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On $\mathbf{A}^1$-bundles of affine morphisms

By

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1. Introduction

Let $\phi : X \to Y$ be an affine faithfully flat morphism of finite type between locally noetherian schemes. The aim of this paper is to investigate sufficient fibre conditions (which would in some sense be minimal) for X to be an $\mathbf{A}^1$-bundle over Y relative to the Zariski topology, or at least an $\mathbf{A}^1$-fibration over Y (i.e., the fibre of ϕ at each point of Y is $\mathbf{A}^1$).

In ([B-D]) this problem has been investigated in detail under the additional hypotheses that Y is affine and X is dominated by the affine n-space over Y.

Recall that by a result of Bass, Connell and Wright ([B-C-W], 4.4) an $\mathbf{A}^1$-bundle over an affine scheme is actually a line bundle.

We shall first prove the following result (see 3.4) in Section 3.

Theorem A. *Let Y be a locally noetherian normal integral scheme and let ϕ: $X \to Y$ be a faithfully flat affine morphism of finite type such that*

(i) *The fibre of ϕ at the generic point of Y is $\mathbf{A}^1$.*

(ii) *The fibre of ϕ at the generic point of each irreducible reduced closed subscheme of Y of codimension one is geometrically integral.*

Then X is an $\mathbf{A}^1$-bundle over Y. In particular if Y is an affine scheme then X is a line bundle over Y.

This result has been proved earlier in ([K-M], Theorem 1) by Kambayashi and Miyanishi under the additional assumptions that Y is locally *factorial* and the fibres of ϕ at *all* points of Y are gemetrically integral. The other result in this direction is the following theorem due to Kambayashi and Wright ([K-W]).

Theorem. *Let Y be a noetherian normal integral scheme and let $\phi : X \to Y$ be a faithfully flat morphism of finite type such that the fibre of ϕ at every point of Y is $\mathbf{A}^1$. Then X is an $\mathbf{A}^1$-bundle over Y.*

The proof of this result is quite involved and difficult. Our result (Theorem A), apart from giving minimal sufficient fibre conditions for an affine faithfully flat morphism over a locally noetherian normal integral

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scheme to be an $\mathbf{A}^1$-bundle, also provides in the process, an alternative and simpler proof of the Kambayashi-Wright theorem when ϕ is assumed to be affine.

Using Theorem A we next prove (see 3.5) :

Theorem B. *Let Y be a locally noetherian scheme, $\phi : X \rightarrow Y$ an affine faithfully flat morphism of finite type such that*

(*i*) *The fibre of ϕ at the genric point of every irreducible component of Y is $\mathbf{A}^1$.*

(*ii*) *The fiber of ϕ at the generic point of each irreducible reduced closed subscheme of Y of codimension one is gemetrically integral.*

Then all the fibres of ϕ are $\mathbf{A}^1$-forms. Thus if Y is a $\mathbf{Q}$-scheme then ϕ is actually an $\mathbf{A}^1$-fibration.

Examples in Section 4 would illustrate that the conditions in Theorems A and B are the best possible.

Section 2 of this paper is on preliminaries. In Section 3 we prove our main theorems. We make further discussion about our results with the help of examples in Section 4.

2. Preliminaries

Notations. For a commutative ring R, $R^{[n]}$ denotes a polynomial ring in n variables over R and R^n a free module of rank n over R.

Definition 2.1. A flat affine morphism $\phi : X \rightarrow Y$ of finite type will be called an *$\mathbf{A}^n$-fibration* if at ever point P of Y the fibre $\phi^{-1}(P)$ is isomorphic to the affine n-space $\mathbf{A}^n$ over the residue field of P on Y. In this situation X will also be called an $\mathbf{A}^n$-fibration over Y.

Definition 2.2. Let k be a field. A k-scheme X is said to be an *$\mathbf{A}^1$-form* (*over k*) if there exists a $\bar{k}$-isomorphism $X \times_k \bar{k} \cong \mathbf{A}^1_{\bar{k}}$, where $\bar{k}$ denotes the algebraic closure of k.

Now we recall the lemma ([K-M], 1.3) of Kambayashi and Miyanishi.

Lemma 2.3. *Let (R, π) be a discrete valuation ring and A a finitely generated flat R-algebra such that $A[1/\pi] = R[1/\pi]^{[1]}$ and $A/\pi A$ is gemetrically integral. Then $A = R^{[1]}$.*

Using the result ([B-C-W], 4.4) which asserts that every locally polynomial algebra is the symmetric algebra of a finitely generated projective module and the fact that the Picard group of a factorial domain is trivial it follows from the above lemma :

Corollary 2.4. *Let R be a principal ideal domain with quotient field K and*

A a finitely generated flat R-algebra such thet $A \otimes_R K = K^{[1]}$ *and* $A \otimes_R k(P)$ *are geometrically integral for all prime ideals P of R. Then* $A = R^{[1]}$.

3. Main Theorems

In this section we prove our main theorems (3.4 and 3.5). The crucial step in the proof of Theorem 3.4 is Lemma 3.3 which uses the lemmas 3.1 and 3.2.

Lemma 3.1. *Let A be a faithfully flat R-algebra,* $\mathbf{c}$ *an element of* R^n *and M an* $m \times n$*-matrix with coefficients in R. Suppose that the system of linear equations*

$$\mathbf{c} = \mathbf{x}M \tag{3.1.1}$$

has a solution for $\mathbf{x}$ *in* $\mathbf{A}^m$. *Then the system* (3.1.1) *also has a solution in* $\mathbf{R}^m$.

Proof. Let L be the cokernel of the map $f: R^m \to R^n$ defined by $\mathbf{y} \to \mathbf{y}M$, g the canonical map $R^n \to L$ and let f', g' be the induced maps $A^m \to A^n$ and $A^n \to L \otimes_R A$ respectively. We thus have the commutative diagram

$$\begin{array}{ccccccc} R^m & \xrightarrow{f} & R^n & \xrightarrow{g} & L & \longrightarrow & 0 \\ \downarrow & & \downarrow j & & \downarrow \rho & & \\ A^m & \xrightarrow{f'} & A^n & \xrightarrow{g'} & L \otimes_R A & \longrightarrow & 0 \end{array}$$

where both the horizontal rows are exact. Since the system (3.1.1) has a solution over A^m, i. e., $j(\mathbf{c}) \in im(f')$, we have $g'(j(\mathbf{c})) = 0$, i. e., $\rho(g(\mathbf{c})) = 0$. But A being faithfully flat over R, by a well-known result (see [A-K], pg. 85, Theorem 1.9) the map $\rho: L \to L \otimes_R A$ is injective so that $g(\mathbf{c}) = 0$. That shows that $\mathbf{c} \in im(f)$, i. e., the system (3.1.1) has a solution in R^m. Hence the result.

The following lemma has been proved in ([B-D], 3.9).

Lemma 3.2. *Let R be a noetherian local ring and A a finitely generated flat R-algebra. Suppose that there exists a regular sequence a, b in R and an elemnt* $\sigma \in \mathbf{GL}_n(R[1/ab])$ *such that*

(*i*) $A[1/a] = R[1/a][F_1, \cdots, F_n] = R[1/a]^{[n]}$.
(*ii*) $A[1/b] = R[1/b][G_1, \cdots, G_n] = R[1/b]^{[n]}$.
(*iii*) $[F_1, \cdots, F_n]\sigma = [G_1, \cdots, G_n]$.

Then $A = R^{[n]}$.

The next lemma shows that for a faithfully flat R-algebra A, Lemma 3.2 would be valid even if σ is an affine transformation of $R[1/ab]$.

Lemma 3.3. *Let R be a noetherian local ring and A a finitely generated* *faithfully flat R-algebra. Suppose that there exists a regular sequence a, b in R such that*

(i) $A[1/a]=R[1/a][F_1,\cdots,F_n]=R[1/a]^{[n]}$.
(ii) $A[1/b]=R[1/b][G_1,\cdots,G_n]=R[1/b]^{[n]}$.
(iii) $G_i=\left(\sum_{1\leq j\leq n}\lambda_{ij}F_j\right)+\mu_i$, *where* λ_{ij}, $\mu_i\in R[1/ab]$, $1\leq i,j\leq n$.

Then $A=R^{[n]}$.

Proof. Without loss of generality we may assume that F_i, $G_i\in A$ for all i so that λ_{ij}, $\mu_i\in R[1/a]$ for all i, j. Now in view of Lemma 3.2 it is enough to show that there exists $U_1,\cdots,U_n\in A[1/a]$, $V_1,\cdots,V_n\in A[1/b]$ and a matrix $\Lambda\in\mathbf{GL}_n(R[1/ab])$ such that

(i) $A[1/a]=R[1/a][U_1,\cdots,U_n]$.
(ii) $A[1/b]=R[1/b][V_1,\cdots,V_n]$.
(iii) $[U_1,\cdots,U_n]\Lambda=[V_1,\cdots,V_n]$.

Let $\Lambda=((\lambda_{ij}))^T_{i,j}$ (the transpose matrix). Note that $R[1/ab][F_1,\cdots,F_n]=R[1/ab][G_1,\cdots,G_n]$ so that the matrix Λ defines an element of $\mathbf{GL}_n(R[1/ab])$.

Now the given relations may be expressed as

$$G_i=\left(\sum_{1\leq j\leq n} b_{ij}F_j+c_j\right)\Big/a^m$$

for b_{ij}, $c_i\in R$, $m\in\mathbf{Z}^+$, $1\leq i\leq n$. Thus

$$c_i=-\sum_{1\leq j\leq n} b_{ij}F_i+a^m G_i. \tag{3.3.1}$$

Now let $\mathbf{c}$ denote the vector $[c_1,\cdots,c_n]$ of R^n and let $\mathbf{F}$, $\mathbf{G}$ denote respectively the vectors $[F_1,\cdots,F_n]$, $[G_1,\cdots,G_n]$ of A^n. Let B denote the $n\times n$-matrix whose (i,j)th entry is b_{ij} and D the scalar matrix a^mI, where I is the $n\times n$-identity matrix. From (3.3.1) we have the matrix equation

$$\mathbf{c}=[-F:G][B:D]^T \tag{3.3.2}$$

Therefore by Lemma 3.1 there exist $\mathbf{r}=[r_1,\cdots,r_n]$ and $\mathbf{s}=[s_1,\cdots,s_n]$ in R^n such that

$$\mathbf{c}=[-\mathbf{r}:\mathbf{s}][B:D]^T. \tag{3.3.3}$$

By (3.3.2) and (3.3.3) we have $[\mathbf{F}-\mathbf{r}]B^T=[\mathbf{G}-\mathbf{s}]D$. Since $\lambda_{ij}=b_{ij}/a^m$ this equation implies

$$(\mathbf{F}-\mathbf{r})\Lambda=\mathbf{G}-\mathbf{s}. \tag{3.3.4}$$

Now let $U_i=F_i-r_i$ and $V_i=G_i-s_i$, $1\leq i\leq n$. Since r_i, $s_i\in R$ for all i,

$$A[1/a]=R[1/a][U_1,\cdots,U_n],\ A[1/b]=R[1/b][V_1,\cdots,V_n]$$

and by (3.3.4)

$$[U_1,\cdots, U_n]\,\Lambda=[V_1,\cdots, V_n]$$

Hence by Lemma 3.2, $A=R^{[n]}$.

The following result will prove Theorem A.

Theorem 3.4. *Let Y be a locally noetherian normal integral scheme and let $\phi: X\to Y$ be a faithfully flat affine morphism of finite type such that*

(*i*) *The fibre of ϕ at the generic point of Y is $\mathbf{A}^1$.*

(*ii*) *The fibre of ϕ at the generic point of every irreducible reduced closed subscheme of Y of codimension one is geometrically integral.*

Then X is an $\mathbf{A}^1$-bundle over Y.

Proof. Let P be a closed point of Y and let $R=\mathcal{O}_{Y,P}$. Replacing X by $X\times_Y$ (*Spec* R) we assume that $Y=$ *Spec* R for a noetherian normal *local* domain R and $X=$ *Spec* A for some finitely generated faithfully flat R-algebra A and prove that $A=R^{[1]}$.

Let K be the quotient field of R. By conditions (i) and (ii) we have

(I) $A\otimes_R K=K^{[1]}$.

(II) $A\otimes_R k(P)$ *are geometrically integral for all prime ideals of R of height one.*

The case $\dim R=0$ (i.e., when R is the field K) follows from (I) and the case $\dim R=1$ (i.e., R is a discrete valuation ring) follows from (2.3). So we assume that $\dim R\geq 2$.

Since A is finitely generated over R from (I) it is easy to see that there exists a non-zero element $a\in R$ such that

$$A[1/a]=R[1/a][F]\ (=R[1/a]^{[1]})\ \text{for some } F\in A.$$

Let $P_1,\cdots, P_t$ be the associated prime ideals of aR. Since R is a noetherian normal domain, ht $P_i=1$ for all i, $1\leq i\leq t$. Let

$$S=R\setminus\left(\bigcup_{i=1}^{t} P_i\right),\quad R_1=S^{-1}R,\quad A_1=S^{-1}A.$$

Then R_1 being a semi-local Dedekind domain is a P.I.D. and hence by (2.4), $A_1=R_1^{[1]}$. Therefore there exists $b\in S$ such that

$$A[1/b]=R[1/b][G]\ (=R[1/b]^{[1]})\ \text{for some } G\in A.$$

Note that by construction (a, b) form a sequence in R. Since

$$R[1/ab][F]=A[1/ab]=R[1/ab][G],$$

we have a relation

$$G=\lambda F+\mu \text{ for some } \lambda, \mu\in R[1/ab].$$

Therefore by Lemma 3.3, $A=R^{[1]}$.

We now prove Theorem B.

Theorem 3.5. *Let Y be a locally noetherian scheme, $\phi: X\to Y$ an affine faithfully flat morphism of finite type such that*

(i) *The fibres of ϕ at the generic points of all the irreducible components of Y are $\mathbf{A}^1$.*

(ii) *The fibres of ϕ at the generic points of all irreducible redced cloed subschemes of Y of codimension one are geometrically integral.*

Then all the fibres of ϕ are $\mathbf{A}^1$-forms. In particular if Y is a $\mathbf{Q}$-scheme then X is an $\mathbf{A}^1$-fibration over Y.

Proof. Fix a point $P\in Y$. We show that the fibre at P is an $\mathbf{A}^1$-form. Let $R=\mathcal{O}_{Y,P}$. As before we may assume $Y=Spec\, R$ where R is a local noetherian ring with maximal ideal P and $X=Spec\, A$ where A is a finitely generated faithfully flat R-algebra. We prove that the fibre $A\otimes_R k(P)$ is an $\mathbf{A}^1$-form over $k(P)$ by induction on ht $P(=\dim R)$. The case ht $P=0$ is trivial.

Consider the case ht $P=1$, i. e., dim $R=1$. Replacing R by R/Q_0 where Q_0 is a minimal prime ideal of R, we may assume R to be an integral domain with quotient field K. Note that condition (ii) now means that $A\otimes_R k(P)$ is geometrically integral. Let $\widetilde{R}$ be the normalisation of R and let $\widetilde{A}=A\otimes_R\tilde{R}$. Then $\widetilde{A}$ is finitely generated faithfully flat $\widetilde{R}$-algebra. By the Krull-Akizuki theorem $\widetilde{R}$ is a Dedekind domain and since R is local, $\widetilde{R}$ is in fact a P.I.D.. Moreover $k(\widetilde{P})$ are algebraic (in fact finite) extensions of $k(P)$ for every maximal ideal $\widetilde{P}$ of $\widetilde{R}$. Thus by (i) and (ii)

(1) $\widetilde{A}\otimes_{\widetilde{R}}K=K^{[1]}$.

(2) $\widetilde{A}\otimes_{\widetilde{R}}k(\widetilde{P})$ are geometrically integral for every maximal ideal $\widetilde{P}$ of $\widetilde{R}$.

Hence by (2.4), $\widetilde{A}=\widetilde{R}^{[1]}$. In particular $\widetilde{A}\otimes_{\widetilde{R}}k(\widetilde{P})=k(\widetilde{P})^{[1]}\ \forall\, \widetilde{P}\in Max\ \widetilde{R}$ showing that $A\otimes_R k(P)$ is an $\mathbf{A}^1$-form over $k(P)$.

We now consider the case ht $P\geq 2$. By induction hypothesis we assume that $A\otimes_R k(Q)$ are $\mathbf{A}^1$-forms over $k(Q)$ for all non-maximal prime ideals Q of R. Let $\widehat{R}$ denote the completion of R and let $\widehat{A}=A\otimes_R\hat{R}$. Then $\widehat{R}$ is a complete local ring with maximal ideal $\widehat{P}$ such that $R/P\cong\widehat{R}/\widehat{P}$. Now $\widehat{A}$ is a finitely generated faithfully flat $\widehat{R}$-algebra. Moreover for any non-maximal prime ideal $\overline{Q}$ of $\widehat{R}$, $\overline{Q}\cap R\neq P$ so that $\widehat{A}\otimes_{\widehat{R}}k(\overline{Q})$ is an $\mathbf{A}^1$-form over $k(\overline{Q})$. Thus replacing R by $\widehat{R}$ we may assume R to be a complete local noetherian ring and further replacing R by R/Q_0 where Q_0 is a mininal prime ideal of R we may in fact assume R to be a *complete local noetherian domain* with maximal ideal P such that the fibres of ϕ at all non-closed points are $\mathbf{A}^1$-forms and the fibre at the generic point is $\mathbf{A}^1$.

Let K be the quotient field of R. Since R is a *complete* local ring the normalisation $\widetilde{R}$ of R is a finite R-module and hence is a *noetherian normal local domain*. Let $\widetilde{A}=A\otimes_R\widetilde{R}$. Clearly $\widetilde{A}$ is a finitely generated faithfully flat $\widetilde{R}$-module and by (i), $\widetilde{A}\otimes_{\widetilde{R}}K$ $(=A\otimes_R K)=K^{[1]}$. Now let $\widetilde{Q}$ be a prime ideal of $\widetilde{R}$ of height one. Then $Q=\widetilde{Q}\cap R\neq P$ and hence $A\otimes_R k(Q)$ and therefore $\widetilde{A}\otimes_{\widetilde{R}}k(\widetilde{Q})$ is an $\mathbf{A}^1$-form over $k(\widetilde{Q})$. In particular $\widetilde{A}\otimes_{\widetilde{R}}k(\widetilde{Q})$ is geometrically integral. Thus by Theorem 3.4, $\widetilde{A}=\widetilde{R}^{[1]}$. In particular $\widetilde{A}\otimes_{\widetilde{R}}k(\widetilde{P})=\mathrm{k}(\widetilde{P})^{[1]}$, where $\widetilde{P}$ is the unique maximal ideal of $\widetilde{R}$. Hence $A\otimes_R k(P)$ is an $\mathbf{A}^1$-form over $k(P)$.

Thus all fibres of ϕ are $\mathbf{A}^1$-forms. Since separable $\mathbf{A}^1$-forms over a field are trivial it follows that if Y is a $\mathbf{Q}$-scheme then X is actually an $\mathbf{A}^1$-fibration over Y. This completes the proof of the theorem.

Remark 3.6. The above proof shows that in the statement of Theorem 3.5 it is enough to assume in (i) that the generic fibres are $\mathbf{A}^1$-forms. (In the proof take $\widetilde{R}$ to be the integral closure of R in L where L is a finite extension of K such that $A\otimes_R L=L^{[1]}$).

Remark 3.7. In the situation of Theorem 3.5 if Y is neither normal nor a $\mathbf{Q}$-scheme then all fibres being A^1-forms does not imply that all fibres are actually $\mathbf{A}^1$ (see [B-D], 4.2). Also it is well-known that if Y is a non-normal scheme then an $\mathbf{A}^1$-fibration need not be an $\mathbf{A}^1$-bundle even if Y is a $\mathbf{Q}$-scheme (see [K-W], 3.4).

Remark 3.8. S.M. Bhatwadekar has pointed out as a corollary to Theorem 3.5 that *when Y is a reduced affine scheme then under the hypotheses of Theorem B, there actually exists a finite surjective morphism $\psi: Y'\to Y$ from an affine noetherian scheme Y' such that $X\times_Y Y'$ is a line bundle over Y'.*

4. Examples

In this section we mention some examples to illustrate that the hypotheses in our theorems cannot be relaxed.

Note that the example ([K-W], 3.2) of A. Bialyniki-Birula shows that the assumption that the morphism $\phi: X\to Y$ is "affine" is necessary in our results. It is also easy to construct examples to show that the condition of "faithful flatness" is essential. For instance, let Y be the affine plane *Spec* $(k[x, y])$, k a field, and let $X=$ *Spec* A where $A=k[x, y][u, v]/(xu+yv-1)$. Then the fibres of A at all prime ideals of $k[x, y]$ *of* height≤ 1 are $\mathbf{A}^1$ but $(x, y)A=A$.

We now discuss the finiteness condition. Note that in ([B-D], 3.11) it was shown that if Y is an affine scheme and X is a flat affine Y-scheme *dominated by some affine n-space over Y*, then the condition that "$\phi: X\to Y$ is a morphism of finite type" can be deduced from appropriate fibre conditions. But below we quote an interesting example of Bhatwadekar which illustrates that in our situation (i.e., without the additional assumption that X is dominated by $\mathbf{A}^n_Y$), the condition that ϕ is a "morphism of finite type" cannot be deduced from other conditions even when Y is the affine line and X an affine *noetherian* regular factorial integral scheme dominated by another affine noetherian regular factorial integral scheme Z which is of finite type over Y.

Example 4.1 (Bhatwadekar). Let $R=\mathbf{C}[x]$ and $K=\mathbf{C}(x)$. Choose an element y in $\mathbf{C}[[x]]$ which is transcendental over $\mathbf{C}(x)$ and let $B=\mathbf{C}[x, 1/x, y]$, $A=B\cap\mathbf{C}[[x]]$. Let $Y=Spec\ R$, $X=Spec\ A$ *and* $Z=Spec\ B$.

We first show that A is a factorial domain. Note that $R[y, 1/x]=A[1/x]=B$ are all factorial domains. Now it is easy to see that $x\mathbf{C}[[x]]\cap A=xA$ so that $A/xA\cong\mathbf{C}$, in particular, x is a prime element of A. Therefore, as $\bigcap_{n\geq 0} x^nA\hookrightarrow\bigcap_{n\geq 0} x^n\mathbf{C}[[x]]=(0)$, it follows by Nagata's criterion that A is a factorial domain.

We now show that A is noetherian. By a theorem of Cohen it is enough to show that every prime ideal of A is finitely generated. Fix $P\in Spec\ A$. If $x\in P$ then as xA is a maximal ideal of A, $P=xA$ is in fact principal. So assume $x\notin P$. In this case xA and P are comaximal so that there exists $c\in P$ such that $(x, c)A=A$. Thus $A/cA=A[1/x]/cA[1/x]=B/cB$ which is of course noetherian. Therefore P/cA is finitely generated and hence P is finitely generated. Thus A is a noetherian factorial domain.

Now R being a P.I.D., A is obviously faithfully flat over R. As $A[1/x]=R[1/x, y]$ it follows that $A\otimes_R K=K^{[1]}$ and $A/(x-a)A\neq\mathbf{C}^{[1]}$ for all $a\neq 0$. We already showed that $A/xA\cong\mathbf{C}$. Thus all the hypotheses of (2.4) (except that A is a finitely generated R-algebra) are satisfied. But as $A/xA\cong\mathbf{C}$, clearly A is not finitely generated over R.

In ([B-D], 3.10 and 3.12) it was shown that if Y is an affine integral scheme which is either normal or a $\mathbf{Q}$-scheme and if X is a flat affine Y-scheme *dominated by the affine n-spac* $\mathbf{A}^n_Y$ *over* Y such that fibre at the generic point of Y is $\mathbf{A}^1$ and the fibres at the generic points of all closed subschemes of Y of codimension one are *integral*, then X is an $\mathbf{A}^1$-fibration over Y. However the following example shows that in our situation the condition of integrality on height one fibres does not imply geometric integrality.

Example 4.2. Let Y be the affine plane *Spec* R where $R=\mathbf{C}[x, y]$. Let $A=R[u, v]/(yu+x-v^2)$ and $X=Spec\ A$. A is a free module over $R[u]$ and hence over R. The generic fibre is $\mathbf{A}^1$. For any prime ideal P of R with $y\notin P$, $A\otimes_R k(P)\cong k(P)^{[1]}$. However for $P=yR$ the fibre is integral but not geometrically integral.

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