otimes x\otimes\partial-q^{-1}x^{m-s}\partial^{m-s}\otimes\partial\otimes x)$ *and* $\partial^{m-1}\otimes x^m\otimes\partial-\partial^{m-1}\otimes\partial\otimes x^m$ *of* $\overline{C}_*(B_q)$ *are homologous.*

Proof. It is straightforward from the definition of b, the fact that q is a primitive m-th root of unity and the equality

$$\partial^j x^n=\sum_{r+s=j}\binom{j}{r}_q\binom{n}{s}_q(s)!_q\,q^{nr-sr}x^{n-s}\partial^r$$

that

$$b\Big(\sum_{i=1}^{m-1}(x^{m-i-1}\partial^{m-1}\otimes\partial\otimes x^i\otimes x-q^ix^{m-i-1}\partial^{m-1}\otimes x^i\otimes\partial\otimes x$$
$$+q^{i+1}x^{m-i-1}\partial^{m-1}\otimes x^i\otimes x\otimes\partial)\Big)$$

equals to

$$\partial^{m-1}\otimes\partial\otimes x^m-\partial^{m-1}\otimes x^m\otimes\partial$$
$$+q\sum_{s=0}^{m-1}\Big(\sum_{i=0}^{m-1}\binom{m-1}{m-s-1}_q\binom{i}{s}_q(s)!_q\,q^{s(s+1-i)}\Big)$$
$$.\Big(q^{-1}x^{m-s-1}\partial^{m-s-1}\otimes\partial\otimes x-x^{m-s-1}\partial^{m-s-1}\otimes x\otimes\partial\Big)\ .$$

So, to finish the proof it suffices to see that

$$\sum_{i=0}^{m-1}\binom{m-1}{m-s-1}_q\binom{i}{s}_q(s)!_q\,q^{s(s+1-i)}=m\Big(\frac{q}{1-q}\Big)^s\qquad(0\leq s<m)\ .$$

This fact can be proved easily by induction on s using that:

$$A(s+1,h)=\frac{q^h}{1-q}(qA(s,h)-A(s,h+1))\quad(h\geq 0,\ 0\leq s<m-1)\ ,$$

where

$$A(s,h)=\sum_{i=0}^{m-1}\binom{m-1}{m-s-1}_q\binom{i}{s}_q(s)!_q\,q^{(s+h)(s+1-i)}\qquad(h\geq 0,\ 0\leq s<m)\ .$$

Proof of Proposition 2.3. By Lemma 2.5 the first and third cycles are homologous. Let $B_{q^{-1}}$ be the algebra generated by y, δ and the relation $\delta y-q^{-1}y\delta=1$. Let us consider the morphism of algebras f: $B_{q^{-1}}\rightarrow B_q$ defined by $f(\delta)=-qx$ and $f(y)=\partial$. This map induces a morphism $HH_*(f)$ from $HH_*(B_{q^{-1}})$ to $HH_*(B_q)$. Applying Lemma 2.5 to $B_{q^{-1}}$ we deduce that $x^{m-1}\otimes x^m\otimes\partial^m-x^{m-1}\otimes\partial^m\otimes x=H_*(f)\,((-1)^{m+1}(\delta^{m-1}\otimes y^m\otimes\delta-\delta^{m-1}\otimes\delta\otimes y^m))$ is homologous to

$$HH_*(f)\Big((-1)^{m+1}mq\sum_{s=1}^{m}\Big(\frac{q^{-1}}{1-q^{-1}}\Big)^{s-1}(y^{m-s}\delta^{m-s}\otimes y\otimes\delta$$
$$-qy^{m-s}\delta^{m-s}\otimes\delta\otimes y)\Big)=mq\sum_{s=1}^{m}\Big(\frac{1}{1-q}\Big)^{s-1}q^{m-s+1}(\partial^{m-s}x^{m-s}\otimes x\otimes\partial$$
$$-q^{-1}\partial^{m-s}x^{m-s}\otimes\partial\otimes x)=\gamma_2\Big(m\sum_{s=1}^{m}\Big(\frac{1}{1-q}\Big)^{s-1}q^{m-s+1}\partial^{m-s}x^{m-s}dx\Big)\ ,$$

where $\gamma_2\colon D_q\Omega^1_{A/k}\to D_q\otimes\overline{D}_q^{\otimes 2}$ is the morphism defined at the beginning of section 1. Hence, by Lemma 2.4, $x^{m-1}\otimes x\otimes\partial^m - x^{m-1}\otimes\partial^m\otimes x$ is homologous to

$$\gamma_2\Big(mq\sum_{s=1}^{m}\Big(\frac{q}{1-q}\Big)^{s-1}x^{m-s}\partial^{m-s}dx\Big)$$
$$=mq\sum_{s=1}^{m}\Big(\frac{q}{1-q}\Big)^{s-1}(x^{m-s}\partial^{m-s}\otimes x\otimes\partial - q^{-1}x^{m-s}\partial^{m-s}\otimes\partial\otimes x)\ .$$

Computation of the cyclic homology

2.6. Notation. Given a cycle z of degree n of a chain complex (C_*, d_*) we will denote $[z]$ the class of z in $H_n(C_*)$.

2.7. Theorem. *Let q be a primitive root of unity with $m>1$, $n, u\in\mathbf{Z}$ and $v\in\mathbf{N}$. We have:*
1) *The morphism $B_*\colon HH_0(D_q)\to HH_1(D_q)$ is given by:*

$$B_*([x^n]) = [nx^{n-1}\otimes x] \quad \text{and}$$
$$B_*([x^{um}\partial^{vm}]) = [vmx^{um}\partial^{vm-1}\otimes\partial + umx^{um-1}\partial^{vm}\otimes x]\ .$$

2) *The morphism $B_*\colon HH_1(D_q)\to HH_2(D_q)$ is given by:*

$$B_*([x^n\otimes x])=0\ ,\quad B_*([x^{um}\partial^{vm-1}\otimes\partial])=umq.Y_{3,u,v}\ ,$$
$$B_*([x^{um-1}\partial^{vm}\otimes x]) = -vmq.Y_{3,u,v}\ ,$$

where $Y_{3,u,v}$ is as in Theorem 1.2.

Proof. 1) It is a direct consequence of the following equalities:

$$1\otimes x^n = nx^{n-1}\otimes x - b\Big(\sum_{j=0}^{n-1}x^j\otimes x\otimes x^{n-j-1}\Big)\qquad \forall n\geq 0\ ,$$
$$1\otimes x^n = nx^{n-1}\otimes x + b\Big(\sum_{j=1}^{-n}x^{-j}\otimes x\otimes x^{n+j-1}\Big)\qquad \forall n<0\ ,$$
$$1\otimes x^{um}\partial^{vm} = vmx^{um}\partial^{vm-1}\otimes\partial - umx^{um-1}\partial^{vm}\otimes x - b\Big(\sum_{j=0}^{um-1}Z_{1,j,u,v}\Big) - b(Z_{2,u,v})\qquad \forall u\geq 0\ ,$$
$$1\otimes x^{um}\partial^{vm} = vmx^{um}\partial^{vm-1}\otimes\partial - umx^{um-1}\partial^{vm}\otimes x + b\Big(\sum_{j=um}^{-1}Z_{1,j,u,v}\Big) - b(Z_{2,u,v})\qquad \forall u<0\ ,$$

where $Z_{1,j,u,v}=x^j\otimes x\otimes x^{um-j-1}\partial^{vm}$ and $Z_{2,u,v}=\sum_{j=0}^{vm-1}x^{um}\partial^j\otimes\partial\otimes\partial^{vm-j-1}$.

2) Let $f\colon k[x] \to D_q$ be the canonical inclusion. Since $HH_2(k[x])$ is null and $B_*([x^n \otimes x]) \in HH_2(f)(HH_2(k[x]))$, it is clear that $B_*([x^n \otimes x]) = 0$. Let $Z(D_q)$ be the center of D_q. Recall that $HH_2(D_q)$ is a $Z(D_q)$-module. By Proposition 2.2, the fact that $x^{(u-1)m}\partial^{(v-1)m} \in Z(D_q)$ and Proposition 2.3, we have that

$$\begin{aligned}
&B_*([x^{um-1}\partial^{vm} \otimes x]) \\
&\quad = [v(x^{um-1}\partial^{(v-1)m} \otimes \partial^m \otimes x - x^{um-1}\partial^{(v-1)m} \otimes x \otimes \partial^m)] \\
&\quad = vx^{(u-1)m}\partial^{(v-1)m}[x^{m-1} \otimes \partial^m \otimes x - x^{m-1} \otimes x \otimes \partial^m] \\
&\quad = -vx^{(u-1)m}\partial^{(v-1)m}mq.Y_{3,1,1} = -vmq.Y_{3,u,v}
\end{aligned}$$

and

$$\begin{aligned}
&B_*([x^{um}\partial^{vm-1} \otimes \partial]) \\
&\quad = [u(x^{(u-1)m}\partial^{vm-1} \otimes x^m \otimes \partial - x^{(u-1)m}\partial^{vm-1} \otimes \partial \otimes x^m)] \\
&\quad = ux^{(u-1)m}\partial^{(v-1)m}[\partial^{m-1} \otimes x^m \otimes \partial - \partial^{m-1} \otimes \partial \otimes x^m] \\
&\quad = ux^{(u-1)m}\partial^{(v-1)m}mq.Y_{3,1,1} = umq.Y_{3,u,v}\ .
\end{aligned}$$

2.8. Theorem. *Let $p \geq 0$ be the characteristic of k. Then,*

$$HC_0(D_q) = \bigoplus_{n \in \mathbf{Z}} kx^n \oplus \bigoplus_{u \in \mathbf{Z}, v > 0} kx^{um}\partial^{vm}$$

$$HC_1(D_q) = \bigoplus_{n \in \mathbf{Z}, p/n} k.(x^{n-1} \otimes x) \oplus \bigoplus_{u \in \mathbf{Z}, v > 0} \frac{k.Y_{1,u,v} \oplus k.Y_{2,u,v}}{\langle vm.Y_{1,u,v} + um.Y_{2,u,v} \rangle}$$

$$HC_2(D_q) = \bigoplus_{n \in \mathbf{Z}, p/n} kx^n \oplus \bigoplus_{\substack{u \in \mathbf{Z}, v > 0 \\ p/u, v}} kx^{um}\partial^{vm} \oplus \bigoplus_{\substack{u \in \mathbf{Z}, v > 0 \\ p/u, v}} k.Y_{3,u,v}$$

$$HC_3(D_q) = \bigoplus_{u \in \mathbf{Z}, p/n} k.(x^{n-1} \otimes x) \oplus \bigoplus_{\substack{u \in \mathbf{Z}, v > 0 \\ p/u, v}} k.Y_{1,u,v} \oplus \bigoplus_{\substack{u \in \mathbf{Z}, v > 0 \\ p/u, v}} k.Y_{2,u,v}$$

$$HC_{2i}(D_q) = HC_2(D_q) \quad \text{and} \quad HC_{2i+1}(D_q) = HC_3(D_q) \qquad \forall i > 1\ ,$$

where $Y_{1,u,v}$, $Y_{2,u,v}$ and $Y_{3,u,v}$ are as in Theorem 1.2. In particular, if $p=0$, then $HC_1(D_q) = k \oplus \bigoplus_{u \in \mathbf{Z}, v > 0} k.(x^{um-1}\partial^{vm} \otimes x)$ and $HC_n(D_q) = k$ for all $n > 1$.

Proof. From the Gysin-Connes exact sequence we obtain the exact sequences

$$HH_0(D_q) \xrightarrow{B_*} HH_1(D_q) \longrightarrow HC_1(D_q) \longrightarrow 0\ ,$$

$$HH_1(D_q) \xrightarrow{B_*} HH_2(D_q) \longrightarrow HC_2(D_q) \longrightarrow HH_0(D_q) \xrightarrow{B_*} HH_1(D_q)\ ,$$

$$0 \longrightarrow HC_3(D_q) \longrightarrow HC_1(D_q) \xrightarrow{B} HH_2(D_q)\ ,$$

$$0 \longrightarrow HC_n(D_q) \xrightarrow{S} HC_{n-2}(D_q) \longrightarrow 0 \qquad \forall n > 3$$

Now the result follows easily from Theorems 1.2 and 2.7.

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