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Homology and cohomology of Lie superalgebra $\mathfrak{sl}(2, 1)$ with coefficients in the spaces of finite-dimensional irreducible reprsentations

By

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Introduction

In this paper we give a method of calculation for homology groups and cohomology groups with coefficients in a space of a representation of a Lie superalgebra, and carry out the calculation to determine them for finite-dimensional irreducible $\mathfrak{sl}(2, 1)$-modules.

A motivation of this work is to understand to what extent the idea of the cohomological induction is useful for representations of Lie superalgebras. In the case of reductive Lie algebras, the theory of cohomological induction was introduced by Vogan and Zuckerman and was developed by Vogan, Knapp and others (see e.g. [8]). Studies of the structures of cohomologically induced modules, vanishing theorems and Blattner's multiplicity formula are very important, and there, the Poincaré duality plays a decisive role.

In the case of Lie superalgebras, Chemla [2] proved a Poincaré duality under a restrictive condition that the representation in question has a finite projective dimension. However, we do not know if this restrictive condition is really neccessary and when it holds. In author's previous work [13] (see also [15]), we see that Poincaré duality does not hold, except the trivial case, for finite-dimensional representations of $\mathfrak{gl}(1, 1)$. This situation is also true for the present case of $\mathfrak{sl}(2, 1)$.

In [6], Kac constructed all finite-dimensional irreducible representations of Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{sl}(m, n)$ as the quotient module $V(\Lambda) = \bar{V}(\Lambda)/I(\Lambda)$ of a standard induced module $\bar{V}(\Lambda)$ by its maximal proper submodule $I(\Lambda)$. Furutsu studied these modules in detail for $\mathfrak{sl}(2, 1)$ and $\mathfrak{sl}(3, 1)$ in [5]. In the case of $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{sl}(2, 1)$, $\bar{V}(\Lambda)$ is constructed starting from an irreducible highest weight $\mathfrak{g}_{\bar{0}}$-module with highest weight Λ. According to $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(2, \mathbf{C}) \oplus \mathbf{C} \cdot C$, the highest weight Λ is given as $\Lambda = (\lambda, c)$, where $\lambda \in \mathbf{Z}_{\geq 0}$ is a highest weight for $\mathfrak{sl}(2, \mathbf{C})$ and c is a scalar for C which is in the center of $\mathfrak{g}_{\bar{0}}$. We can construct all finite-dimensional irreducible modules of $\mathfrak{sl}(2, 1)$ quite explicitly. Further, we find that any such irreducible module is equivalent to one of $\bar{V}(\Lambda)$ or $I(\Lambda)$, and that as $\mathfrak{g}_{\bar{0}}$-modules $\bar{V}(\Lambda)$ is a direct sum of four

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irreducible $\mathfrak{g}_{\bar{0}}$-submodules while $I(\Lambda)$ is that of two such ones.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over $\mathbf{C}$. We have a projective resolution of the trivial $\mathfrak{g}$-module $\mathbf{C}$ in a category of $\mathfrak{g}$-modules, which is an analogue of Koszul resolution for Lie algebras (cf. [10] and also [12]). In [12] we introduced a method of calculating homology and cohomology for representations of a Lie superalgebra. This method is rather complicated, but we give in this paper a short cut via Lemma 1.6. This lemma reduces the chain complex (B, ∂) to be studied to a simpler complex $(B^{\mathfrak{q}}, \partial|_{B^{\mathfrak{q}}})$ consisting of $\mathfrak{q}$-invariants, where $\mathfrak{q}$ is a subalgebra of $\mathfrak{g}$ which acts on (B, ∂) semisimply. Cohomology groups are obtained by the duality between homology and cohomology in Lemma 1.7 after calculating homology groups.

This paper is organized as follows. In §1, we first recall an explicit description of finite-dimensional irreducible modules of $\mathfrak{sl}(2, 1)$. Then we introduce a practical method of calculating homology through a chain complex (1.4–6).

The induced module $\bar{V}(\Lambda)$ with highest weight $\Lambda = (\lambda, c)$ is irreducible if and only if $(\lambda - c)(\lambda + c + 2) \neq 0$ (cf. Corollary 1.2). We discuss according to the three cases $(\lambda - c)(\lambda + c + 2) \neq 0$, $\lambda - c = 0$, and $\lambda + c + 2 = 0$. In §2, we calculate the homology $\mathrm{H}_n(\mathfrak{g}, V(\Lambda))$, for $\Lambda = (\lambda, c)$ with $(\lambda - c)(\lambda + c + 2) \neq 0$. In this case $V(\Lambda) = \bar{V}(\Lambda)$. Studying $\mathfrak{g}_{\bar{0}}$-module structures in the complex (1.4) for $\bar{V}(\Lambda)$ and applying Lemma 1.6 we find that homology groups $\mathrm{H}_n(\mathfrak{g}, \bar{V}(\Lambda))$ vanish for any n.

In §3, we describe the homology $\mathrm{H}_n(\mathfrak{g}, V(\Lambda))$ in case $\lambda = c \geq 0$. When $\lambda \geq 1$, $V(\Lambda) \cong I(\Lambda')$ with $\Lambda' = (\lambda - 1, c - 1)$. For sufficiently large n, the spaces of n-chains in the chain complex of $\mathfrak{g}_{\bar{0}}$-invariants in (1.8) for $I(\Lambda')$ are always 8-dimensional. Fixing standard bases of these spaces, we can express the boundary operators by (8×8)-matrices. Calculating the ranks of these matrices, we get the dimensions of homology groups $\mathrm{H}_n(\mathfrak{g}, I(\Lambda'))$. When $\lambda = 0$, $V(\Lambda)$ is a trivial $\mathfrak{sl}(2, 1)$-module and the n-chains are 4-dimensional for $n \geq 4$.

In §4, the case $\lambda + c + 2 = 0$ is treated. Here we have a similar result as in §3.

Summarizing the results in §§2–4, we get the main result of this article as follows.

Theorem (see Theorem 5.1 and Theorem 5.3). *Let $V(\Lambda)$ be a finite-dimensional irreducible representation of $\mathfrak{g} = \mathfrak{sl}(2, 1)$ with highest weight $\Lambda = (\lambda, c)$, $\lambda \in \mathbf{Z}_{\geq 0}$, $c \in \mathbf{C}$.*
If $\lambda = c$, then

$$\dim \mathrm{H}_n(\mathfrak{g}, V(\Lambda)) = \dim \mathrm{H}^n(\mathfrak{g}, V(\Lambda)) = \begin{cases} 1 & \text{for } n = \lambda, \lambda + 3 \\ 0 & \text{otherwise.} \end{cases}$$

If $\lambda + c + 2 = 0$, then

$$\dim \mathrm{H}_n(\mathfrak{g}, V(\Lambda)) = \dim \mathrm{H}^n(\mathfrak{g}, V(\Lambda)) = \begin{cases} 1 & \text{for } n = \lambda + 1, \lambda + 4 \\ 0 & \text{otherwise.} \end{cases}$$

If $(\lambda - c)(\lambda + c + 2) \neq 0$, *then*

$$\mathrm{H}_n(\mathfrak{g}, V(\Lambda)) = \mathrm{H}^n(\mathfrak{g}, V(\Lambda)) = (0) \qquad \textit{for any} \quad n \geq 0.$$

§1. Preliminaries

1.1. Definitions and notations for Lie superalgebras. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbf{Z}_2$-graded vector space over $\mathbf{C}$, where $\mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$. The algebra End V of all linear maps from V into itself becomes an associative superalgebra if we define a gradation as

$$\mathrm{End}_i V := \{X \in \mathrm{End}\, V \mid X V_k \subset V_{i+k},\ k \in \mathbf{Z}_2\} \qquad \text{for } i \in \mathbf{Z}_2.$$

Then introducing a bracket operation in $\mathfrak{g} = \mathrm{End}\, V$ as

$$[X, Y] := XY - (-1)^{|X||Y|} YX \qquad \text{for homogeneous elements } X, Y \in \mathfrak{g},$$

where $|X|$ means the degree of X, we get a Lie superalgebra, that is, this operation satisfies super-antisymmetry and Jacobi identity for a Lie superalgebra:

$$[X, Y] + (-1)^{|X||Y|}[Y, X] = 0,$$
$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|}[Y, [X, Z]],$$

for $X, Y, Z \in \mathfrak{g}$, homogeneous. From now on, if the notation $|X|$ appears, the element X is assumed to be homogeneous. Let dim $V_{\bar{0}} = m$ and dim $V_{\bar{1}} = n$, then this Lie superalgebra is denoted by $\mathfrak{gl}(m, n)$. In a natural basis of V consistent with the $\mathbf{Z}_2$-gradation, $\mathfrak{gl}(m, n)$ consists of matrices of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, and $\mathfrak{gl}(m, n) = \mathfrak{gl}(m, n)_{\bar{0}} \oplus \mathfrak{gl}(m, n)_{\bar{1}}$ with

$$\mathfrak{gl}(m, n)_{\bar{0}} = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\right\}, \qquad \mathfrak{gl}(m, n)_{\bar{1}} = \left\{\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right\},$$

where $\alpha \in \mathscr{M}(m, m)$, $\beta \in \mathscr{M}(m, n)$, $\gamma \in \mathscr{M}(n, m)$, $\delta \in \mathscr{M}(n, n)$. Here $\mathscr{M}(m, n)$ denotes the set of $m \times n$-matrices over $\mathbf{C}$. Moreover, let

$$\mathfrak{gl}(m, n)_{-1} = \left\{\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}\right\},\ \mathfrak{gl}(m, n)_0 = \mathfrak{gl}(m, n)_{\bar{0}},\ \mathfrak{gl}(m, n)_1 = \left\{\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\right\},$$

and $\mathfrak{gl}(m, n)_k = (0)$ for $|k| \geq 2$. Then we have a $\mathbf{Z}$-gradation on $\mathfrak{gl}(m, n)$ consistent with the $\mathbf{Z}_2$-gradation:

$$\mathfrak{gl}(m, n) = \mathfrak{gl}(m, n)_{-1} \oplus \mathfrak{gl}(m, n)_0 \oplus \mathfrak{gl}(m, n)_1,$$
$$[\mathfrak{gl}(m, n)_a, \mathfrak{gl}(m, n)_b] \subset \mathfrak{gl}(m, n)_{a+b} \ (a, b \in \mathbf{Z}).$$

On $\mathfrak{gl}(m, n)$, define the supertrace str: $\mathfrak{gl}(m, n) \to \mathbf{C}$ by $\mathrm{str}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \mathrm{tr}\, \alpha - \mathrm{tr}\, \delta$, and then define a subalgebra $\mathfrak{sl}(m, n)$ as $\mathfrak{sl}(m, n) := \{X \in \mathfrak{gl}(m, n) \mid \mathrm{str}\, X = 0\}$.

For our later use, we fix the following basis of a Cartan subalgebra $\mathfrak{h}$ of

$\mathfrak{sl}(m, 1)$:

$$H_i = E_{ii} - E_{i+1,i+1} (1 \leq i \leq m-1), \quad C = \sum_{j=1}^{m} E_{jj} + mE_{m+1,m+1},$$

and a basis on the odd part of $\mathfrak{sl}(m, 1)$:

$$X_i = E_{i,m+1}, \ Y_i = E_{m+1,i} \quad (1 \leq i \leq m),$$

where E_{ij} denotes the elementary matrix with 1 as (i, j)-component and 0 elsewhere. In particular, in the case of $\mathfrak{g} = \mathfrak{sl}(2, 1)$, the even part $\mathfrak{g}_{\bar{0}}$ is generated by its Cartan subalgebra $\mathfrak{h} = \langle H, C \rangle_{\mathbf{C}}$ with $H = H_1$, and two elements $Z_+ = E_{12}$ and $Z_- = E_{21}$. Further $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are generated by $\{X_i\}_{i=1,2}$ and $\{Y_i\}_{i=1,2}$ respectively. Here $\langle \mathfrak{A} \rangle_{\mathbf{C}}$ denotes the vector space spanned over $\mathbf{C}$ by a set of vectors $\mathfrak{A}$.

The Grassmann algebra $\wedge \mathfrak{g}$ is defined for a Lie superalgebra $\mathfrak{g}$ as the quotient of the tensor algebra of $\mathfrak{g}$ by the two-sided ideal generated by

$$\{X \otimes Y + (-1)^{|X||Y|} Y \otimes X \mid X, Y \in \mathfrak{g}, \text{ homogeneous}\}.$$

It is a $\mathfrak{g}$-module by the action given as

$$X \cdot (X_1 \wedge \cdots \wedge X_n) = \sum_i (-1)^{|X|(|X_1| + \cdots + |X_{i-1}|)} X_1 \wedge \cdots \wedge [X, X_i] \wedge \cdots \wedge X_n,$$

where $X, X_1, \cdots, X_n \in \mathfrak{g}$, and its $\mathbf{Z}_2$-gradation is determined by

$$|X_1 \wedge \cdots \wedge X_n| = |X_1| + \cdots + |X_n|.$$

We remark that the subalgebra $\wedge \mathfrak{g}_{\bar{1}}$ here is what we usually call a symmetric algebra for a vector space. The universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ is defined as the quotient of the tensor algebra by the two-sided ideal generated by

$$\{X \otimes Y - (-1)^{|X||Y|} Y \otimes X - [X, Y] \mid X, Y \in \mathfrak{g}, \text{ homogeneous}\},$$

and $\mathfrak{g}$-action on it is given as $X \cdot (X_1 \cdots X_n) = X X_1 \cdots X_n$.

1.2. Irreducible modules $V(\Lambda)$ with highest weight Λ. Here we consider $\mathfrak{g} = \mathfrak{sl}(2, 1)$. Note that $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}(2, \mathbf{C}) \oplus \mathbf{C} \cdot C$. Let V_λ be a finite-dimensional highest weight representation of $\mathfrak{sl}(2, \mathbf{C})$ with highest weight $\lambda \in \mathbf{Z}_{\geq 0}$. We can fix a basis $\{v_0, v_1, \cdots, v_\lambda\}$ of V_λ such that

$$Hv_i = (\lambda - 2i)v_i, \ Z_+ v_i = i(\lambda + 1 - i)v_{i-1}, \ Z_- v_i = v_{i+1},$$
$$(i = 0, 1, \cdots, \lambda; \ v_{-1} = v_{\lambda+1} = 0), \tag{1.1}$$

and call this basis standard.

Take a $\mathbf{Z}$-graded subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}$ and extend $\mathfrak{sl}(2, \mathbf{C})$-module V_λ to a $\mathfrak{p}$-module $L(\Lambda)$ by putting

$$Cv = cv, \ Xv = 0 \ (v \in V_\lambda, \ X \in \mathfrak{g}_1),$$

where $c \in \mathbf{C}$ is a fixed constant and $\Lambda = (\lambda, c) \in \mathfrak{h}^*$. We define an induced module $\bar{V}(\Lambda) = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L(\Lambda) := \mathscr{U}(\mathfrak{g}) \otimes_{\mathfrak{p}} L(\Lambda)$, where the $\mathfrak{g}$-module structure is given by

$$X(u \otimes v) = (Xu) \otimes v \qquad (X \in \mathfrak{g},\ u \in \mathscr{U}(\mathfrak{g}),\ v \in L(\Lambda)).$$

Then $\bar{V}(\Lambda)$ has a unique maximal proper submodule, say $I(\Lambda)$, and the quotient is a unique (up to isomorphisms) irreducible representation of $\mathfrak{sl}(2, 1)$

$$V(\Lambda) = \bar{V}(\Lambda)/I(\Lambda)$$

with highest weight Λ (cf. Kac [6]). Therefore we consider this quotient $\bar{V}(\Lambda)/I(\Lambda) = V(\Lambda)$ later on.

1.3. Finite dimensional irreducible representations of $\mathfrak{sl}(2, 1)$. We have the following thorem from general theory of Kac [6] (see also Furutsu [5]).

Theorem 1.1. *The $\mathfrak{sl}(m, 1)$-module $\bar{V}(\Lambda)$ is irreducible if and only if*

$$\prod_{1 \le k \le m} (\Lambda(H_k') + m - k) \neq 0.$$

Here $H_k' = E_{k,k} - E_{k+1,k+1} \in \mathfrak{sl}(m, 1)$ and Λ is a highest weight of $\bar{V}(\Lambda)$.

From this theorem, we get the following.

Corollary 1.2. *For $\mathfrak{sl}(2, 1)$, the induced module $\bar{V}(\Lambda)$ with $\Lambda = (\lambda, c)$ is irreducible if and only if*

$$(\lambda - c)(\lambda + c + 2) \neq 0.$$

From this corollary, we can list up all finite-dimensional irreducible $\mathfrak{sl}(2, 1)$-modules as follows. They have highest weights $\Lambda = (\lambda, c)$ with $\lambda \in \mathbf{Z}_{\geq 0}$, $c \in \mathbf{C}$.

If $(\lambda - c)(\lambda + c + 2) \neq 0$, we have $V(\Lambda) = \bar{V}(\Lambda)$.

If $\lambda - c = 0$, define two irreducible $\mathfrak{sl}(2, \mathbf{C})$-modules as

$$\begin{aligned} V_{\lambda+1}' &:= \langle v_i' \mid 0 \le i \le \lambda + 1 \rangle_{\mathbf{C}}, & v_i' &:= -i(Y_1 \otimes v_{i-1}) + Y_2 \otimes v_i, \\ V_\lambda'' &:= Y_1 Y_2 \otimes L(\Lambda), & v_i'' &:= Y_1 Y_2 \otimes v_i \ (0 \le i \le \lambda). \end{aligned} \tag{1.2}$$

Then $I(\Lambda) = V_{\lambda+1}' + V_\lambda''$ and it is isomophic to $L(\lambda + 1, c + 1) \oplus L(\lambda, c + 2)$ as $\mathfrak{g}_{\bar{0}}$-module while $V(\Lambda) \cong L(\lambda, c) \oplus L(\lambda - 1, c + 1)$. If $\lambda = c = 0$, then $V(0, 0) = \mathbf{C}$ is a trivial $\mathfrak{g}$-module ($L(-1, 1) = (0)$ by convention).

If $\lambda + c + 2 = 0$, $I(\Lambda)$ is a direct sum of two $\mathfrak{sl}(2, \mathbf{C})$-modules: $I(\Lambda) = \tilde{V}_{\lambda-1}' + V_\lambda''$,

$$\begin{aligned} \tilde{V}_{\lambda-1}' &:= \langle \tilde{v}_i' \mid 0 \le i \le \lambda - 1 \rangle_{\mathbf{C}}, & \tilde{v}_i' &:= (\lambda - i) Y_1 \otimes v_i + Y_2 \otimes v_{i+1} \\ V_\lambda'' &:= Y_1 Y_2 \otimes L(\Lambda), & v_i'' &:= Y_1 Y_2 \otimes v_i \end{aligned} \tag{1.3}$$

and $I(\Lambda) \cong L(\lambda - 1, c + 1) \oplus L(\lambda, c + 2)$, $V(\Lambda) \cong L(\lambda, c) \oplus L(\lambda + 1, c + 1)$. If $\lambda - c = 0$, or $\lambda + c + 2 = 0$, $I(\Lambda)$ is easily seen to be irreducible, so there must exist a highest weight $\Lambda' = (\lambda', c')$ such that $I(\Lambda) \cong V(\Lambda')$. We have $\Lambda' = (\lambda + 1, c + 1)$ in the former case and $\Lambda' = (\lambda - 1, c + 1)$ in the latter case.

So we can realize finite-dimensional irreducible $\mathfrak{sl}(2, 1)$-modules as follows and we use this realization later on.

Lemma 1.3. *The finite-dimensional irreducible* $\mathfrak{sl}(2, 1)$*-modules* $V(\Lambda)$, $\Lambda = (\lambda, c)$, *is equivalent to one of the following*:

$$\begin{array}{ll} \bar{V}(\lambda, c) & \text{if } (\lambda - c)(\lambda + c + 2) \neq 0, \\ I(\lambda - 1, c - 1) & \text{if } \lambda = c \geq 1, \\ I(\lambda + 1, c - 1) & \text{if } \lambda = -c - 2 \geq 0, \end{array}$$

and $V(0, 0) = \mathbf{C}$, *a trivial representation.*

1.4. Killing form. A bilinear form B on a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ given by

$$B(X, Y) := \mathrm{str}(\mathrm{ad}\, X\, \mathrm{ad}\, Y), \qquad X, Y \in \mathfrak{g},$$

is called the Killing form on $\mathfrak{g}$. If B is non-degenerate, the Casimir element Ω for $\mathfrak{g}$ is defined as

$$\Omega = \sum_{1 \leq i_1, i_2 \leq d} B(E_{i_1}, E_{i_2}) F_{i_1} F_{i_2},$$

where $d = \dim \mathfrak{g}$, and $(E_i)_{1 \leq i \leq d}$, $(F_i)_{1 \leq i \leq d}$ are dual bases of $\mathfrak{g}$ such that $B(E_i, F_j) = \delta_{ij}$ $(1 \leq i, j \leq d)$.

In our case of $\mathfrak{sl}(2, 1)$, the Casimir element Ω is

$$\Omega = -\frac{1}{4} C^2 + \frac{1}{4} H^2 + Z_- Z_+ + \frac{1}{2} H - \frac{1}{2} C + X_1 Y_1 + X_2 Y_2.$$

It acts on $\bar{V}(\Lambda)$ as a scalar multiple by $\frac{1}{4}(\lambda - c)(\lambda + c + 2)$.

1.5. Koszul resolution and its application. To calculate the homology groups of finite-dimensional irreducible representations of Lie superalgebra $\mathfrak{g} = \mathfrak{sl}(2, 1)$, we start with recalling some general results in the theory of the homology of representations. Let $\mathfrak{g}$ be a Lie superalgebra and V a $\mathfrak{g}$-module. Take a projective resolution of V:

$$0 \longleftarrow V \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow P_3 \longleftarrow \cdots.$$

The homology $\mathrm{H}_n(\mathfrak{g}, V)$ is by definition the homology of the derived functor of the functor $(\cdot) \otimes_{\mathfrak{g}} \mathbf{C}$, where $\mathbf{C}$ is the trivial $\mathfrak{g}$-module. In other words, denote by $P_i{}^{\mathfrak{g}}$ the space of $\mathfrak{g}$-invariants in P_i, then $\mathrm{H}_n(\mathfrak{g}, V)$ is the n-th homology of the chain complex

$$0 \longleftarrow P_0^{\mathfrak{g}} \longleftarrow P_1^{\mathfrak{g}} \longleftarrow P_2^{\mathfrak{g}} \longleftarrow P_3^{\mathfrak{g}} \longleftarrow \cdots.$$

The Koszul resolutions of $\mathbf{C}$ in the categories of modules of Lie algebras are well-known. In the case of Lie superalgebras, the author introduced a similar resolution, given below, for Lie superalgebras in her master's thesis [12], which is another expression of the Koszul complex in [10, p. 171].

Theorem 1.4 ([12]). *Let* $\mathfrak{g}$ *be a Lie superalgebra. The following complex* (A, ∂) *is a projective resolution of the trivial module* $\mathbf{C}$ *in the category of* $\mathfrak{g}$*-modules*:

$$0 \longleftarrow \mathbf{C} \xleftarrow{\partial_{-1}} A_0 \xleftarrow{\partial_0} A_1 \xleftarrow{\partial_1} A_2 \xleftarrow{\partial_2} \cdots,$$

with $A_n := \mathscr{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \wedge^n \mathfrak{g}$, $\partial_{-1}(u) :=$ (*the constant term of* u), *and for* $n \geq 0$,

$$\partial_{n-1}(u \otimes X_1 \wedge \cdots \wedge X_n) =$$
$$= \sum_{i=1}^{n} (-1)^{i+1+\eta_i} uX_i \otimes X_1 \wedge \cdots \hat{i} \cdots \wedge X_n$$
$$+ \sum_{k<l} (-1)^{k+l+\eta_k+\eta_l+\xi_k\xi_l} u \otimes [X_k, X_l] \wedge X_1 \wedge \cdots \hat{k} \cdots \hat{l} \cdots \wedge X_n,$$

where $u \in \mathscr{U}(\mathfrak{g})$, $X_i \in \mathfrak{g}$, $\xi_i = |X_i|$, $\eta_i = \xi_i(\xi_1 + \cdots + \xi_{i-1})$, *and the symbol* $\hat{i}$ *indicates a term* X_i *to be omitted.*

If V is a $\mathfrak{g}$-module, the functor $(\cdot) \otimes_{\mathbf{C}} V$ gives a projective resolution of V induced from the resolution in Theorem 1.4. Further, by the functor $(\cdot) \otimes_{\mathfrak{g}} \mathbf{C}$, we have the following complex (B, ∂):

$$0 \longleftarrow B_0 \xleftarrow{\partial_0} B_1 \xleftarrow{\partial_1} B_2 \xleftarrow{\partial_2} B_3 \xleftarrow{\partial_3} \cdots, \tag{1.4}$$

$$B_n = \wedge^n \mathfrak{g} \otimes V, \tag{1.5}$$

$$\partial_{n-1}(X_1 \wedge \cdots \wedge X_n \otimes v) =$$
$$= \sum_{i=1}^{n} (-1)^{i+\eta'_i} X_1 \wedge \cdots \hat{i} \cdots \wedge X_n \otimes X_i v \tag{1.6}$$
$$+ \sum_{k<l} (-1)^{k+l+\eta_k+\eta_l+\xi_k\xi_l} [X_k, X_l] \wedge X_1 \wedge \cdots \hat{k} \cdots \hat{l} \cdots \wedge X_n \otimes v,$$

where $X_i \in \mathfrak{g}$, $v \in V$, $\xi_i = |X_i|$, $\eta_i = \xi_i(\xi_1 + \cdots + \xi_{i-1})$, $\eta'_i = \xi_i(\xi_{i+1} + \cdots + \xi_n)$. The module structure of B_n is given by a natural action ρ_n:

$$\rho_n(X)(\theta \otimes v) = (X\theta) \otimes v + (-1)^{|X||\theta|} \theta \otimes (Xv)$$

with $X \in \mathfrak{g}$, $\theta \in \wedge^n \mathfrak{g}$, $v \in V$. The system $\{\rho_n\}$ commutes with $\{\partial_n\}$: $\partial_n \circ \rho_{n+1} = \rho_n \circ \partial_n$. The homology group $\mathrm{H}_n(\mathfrak{g}, V)$ can be computed as $\mathrm{Ker}\, \partial_{n-1}/\mathrm{Im}\, \partial_n$.

The following lemma is known as Shapiro's lemma in the case of Lie algebras, and we can prove it for Lie superalgebras similarly.

Lemma 1.5. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie superalgebra and* $\mathfrak{p}$ *its subalgebra. If* V *is a* $\mathfrak{p}$*-module, there is the following natural isomorphism of vector spaces*:

$$\mathrm{H}_n(\mathfrak{g}, \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V) \cong \mathrm{H}_n(\mathfrak{p}, V) \qquad (n=0, 1, 2, \cdots). \tag{1.7}$$

We appeal to the following lemma to calculate the homology groups.

Lemma 1.6. *Let $\mathfrak{g}$ be a Lie superalebra, V a $\mathfrak{g}$-module, and (B, ∂) the chain complex introduced just before Lemma 1.5. Let $\mathfrak{q}$ be a subalgebra of $\mathfrak{g}$ such that its natural representation $\rho_n|_{\mathfrak{q}}$ on the n-th chain B_n are all semisimple. Then the homology $\mathrm{H}_n(\mathfrak{g}, V)$ can be obtained from the following subcomplex $(B^{\mathfrak{q}}, \partial|_{B^{\mathfrak{q}}})$ as* $\mathrm{Ker}\,(\partial_{n-1}|_{B^{\mathfrak{q}}})/\mathrm{Im}\,(\partial_n|_{B^{\mathfrak{q}}})$:

$$0 \longleftarrow B_0{}^{\mathfrak{q}} \overset{\partial_0}{\longleftarrow} B_1{}^{\mathfrak{q}} \overset{\partial_1}{\longleftarrow} B_2{}^{\mathfrak{q}} \overset{\partial_2}{\longleftarrow} B_3{}^{\mathfrak{q}} \overset{\partial_3}{\longleftarrow} \cdots, \tag{1.8}$$

where $B_n{}^{\mathfrak{q}}$ is the space of $\mathfrak{q}$-invariants in B_n, and $\partial_n|_{B^{\mathfrak{q}}}$ is denoted again by ∂_n.

Proof. Since the representation ρ_n is semisimple, we can take $\mathfrak{q}$-stable subspace B'_n such that $B_n = B_n{}^{\mathfrak{q}} \oplus B'_n$, and then $\partial_n(B'_{n+1}) \subset B'_n$. The complex (B, ∂) is the direct sum of two subcomplexes $(B^{\mathfrak{q}}, \partial|_{B^{\mathfrak{q}}})$ and $(B', \partial|_{B'})$.

Consider maps $s_n(X)\colon B_n \to B_{n+1}$, given as

$$s_n(X)(X_1 \wedge \cdots \wedge X_n \otimes v) = X \wedge X_1 \wedge \cdots \wedge X_n \otimes v.$$

By a simple calculation, we get an equality $s_{n-1}(X) \circ \partial_{n-1} + \partial_n \circ s_n(X) = \rho_n(X)$. So the maps $\rho_n(X)$ are homotopic to 0 with homotopies $s_n(X)$, or, they induce 0-maps on homology groups $\mathrm{H}_n(\mathfrak{g}, V)$. This means that $\rho_n(X)(\mathrm{Ker}\,\partial_{n-1}) \subset \mathrm{Im}\,\partial_n$ for $X \in \mathfrak{q}$. Take an invariant complement T_n of the invariant subspace $\mathrm{Im}\,\partial_n$ in the module $\mathrm{Ker}\,\partial_{n-1}$. We have $\rho_n(X)T_n = (0)$ because $\rho_n(X)T_n \subset T_n \cap \mathrm{Im}\,\partial_n$. So $T_n \subset B_n{}^{\mathfrak{q}}$, and every element in homology is represented by some element in $T_n \subset B_n{}^{\mathfrak{q}}$, which is contained in the subcomplex $(B^{\mathfrak{q}}, \partial|_{B^{\mathfrak{q}}})$. Q.E.D.

1.6. Cohomology groups of Lie superalgebras. The cohomology $\mathrm{H}^n(\mathfrak{g}, V)$ is by definition the cohomology of the derived functor of the functor $\mathrm{Hom}_{\mathfrak{g}}(\cdot, V)$. Similarly to homology, the cohomology group $\mathrm{H}^n(\mathfrak{g}, V)$ can be obtained as $\mathrm{Ker}\,d_n/\mathrm{Im}\,d_{n-1}$ of the following complex (C, d):

$$0 \longrightarrow C_0 \overset{d_0}{\longrightarrow} C_1 \overset{d_1}{\longrightarrow} C_2 \overset{d_2}{\longrightarrow} C_3 \overset{d_3}{\longrightarrow} \cdots,$$

$$C_n = \mathrm{Hom}\,(\wedge^n \mathfrak{g}, V),$$

$$\begin{aligned}(d_{n-1}\phi)(X_1 \wedge \cdots \wedge X_n) &= \\ &= \sum_{i=1}^{n} (-1)^{i+\eta_i+\xi_i|\phi|} X_i \phi(X_1 \wedge \cdots \hat{i} \cdots \wedge X_n) \\ &\quad + \sum_{k<l} (-1)^{k+l+\eta_k+\eta_l+\xi_k\xi_l} \phi([X_k, X_l] \wedge X_1 \wedge \cdots \hat{k} \cdots \hat{l} \cdots \wedge X_n),\end{aligned}$$

where $X_i \in \mathfrak{g}$, $\xi_i = |X_i|$, $\eta_i = \xi_i(\xi_1 + \cdots + \xi_{i-1})$, $\phi \in C_{n-1}$, homogeneous. In our present case, where the $\mathfrak{g}$-module V is finite-dimensional, we have the following duality between homology and cohomology (cf. [8, p. 288]).

Lemma 1.7 (Duality). *Assume that a Lie superalgebra* $\mathfrak{g}$ *and a* $\mathfrak{g}$*-module* V *are both finite-dimensional. Let* V^* *be the dual* $\mathfrak{g}$*-module of* V*. Then there is a natural* $\mathfrak{g}$*-module isomorphism*

$$\mathrm{H}^n(\mathfrak{g}, V)^* \cong \mathrm{H}_n(\mathfrak{g}, V^*).$$

§2. Homology groups for the induced modules $\bar{V}(\Lambda)$

2.1. Application of Lemma 1.6. In this section, we calculate the homology $\mathrm{H}_n(\mathfrak{g}, \bar{V}(\Lambda))$ with $\Lambda = (\lambda, c)$ for $(\lambda - c)(\lambda + c + 2) \neq 0$. First of all, Lemma 1.5 gives the following isomorphism:

$$\mathrm{H}_n(\mathfrak{g}, \bar{V}(\Lambda)) = \mathrm{H}_n(\mathfrak{g}, \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L(\Lambda)) \cong \mathrm{H}_n(\mathfrak{p}, L(\Lambda)). \tag{2.1}$$

Here $L(\Lambda)$ is the $\mathfrak{p}$-module given in §1.3.

We can calculate the homology $\mathrm{H}_n(\mathfrak{p}, L(\Lambda))$ from the following complex:

$$0 \longleftarrow B_0 \xleftarrow{\partial_0} B_1 \xleftarrow{\partial_1} B_2 \xleftarrow{\partial_2} B_3 \xleftarrow{\partial_3} \cdots, \tag{2.2}$$

where $B_n = \wedge^n \mathfrak{p} \otimes L(\Lambda)$.

Let us apply Lemma 1.6. We take a subalgebra $\mathfrak{g}_{\bar{0}} = \mathbf{C} \cdot C \oplus \mathfrak{sl}(2, \mathbf{C})$ as $\mathfrak{q}$, and consider its representations ρ_n on B_n given by (1.7), which are semisimple for all n. First we decompose B_n into a direct sum of $\mathfrak{g}_{\bar{0}}$-submodules

$$B_n = \wedge^n \mathfrak{p} \otimes L(\Lambda) \cong \bigoplus_{a+b=n} \wedge^a \mathfrak{g}_{\bar{0}} \otimes \wedge^b \mathfrak{g}_1 \otimes L(\Lambda) \ (a = 0, 1, 2, 3, 4;\ b \in \mathbf{Z}_{\geq 0})$$

as $\mathfrak{g}_{\bar{0}}$-modules. Since the eigenvalue of $\rho_n(C)$ on $\wedge^a \mathfrak{g}_{\bar{0}} \otimes \wedge^b \mathfrak{g}_1 \otimes L(\Lambda)$ is equal to $-b + c$, the space of $\mathfrak{g}_{\bar{0}}$-invariants $B_n^{\mathfrak{g}_{\bar{0}}}$ is contained in the direct sum of submodules with $-b + c = 0$.

We construct a subcomplex of (2.2) consisting of submodules B_n^C of $\rho_n(C)$-invariants in B_n:

$$0 \longleftarrow B_c^C \xleftarrow{\partial_c} B_{c+1}^C \xleftarrow{\partial_{c+1}} B_{c+2}^C \xleftarrow{\partial_{c+2}} B_{c+3}^C \xleftarrow{\partial_{c+3}} B_{c+4}^C \longleftarrow 0. \tag{2.3}$$

Here, since $-b + c = 0$, B_{c+a}^C is given by

$$B_{c+a}^C = (\wedge^a \mathfrak{g}_{\bar{0}} \otimes \wedge^c \mathfrak{g}_1) \otimes L(\Lambda) \cong \wedge^a \mathfrak{g}_{\bar{0}} \otimes \{\wedge^c \mathfrak{g}_1 \otimes L(\Lambda)\} \qquad (a = 0, 1, 2, 3, 4).$$

Next let us consider the subalgebra $\mathfrak{sl}(2, \mathbf{C}) \subset \mathfrak{q} = \mathfrak{g}_{\bar{0}}$ and its action on $\wedge^n \mathfrak{p}$. As $\mathfrak{sl}(2, \mathbf{C})$-modules, we have isomorphisms:

$$\begin{gathered} \mathfrak{g}_{\bar{0}} \cong \mathbf{C} \oplus V_2,\ \wedge^2 \mathfrak{g}_{\bar{0}} \cong [2] V_2,\ \wedge^3 \mathfrak{g}_{\bar{0}} \cong \mathbf{C} \oplus V_2,\ \wedge^4 \mathfrak{g}_{\bar{0}} \cong \mathbf{C}, \\ \wedge^c \mathfrak{g}_1 \cong V_c, \end{gathered} \tag{2.4}$$

where the symbol $[\cdot]$ expresses the multiplicity, and $\mathbf{C}$ denotes the trivial $\mathfrak{sl}(2, \mathbf{C})$-module $(= V_0)$. We introduce eight subspaces of $\wedge \mathfrak{g}_{\bar{0}}$ as follows.

$$\mathscr{X}^0(1)_2 = \langle Z_+,\ -H,\ -2Z_-\rangle_{\mathbf{C}}, \qquad \mathscr{X}^1(1)_0 = \mathbf{C}\cdot C,$$
$$\mathscr{X}^0(2)_2 = \langle H\wedge Z_+,\ -2Z_+\wedge Z_-,\ 2H\wedge Z_-\rangle_{\mathbf{C}}, \quad \mathscr{X}^0(3)_0 = \mathbf{C}\cdot H\wedge Z_+\wedge Z_-,$$
$$\mathscr{X}^1(2)_2 = \langle C\wedge Z_+,\ -C\wedge H,\ -2C\wedge Z_-\rangle_{\mathbf{C}}, \quad \mathscr{X}^1(4)_0 = \mathbf{C}\cdot C\wedge H\wedge Z_+\wedge Z_-.$$
$$\mathscr{X}^1(3)_2 = \langle C\wedge H\wedge Z_+,\ -2C\wedge Z_+\wedge Z_-,\ 2C\wedge H\wedge Z_-\rangle_{\mathbf{C}}, \qquad \mathscr{X}^0(0)_0 = \mathbf{C}, \tag{2.5}$$

Here $\mathscr{X}^j(i)_2$ and $\mathscr{X}^m(n)_0$ are 3-dimensional and trivial $\mathfrak{sl}(2, \mathbf{C})$-modules and their standard bases will be denoted by $\{z^j(i)_0, z^j(i)_1, z^j(i)_2\}$ and $\{z^m(n)_0\}$ respectively.

We put

$$x_i^c := \frac{1}{(c-i)!}\cdot X_1^{(c-i)}\wedge X_2^{(i)} \quad \text{and} \quad \mathscr{X}_c := \wedge^c \mathfrak{g}_1 = \langle x_i^c \mid 0 \le i \le c\rangle_{\mathbf{C}}. \tag{2.6}$$

The space $\mathscr{X}_c$ is a $(c+1)$-dimensional irreducible $\mathfrak{sl}(2, \mathbf{C})$-module.

2.2. Construction of $\mathfrak{sl}(2, \mathbf{C})$-invariant vectors in the tensor product.

Lemma 2.1. *Let V_n $(n \in \mathbf{Z}_{\ge 0})$ denote an $(n+1)$-dimensional irreducible $\mathfrak{sl}(2, \mathbf{C})$-module. For $k, l \in \mathbf{Z}_{\ge 0}$, the tensor product of two modules V_k and V_l is a direct sum of* $\min(k, l) + 1$ *number of $\mathfrak{sl}(2, \mathbf{C})$-modules as*

$$V_k \otimes V_l \cong \bigoplus_{0 \le r \le \min(k,l)} V_{k+l-2r}, \tag{2.7}$$

and the highest weight vector of V_{k+l-2r} is given as

$$\sum_{i=0}^{r} (-1)^i \frac{(k-i)!\,(l-r+i)!}{i!\,(r-i)!}\cdot v_i \otimes w_{r-i}, \tag{2.8}$$

where $\{v_i\}$ and $\{w_i\}$ are standard bases in V_k and in V_l respectively.

Using this lemma, we see that the module B_n contains non-zero $\mathfrak{g}_{\bar{0}}$-invariant vectors only in the cases $c = \lambda, \lambda \pm 2$. We have excluded the case $c = \lambda$ by assumption, so we treat the cases $c = \lambda \pm 2$.

Let h_0 be a highest weight vector with weight 2 in $\mathscr{X}_{\lambda+2} \otimes V_\lambda$. Further let $\tilde{z}^j(i)$ be a $\mathfrak{g}_{\bar{0}}$-invariant vector in $\mathscr{X}^j(i)_0 \otimes \mathscr{X}_{\lambda+2} \otimes V_\lambda \subset B_{i+\lambda+2}$ with $j \in \{0, 1\}$, $i \in \{1, 2, 3\}$, which is unique up to a scalar multiplication. The vectors h_0 and $\tilde{z}^j(i)$ can be written as

$$h_0 = \sum_{k=0}^{\lambda} (-1)^k (\lambda + 1 - k)(\lambda + 2 - k)\, x_k^{\lambda+2} \otimes v_{\lambda-k},$$
$$\tilde{z}^j(i) = \sum_{k=0}^{2} (-1)^k z^j(i)_k \otimes h_{2-k}, \tag{2.9}$$

where $h_i := Z_- h_{i-1} \in \mathscr{X}_{\lambda+2} \otimes V_\lambda$ is defined inductively (cf. (1.1)). Taking $\mathfrak{g}_{\bar{0}}$-invariants, we can reduce the complex (2.3) to the following:

$$0 \longleftarrow \mathbf{C} \cdot \tilde{z}^0(1) \overset{\partial_3}{\longleftarrow} \langle \tilde{z}^0(2), \tilde{z}^1(2) \rangle \overset{\partial_4}{\longleftarrow} C \cdot \tilde{z}^1(3) \longleftarrow 0. \tag{2.10}$$

2.3. Homology $\mathrm{H}_n(\mathfrak{g}, \bar{V}(\Lambda))$. Now, let us write down the map ∂ precisely. If $x \in \wedge \mathfrak{sl}(2, \mathbf{C}) \otimes \mathscr{X}_c \otimes V(\Lambda)$, then we have $\partial(C \otimes x) = -C \otimes \partial(x)$. Therefore the equality $\partial(\tilde{z}^0(1)) = 0$ gives that $\partial(\tilde{z}^1(2)) = \partial(C \otimes \tilde{z}^0(1)) = 0$. From some calculations, we deduce

$$\partial(\tilde{z}^0(2)) = 2\tilde{z}^0(1), \quad \partial(\tilde{z}^1(3)) = -2\tilde{z}^1(2).$$

Thus, dimensions of $\mathrm{Im}\, \partial_n$ and $\mathrm{Ker}\, \partial_n$ are given in Table 2.11 below. From this we see that the homology of $\bar{V}(\Lambda)$ vanishes if $c = \lambda + 2$.

n	$\cdots$	$\lambda+3$	$\lambda+4$	$\lambda+5$	$\cdots$
$\dim D_n$	0	1	2	1	$\cdots$
$\dim(\mathrm{Ker}\, \partial_{n-1})$	0	1	1	0	$\cdots$
$\dim(\mathrm{Im}\, \partial_n)$	0	1	1	0	$\cdots$

Table 2.11

Similar caluculations can be carried out for $c = \lambda - 2$.

Now we have the following result.

Theorem 2.2. *All the homology groups* $\mathrm{H}_n(\mathfrak{g}, \bar{V}(\Lambda))$ *with* $\Lambda = (\lambda, c)$ *vanish if* $(\lambda - c)(\lambda + c + 2) \neq 0$.

§ 3. Homology groups for the maximal submodules $I(\Lambda)$ for $\lambda = c \geq 0$

In §§ 3.1–3.3, we compute the homology $\mathrm{H}_n(\mathfrak{g}, I(\Lambda))$ for $I(\Lambda) \subset \bar{V}(\Lambda)$ for $\lambda = c \geq 0$, and in § 3.4, the homology $\mathrm{H}_n(\mathfrak{g}, \mathbf{C})$ with trivial $\mathfrak{sl}(2, 1)$-module $\mathbf{C}$.

3.1. Space of $\mathfrak{g}_{\bar{0}}$-invariants in $\wedge \mathfrak{g} \otimes I(\Lambda)$. We consider the following complex:

$$0 \longleftarrow I(\Lambda) \longleftarrow \mathfrak{g} \otimes I(\Lambda) \longleftarrow \wedge^2 \mathfrak{g} \otimes I(\Lambda) \longleftarrow \wedge^3 \mathfrak{g} \otimes I(\Lambda) \longleftarrow \cdots, \tag{3.1}$$

obtained from (1.4) by putting $V = I(\Lambda)$. To apply Lemma 1.6, we take $\mathfrak{q} = \mathfrak{g}_{\bar{0}}$ and then determine $\mathfrak{g}_{\bar{0}}$-invariants in $\wedge^n \mathfrak{g} \otimes I(\Lambda)$. So the complex to be studied is the following:

$$0 \longleftarrow D_{\lambda+1} \overset{\partial_{\lambda+1}}{\longleftarrow} D_{\lambda+2} \overset{\partial_{\lambda+2}}{\longleftarrow} D_{\lambda+3} \longleftarrow \cdots, \tag{3.2}$$

where

$$D_n := (\wedge^n \mathfrak{g} \otimes I(\Lambda))^{\mathfrak{g}_{\bar{0}}}. \tag{3.3}$$

Put

$$V'_{\lambda+1} := \langle v'_i \,|\, 0 \leq i \leq \lambda + 1 \rangle_{\mathbf{C}}, \qquad v'_i := -i(Y_1 \otimes v_{i-1}) + Y_2 \otimes v_i,$$

$$\begin{aligned} &V_\lambda'' := Y_1 Y_2 \otimes L(\Lambda), && v_i'' := Y_1 Y_2 \otimes v_i \ (0 \le i \le \lambda), \\ &\mathscr{X}_k := \wedge^k \mathfrak{g}_1 = \langle x_i^k \mid 0 \le i \le k \rangle_{\mathbf{C}}, && x_i^k := \frac{1}{(k-i)!} \cdot X_1^{(k-i)} \wedge X_2^{(i)}, \\ &\mathscr{Y}_l := \wedge^l \mathfrak{g}_{-1} = \langle y_i^l \mid 0 \le i \le l \rangle_{\mathbf{C}}, && y_i^l := \frac{(-1)^i}{(l-i)!} \cdot Y_1^{(i)} \wedge Y_2^{(l-i)} \ (l \in \mathbf{Z}_{\ge 0}). \end{aligned} \tag{3.4}$$

The eigenvalue of $\rho_n(C)$ on the module $\wedge \mathfrak{g}_{\bar 0} \otimes \mathscr{X}_k \otimes \mathscr{Y}_l \otimes V$ is equal to $-k + l + \varepsilon + \lambda$, where $\varepsilon = 1$ for $V = V'_{\lambda+1}$ and $\varepsilon = 2$ for $V = V''_\lambda$. To be $\mathfrak{g}_{\bar 0}$-trivial, we should have $-k + l + \varepsilon + \lambda = 0$. So it is enough to look for $\mathfrak{g}_{\bar 0}$-invariants in the module

$$\wedge \mathfrak{g}_{\bar 0} \otimes \mathscr{X}_{\lambda+l+1} \otimes \mathscr{Y}_l \otimes V'_{\lambda+1} + \wedge \mathfrak{g}_{\bar 0} \otimes \mathscr{X}_{\lambda+l+2} \otimes \mathscr{X}_{\lambda+l+2} \otimes \mathscr{Y}_l \otimes V''_\lambda. \tag{3.5}$$

3.2. Description of basis vectors. Now we apply Lemma 2.1 taking into account the isomorphism (2.4). Let $\mathscr{Z}_0$ and $\mathscr{Z}_2$ be one of 1-dimensional and 3-dimensional irreducible $\mathfrak{sl}(2, \mathbf{C})$-modules in $\wedge \mathfrak{g}_{\bar 0}$ respectively. Then it is easy to see that each of modules $\mathscr{Z}_0 \otimes (\mathscr{X}_{\lambda+l+1} \otimes \mathscr{Y}_l) \otimes V'_{\lambda+1}$ and $\mathscr{Z}_2 \otimes (\mathscr{X}_{\lambda+l+2} \otimes \mathscr{Y}_l) \otimes V''_\lambda$ has exactly one $\mathfrak{g}_{\bar 0}$-trivial vector up to constant multiples. Moreover, the module $\mathscr{Z}_2 \otimes (\mathscr{X}_{\lambda+l+1} \otimes \mathscr{Y}_l) \otimes V'_{\lambda+1}$ has two linearly independent $\mathfrak{g}_{\bar 0}$-trivial vectors, because the space $\mathscr{X}_{\lambda+l+1} \otimes \mathscr{Y}_l$ contains exactly once modules with highest weights $\lambda + 1$ and $\lambda + 3$ and both produce trivial vectors after tensoring. Thus the module $D_{\lambda+2l+2}$ is spanned by invariant elements, $w(1), w(2), \cdots, w(8)$, for $l \ge 2$, which belong to the spaces given in the second column in Table 3.6.

$w(k)$	module	highest weight	minimum value of l
$w(1)$	$\mathscr{Z}^1(1)_0 \otimes (\mathscr{X}_{\lambda+l+1} \wedge \mathscr{Y}_l) \otimes V'_{\lambda+1}$	$\lambda + 1$	0
$w(2)$	$\mathscr{Z}^0(1)_2 \otimes (\mathscr{X}_{\lambda+l+1} \wedge \mathscr{Y}_l) \otimes V'_{\lambda+1}$	$\lambda + 1$	0
$w(3)$	$\mathscr{Z}^0(1)_2 \otimes (\mathscr{X}_{\lambda+l+1} \wedge \mathscr{Y}_l) \otimes V'_{\lambda+1}$	$\lambda + 3$	1
$w(4)$	$\mathscr{Z}^1(3)_2 \otimes (\mathscr{X}_{\lambda+l} \wedge \mathscr{Y}_{l-1}) \otimes V'_{\lambda+1}$	$\lambda + 1$	1
$w(5)$	$\mathscr{Z}^1(3)_2 \otimes (\mathscr{X}_{\lambda+l} \wedge \mathscr{Y}_{l-1}) \otimes V'_{\lambda+1}$	$\lambda + 3$	2
$w(6)$	$\mathscr{Z}^0(3)_0 \otimes (\mathscr{X}_{\lambda+l} \wedge \mathscr{Y}_{l-1}) \otimes V'_{\lambda+1}$	$\lambda + 1$	1
$w(7)$	$\mathscr{Z}^1(2)_2 \otimes (\mathscr{X}_{\lambda+l+1} \wedge \mathscr{Y}_{l-1}) \otimes V''_\lambda$	$\lambda + 2$	1
$w(8)$	$\mathscr{Z}^0(2)_2 \otimes (\mathscr{X}_{\lambda+l+1} \wedge \mathscr{Y}_{l-1}) \otimes V''_\lambda$	$\lambda + 2$	1

Table 3.6

Here the third column represents the highest weight of $\mathfrak{g}_{\bar 0}$-irreducible submodules of $\mathscr{X}_p \otimes \mathscr{Y}_q$ with which $w(k)$ is produced. The last column indicates the minimum value of l for which the corresponding vector $w(k) \in D_{\lambda+2l+2}$ exists. So that $D_{\lambda+4} = \langle w(k) \mid 1 \le k \le 8,\ k \ne 5 \rangle_{\mathbf{C}}$ and $D_{\lambda+2} = \langle w(k) \mid k = 1, 2 \rangle_{\mathbf{C}}$.

A basis of the module $D_{\lambda+2l+1}$ is also given in the following Table 3.7.

$u(k)$	module	highest weight	minimum value of l
$u(1)$	$\mathscr{X}^0(0)_0 \otimes (\mathscr{X}_{\lambda+l+1} \wedge \mathscr{Y}_l) \otimes V'_{\lambda+1}$	$\lambda + 1$	0
$u(2)$	$\mathscr{X}^1(2)_2 \otimes (\mathscr{X}_{\lambda+l} \wedge \mathscr{Y}_{l-1}) \otimes V'_{\lambda+1}$	$\lambda + 1$	1
$u(3)$	$\mathscr{X}^1(2)_2 \otimes (\mathscr{X}_{\lambda+l} \wedge \mathscr{Y}_{l-1}) \otimes V'_{\lambda+1}$	$\lambda + 3$	2
$u(4)$	$\mathscr{X}^0(2)_2 \otimes (\mathscr{X}_{\lambda+l} \wedge \mathscr{Y}_{l-1}) \otimes V'_{\lambda+1}$	$\lambda + 1$	1
$u(5)$	$\mathscr{X}^0(2)_2 \otimes (\mathscr{X}_{\lambda+l} \wedge \mathscr{Y}_{l-1}) \otimes V'_{\lambda+1}$	$\lambda + 3$	2
$u(6)$	$\mathscr{X}^1(4)_0 \otimes (\mathscr{X}_{\lambda+l-1} \wedge \mathscr{Y}_{l-2}) \otimes V'_{\lambda+1}$	$\lambda + 1$	2
$u(7)$	$\mathscr{X}^0(1)_2 \otimes (\mathscr{X}_{\lambda+l+1} \wedge \mathscr{Y}_{l-1}) \otimes V''_{\lambda}$	$\lambda + 2$	1
$u(8)$	$\mathscr{X}^1(3)_2 \otimes (\mathscr{X}_{\lambda+l} \wedge \mathscr{Y}_{l-2}) \otimes V''_{\lambda}$	$\lambda + 2$	2

Table 3.7

Now we fix the integer $l \geq 0$. Take a basis $\{w'(k) \mid 1 \leq k \leq 8\}$ of $D_{\lambda+2l}$ similarly as $w(k)$'s of $D_{\lambda+2l+2}$. Then derivations $\partial_{\lambda+2l+1}$: $D_{\lambda+2l+2} \to D_{\lambda+2l+1}$, and $\partial_{\lambda+2l}$: $D_{\lambda+2l+1} \to D_{\lambda+2l}$, are expressed by (8×8)-matrices, which we denote again by $\partial_{\lambda+2l+1}$ and $\partial_{\lambda+2l}$ respectively.

Let $t(1)_0$ be a highest weight vector in $\mathscr{X}_{\lambda+l+1} \otimes \mathscr{Y}_l$, given as

$$t(1)_0 = \sum_{i=0}^{l} (-1)^i \frac{(\lambda + l - i + 1)!}{(l - i)!} x_i^{\lambda+l+1} \wedge y_{l-i}^l$$

by (2.8) in Lemma 2.1. And let $\{t(1)_i\}$ be a standard basis, starting from $t(1)_0$, of the $\mathfrak{sl}(2, \mathbf{C})$-submodule of $\mathscr{X}_{\lambda+l+1} \otimes \mathscr{Y}_l$ generated by $t(1)_0$. Then $u(1)$ can be represented with $\{t(1)_i\}$ as

$$u(1) = \sum_{i=0}^{\lambda+1} (-1)^i \, t(1)_i \otimes v'_{\lambda+1-i}, \tag{3.8}$$

where $\{v'_j\}$ is given in (3.4). Let us write down other elements in $D_{\lambda+l+2}$ and $D_{\lambda+l+1}$. Put

$$\begin{aligned} \alpha_0 &= \sum_{i=0}^{l} a_i x_i^{\lambda+l+1} \wedge y_{l-i}^l, & a_i &= (-1)^i \frac{(\lambda + l - i + 1)!}{(l - i)!}, \\ \beta_0 &= \sum_{i=0}^{l-1} b_i x_i^{\lambda+l+1} \wedge y_{l-i-1}^l, & b_i &= (-1)^i \frac{(i + 1)(\lambda + l - i + 1)!}{(l - i - 1)!}, \\ \gamma_0 &= \sum_{i=0}^{l-1} c_i x_i^{\lambda+l+1} \wedge y_{l-i-1}^{l-1}, & c_i &= (-1)^i \frac{(\lambda + l - i + 1)!}{(l - i - 1)!}, \end{aligned} \tag{3.9}$$

and let the symbol $'$ (prime) mean substitution $l - 1$ for l. For example, if we regard $\alpha_0 = \alpha_0(l)$ as a function in l, then $\alpha'_0 = \alpha_0(l - 1)$, $\alpha''_0 := (\alpha'_0)' = \alpha_0(l - 2)$. Moreover, let

$$s(1)_0 = C \wedge \alpha_0$$

$$s(2)_0 = H \wedge \alpha_0 + \frac{2}{\lambda + 1} Z_+ \wedge \alpha_1$$

$$s(3)_0 = -2Z_- \wedge \beta_0 + \frac{2}{\lambda + 3} H \wedge \beta_1 + \frac{2}{(\lambda + 2)(\lambda + 3)} Z_+ \wedge \beta_2$$

$$
\begin{aligned}
s(4)_0 &= 2C \wedge Z_+ \wedge Z_- \wedge \alpha_0' + \frac{2}{\lambda+1} C \wedge H \wedge Z_+ \wedge \alpha_1' \\
s(5)_0 &= 2C \wedge H \wedge Z_- \wedge \beta_0' + \frac{4}{\lambda+3} C \wedge Z_+ \wedge Z_- \wedge \beta_1' \\
&\quad + \frac{2}{(\lambda+2)(\lambda+3)} C \wedge H \wedge Z_+ \wedge \beta_2' \\
s(6)_0 &= H \wedge Z_+ \wedge Z_- \wedge \alpha_0', \\
s(7)_0 &= -2C \wedge Z_- \wedge \gamma_0 + \frac{2}{\lambda+2} C \wedge H \wedge \gamma_1 + \frac{2}{(\lambda+1)(\lambda+2)} C \wedge Z_+ \wedge \gamma_2 \\
s(8)_0 &= 2H \wedge Z_- \wedge \gamma_0 + \frac{4}{\lambda+2} Z_+ \wedge Z_- \wedge \gamma_1 + \frac{2}{(\lambda+1)(\lambda+2)} H \wedge Z_+ \wedge \gamma_2
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
t(1)_0 &= \alpha_0 \\
t(2)_0 &= C \wedge H \wedge \alpha_0' + \frac{2}{\lambda+1} C \wedge Z_+ \wedge \alpha_1' \\
t(3)_0 &= -2C \wedge Z_- \wedge \beta_0' + \frac{2}{\lambda+3} C \wedge H \wedge \beta_1' + \frac{2}{(\lambda+2)(\lambda+3)} C \wedge Z_+ \wedge \beta_2' \\
t(4)_0 &= 2Z_+ \wedge Z_- \wedge \alpha_0' + \frac{2}{\lambda+1} H \wedge Z_+ \wedge \alpha_1' \\
t(5)_0 &= 2H \wedge Z_- \wedge \beta_0' + \frac{4}{\lambda+3} Z_+ \wedge Z_- \wedge \beta_1' + \frac{2}{(\lambda+2)(\lambda+3)} H \wedge Z_+ \wedge \beta_2' \\
t(6)_0 &= C \wedge H \wedge Z_+ \wedge Z_- \wedge \alpha_0'' \\
t(7)_0 &= -2Z_- \wedge \gamma_0 + \frac{2}{\lambda+2} H \wedge \gamma_1 + \frac{2}{(\lambda+1)(\lambda+2)} Z_+ \wedge \gamma_2 \\
t(8)_0 &= 2C \wedge H \wedge Z_- \wedge \gamma_0' + \frac{2}{\lambda+2} C \wedge Z_+ \wedge Z_- \wedge \gamma_1' \\
&\quad + \frac{2}{(\lambda+1)(\lambda+2)} C \wedge H \wedge Z_+ \wedge \gamma_2'
\end{aligned}
\tag{3.11}
$$

Here $\alpha_1 = Z_- \cdot \alpha_0$, $\alpha_2 = Z_- \cdot \alpha_1$, and $\beta_1, \beta_2, \gamma_1, \gamma_2$ are similarly defined: $\beta_j = Z_-^j \cdot \beta_0$, $\gamma_j = Z_-^j \cdot \gamma_0$ $(j = 1, 2)$. Similarly as $\{t(1)_i\}$ given by $t(1)_0$, we determine $\{s(k)_i\}$ from $s(k)_0$ and $\{t(k)_i\}$ from $t(k)_0$.

Proposition 3.1. *The bases* $w(k)$ $(1 \le k \le 8)$ *for* $D_{\lambda+2l+2}$ *and* $u(k)$ $(1 \le k \le 8)$ *for* $D_{\lambda+2l+1}$ *are expressed as follows*:

$$
\begin{aligned}
w(k) &= \sum_{i=0}^{\lambda+1} (-1)^i s(k)_i \otimes v'_{\lambda+1-k} \qquad \text{for } k = 1, \cdots, 6, \\
w(k) &= \sum_{i=0}^{\lambda} (-1)^i s(k)_i \otimes v''_{\lambda-k} \qquad \text{for } k = 7, 8, \\
u(k) &= \sum_{i=0}^{\lambda+1} (-1)^i t(k)_i \otimes v'_{\lambda+1-k} \qquad \text{for } k = 1, \cdots, 6,
\end{aligned}
\tag{3.12}
$$

$$u(k) = \sum_{i=0}^{\lambda} (-1)^i t(k)_i \otimes v''_{\lambda-k} \qquad \textit{for } k = 7, 8.$$

3.3. Matrices for boundary operators. Let us now decompose the boundary map ∂_{n-1} in (1.6) as $\partial_{n-1} = \sum_{i=1}^{4} \partial_{n-1}^{(i)}$, where

$$\begin{aligned}
\partial_{n-1}^{(1)}(X_1 \wedge \cdots \wedge X_n \otimes v) &= - \sum_{X_j \in \mathfrak{g}_1, X_k \in \mathfrak{g}_{-1}} [X_j, X_k] \wedge X_1 \wedge \cdots \hat{j} \cdots \hat{k} \cdots \wedge X_n \otimes v, \\
\partial_{n-1}^{(2)}(X_1 \wedge \cdots \wedge X_n \otimes v) &= \sum_{X_i \in \mathfrak{g}_{\bar{0}}} (-1)^i X_1 \wedge \cdots \hat{i} \cdots \wedge X_n \otimes X_i v \\
&\quad - \sum_{X_j \in \mathfrak{g}_{\bar{0}}, X_k \in \mathfrak{g}} (-1)^{j+k+\eta_k} [X_j, X_k] \wedge X_1 \wedge \cdots \hat{j} \cdots \hat{k} \cdots \wedge X_n \otimes v, \qquad (3.13) \\
\partial_{n-1}^{(3)}(X_1 \wedge \cdots \wedge X_n \otimes v) &= (-1)^n \sum_{X_i \in \mathfrak{g}_1} X_1 \wedge \cdots \hat{i} \cdots \wedge X_n \otimes X_i v, \\
\partial_{n-1}^{(4)}(X_1 \wedge \cdots \wedge X_n \otimes v) &= (-1)^n \sum_{X_i \in \mathfrak{g}_{-1}} X_1 \wedge \cdots \hat{i} \cdots \wedge X_n \otimes X_i v.
\end{aligned}$$

From the beginning, we know that some of elements in the matrices $\partial_{\lambda+2l+1}$ and $\partial_{\lambda+2l}$ are equal to zero. In fact, the derivation ∂ has the following property:

(i) $\partial(\wedge^i \mathfrak{sl}(2, \mathbf{C}) \otimes \wedge \mathfrak{g}_{\bar{1}} \otimes I(\Lambda))$
$\subset \{(\bigoplus_{j=i,i\pm 1} \wedge^j \mathfrak{sl}(2, \mathbf{C})) \oplus (C \otimes \wedge^i \mathfrak{sl}(2, \mathbf{C}))\} \otimes \wedge \mathfrak{g}_{\bar{1}} \otimes I(\Lambda)$

(ii) $\partial(C \otimes \wedge^i \mathfrak{sl}(2, \mathbf{C}) \otimes \wedge \mathfrak{g}_{\bar{1}} \otimes I(\Lambda)) \subset C \otimes \{\bigoplus_{j=i,i\pm 1} \wedge^j \mathfrak{sl}(2, \mathbf{C})\} \otimes \wedge \mathfrak{g}_{\bar{1}} \otimes I(\Lambda)$

(iii) $\partial^{(2)}((\mathfrak{sl}(2, \mathbf{C}) \otimes \wedge \mathfrak{g}_{\bar{1}} \otimes I(\Lambda))^{\mathfrak{g}_{\bar{0}}}) = \{0\}$,
$\partial^{(2)}((C \otimes \mathfrak{sl}(2, \mathbf{C}) \otimes \wedge \mathfrak{g}_{\bar{1}} \otimes I(\Lambda))^{\mathfrak{g}_{\bar{0}}}) = \{0\}$.

Let us now compute the matrix elements corresponding to each $\partial^{(j)}$ $(1 \leq j \leq 4)$.

First consider $\partial^{(1)}$. Let $x_i^r \in \mathscr{X}_r$ and $y_j{}^s \in \mathscr{Y}_s$ be as in (3.4) respectively:

$$\begin{aligned}
x_i^r &:= \frac{1}{(r-i)!} \cdot X_1^{(r-i)} \wedge X_2^{(i)}, \\
y_j^s &:= \frac{(-1)^j}{(s-j)!} Y_1^{(j)} \wedge Y_2^{(s-j)}.
\end{aligned} \qquad (3.14)$$

Then we have

$$\begin{aligned}
\partial(x_i^r \wedge y_j^s) &= \frac{1}{2} C \wedge (j x_i^{r-1} \wedge y_{j-1}^{s-1} - i x_{i-1}^{r-1} \wedge y_j{}^{s-1}) - Z_+ \wedge x_i^{r-1} \wedge y_j{}^{s-1} \\
&\quad + \frac{1}{2} H \wedge (j x_i^{r-1} \wedge y_{j-1}^{s-1} + i x_{i-1}^{r-1} \wedge y_j{}^{s-1}) + ij Z_- \wedge x_{i-1}^{r-1} \wedge y_{j-1}^{s-1}.
\end{aligned} \qquad (3.15)$$

Put

$$p_1 = \frac{\lambda+1}{\lambda+3},\ p_2 = p_2(l) = \frac{\lambda+l+2}{\lambda+3},\ p_3 = p_3(l) = \frac{1}{2}(\lambda+l+3),\ \varepsilon = (-1)^{\lambda};$$

$$p_2' = p_2(l-1),\ p_3' = p_3(l-1),\ p_3'' = p_3(l-2). \tag{3.16}$$

Then the following equality holds for $\partial t(1)_0 := \partial(t(1)_0)$

$$\partial t(1)_0 = p_3' s'(1)_0 + p_1 p_3' s'(2)_0 + \frac{1}{2} s'(3)_0. \tag{3.17}$$

On the other hand, we know that $\partial_{n-1}^{(j)}(u(1)) = 0$ for $j = 2, 3, 4$, according to Tables 3.6 and 3.7, then,

$$\begin{aligned}\partial u(1) &= \partial^{(1)}\Big(\sum_{i=0}^{\lambda+1} t(1)_i \otimes v'_{\lambda+1-i}\Big) = \sum_{i=0}^{\lambda+1} (\partial t(1)_i) \otimes v'_{\lambda+1-i} \\ &= \left\{p_3' s'(1)_0 + p_1 p_3' s'(2)_0 + \frac{1}{2} s'(3)_0\right\} \otimes v'_{\lambda+1} + \sum_{i=1}^{\lambda+1} (\partial t(1)_i) \otimes v'_{\lambda+1-i}.\end{aligned}$$

In the last expression, the first and the seond terms are linearly independent. This means that $\partial u(1) = p_3' w'(1) + p_1 p_3' w'(2) + \frac{1}{2} w'(3)$. By similar calculations, we get matrix elements related to $\partial^{(1)}$ of the matrices $\partial = \partial_{\lambda+2l+1}$ and $\partial = \partial_{\lambda+2l}$ in.

Secondly we discuss about $\partial^{(2)}$. We constructed the vectors $w(i)$'s and $u(i)$'s $(1 \le i \le 8)$ as elements in the module $(\wedge \mathfrak{g}_{\bar 0} \otimes \wedge \mathfrak{g}_{\bar 1}) \otimes I(\Lambda)$. There is an isomorphism

$$(\wedge \mathfrak{g}_{\bar 0} \otimes \wedge \mathfrak{g}_{\bar 1}) \otimes I(\Lambda) \cong \wedge \mathfrak{g}_{\bar 0} \otimes (\wedge \mathfrak{g}_{\bar 1} \otimes I(\Lambda))$$

as $\mathfrak{g}^{\bar 0}$-modules. So we can express the vectors $u(4)$ and $w'(2)$ as

$$\begin{aligned} u(4) &= 2H \wedge Z_- \wedge h_0 + 2Z_+ \wedge Z_- \wedge h_1 + H \wedge Z_+ \wedge h_2, \\ w'(2) &= -2Z_- \wedge h_0 + H \wedge h_0 + H \wedge h_1 + Z_+ \wedge h_2, \end{aligned} \tag{3.18}$$

where $\{h_0, h_1, h_2\}$ is a standard basis in an irreducible $\mathfrak{sl}(2, \mathbf{C})$-module with a highest weight 2 in the module $(\mathscr{X}_{\lambda+l} \wedge \mathscr{Y}_{l-1}) \otimes V'_{\lambda+1}$. We get

$$\partial^{(2)}(u(4)) = -4Z_- \wedge h_0 + 2H \wedge h_1 + 2Z_+ \wedge h_2 = 2w'(2).$$

Similarly, we get the action of $\partial^{(2)}$ on other vectors.

Thirdly, we discuss about $\partial^{(3)}$ and $\partial^{(4)}$ at the same time. For $z(k) \otimes x_j^p \otimes y_m^q \otimes v_i'' \in \wedge^k \mathfrak{g}_{\bar 0} \otimes \mathscr{X}_p \otimes \mathscr{Y}_q \otimes V_\lambda''$ and for $z(k) \otimes x_j^p \otimes y_m^q \otimes v_i' \in \wedge^k \mathfrak{g}_{\bar 0} \otimes \mathscr{X}_p \otimes \mathscr{Y}_q \otimes V'_{\lambda+1}$, with $j + k + m = n$, the derivations $\partial^{(3)}$ and $\partial^{(4)}$ act as follows:

$$\begin{aligned} &\partial^{(3)}(z(k) \otimes x_j^{p-1} \otimes y_m^q \otimes v_i'') \\ &\quad = (-1)^n z(k) \otimes \{(\lambda - i + 1) x_j^{p-1} \otimes y_m^q \otimes v_i' + j x_{j-1}^{p-1} \otimes y_m^q \otimes v'_{i+1}\}, \\ &\partial^{(4)}(z(k) \otimes x_j^p \otimes y_m^q \otimes v_i') = (-1)^n z(k) \otimes x_j^p \otimes \{-j y_{m-1}^{q-1} \otimes v_i'' + i y_j^{q-1} \otimes v''_{i-1}\}. \end{aligned} \tag{3.19}$$

Derivations $\partial^{(3)}$ and $\partial^{(4)}$ can be calculated using these equalities.

Finally, as a result, we get the following matrices for $l \ge 2$:

$$\partial_{\lambda+2l+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_1p_3' & p_3' & 0 & -2 & 0 & 0 & 2\varepsilon p_2 & 0 \\ -\frac{1}{2} & 0 & p_3 & 0 & -2 & 0 & -\varepsilon & 0 \\ 0 & -p_2 & 2p_1p_2p_3 & 0 & 0 & 0 & 0 & 2\varepsilon p_2 \\ 0 & \frac{1}{2} & -p_1p_3 & 0 & 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & 0 & -2p_3'' & -4p_2p_3'' & p_3'' & 0 & 0 \\ 0 & -\varepsilon & 2\varepsilon p_1p_3 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -\varepsilon & 2\varepsilon p_1p_3' & 0 & p_3' & p_3' \end{pmatrix}, \tag{3.20}$$

$$\partial_{\lambda+2l} = \begin{pmatrix} p_3' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_1p_3' & 0 & 0 & 2 & 0 & 0 & -2\varepsilon p_2 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 2 & 0 & \varepsilon & 0 \\ 0 & p_2' & -2p_1p_2'p_3' & p_3'' & 0 & 0 & 0 & -2\varepsilon p_2' \\ 0 & -\frac{1}{2} & p_1p_3' & 0 & p_3' & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 2p_3'' & 4p_2p_3'' & 0 & 0 & 0 \\ 0 & \varepsilon & -2\varepsilon p_1p_3' & 0 & 0 & 0 & p_3' & -2 \\ 0 & 0 & 0 & \varepsilon & -2\varepsilon p_1p_3' & 0 & -p_3' & 0 \end{pmatrix}. \tag{3.21}$$

When $l = 1$, we can calculate $\partial_{\lambda+2l+1}$ and $\partial_{\lambda+2l}$ similarly. In this degenerate case, there vanish vectors $w(5)$, $u(3)$, $u(5)$, $u(6)$, $u(8)$, and $w'(i)$ $(3 \le i \le 8)$. So the matrices are given by

$$\partial_{\lambda+3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon & 2\varepsilon p_1p_3 & 0 & 0 & 0 & 2 \\ 0 & -p_2 & 2p_1p_2p_3 & 0 & 0 & 0 & 2\varepsilon p_2 \\ -p_1p_3' & p_3' & 0 & -2 & 0 & 2\varepsilon p_2 & 0 \end{pmatrix},$$

$$\partial_{\lambda+2} = \begin{pmatrix} p_3' & 0 & 0 & 0 \\ p_1p_3' & -2\varepsilon p_2 & 0 & 0 \end{pmatrix}.$$

When $l = 0$, we have

$$\partial_{\lambda+1} = (0 \quad 0) \text{ and } \partial_\lambda = (0).$$

Now, calculating the rank of these matrices, we get $\dim(\operatorname{Im} \partial_{\lambda+n}) = \operatorname{rank} \partial_{\lambda+n}$ and $\dim(\operatorname{Ker} \partial_{\lambda+n}) = \dim D_{\lambda+n+1} - \dim(\operatorname{Im} \partial_{\lambda+n})$ as follows.

n	$\lambda+1$	$\lambda+2$	$\lambda+3$	$\lambda+4$	$\lambda+5$	$\lambda+6$	$\cdots$
$\dim D_n$	1	2	4	7	8	8	$\cdots$
$\dim(\mathrm{Ker}\,\partial_{n-1})$	1	2	2	5	4	4	$\cdots$
$\dim(\mathrm{Im}\,\partial_n)$	0	2	2	4	4	4	$\cdots$

Table 3.22

From this result, we have the following theorem.

Theorem 3.2. *Let $\Lambda = (\lambda, c)$ with $\lambda = c \geq 0$. Then the dimensions of homology groups of the irreducible $\mathfrak{g}$-module $I(\Lambda)$ are*

$$\dim \mathrm{H}_n(\mathfrak{g}, I(\Lambda)) = \begin{cases} 1 & \text{for } n = \lambda+1, \lambda+4, \\ 0 & \text{otherwise.} \end{cases}$$

3.4. Homology groups for the trivial module C. If $V(\Lambda) = \mathbf{C}$, the complex to be studied is as following:

$$0 \longleftarrow D_0 \xleftarrow{\partial_0} D_1 \xleftarrow{\partial_1} D_2 \xleftarrow{\partial_2} D_3 \xleftarrow{\partial_3} D_4 \longleftarrow 0, \tag{3.23}$$

where $D_n = (\wedge^n \mathfrak{g} \otimes \mathbf{C})^{\mathfrak{g}_0} = (\wedge^n \mathfrak{g})^{\mathfrak{g}_0}$. The basis vectors in D_{2l+1} and those in D_{2l} are given by letting $\lambda = -1$ in Table 3.6 and in Table 3.7 respectively, and thus we get the following tables correnspondingly.

$w(k)$	module	highest weight	minimum value of l
$w(1)$	$\mathscr{Z}^1(1)_0 \otimes (\mathscr{X}_l \wedge \mathscr{Y}_l)$	0	0
$w(3)$	$\mathscr{Z}^0(1)_2 \otimes (\mathscr{X}_l \wedge \mathscr{Y}_l)$	2	1
$w(5)$	$\mathscr{Z}^1(3)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{l-1})$	2	2
$w(6)$	$\mathscr{Z}^0(3)_0 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{l-1})$	0	1

Table 3.24

$u(k)$	module	highest weight	minimum value of l
$u(1)$	$\mathscr{Z}^0(0)_0 \otimes (\mathscr{X}_l \wedge \mathscr{Y}_l)$	0	0
$u(3)$	$\mathscr{Z}^1(2)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{l-1})$	2	2
$u(5)$	$\mathscr{Z}^0(2)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{l-1})$	2	2
$u(6)$	$\mathscr{Z}^1(4)_0 \otimes (\mathscr{X}_{l-2} \wedge \mathscr{Y}_{l-2})$	0	2

Table 3.25

By similar calculations, we get the following matrices for boundary operators.

$$\partial_{2l} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2}(l+2) & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -l(l+1) & \frac{1}{2}l \end{pmatrix} \qquad (l \geq 2),$$

$$\partial_{2l-1} = \begin{pmatrix} \frac{1}{2}(l+1) & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2}(l+1) & 0 \\ 0 & 0 & l & l(l+1) \end{pmatrix} \qquad (l \geq 2),$$

$$\partial_2 = (0 \ \ 0 \ \ 0 \ \ 0), \quad \partial_1 = (1), \quad \partial_0 = (0),$$

for $l = 1$ and 0. For the dimensions of D_n, $\mathrm{Ker}\,\partial_{n-1}$ and $\mathrm{Im}\,\partial_n$, we get Table 3.26 bellow.

n	0	1	2	3	4	5	$\cdots$
$\dim D_n$	1	1	1	3	4	4	$\cdots$
$\dim(\mathrm{Ker}\,\partial_{n-1})$	1	1	0	3	2	2	$\cdots$
$\dim(\mathrm{Im}\,\partial_n)$	0	1	0	2	2	2	$\cdots$

Table 3.26

Now we have the following result for the trivial module.

Theorem 3.3. *Let $V(0, 0) = \mathbf{C}$ be a trivial representation of $\mathfrak{g} = \mathfrak{sl}(2, 1)$. Then,*

$$\dim \mathrm{H}_n(\mathfrak{g}, \mathbf{C}) = \begin{cases} 1 & \textit{for } n = 0, 3 \\ 0 & \textit{otherwise.} \end{cases}$$

4. Homology groups for the maximal submodules $I(\Lambda)$ for $\lambda + c + 2 = 0$

In this case, the maximal submodle $I(\Lambda)$ of the module $\bar{V}(\Lambda)$ is decomposed as

$$I(\Lambda) = \tilde{V}'_{\lambda-1} + \tilde{V}''_{\lambda},$$
$$\tilde{V}'_{\lambda-1} := \langle v'_i \mid 0 \leq i \leq \lambda - 1 \rangle_{\mathbf{C}}, \quad v'_i := (\lambda - i) Y_1 \otimes v_i + Y_2 \otimes v_{i+1}, \qquad (4.1)$$
$$V''_{\lambda} := Y_1 Y_2 \otimes L(\Lambda),$$

where $\{v_i\}_{i=0}^{\lambda}$ is the standard bases in (1.1) of $L(\Lambda) \cong V_\lambda$, $\tilde{V}'_{\lambda-1}$ and V''_λ are irreducible as $\mathfrak{g}_{\bar{0}}$-modules.

In place of Tables 3.6 and 3.7, we have Table 4.2 for $D_{\lambda+2l+1}$ and Table 4.3 for $D_{\lambda+2l}$ as follows:

$\tilde{w}(k)$	module	highest weight	minimum value of l
$\tilde{w}(1)$	$\mathscr{Z}^1(1)_0 \otimes (\mathscr{X}_l \wedge \mathscr{Y}_{\lambda+l}) \otimes V_\lambda''$	λ	0
$\tilde{w}(2)$	$\mathscr{Z}^0(1)_2 \otimes (\mathscr{X}_l \wedge \mathscr{Y}_{\lambda+l}) \otimes V_\lambda''$	λ	0
$\tilde{w}(3)$	$\mathscr{Z}^0(1)_2 \otimes (\mathscr{X}_l \wedge \mathscr{Y}_{\lambda+l}) \otimes V_\lambda''$	$\lambda+2$	1
$\tilde{w}(4)$	$\mathscr{Z}^1(3)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l-1}) \otimes V_\lambda''$	λ	1
$\tilde{w}(5)$	$\mathscr{Z}^1(3)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l-1}) \otimes V_\lambda''$	$\lambda+2$	2
$\tilde{w}(6)$	$\mathscr{Z}^0(3)_0 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l-1}) \otimes \tilde{V}_{\lambda-1}'$	λ	1
$\tilde{w}(7)$	$\mathscr{Z}^1(2)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l}) \otimes \tilde{V}_{\lambda-1}'$	$\lambda+1$	1
$\tilde{w}(8)$	$\mathscr{Z}^0(2)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l}) \otimes \tilde{V}_{\lambda-1}'$	$\lambda+1$	1

Table 4.2

$\tilde{u}(k)$	module	highest weight	minimum value of l
$\tilde{u}(1)$	$\mathscr{Z}^0(0)_0 \otimes (\mathscr{X}_l \wedge \mathscr{Y}_{\lambda+l}) \otimes V_\lambda''$	λ	0
$\tilde{u}(2)$	$\mathscr{Z}^1(2)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l-1}) \otimes V_\lambda''$	λ	1
$\tilde{u}(3)$	$\mathscr{Z}^1(2)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l-1}) \otimes V_\lambda''$	$\lambda+2$	2
$\tilde{u}(4)$	$\mathscr{Z}^0(2)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l-1}) \otimes V_\lambda''$	λ	1
$\tilde{u}(5)$	$\mathscr{Z}^0(2)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l-1}) \otimes V_\lambda''$	$\lambda+2$	2
$\tilde{u}(6)$	$\mathscr{Z}^1(4)_0 \otimes (\mathscr{X}_{l-2} \wedge \mathscr{Y}_{\lambda+l-2}) \otimes V_\lambda''$	λ	2
$\tilde{u}(7)$	$\mathscr{Z}^0(1)_2 \otimes (\mathscr{X}_{l-1} \wedge \mathscr{Y}_{\lambda+l}) \otimes \tilde{V}_{\lambda-1}'$	$\lambda+1$	1
$\tilde{u}(8)$	$\mathscr{Z}^1(3)_2 \otimes (\mathscr{X}_{l-2} \wedge \mathscr{Y}_{\lambda+l-1}) \otimes \tilde{V}_{\lambda-1}'$	$\lambda+1$	2

Table 4.3

In place of (3.9), we take the following (4.4).

$$\begin{aligned} \tilde{\alpha}_0 &= \sum_{i=0}^{l} a_i x_{l-1}^l \wedge y_i^{\lambda+l+1}, \quad a_i = (-1)^i \frac{(\lambda+l-i+1)!}{(l-i)!}, \\ \tilde{\beta}_0 &= \sum_{i=0}^{l-1} b_i x_{l-i-1}^l \wedge y_i^{\lambda+l+1}, \quad b_i = (-1)^i \frac{(\lambda+l-i+1)!}{(l-i-1)!}(i+1), \\ \tilde{\gamma}_0 &= \sum_{i=0}^{l-1} c_i x_{l-i-1}^{l-1} \wedge y_i^{\lambda+l+1}, \quad c_i = (-1)^i \frac{(\lambda+l-i+1)!}{(l-i-1)!}. \end{aligned} \tag{4.4}$$

In place of α_j, β_j and γ_j in (3.10) and (3.11), we have $\tilde{\alpha}_j$, $\tilde{\beta}_j$ and $\tilde{\gamma}_j$ respectively, defined as $\tilde{\alpha}_j = Z_-^j \tilde{\alpha}_0$, $\tilde{\beta}_j = Z_-^j \tilde{\beta}_0$, $\tilde{\gamma}_j = Z_-^j \tilde{\gamma}_0$. We define $\tilde{s}(k)$ and $\tilde{t}(k)$ by substituting $\lambda - 1$ to λ in $s(k)$ in (3.10) and $t(k)$ in (3.11) respectively.

By similar calculations as in §3, we obtain the following matrices:

$$\partial_{\lambda+2l+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q_1q_3' & -q_3' & 0 & -2 & 0 & 0 & 2\varepsilon q_2 & 0 \\ -\frac{1}{2} & 0 & -q_3 & 0 & -2 & 0 & -\varepsilon & 0 \\ 0 & -q_2 & 2q_1q_2q_3 & 0 & 0 & 0 & 0 & 2\varepsilon q_2 \\ 0 & \frac{1}{2} & -q_1q_3 & 0 & 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & 0 & -2q_3'' & -4q_2q_3'' & -q_3'' & 0 & 0 \\ 0 & -\varepsilon & 2\varepsilon q_1q_3 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -\varepsilon & 2\varepsilon q_1q_3' & 0 & q_3' & -q_3' \end{pmatrix}, \tag{4.7}$$

$$\partial_{\lambda+2l} = \begin{pmatrix} -q_3' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_1q_3' & 0 & 0 & 2 & 0 & 0 & -2\varepsilon q_2 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 2 & 0 & \varepsilon & 0 \\ 0 & q_2' & -2q_1q_2'q_3' & -q_3'' & 0 & 0 & -2\varepsilon q_2' & 0 \\ 0 & -\frac{1}{2} & q_1q_3' & 0 & -q_3' & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 2q_3'' & 4q_2q_3'' & 0 & 0 & 0 \\ 0 & \varepsilon & -2\varepsilon q_1q_3' & 0 & 0 & 0 & -q_3' & -2 \\ 0 & 0 & 0 & \varepsilon & -2\varepsilon q_1q_3' & 0 & -q_3' & 0 \end{pmatrix}, \tag{4.8}$$

where

$$q_1 = \frac{\lambda}{\lambda+2},\ q_2 = q_2(l) = \frac{\lambda+l+1}{\lambda+2},\ q_3 = q_3(l) = \frac{1}{2}(\lambda+l+2),\ \varepsilon = (-1)^{\lambda},$$
$$\text{and } q_3' = q_3(l-1),\ q_3'' = q_3(l-2),\ q_2' = q_2(l-1). \tag{4.9}$$

For exceptional values $l = 1$ and 0, these matrices degenarate as in § 3.4. Calculating the ranks of the above matrices, we get the next table.

n	λ	$\lambda+1$	$\lambda+2$	$\lambda+3$	$\lambda+4$	$\lambda+5$	$\cdots$
$\dim D_n$	1	2	4	7	8	8	$\cdots$
$\dim(\mathrm{Ker}\,\partial_{n-1})$	1	2	2	5	4	4	$\cdots$
$\dim(\mathrm{Im}\,\partial_n)$	0	2	2	4	4	4	$\cdots$

Table 4.10

From these results, we get the following theorem.

Theorem 4.1. *Let* $\Lambda = (\lambda, c)$ *with* $\lambda = -c - 2 \in \mathbf{Z}_{\geq 0}$. *Then dimensions of homology groups of irreducible module* $I(\Lambda)$ *are*

$$\dim \mathrm{H}_n(\mathfrak{g}, I(\Lambda)) = 1 \qquad \text{for } n = \lambda, \lambda+3,$$

$$\mathrm{H}_n(\mathfrak{g}, I(\Lambda)) = 0 \qquad \textit{otherwise.}$$

§5. Homology and cohomology for the irreducible modules $V(\Lambda)$

5.1. Homology groups. In §§2–4, we have the results about homology groups of the induced module $\bar{V}(\Lambda)$ and the maximal submodule $I(\Lambda)$ in respective cases. Summarizing them, we obtain all the homology groups as follows.

Theorem 5.1. *Let $V(\Lambda)$ be a finite-imensional irreducible representation of $\mathfrak{g} = \mathfrak{sl}(2, 1)$ with highest weight $\Lambda = (\lambda, c)$, $\lambda \in \mathbf{Z}_{\geq 0}$, $c \in \mathbf{C}$. If $\lambda = c$, then*

$$\dim \mathrm{H}_n(\mathfrak{g}, V(\Lambda)) = \begin{cases} 1 & \textit{for } n = \lambda, \lambda + 3 \\ 0 & \textit{otherwise.} \end{cases}$$

If $\lambda + c + 2 = 0$, then

$$\dim \mathrm{H}_n(\mathfrak{g}, V(\Lambda)) = \begin{cases} 1 & \textit{for } n = \lambda + 1, \lambda + 4 \\ 0 & \textit{otherwise.} \end{cases}$$

If $(\lambda - c)(\lambda + c + 2) \neq 0$, then

$$\mathrm{H}_n(\mathfrak{g}, V(\Lambda)) = (0) \qquad \textit{for any } n \geq 0.$$

We compute cohomology groups from the result on homology groups using Lemma 1.7. To apply Lemma 1.7, we should specify the dual representation $V(\Lambda)^*$.

As we have seen in §1.3, when $\lambda = c$, $V(\Lambda) \cong L(\lambda, c) \oplus L(\lambda - 1, c + 1)$ as $\mathfrak{g}_{\bar{0}}$-modules, and when $\lambda + c + 2 = 0$, $V(\Lambda) \cong L(\lambda, c) \oplus L(\lambda + 1, c + 1)$. On the other hand, $L(\lambda, c)^* \cong L(\lambda, -c)$ as $\mathfrak{g}_{\bar{0}}$-modules. We have $\mathfrak{g}_{\bar{0}}$-module isomorphisms, in case $\lambda = c \in \mathbf{Z}_{>0}$.

$$\begin{aligned} V(\lambda, c)^* &\cong (L(\lambda, c) + L(\lambda - 1, c + 1))^* \cong L(\lambda, -c) \oplus L(\lambda - 1, -c - 1) \\ &\cong V(\lambda - 1, -c - 1), \end{aligned}$$

where we have $\lambda' + c' + 2 + 0$ with $\lambda' = \lambda - 1$, $c' = -c - 1$. On the contrary, if $\lambda' + c' + 2 = 0$, then $V(\lambda', c')^* \cong V(\lambda' + 1, -c' - 1)$. Therefore, $V(\lambda', c')^* \cong V(\lambda' - 1, -c' - 1)$ as $\mathfrak{g}$-modules if $\lambda = c \geq 1$. Thus we see that $\mathfrak{g}_{\bar{0}}$-module structures distiguish $\mathfrak{g}$-modules in these cases.

By Lemma 1.7, we have the following lemma.

Lemma 5.2. *Let the notations be the same in Theorem 5.1. if $\lambda = c \geq 1$,*

$$\mathrm{H}^n(\mathfrak{g}, V(\lambda, c)) \cong \mathrm{H}_n(\mathfrak{g}, V(\lambda - 1, c - 1))^*$$

and if $\lambda = -c - 2 \geq 0$,

$$\mathrm{H}^n(\mathfrak{g}, V(\lambda, c)) \cong \mathrm{H}_n(\mathfrak{g}, V(\lambda + 1, c - 1))^*$$

Finally we have the result about cohomology groups. Note that $V(0, 0) = \mathbf{C}$ is the trivial $\mathfrak{g}$-module.

Theorem 5.3. *Let the notations are the same as in Theorem* 5.1 *again. If* $\lambda = c$, *then*

$$\dim H^n(\mathfrak{g}, V(\Lambda)) = \begin{cases} 1 & \text{for } n = \lambda, \lambda + 3 \\ 0 & \text{otherwise.} \end{cases}$$

If $\lambda + c + 2 = 0$, *then*

$$\dim H^n(\mathfrak{g}, V(\Lambda)) = \begin{cases} 1 & \text{for } n = \lambda + 1, \lambda + 4 \\ 0 & \text{otherwise.} \end{cases}$$

If $(\lambda - c)(\lambda + c + 2) \neq 0$, *then*

$$H^n(\mathfrak{g}, V(\Lambda)) = (0) \qquad \text{for any } n \geq 0.$$

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