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Ornstein-Uhlenbeck semigroup and fourier transform acting on positive finite measures on the schwartz space

By

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1. Introduction

Let $(\mathcal{S}^*, \mu)$ be the White noise space, that is, $\mathcal{S}^*$ is the space of Schwartz distributions on $\mathbf{R}$, $\mathcal{S}$ is the space of its testing functions ($\mathcal{S} \subset L_2(\mathbf{R}, du) \subset \mathcal{S}^*$) and μ is a Gaussian measure on $\mathcal{S}^*$ defined by

$$\int_{\mathcal{S}^*} e^{\sqrt{-1}(\xi, x)} \mu(dx) = e^{-\frac{1}{2}|\xi|^2} = e^{-\frac{1}{2}\int_{\mathbf{R}} \xi(u)^2 du}, \quad \xi \in \mathcal{S},$$

where $(\cdot, \cdot)$ is the canonical bilinear from on $\mathcal{S} \otimes \mathcal{S}^*$. We consider the following semigroup on $L^2(\mu)$.

$$T_t^A F(x) = \int_{\mathcal{S}^*} F(e^{-tA}x + \sqrt{1-e^{-2tA}}y) \mu(dy),$$

where $A = 1 + u^2 - \frac{d^2}{du^2}$, which is a positive definite self adjoint operator on $L_2(\mathbf{R}, du)$. Let $(B, H, \tilde{\mu})$ be an abstract Wiener space. Then $\{T_t^A\}_{t>0}$ is a special case of a generalized Ornstein-Uhlenbeck semigroup introduced by I. Shigekawa [5] when $(\mathcal{S}^*, \mu)$ is replaced by $(B, \tilde{\mu})$, and the original Ornstein-Uhlenbeck semigroup $\{T_t\}_{t>0}$ is the case where A is replaced by the identity operator. It is an interesting feature of these semigroups that, roughly speaking, $T_t^A F(x)$ (or $T_t F(x)$) is a smooth function in x for any $t > 0$ and that F is approximated by $T_t^A F$ (or $T_t F$) (see for example H. Sugita [7] Lemma 2.2). In this article, we will show that $\{T_t^A\}_{t>0}$ satisfies these properties when they act on positive finite measures on $\mathcal{S}^*$, and using this semigroup, we shall give inversion formulae of Fourier transform of positive finite measures on $\mathcal{S}^*$.

Let $(\mathcal{S})^*$ be the space of Generalized White noise functionals, $(\mathcal{S})$ be the space of its testing functionals and $\langle \cdot, \cdot \rangle$ denote the canonical bilinear form on $(\mathcal{S}) \times (\mathcal{S})^*$. They were defined in [1] or [2] for example, and we will de-

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fine them in a generalized form in Section 2. On the smoothness of $F\in(\mathcal{S})$, Yu.-J. Lee [1] proved that F is $\mathcal{S}^*$-continuous and, for any $x, y\in\mathcal{S}^*$, $y(\in\mathbf{R})$ $\mapsto F(x+uy)$ can be extended to an entire function in $u\in\mathbf{C}$. We call such a function *$\mathcal{S}^*$-analytic function*. For any $t>0$, the operator T_t^A can be continuously extended on $(\mathcal{S})^*$, but for a generalized White noise functional $\phi\in(\mathcal{S})^*$, $T_t^A\phi$ is far from being a smooth function, it is not a measure on $\mathcal{S}^*$ in general (Example 4.1). However if $\phi\in(\mathcal{S})^*$ is a positive finite measure (if there exists a positive finite measure ν on $\mathcal{S}^*$ such that $\langle F, \phi\rangle=\int_{\mathcal{S}^*}F(x)\nu(dx)$ for any $F\in(\mathcal{S})$), we will show that $T_t^A\phi$ is $\mathcal{S}^*$-analytic for any $t>0$ (Theorem 2.2).

Next we extend the semigroup $\{T_t^A\}_{t>0}$ for a general positive finite measure ν on $\mathcal{S}^*$. In Prososition 2.1, we will prove that, for any $t>0$, there exists a continuous function $D_t(\cdot,\cdot):\mathcal{S}^*\times\mathcal{S}^*\longrightarrow\mathbf{R}^+$ such that $\|D_t(\cdot, y)\|_{L_1(\mu)}=1$ for any $y\in\mathcal{S}^*$ and

$$\int_{\mathcal{S}^*}F(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\mu(dy)=\int_{\mathcal{S}^*}F(y)D_t(x, y)\mu(dy)$$

for any bounded $\mathcal{S}^*$-continuous function F. Then we define, for a positive finite measure ν on $\mathcal{S}^*$ and $t>0$,

$$T_t^A\nu(x)\equiv\int_{\mathcal{S}^*}D_t(x, y)\nu(dy)$$

as an $L_1(\mu)$-function (Definition 2.1). When $\nu(dx)=F(x)\mu(dx)$ ($F\in L_p(\mu)$, $p\geq 1$), $T_t^A\nu$ will be also denoted by T_t^AF.

For any $t>0$, T_t^A satisfies the following (St-1) ~ (St-3) as an operator on $L_p(\mu)$ ($p>1$).

(St-1) $\|T_t^AF\|_{L_p(\mu)}\leq\|F\|_{L_p(\mu)}$ for any $F\in L_p(\mu)$.

(St-2) T_t^A is a self-adjoint operator on $L^2(\mu)$.

(St-3) $T_t^AF\geq 0$ if $F\geq 0$.

(St-4) $T_t^A1=1$.

In the case where $F\in L_p(\mu)$ ($p>1$), E. M. Stein [6] proved that T_t^AF converges to F μ-almost everywhere if the $L_2(\mu)$-continuous semigroup $\{T_t^A\}_{t>0}$ satisfies the above (St-1) ~ (St-4).

Let N be a fixed natural number, μ_N be a standard Gaussian measure on $\mathbf{R}^N$ and $\tilde{A}$ be a positive definite symmetric matrix. Then there exists $\tilde{D}_t(x, y)$ ($x, y\in\mathbf{R}^N$) such that

$$\int_{\mathbf{R}^N}F(e^{-t\tilde{A}}x+\sqrt{1-e^{-2t\tilde{A}}}y)\mu_N(dy)=\int_{\mathbf{R}^N}F(y)\tilde{D}_t(x, y)\mu_N(dy)$$

for any bounded continuous function F on $\mathbf{R}^N$ and $t>0$. As an analogy of T_t^A, we define for a positive finite measure $\tilde{\nu}$ on $\mathbf{R}^N$ and $t>0$,

$$\widetilde{T}_t^A \tilde{\nu}(x) \equiv \int_{\mathbf{R}^N} \tilde{D}_t(x, y)\, \tilde{\nu}(dy) \in L_1(\mu).$$

Let $\frac{d\tilde{\nu}}{d\mu_N}$ be the Radon Nikodym derivative in the sense of the Lebesgue decomposition. Then, using the Global Density Theorem proved by H. Sato [4], we can show that $\widetilde{T}_t^A \tilde{\nu}(x)$ converges to $\frac{d\tilde{\nu}}{d\mu_N}(x)$ μ_N-almost everywhere.

For a positive finite measure on $\mathcal{S}^*$, we shall prove the following theorem.

Theorem 1.1. *Let ν be a positive finite measure on $\mathcal{S}^*$. Then*

(a) $T_t^A \nu(x) \mu(dx)$ *converges to* $\nu(dx)$ *weakly as* $t \longrightarrow 0$.

(b) $T_t^A \nu(x)$ *converges to* $\frac{d\nu}{d\mu}(x)$ *in the measure* μ *as* $t \longrightarrow 0$.

It is still an open problem that $T_t^A \nu(x)$ converges to $\frac{d\nu}{d\mu}(x)$ μ-almost everywhere or not.

Next we consider the following Fourier transform.

Definition 1.1. *Let $\mathcal{M}_+$ be the family of all positive finite measures on $\mathcal{S}^*$. We define a Fourier transform $\mathcal{F}$ on $\mathcal{M}_+ \cup (\mathcal{S})^*$ as follows.*

(a) *For* $\nu \in \mathcal{M}_+$,

$$\mathcal{F}[\nu](x) \equiv e^{\frac{1}{2}|x|^2} \int_{\mathcal{S}^*} e^{\sqrt{-1}(x,y)} \nu(dy), \quad x \in \mathcal{S}.$$

(b) *For* $\Phi \in (\mathcal{S})^*$,

$$\mathcal{F}[\Phi](\mathrm{x}) \equiv e^{\frac{1}{2}|x|^2} \langle e^{\sqrt{-1}(x,\cdot)}, \Phi \rangle.$$

When $\nu(dy) = F(y)\mu(dy)$, $\mathcal{F}[\nu]$ *is also denoted by* $\mathcal{F}[F]$.

It is easy to show that $\mathcal{F}[\nu](x)$ (in (a)) and $\mathcal{F}[\phi](x)$ (in (b)) are $\mathcal{S}$-continuous functions in x for any $\nu \in \mathcal{M}_+$ and $\phi \in (\mathcal{S})^*$. If a positive finite measure ν belongs to $(\mathcal{S})^*$, (a) and (b) are identical with each other (Proposition 3.1). Thus $\mathcal{F}$ is well defined on $\mathcal{M}_+ \cup (\mathcal{S})^*$. The above $\mathcal{F}$ is an extension of the Fourier transform on $L_1(\mu)$ defined by H. Sato [3]. He gave an inversion formula of this transform for an element of $L_1(\mu)$ ([3]), and Yu.-J. Lee [1] proved that $(\mathcal{S}) \oplus \sqrt{-1}(\mathcal{S})$ is invariant under this transform.

Let $\mathbf{F}$ be a finite dimensional Fourier transform for a positive finite measure $\tilde{\nu}$ on $\mathbf{R}^N$ defined by

$$\mathbf{F}[\tilde{\nu}](u) = \int_{\mathbf{R}^N} e^{\sqrt{-1}(u,v)} \nu(du).$$

Let λ be the Lebesgue measure on $\mathbf{R}^N$. Then $\mathbf{F}[\tilde{\nu}](u) e^{-\varepsilon|u|^2}$ is λ-integrable for any $\varepsilon > 0$. Therefore we can define

$$\tilde{\nu}^{(\varepsilon)}(u) \equiv \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} e^{-\sqrt{-1}(u,v)} \mathbf{F}[\tilde{\nu}](v) e^{-\varepsilon|u|^2} \lambda(dv).$$

Then $\tilde{\nu}^{(\varepsilon)}(u)\lambda(du)$ converges weakly to $\tilde{\nu}(du)$, and using the Global Density Theorem again (H. Sato [4]), we obtain the λ-almost everywhere convergence of $\tilde{\nu}^{(\varepsilon)}(u)$ to $\frac{d\tilde{\nu}}{d\lambda}(u)$ when $\varepsilon \longrightarrow 0$.

In our infinite dimensional case, we have the following theorem.

Theorem 1.2. *Let ν be a positive finite measure on $\mathcal{S}^*$, $\{e_n\}_{n\in\mathbf{N}}$ be the CONS composed of the eigenvectors of A and set $P_N x \equiv \sum_{n=0}^N (e_n, x) e_n$ for any $N \in \mathbf{N}$. Then we have*

$$T_t^A \nu(x) = L_1 - \lim_{N\to\infty} \int_{\mathcal{S}^*} e^{-\sqrt{-1}(P_N x, y)} \mathcal{F}[\nu](e^{-tA} y) \mu(dy)$$

for any $t > 0$.

Summing up Theorems 1.1 and 1.2, we have obtained inversion formulae of the Fourier transform $\mathcal{F}$ for a positive finite measure on $\mathcal{S}^*$.

Recall that $(\mathcal{S}) \oplus \sqrt{-1}\,(\mathcal{S})$ is invariantunder $\mathcal{F}$. However $(\mathcal{S})^* \oplus \sqrt{-1}\,(\mathcal{S})^*$ can not be invariant under this transform. We will give an example of $\Phi \in (\mathcal{S})^*$ which satisfies that $\mathcal{F}[\Phi]$ is a nonnegative $\mathcal{S}^*$-continuous function but $\mathcal{F}[\Phi]$ does not belong to $(\mathcal{S})^*$ (Example 4.1).

2. Smoothness of $T_t^A \nu$

In order to simplify the arguments, we generalize the spaces $\mathcal{S} \subset L_2(\mathbf{R}) \subset \mathcal{S}^*$ and $(\mathcal{S}) \oplus L_2(\mu) \subset (\mathcal{S})^*$. Let $H = (H, |\cdot|)$ be a real separable Hilbert space and $\{e_n\}_{n\in\mathbf{N}}$ be a CONS of H. Define a symmetric positvie definite operator A on H as follows.

$$Ax \equiv \sum_{n=0}^{\infty} \lambda_n (e_n, x) e_n$$

where $(\cdot, \cdot)$ is the inner product of H and $\{\lambda_n\}_{n\in\mathbf{N}}$ is a sequence of positive numbers satisfying

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} < \infty \quad \text{and} \quad \inf_{n\geq 0} \lambda_n > 1.$$

Let $\mathcal{P}$ be the linear span of $\{e_n\}_{n\in\mathbf{N}}$ and consider the norms $\{|\cdot|_p\}_{p\in\mathbf{Z}}$ defined by

$$|x|_p \equiv |A^p x| \quad (x \in \mathcal{P}).$$

Define

$$\mathcal{S}_p \equiv \overline{\mathcal{P}}^{|\cdot|_p} \quad (p \in \mathbf{Z}),$$

$$\mathcal{S} \equiv \cap_{p \in \mathbf{Z}} \mathcal{S}_p, \quad \mathcal{S}^* \equiv \cup_{p \in \mathbf{Z}} \mathcal{S}_p$$

and let μ be a Gaussian measure on $\mathcal{S}^*$ defined by

$$\int_{\mathcal{S}^*} e^{\sqrt{-1}(\xi, x)} \mu(dx) = e^{-\frac{1}{2}|\xi|^2}, \quad \zeta \in \mathcal{S}.$$

Let $\mathbf{N}_0^\infty$ be the family of all sequences of non-negative integers $\alpha = \{\alpha_j\}_{j \in \mathbf{N}}$ such that $\alpha_j = 0$ except for finite j's. We prepare the following notations for a multi index $\alpha = \{\alpha_j\}_{j \in \mathbf{N}} \in \mathbf{N}_0^\infty$.

$$|\alpha| \equiv \sum_j \alpha_j, \quad \alpha! \equiv \prod_j \alpha_j!.$$

For any $n \in \mathbf{N}$, let $h_n(u)$ be a Hermitian polynomial defined by

$$h_n(u) = (-1)_n \left(\frac{d^n}{du^n} e^{-\frac{1}{2}u^2} \right) e^{\frac{1}{2}u^2},$$

and define

$$\mathbf{h}_\alpha(x) \equiv \frac{1}{\sqrt{\alpha!}} \prod_j h_{\alpha_j}((e_j, x))$$

for any $\alpha \in \mathbf{N}_0^\infty$. Then $\{\mathbf{h}_\alpha\}_{\alpha \in \mathbf{N}_0^\infty}$ is a CONS of $L_2(\mu)$.

We consider the operator $\mathbf{A}$ on $L_2(\mu)$ defined by

$$\mathbf{A}F \equiv \sum_{\alpha \in \mathbf{N}_0^\infty} (\prod_j \lambda_j^{\alpha_j}) \langle \mathbf{h}_\alpha, F \rangle \mathbf{h}_\alpha$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L_2(\mu)$. Let $(\mathcal{P})$ be the linear span of $\{\mathbf{h}_\alpha\}_{\alpha \in \mathbf{N}_0^\infty}$. Then $\mathbf{A}F$ is well defined for any $F \in (\mathcal{P})$. Define

$$\|F\|_p \equiv \|\mathbf{A}^p F\|_{L_2(\mu)} \quad (p \in \mathbf{Z}),$$
$$(\mathcal{S}_p) \equiv \overline{(\mathcal{P})}^{\|\cdot\|_p}, \quad (\mathcal{S}) \equiv \cap_p (\mathcal{S}_p), \quad (\mathcal{S})^* \equiv \cup_p (\mathcal{S}_p).$$

When $H = L_2(\mathbf{R}, du)$ and $A = 1 + u^2 - \frac{d^2}{du^2}$, $(\mathcal{S})^*$ is identical with the space of generalized White noise functional defined in, for example, [1] or [2]. On the spaces $\mathcal{S}$ and $(\mathcal{S})$, we consider the projective limit topology, and the inductive limit topology on the spaces $\mathcal{S}^*$ and $(\mathcal{S})^*$.

Before giving the definition of T_t^A for a positive finite measure on $\mathcal{S}^*$, we prepare the following proposition.

Proposition 2.1. *Fix $t > 0$. Then for any $F \in (\mathcal{P})$, we have*

$$\int_{\mathcal{S}^*} F(e^{-tA}x + \sqrt{1 - e^{-2tA}}y) \mu(dy) = \int_{\mathcal{S}^*} F(y) D_t(x, y) \mu(dy), \tag{1}$$

where

$$D_t(x, y) = \left(\prod_j \frac{1}{\sqrt{1 - e^{-2t\lambda_j}}} \right) e^{-\frac{1}{2}\left|\frac{e^{-tA}}{\sqrt{1-e^{-2tA}}}x\right|^2 - \frac{1}{2}\left|\frac{e^{-tA}}{\sqrt{1-e^{-2tA}}}y\right|^2 + \left(\frac{e^{-tA}}{1-e^{-2tA}}x, y\right)},$$

which satisfies $\|D_t(\cdot, y)\|_{L_1(\mu)}=1$ *for any* $y\in\mathcal{S}^*$.

Proof. It is sufficient to prove the equality for F defined by

$$F(x)\equiv\prod_j f_j((e_j, x))$$

where $f_j=1$ except for finite j's. Fix $x\in\mathcal{S}^*$ and set $u_j\equiv(e_j, x)$, then

$$\int_{\mathcal{S}^*}F(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\mu(dy)$$
$$=\prod_j\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f_j(e^{-t\lambda_j}u_j+\sqrt{1-e^{-2t\lambda_j}}v_j)e^{-\frac{1}{2}v_j^2}dv_j\right\}$$
$$=\prod_j\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f_j(w)e^{-\frac{1}{2}\left(\frac{w}{\sqrt{1-e^{-2t\lambda_j}}}-\frac{e^{-t\lambda_j}u_j}{\sqrt{1-e^{-2t\lambda_j}}}\right)^2}\frac{1}{\sqrt{1-e^{-2t\lambda_j}}}dw\right\}$$
$$=\prod_j\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f_j(w)e^{-\frac{1}{2}\left(\frac{e^{-t\lambda_j}u_j}{\sqrt{1-e^{-2t\lambda_j}}}\right)^2-\frac{1}{2}\left(\frac{e^{-t\lambda_j}w}{\sqrt{1-e^{-2t\lambda_j}}}\right)^2+\frac{e^{-t\lambda_j}u_jw}{1-e^{-2t\lambda_j}}}\frac{1}{\sqrt{1-e^{-2t\lambda_j}}}e^{-\frac{1}{2}w^2}dw\right\}$$
$$=\int_{\mathcal{S}^*}F(y)\left(\prod_j\frac{1}{\sqrt{1-e^{-2t\lambda_j}}}\right)e^{-\frac{1}{2}\left|\frac{e^{-tA}}{\sqrt{1-e^{-2tA}}}x\right|^2-\frac{1}{2}\left|\frac{e^{-tA}}{\sqrt{1-e^{-2tA}}}y\right|^2+\left(\frac{e^{-tA}}{1-e^{-2tA}}x,\, y\right)}\mu(dy).$$

For an arbitrary $a\in(0, 1)$,

$$\frac{1}{\sqrt{2\pi(1-a^2)}}\int_{-\infty}^{\infty}e^{-\frac{a^2u^2}{2(1-a^2)}-\frac{a^2v^2}{2(1-a^2)}+\frac{auv}{1-a^2}}e^{-\frac{1}{2}u^2}du$$
$$=\frac{1}{\sqrt{2\pi(1-a^2)}}\int_{-\infty}^{\infty}e^{-\frac{(u-av)^2}{2(1-a^2)}}du=1.$$

We therefore have, for any $t>0$ and $y\in\mathcal{S}^*$,

$$\int_{\mathcal{S}^*}|D_t(x, y)|\mu(dx)=\int_{\mathcal{S}^*}D_t(x, y)\mu(dx)=1$$

Then we define $T_t^A\nu$ for a positive finite measure ν on $\mathcal{S}^*$ as follows.

Definition 2.1. *For any $t>0$ and a positive finite measure ν on $\mathcal{S}^*$, define*

$$T_t^A\nu(x)\equiv\int_{\mathcal{S}^*}D_t(x, y)\nu(dy).$$

Remark. $T_t^A\nu$ is determined μ-almost everywhere as an element of $L_1(\mu)$.

In order to discuss the smoothness of the function $x\mapsto T_t^A\nu(x)$, we define the norms $\{\|\cdot\|_{a(p,K)}\}_{p\in\mathbf{Z},K=0}$, which was defined by Yu.-J. Lee[1] when $K=1$.

Defintion 2.2. *For $p\in\mathbf{Z}$ and $K>0$ we define*

$$\|F\|_{a(p,K)}\equiv\sup_{x,y\in\mathbf{S}_{-p}}e^{-\frac{K}{2}(|x|_{-p}^2+|y|_{-p}^2)}|F(x+\sqrt{-1}y)|,\quad F\in(\mathcal{P}),$$

and set

$$\mathscr{A}_{p,K} \equiv (\mathscr{P})^{\|\cdot\|_{a(p,K)}}.$$

Remark. (a) In the case where $F(x)=(e_1, x)^n$, for example, we define $F(x+\sqrt{-1}y)=\{(e_1, x)+\sqrt{-1}(e_1, y)\}^n$, and for general $F\in(\mathscr{P})$ we define $F(x+\sqrt{-1}y)$ in the same manner.
(b) An arbitrary element F of $\mathscr{A}_{p,K}$ is $\mathscr{S}_{-p}$-continuous and, for any $x, y\in\mathscr{S}^*$, $u(\in\mathbf{R})\mapsto F(x+uy)$ can be extended to an entire function in $u\in\mathbf{C}$. We call such a function *$\mathscr{S}_{-p}$-analytic function.*

Yu.-J. Lee [1] proved that

$$(\mathscr{S})=\cap_p \mathscr{A}_{p1}$$

so that any $F\in(\mathscr{S})$ is $\mathscr{S}^*$-analytic. As an application of this property, he proved that a positive finite measure ν on $\mathscr{S}^*$ belongs to $(\mathscr{S})^*$ (the functional $F(\in(\mathscr{S}))\mapsto\int_{\mathscr{S}^*}F(x)\,\nu(dx)$ is continuous) if and only if there exists $p_0\in\mathbf{N}$ such that $\nu(\mathscr{S}^c_{-p_0})=0$ and

$$\int_{\mathscr{S}^*} e^{\frac{1}{2}|x|^2_{-p_0}}\nu(dx)<\infty.$$

Next we shall show that $T^A_t\nu(x)$ is $\mathscr{S}^*$-analytic in x for any $t>0$ if a positive finite measure ν on $\mathscr{S}^*$ belongs to $(\mathscr{S})^*$.

Theorem 2.2. *Let ν be a positive finite measure on $\mathscr{S}^*$. Assume that ν belongs to $(\mathscr{S})^*$. Then, for any $t>0$ and $p\in\mathbf{Z}$, there exists $K>0$ such that $T^A_t\nu\in\mathscr{A}_{p,K}$.*

Proof. Since ν belongs to $(\mathscr{S})^*$, by the result of Yu.-J. Lee (Theorem 5.1 in [1]), there exists $p_0\in\mathbf{N}$ Such that

$$\int_{\mathscr{S}^*} e^{\frac{1}{2}|x|^2_{-p_0}}\nu(dx)<\infty.$$

Fix arbitrary $t>0$ and $p\in\mathbf{Z}$, and set

$$P_N x\equiv\sum_{n=0}^{N}(e_n, x)e_n \quad (N\in\mathbf{N},\ x\in\mathscr{S}^*),$$

$$U_t\equiv\frac{e^{-tA}}{\sqrt{1-e^{-2tA}}},$$

$$V_t\equiv\frac{e^{-tA}}{1-e^{-2tA}},$$

$$\kappa_t\equiv\prod_j\frac{1}{\sqrt{1-e^{-2t\lambda_j}}}.$$

Then $U_t:\mathscr{S}_{-p}\longrightarrow\mathscr{S}_0$ and $V_t:\mathscr{S}_{-p_0}\longrightarrow\mathscr{S}_{p_0}$ are continuous operators. Therefore there exist $K_1, K_2>0$ such that

$$\left|-\frac{1}{2}|U_t(P_N(x_1+\sqrt{-1}x_2))|^2-\frac{1}{2}|U_t y|^2+(V_t(P_N(x_1+\sqrt{-1}x_2)),y)\right|$$
$$\leq\frac{K_1}{2}(|x_1|_{-p}^2+|x_2|_{-p}^2)+\frac{K_2}{2}|y|_{-p_0}^2.$$

Fix $y\in\mathscr{S}^*$ and set

$$F_{N_1,N_2}(x)\equiv\sum_{n=0}^{N_1}\frac{1}{n!}\left\{-\frac{1}{2}|U_t(P_{N_2}x)|^2-\frac{1}{2}|U_t y|^2+(V_t(P_{N_2}x),y)\right\}^n$$

for any $N_1, N_2\in\mathbf{N}$. Then $F_{N_1,N_2}\in(\mathscr{P})$ and

$$\|F_{N_1,N_2}\|_{a(p,K_1)}\leq e^{\frac{K_2}{2}|y|_{-p_0}^2}.$$

for any $N_1, N_2\in\mathbf{N}$. Therefore

$$\lim_{|x_1|_{-p},|x_2|_{-p}\to\infty}\ \sup_{N_1,N_2\in\mathbf{N}}|F_{N_1,N_2}(x_1+\sqrt{-1}x_2)|e^{-\frac{K_1+1}{2}(|x_1|_{-p}^2+|x_2|_{-p}^2)}=0.$$

Thus we have

$$\lim_{N_1,N_2\to\infty}\|\kappa_t F_{N_1,N_2}-D_t(\cdot,y)\|_{a(p,K_1+1)}=0.$$

where K_1 does not depend on $y\in\mathscr{S}_{-p_0}$.

For any $x_1, x_2\in\mathscr{S}_{-p}$ and $y\in\mathscr{S}_{-p_0}$.

$$\mathscr{R}\left(-\frac{1}{2}|U_t(x_1+\sqrt{-1}x_2)|^2-\frac{1}{2}|U_t y|^2+(V_t(x_1+\sqrt{-1}x_2)y)\right)\leq\frac{1}{2}|U_t x_2|^2+(V_t x_1,y),$$

where $\mathscr{R}(\cdots)$ denotes the real part of $(\cdots)$. By the continuity of U_t and V_t, there exists $K_3>0$ such that

$$\frac{1}{2}|U_t x_2|^2+(V_t x_1,y)\leq\frac{K_3}{2}(|x_1|_{-p}^2+|x_2|_{-p}^2)+\frac{1}{2}|y|_{-p_0}^2.$$

Set $K\equiv\max\{K_1+1, K_3\}$. Then $D_t(\cdot,y)\in\mathscr{A}_{p,K}$ for any $y\in\mathscr{S}^*$ and

$$\int_{\mathscr{S}^*}\|D_t(\cdot,y)\|_{a(p,K)}\nu(dy)<\infty.$$

Thus $T_t^A\nu\in\mathscr{A}_{p,K}$.

Remark. For $p\in\mathbf{N}$ and $K>0$, the family $\{F : F$ is $\mathscr{S}_{-p}$-analytic and $\|F\|_{a(p,K)}<\infty\}$ is different from $\mathscr{A}_{p,K}$ and this family is inseparable with respect to $\|\cdot\|_{a(p,K)}$.

We next show that the function $x\mapsto\int_{\mathscr{S}^*}D_t(x,y)\,\nu(dy)$ $(x\in\mathscr{S}_0=H)$ is H-continuous for a general positive finite measure ν on $\mathscr{S}^*$.

Proposition 2.3. *Let* $\|\cdot\|_{\mathscr{H}}$ *be a norm on* $(\mathscr{P})$ *defined by*

$$\|F\|_{\mathcal{H}} \equiv \sup_{x \in H} |F(x)| e^{-\frac{1}{2}|x|^2},$$

and $\mathcal{H}$ be the completion of $(\mathcal{P})$ with respect to $\|\cdot\|_{\mathcal{H}}$. Then, for any $t>0$ and any positive finite measure ν on $\mathcal{S}^*$, we have

$$\int_{\mathcal{S}^*} D_t(x, y)\, \nu(dy) \in \mathcal{H}.$$

Naotation. The function $x(\in H) \mapsto \int_{\mathcal{S}^*} D_t(x, y)\, \nu(dy)$ will be denoted by $\overline{T_t^A}\,\nu$.

Proof. Fix $t>0$ throughout the proof. By the former part of the proof of previous theorem, $D_t(x, y) \in \mathcal{H}$ for any $y \in \mathcal{S}^*$. Then we have only to prove that

$$\sup_{y \in \mathcal{S}^*} \|D_t(\cdot, y)\|_{\mathcal{H}} < \infty.$$

Let U_t and V_t be the operators given in the previous theorem. Then we have

$$-\frac{1}{2}|U_t x|^2 - \frac{1}{2}|U_t y|^2 + (V_t x, y) = -\frac{1}{2}\left|U_t y - \frac{1}{\sqrt{1-e^{-2tA}}}x\right|^2 + \frac{1}{2}|x|^2$$

for any $x \in H$ and $y \in \mathcal{S}^*$. This implies that $\|D_t(\cdot, y)\|_{\mathcal{H}} \leq \kappa_t$ for any $y \in \mathcal{S}^*$ and concludes the proof.

Using the function $\overline{T_t^A}\,\nu$, the $L_1(\mu)$-function $T_t^A \nu$ is approximated as follows.

Proposition 2.4. *Let ν be a positive finite measure on $\mathcal{S}^*$ and fix $t>0$. Set $P_N x \equiv \sum_{n=0}^{N} (e_n, x) e_n$. Then*

$$\lim_{N\to\infty} \|\overline{T_t^A}\,\nu(P_N x) - T_t^A \nu(x)\|_{L_1(\mu)} = 0.$$

Proof.

$$\begin{aligned}\|\overline{T_t^A}\,\nu(P_N \cdot) - T_t^A \nu\|_{L_1(\mu)} &= \int_{\mathcal{S}^*}\left|\int_{\mathcal{S}^*} D_t(P_N x, y) - D_t(x, y)\, \nu(dy)\right| \mu(dx) \\ &\leq \int_{\mathcal{S}^*}\int_{\mathcal{S}^*} |D_t(P_N x, y) - D_t(x, y)|\, \mu(dy)\, \nu(dx).\end{aligned}$$

Fix $y \in \mathcal{S}^*$. Then

$$\begin{aligned}&\sup_N |D_t(P_N x, y) - D_t(x, y)| \\ &\leq \sup_N D_t(P_N x, y) + D_t(x, y) \\ &= \sup_N \kappa_t e^{-\frac{1}{2}|U_t y|^2}\left(\prod_{j=0}^{N} e^{-\frac{1}{2}\frac{e^{-2t\lambda_j}}{1-e^{-2t\lambda_j}}(e_j, x)^2 + \frac{e^{-t\lambda_j}}{1-e^{-2t\lambda_j}}(e_j, y)}\right) + D_t(x, y) \\ &\leq \sup_N \kappa_t e^{-\frac{1}{2}|U_t y|^2}\left(\prod_{j=0}^{\infty} e^{-\frac{1}{2}\frac{e^{-2t\lambda_j}}{1-e^{-2t\lambda_j}}(e_j, x)^2 + \frac{e^{-t\lambda_j}}{1-e^{-2t\lambda_j}}(e_j, y)} \vee 1\right) + D_t(x, y)\end{aligned}$$

$$\leq \kappa_t (e^{-\frac{1}{2}|U_t x|^2 + (V_t x, y)} \vee 1) + D_t(x, y) \tag{2}$$

where U_t, V_t and κ_t is defined in the proof of Theorem 2.2. Since the right hand side is μ-integrable,

$$\lim_{N\to\infty} \int_{\mathscr{S}^*} |D_t(P_N x, y) - D_t(x, y)| \mu(dx) = 0$$

for any $y \in \mathscr{S}^*$. Using the inequality (2), we have

$$\begin{aligned} &\sup_N \int_{\mathscr{S}^*} |D_t(P_N x, y) - D_t(x, y)| \mu(dx) \\ &\leq \sup_N \int_{\mathscr{S}^*} D_t(P_N x, y) \mu(dx) + \int_{\mathscr{S}^*} D_t(x, y) \mu(dx) \\ &\leq \kappa_t + 1. \end{aligned}$$

Thus we have

$$\lim_{N\to\infty} \int_{\mathscr{S}^*} \int_{\mathscr{S}^*} |D_t(P_N x, y) - D_t(x, y)| \mu(dx) \nu(dy) = 0$$

and this concludes the proof.

3. Convergence of T_t^A

Let ν be a positive finite measure on $\mathscr{S}^*$. In the case where $\nu(dx) = F(x) \mu(dx)$ with $F \in L_p(\mu)$ $(p > 1)$, using the theorem of E. M. Stein [6], $T_t^A \nu(x)$ converges to $F(x)$ μ-almost everywhere as $t \longrightarrow 0$. If the dimension of H is finite, by the Global Density Theorem (H. Sato [4]), $T_t^A \nu(x)$ converges to $\frac{d\nu}{d\mu}(x)$ μ-almost everywhere as $t \longrightarrow 0$. In this section, we shall study more about the convergence of $T_t^A \nu$ $(t \longrightarrow 0)$.

First we consider the case where $\nu \in (\mathscr{S})^*$. The operator T_t^A can be continuously extended on $(\mathscr{S})^*$, and we temporally denote the extension by $\mathscr{T}_t^A$. On the other hand $T_t^A \nu$ is defined as an element of $L_1(\mu)$ for a positive finite measure ν on $\mathscr{S}^*$. Next propositions imply that these two extensions are identical with each other when ν belongs to $(\mathscr{S})^*$.

Proposition 3.1. *Assume that a positive finite measure ν on $\mathscr{S}^*$ belongs to $(\mathscr{S})^*$. Then*

$$\int_{\mathscr{S}^*} T_t^A \nu(x) F(x) \mu(dx) = \langle F, \mathscr{T}_t^A \nu \rangle$$

holds for any $F \in (\mathscr{S})$ and $t > 0$.

Proof. Fix $t > 0$ and $F \in (\mathscr{S})$ throughout the proof. Since T_t^A is a symmetric operator on $L_2(\mu)$

$$\langle F, \mathcal{T}_t^A \nu\rangle = \langle T_t^A F, \nu\rangle.$$

In the next proposition, we will show that

$$\int_{\mathcal{S}^*} F(x)\, T_t^A \nu(x)\, \mu(dx) = \int_{\mathcal{S}^*} T_t^A F(x)\, \nu(dx)\ (= \langle T_t^A F, \nu\rangle).$$

The above two equalities imply that

$$\int_{\mathcal{S}^*} T_t^A \nu(x) F(x)\, \mu(dx) = \langle F, \mathcal{T}_t \nu\rangle,$$

and this concludes the proof.

Proposition 3.2. *Let ν be a positive finite measure on $\mathcal{S}^*$.*

(*a*) *For a bounded continuous function F and $t>0$,*

$$\int_{\mathcal{S}^*} F(x)\, T_t^A \nu(x)\, \mu(dx) = \int_{\mathcal{S}^*} T_t^A F(x)\, \nu(dx). \tag{3}$$

(*b*) *In the case where $\nu \in (\mathcal{S})^*$, (3) holds for any $F \in (\mathcal{S})$ and $t>0$.*

Proof. (a) Since $D_t(x, y) = D_t(y, x)$ for any $x, y \in \mathcal{S}^*$, using the Fubini Theorem, we have

$$\begin{aligned} \int_{\mathcal{S}^*} F(x)\, T_t^A \nu(x)\, \mu(dx) &= \int_{\mathcal{S}^*} F(x) \int_{\mathcal{S}^*} D_t(x, y)\, \nu(dy)\, \mu(dx) \\ &= \int_{\mathcal{S}^*}\int_{\mathcal{S}^*} F(x) D_t(x, y)\, \mu(dx)\, \nu(dy) \\ &= \int_{\mathcal{S}^*} T_t^A F(y)\, \nu(dy). \end{aligned}$$

(b) In the case where $\nu \in (\mathcal{S})^*$, for any $F \in (\mathcal{P})$, $T_t^A F(x) \in (\mathcal{P})$ is ν-integrable. Therefore the above proof implies that (3) holds for any $F \in (\mathcal{P})$. Then, using the approximation by certain elements of $(\mathcal{P})$, we obtain (3) for any $F \in (\mathcal{S})$.

Lemma 3.3. *Let ν be a positive finite measure on $\mathcal{S}^*$. Assume that ν belongs to $(\mathcal{S})^*$. Then $T_t^A \nu$ converges to ν as $t \mapsto 0$ with respect to the topology of $(\mathcal{S})^*$ (the inductive limit topology of $\{(\mathcal{S}), \|\cdot\|_p \in \mathbf{z})$.*

Proof. By proposition 3.1, we may regard T_t^A as its own continuous extension on $(\mathcal{S})^*$. Therefore the following expansion implies that $T_t^A \nu$ converges to ν in $(\mathcal{S})_p$ as $t \longrightarrow 0$ if $\nu \in (\mathcal{S}_p)$ $(p \in \mathbf{Z})$:

$$T_t^A \nu(x) = \sum_{\alpha \in \mathbf{N}_0^\infty} \langle \mathbf{h}_\alpha, \nu\rangle e^{-t\Sigma_j \lambda_j \alpha_j} \mathbf{h}_\alpha(x).$$

Next we consider the general case. Before giving the proof of Theorem 1.1, we prepare two propositions.

Proposition 3.4. *Let F be a bounded continuous function on $\mathcal{S}^*$. Then*

$\{T_t^A F(x)\}_{t>0}$ is uniformly bounded and converges to $F(x)$ as $t \longrightarrow 0$ for any $x \in \mathcal{S}^*$.

Proof. It is easily obtained by the original definition of $\{T_t^A\}$ (see the left hand side of (1)).

Proposition 3.5. *For $F \in L_p(\mu)$ $(p \geq 1)$, $T_t^A F$ converges to F in $L_p(\mu)$.*

Proof. Since F is approximated by bounded continuous functions, the proposition is an immediate consequence of the property (St-1) (in Section 1) and Proposition 3.4.

Proof of Theorem 1.1. By Proposition 3.4, Proposition 3.2 (a) and the dominated convergence theorem, the measure $T_t^A \nu(x) \mu(dx)$ converges weakly to $\nu(dx)$. Thus we obtain (a) of Theorem 1.1.

Next we prove (b) of Theorem 1.1. By virtue of the Lebesgue decomposition of ν with respect to μ and Proposition 3.5, we have only to prove that $T_t^A \nu(x)$ converges to 0 in the measure $\mu (t \longrightarrow 0)$ when ν is singular to μ.

Let $\varepsilon > 0$ be an arbitrary positive number. Then there exist two compact subsets K_1, K_2 of $\mathcal{S}^*$ such that $\mu(K_1^c) < \varepsilon$, $\nu(K_2^c) < \varepsilon$ and $K_1 \cap K_2 = \emptyset$. Since the two compact (closed) subsets K_1 and K_2 are disjoint, there exists a bounded continuous function $g(x)$ such that

$$g(x) = \begin{cases} 1 & x \in K_1 \\ 0 & x \in K_2 \\ \in [0, 1] & \text{for any } x \in \mathcal{S}^*. \end{cases}$$

Since $T_t^A \nu(x) \mu(dx)$ converges weakly to $\nu(dx)$,

$$\begin{aligned} &\limsup_{t \to 0} \int_{K_1} T_t^A \nu(x) \mu(dx) \\ &\leq \limsup_{t \to 0} \int_{\mathcal{S}^*} g(x) T_t^A \nu(x) \mu(dx) \\ &= \lim_{t \to 0} \int_{\mathcal{S}^*} g(x) T_t^A \nu(x) \mu(dx) \\ &\leq \nu(K_2^c) < \varepsilon. \end{aligned}$$

This implies that

$$\mu\ (\{x | T_t^A \nu(x) > \sqrt{\varepsilon}\}) < \varepsilon + \sqrt{\varepsilon}$$

if $t > 0$ is small enough. Thus $T_t^A \nu$ converges to 0 in the measure μ.

4. Fourier transform

In this section we will give inversion formulae for the Fourier transform $\mathcal{F}$ when they act on positive finite measure on $\mathcal{S}^*$. The next proposition is easily given by the definitions of $\mathcal{F}$ and e^{-tA}.

Proposition 4.1. (*a*) *$\mathcal{F}[\nu]$ is a continuous function on $\mathcal{S}$.*
(*b*) *For any $t>0$, $\mathcal{F}[\nu](e^{-tA}x)$ is μ-integrable.*

Define the transform $\bar{\mathcal{F}}$ as follows.

$$\bar{\mathcal{F}}[\nu](x) \equiv e^{\frac{1}{2}|x|^2} \int_{\mathcal{S}^*} e^{-\sqrt{-1}(x,y)} \nu(dy).$$

Proposition 4.2. *Let ν be a positive finite measure on $\mathcal{S}^*$. Then we have*

$$\bar{\mathcal{F}}[\mathcal{F}[\nu](e^{-tA}\cdot)](x) = \overline{T_t^A}\,\nu(x)$$

for any $x \in \mathcal{S}$.

Proof. Fix an arbitrary $t>0$ throughout the proof. First we prove that

$$e^{\frac{1}{2}|x|^2} \int_{\mathcal{S}^*} e^{\frac{1}{2}|e^{-tA}z|^2 - \sqrt{-1}(x,z) + \sqrt{-1}(e^{-tA}z,y)} \mu(dz) = D_t(x,y). \tag{4}$$

for any $x \in \mathcal{S}$ and $y \in \mathcal{S}^*$. For any $j \in \mathbf{N}$, set $x_j \equiv (e_j, x)$, $y_j \equiv (e_j, y)$ and $\gamma_j \equiv e^{-t\lambda_j}$. Then

$$e^{\frac{1}{2}|x|^2} \int_{\mathcal{S}^*} e^{\frac{1}{2}|e^{-tA}z|^2 - \sqrt{-1}(x,z) + \sqrt{-1}(e^{-tA}z,y)} \mu(dz)$$
$$= \prod_j \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(1-\gamma_j^2)\left(z_j + \frac{\sqrt{-1}(x_j - \gamma_j y_j)}{1-\gamma_j^2}\right)^2 - \frac{(x_j - \gamma_j y_j)^2}{2(1-\gamma_j^2)} + \frac{x_j^2}{2} \right) dz_j \right\}$$
$$= \left(\prod_j \frac{1}{\sqrt{1-\gamma_j^2}} \right) \exp\left\{ -\frac{1}{2}\left| \frac{e^{-tA}}{\sqrt{1-e^{-2tA}}} x \right|^2 - \frac{1}{2}\left| \frac{e^{-tA}}{\sqrt{1-e^{-2tA}}} y \right|^2 + \left(\frac{e^{-tA}}{\sqrt{1-e^{-2tA}}} x, y \right) \right\}$$
$$= D_t(x,y).$$

Thus we have (4). Therefore

$$\bar{\mathcal{F}}[\mathcal{F}[\nu](e^{-tA}\cdot)](x)$$
$$= e^{\frac{1}{2}|x|^2} \int_{\mathcal{S}^*} e^{-\sqrt{-1}(x,z)} e^{\frac{1}{2}|e^{-tA}z|^2} \int_{\mathcal{S}^*} e^{-\sqrt{-1}(e^{-tA}z,y)} \mu(dy)\, \mu(dz)$$
$$= \int_{\mathcal{S}^*} \nu(dy)\, e^{\frac{1}{2}|x|^2} \int_{\mathcal{S}^*} e^{\frac{1}{2}|e^{-tA}z|^2 - \sqrt{-1}(x,z) + \sqrt{-1}(e^{-tA}z,y)} \mu(dz)$$
$$= \int_{\mathcal{S}^*} D_t(x,y)\, \nu(dy) = \overline{T_t^A}\,\nu(x)$$

Proof of Theorem 1.2. We have only to sum up Proposition 2.4 and Proposition 4.2 to prove the theorem.

Remark. If ν belongs to $(\mathcal{S})^*$, $\bar{\mathcal{F}}[\mathcal{F}[\nu](e^{-tA}\cdot)]$ $(=\overline{T_t^A}\,\nu)$ can be continuously extended to $T_t^A \nu$. Then, besides the convergence in the statement of Theorem 1.2, this also converges to ν with respect to the topology of $(\mathcal{S})^*$.

Notation. For $p \in \mathbf{Z}$ set

$$\mathscr{C}(\mathscr{S}_p)=(\mathscr{S}_p)\oplus\sqrt{-1}\,(\mathscr{S}_p)=\{\phi+\sqrt{-1}\psi|\phi,\ \psi\in(\mathscr{S}_p)\},$$

and the spaces $\mathscr{C}(\mathscr{S})$, $\mathscr{C}(\mathscr{S})^*$ and $\mathscr{C}\mathscr{A}_{p,K}(p\in\mathbf{Z},\ K>0)$ are defined in the same manner. On $\mathscr{C}(\mathscr{S}_p)$ $(p\in\mathbf{Z})$, the norm $\|\cdot\|_p$ is extended as follows.

$$\|\phi+\sqrt{-1}\psi\|_p\equiv\sqrt{\|\phi\|_p^2+\|\psi\|_p^2}.$$

Yu.-J. Lee [1] proved that, for any $F\in\mathscr{C}(\mathscr{S})$, $\mathscr{F}[F]\in\mathscr{C}(\mathscr{S})$ and $\overline{\mathscr{F}}[\mathscr{F}[F]]=F$, and J. Potthoff and L. Streit [2] proved that $\mathscr{F}$ is an isomorphism from $\mathscr{C}(\mathscr{S})^*$ to $\cup_p\mathscr{C}\mathscr{A}_{p,1}$.

In the following proposition we will show that, for any $t>0$,

$$\mathscr{F}[\nu](e^{-tA}x)=\mathscr{F}[T_t^A\nu](x),\quad \text{for any } x\in\mathscr{S}.$$

So that Proposition 4.2 implies that $\mathscr{F}[T_t^A\nu]$ can be extended to a continuous function which belongs to $L_1(\mu)$ and $\overline{\mathscr{F}}[\mathscr{F}[T_t^A\nu]]=T_t^A\nu$. Moreover, when ν belongs to $(\mathscr{S})^*$, $\overline{\mathscr{F}}[\mathscr{F}[T_t^A\nu]]=T_t^A\nu$.

Proposition 4.3. *Fix any $t>0$. Then*

(a) *For a positive finite measure ν on $\mathscr{S}^*$,*

$$\mathscr{F}[\nu](e^{-tA}x)=\mathscr{F}[T_t^A\nu](x),\quad x\in\mathscr{S}.$$

(b) *For $\Phi\in(\mathscr{S})^*$*

$$\mathscr{F}[\Phi](e^{-tA}x)=\mathscr{F}[T_t^A\Phi](x),\quad (x\in\mathscr{S}).$$

Proof. (a) Fix $x\in\mathscr{S}^*$ and set, for any $j\in\mathbf{N}$,

$$x_j\equiv(e_j,x),\quad y_j=y_j(y)\equiv(e_j,y),\quad y\in\mathscr{S},\quad \gamma_j\equiv e^{-t\lambda_j}.$$

Then we have

$$\begin{aligned}
&\mathscr{F}[T_t^A\nu](x)\\
&=e^{\frac{1}{2}|x|^2}\int_{\mathscr{S}^*}e^{\sqrt{-1}(x,z)}\int_{\mathscr{S}^*}D_t(z,y)\,\nu(dy)\,\mu(dz)\\
&=\int_{\mathscr{S}^*}\nu(dy)\Big[\prod_j\Big\{e^{\frac{x_j^2}{2}}\frac{1}{2\pi(1-\gamma_j^2)}\int_{-\infty}^{\infty}\exp\Big(\frac{\gamma_jz_jy_j}{1-\gamma_j^2}-\frac{\gamma_j^2z_j^2}{2(1-\gamma_j^2)}-\frac{\gamma_j^2y_j^2}{2(1-\gamma_j^2)}\\
&\qquad+\sqrt{-1}z_jx_j-\frac{1}{2}z_j^2\Big)dz_j\Big\}\Big]\\
&=\int_{\mathscr{S}^*}\nu(dy)\Big[\prod_j\Big\{e^{\frac{x_j^2}{2}}\frac{1}{2\pi(1-\gamma^2_j)}\int_{-\infty}^{\infty}\exp\Big(-\frac{(x_j-\gamma_jz_j-\sqrt{-1}(1-\gamma_j^2)x_j^2)^2}{2(1-\gamma_j^2)}\\
&\qquad+\sqrt{-1}\gamma_jx_jy_j-\frac{1}{2}(1-\gamma_j^2)x_j^2\Big)dz_j\Big\}\Big]\\
&=\int_{\mathscr{S}^*}\prod_j\{e^{\frac{1}{2}\gamma^2_jx^2_j}e^{\sqrt{-1}\gamma_jx_jy_j}\}\,\nu(dy)\\
&=e^{\frac{1}{2}|e^{-tA}x|^2}\int_{\mathscr{S}^*}e^{\sqrt{-1}(e^{-tA}x,\,y)}\,\nu(dy)\\
&=\mathscr{F}[\nu](e^{-tA}x).
\end{aligned}$$

(b) *In the above proof we have obtained*

$$\mathcal{F}[F](e^{-tA}x)=\mathcal{F}[T_t^A F](x), \quad x\in\mathcal{S},$$

for any $F\in L_2(\mu)=(\mathcal{S}_0)$, which space is dense in $(\mathcal{S})^*$. However, for an element of $(\mathcal{P})$, we shall give its direct proof to be clear the meaning of this equality. Set $F=\sum_\alpha a_\alpha \mathbf{h}_\alpha$ and assume that $a_\alpha=0$ except for finite α's $(\in\mathbf{N}_0^\infty)$. Note that

$$\mathcal{F}[\mathbf{h}_\alpha](x)=(\sqrt{-1})^{|\alpha|}\frac{1}{\sqrt{\alpha!}}(\prod_j(e_j,x)^{\alpha_j}), \quad T_t^A\mathbf{h}_\alpha(x)=e^{-t\Sigma_j\lambda_j\alpha_j}\mathbf{h}_\alpha(x).$$

Therefore

$$\begin{aligned}\mathcal{F}[\sum_\alpha a_\alpha\mathbf{h}_\alpha(\cdot)](e^{-tA}x)&=\sum_\alpha a_\alpha(\sqrt{-1})^{|\alpha|}\frac{1}{\sqrt{\alpha!}}(\prod_j(e_j,e^{-tA}x)^{\alpha_j})\\&=\sum_\alpha a_\alpha(\sqrt{-1})^{|\alpha|}e^{-t\Sigma_j\lambda_j\alpha_j}\frac{1}{\sqrt{\alpha!}}(\prod_j(e_j,x)^{\alpha_j})\\&=\sum_\alpha a_\alpha e^{-t\Sigma_j\lambda_j\alpha_j}\mathcal{F}[\mathbf{h}_\alpha]\\&=\mathcal{F}[\sum_\alpha a_\alpha e^{-t\Sigma_j\lambda_j\alpha_j}\mathbf{h}_\alpha]\\&=\mathcal{F}[T_t^AF]\end{aligned}$$

Since T_t^A is a continuous map from $(\mathcal{S})^*$ to $(\mathcal{S})^*$ and $\mathcal{F}$ is a continuous map from $(\mathcal{S})^*$ to $\cup_p\mathcal{C}\mathcal{A}_{p,1}$ (J. Potthoff and L. Streit [2]), we have the above equality for any summation of infinite $a_\alpha\mathbf{h}_\alpha$'s which converges in $(\mathcal{S})^*$.

At the last of this article we give an example of generalized function $\Psi\in(\mathcal{S})^*$ satisfying following (a) and (b).

(a) $\mathcal{F}[\Psi]$ is a nonnegative $\mathcal{S}^*$-continuous function.
(b) $\mathcal{F}[\Psi]$ does not belong to $(\mathcal{S})^*$.

These illustrate that $(\mathcal{S})^*$ or $\mathcal{C}(\mathcal{S})^*$ can not be invariant under $\mathcal{F}$.

Expamle 4.1. *Fix $k_0\in\mathbf{N}$, $p\in\mathbf{N}\,(p\geq 1)$, set $a_n=\lambda_{k_0}^{pn}$ and set*

$$\Phi=\sum_{n=0}^{\infty}\frac{(-1)^n a_n}{\sqrt{(2n)!}}h_{2n}((e_{k_0},\cdot)).$$

Then $\Phi\in(\mathcal{S}_{-p})$ and

$$\mathcal{F}[T_t^A\Phi]=\mathcal{F}[\Phi](e^{-tA}x)$$

is a $\mathcal{S}^*$-continuous function. On the other hand,

$$\mathcal{F}[T_t^A\Phi](x)=\sum_n a_n\frac{e^{tn\lambda_{k_0}}}{\sqrt{(2n)!}}(e_{k_0},x)^{2n}$$

is non-negative for any $x\in\mathcal{S}^*$. Assume that $\mathcal{F}[T_t^A\Phi]$ belongs to $(\mathcal{S})^*$, then $\mathcal{F}[T_t^A\Phi]$ is μ-integrable since it is nonnegative and $1\in(\mathcal{S})$. However $\mathcal{F}[T_t^A\Phi]$ is *not μ-integrable for some t, p and k_0. Indeed,*

$$\int_{\mathscr{S}^*}\mathscr{F}[T_t^A\Phi](x)\mu(dx)$$
$$=\sum_n a_n e^{-2tn\lambda_{k_0}}\frac{(2n)!}{2^n n!\sqrt{(2n)!}}$$
$$\geq \sum_n e^{n(p\log\lambda_{k_0}-2t\lambda_{k_0}-\frac{1}{2}\log 2)}.$$

Then, when we fix one of the three numbers p, k_0 and t, we can chose other two numbers to make the above value infinity, and this implies that $\mathscr{F}[T_t^A\Phi]$ does not belong to $(\mathscr{S})^*$ in general.

Assume that $T_t^A\Phi$ is a signed measure on $\mathscr{S}^*$, then $\mathscr{F}[\Phi](e^{-tA}x)$ is μ-integrable for any $t>0$. By Proposition 4.2 we have

$$\mathscr{F}[T_t^A\Phi](e^{-tA}x)=\mathscr{F}[\Phi](e^{-2tA}x).$$

Thus the above example shows that $T_t^A\Phi$ is not a measure on $\mathscr{S}^*$ for $\Phi\in(\mathscr{S})^*$ in general.

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