J. Math. Kyoto Univ. (JMKYAZ)
35-4 (1995) 639-651

An example of regular (r, p)-capacity and essential self-adjointness of a diffusion operator in infinite dimensions

Dedicated to Professor Masatoshi Fukushima on his 60th birthday

By

Ichiro SHIGEKAWA

1. Introduction

The general theory of (r, p)-capacity has been developed by Fukushima-Kaneko [8] (see also [10]). In their theory, the regularity condition is fundamental.

To be precise, let X be a separable metric space and m be a finite Borel measure on X. Suppose that a symmetric Markovian semigroup $\{T_t\}$ on $L^2(X; m)$ is given. By the Markovian property, $\{T_t\}$ is a contraction semigroup on $L^p(X; m)$ for any $p \in [1, \infty)$. The Gamma transformation is defined by

$$V_r = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} T_t dt.$$

Set $\mathscr{F}_{r,p} := V_r(L^p(X; m))$. Using $\mathscr{F}_{r,p}$, we can define the (r, p)-capacity $C_{r,p}$ as follows: for an open set G,

$$C_{r,p}(G) := \inf\{\|u\|_{r,p}^p; u \in \mathscr{F}_{r,p}, u \geq 1 \text{ } m\text{-a.e. on } G\} \tag{1.1}$$

and for an arbitrary set $S \subseteq X$,

$$C_{r,p}(S) := \inf\{C_{r,p}(G); G \text{ is open and } G \supseteq S\}. \tag{1.2}$$

In this paper, we say that the (r, p)-capacity is *regular* if the following condition is satisfied:

(R) $\mathscr{F}_{r,p} \cap C_b(X)$ is dense in $\mathscr{F}_{r,p}$.

Here $C_b(X)$ denotes the set of all bounded continuous functions on X. Assuming the condition (R), Fukushima-Kaneko [8] proved the continuity from the below of the (r, p)-capacity.

The purpose of this paper is to give an example satisfying the condition (R). Let (B, H, μ) be an abstract Wiener space. We take a function $\rho \in W^{\infty, \infty-}$ with $\rho > 0$ μ-a.e. and fix it. We consider the following Dirichlet form in $L^2(\rho^2\mu)$:

Received December 9, 1994

$$\mathscr{E}(u, v) = \int_B (Du(x), Dv(x))_{H^*}\rho(x)^2\mu(dx). \tag{1.3}$$

Here Du is the H-derivative. Let $\{T_t\}$ be an associated semigroup. We shall prove that the (r, p)-capacity associated with this semigroup satisfies the condition (R).

In the case of $r = 2$ and $p = 2$, this condition is closely related to essential self-adjointness of the generator. In this case, we shall consider more general Dirichlet form given by

$$\mathscr{E}(u, v) = \int_B (\sqrt{A^*}Du(x), \sqrt{A^*}Dv(x))_{H^*}\rho(x)^2\mu(dx) \tag{1.4}$$

where A^* is a strictly positive definite self-adjoint operator.

The organization of this paper is as follows. In the section 2, we consider the regularity of (r, p)-capacity. We do it in the framework of the Malliavin calculus. To show the regularity, the hypoellipticity of the Ornstein-Uhlenbeck operator plays an important role. In the section 3, we concentrate upon the problem of the essential self-adjointness. The hypoellipticity is crucial as well. In addition, we need the Meyer equivalence for an Ornstein-Uhlenbeck operator with a linear drift.

2. Regularity of $\mathscr{F}_{r,p}$

Let (B, H, μ) be an abstract Wiener space: B is a real separable Banach space, H is a real separable Hilbert space which is embedded densely and continuously in B and μ is a Gaussian measure with

$$\hat{\mu}(l) := \int_B \exp\{\sqrt{-1}\,{}_{B^*}\langle l, x\rangle_B\}\mu(dx) = \exp\left\{-\frac{1}{2}|l|_{H^*}^2\right\}, \qquad l \in B^* \subseteq H^*.$$

Let $\{T_t^{\mathrm{O-U}}\}_{t\geq 0}$ be the Ornstein-Uhlenbeck semigroup and L be its generator, which we call the Ornstein-Uhlenbeck operator. For $p \geq 1$, $\{T_t^{\mathrm{O-U}}\}$ is the strongly continuous contraction semigroup on $L^p(\mu)$. Moreover, $\{T_t^{\mathrm{O-U}}\}$ is Markovian. For $r \geq 0$, the Gamma transformation of $\{T_t^{\mathrm{O-U}}\}$ is defined by

$$V_r^{\mathrm{O-U}} = \frac{1}{\Gamma(r/2)}\int_0^\infty t^{r/2-1}e^{-t}T_t^{\mathrm{O-U}}dt. \tag{2.1}$$

Here we set $V_0^{\mathrm{O-U}} = I$ for $r = 0$ for convention, I being the identity operator. As usual, the Sobolev space is defined by

$$W^{r,p} := \mathrm{Ran}\,(V_r^{\mathrm{O-U}}) = V_r^{\mathrm{O-U}}(L^p(\mu)). \tag{2.2}$$

We denote the dual space of $W^{r,p}$ by $W^{-r,p'}$ where p' is the conjugate exponent of p: $1/p + 1/p' = 1$. In the sequel, we adopt the convention that the conjugate exponent is indicated by adding the prime. For notational simplicity, we use the following notation:

$$W^{r,p+} = \bigcup_{q>p} W^{r,q}, \quad r \in [0, \infty), \ p \in [1, \infty), \tag{2.3}$$

$$W^{\infty,p} = \bigcap_{r \geq 0} W^{r,p}, \quad p \in [1, \infty), \tag{2.4}$$

$$W^{\infty,p+} = \bigcap_{r \geq 0} W^{r,p+}, \quad p \in [1, \infty), \tag{2.5}$$

$$W^{r,p-} = \bigcap_{q<p} W^{r,q}, \quad r \in [0, \infty], \ p \in (1, \infty]. \tag{2.6}$$

We denote by $\mathscr{F}C_b^\infty$ the set of all functions f on B which is expressed as

$$f(x) = F(\langle x, l_1 \rangle, \cdots, \langle x, l_n \rangle), \qquad l_1, \cdots, l_n \in B^*$$

where $F \in C_b^\infty(\mathbf{R}^n)$, $C_b^\infty(\mathbf{R}^n)$ being the space of all bounded C^∞ functions on $\mathbf{R}^n$ whose derivatives are all bounded.

We take a function $\rho \in W^{\infty,\infty-}$ with $\rho > 0$ μ-a.e. and fix it. Let us consider the following Dirichlet form in $L^2(\rho^2\mu)$:

$$\mathscr{E}(u, v) = \int_B (Du(x), Dv(x))_{H^*} \rho(x)^2 \mu(dx). \tag{2.7}$$

Here Du is the H-derivative, i.e.,

$$_H\langle h, Du(x) \rangle_{H^*} = \lim_{\varepsilon \to 0} \frac{u(x + \varepsilon h) - u(x)}{\varepsilon}.$$

The above bilinear form can be extended easily to $W^{\infty,\infty-}$. Moreover $\mathscr{E}$ is closable because

$$\begin{aligned} \mathscr{E}(u, v) &= \int_B D^*(\rho^2 Du) v \mu(dx) \\ &= \int_B \left\{ D^* Du - \left(2\frac{D\rho}{\rho}, Du \right)_{H^*} \right\} v \rho^2 \mu(dx) \\ &= -\int_B \{Lu + 2(D \log \rho, Du)_{H^*}\} v \rho^2 \mu(dx). \end{aligned} \tag{2.8}$$

We also denote the closure of $(\mathscr{E}, \mathscr{F}C_b^\infty)$ by $\mathscr{E}$.

Let $\mathfrak{G}$ and $\{T_t\}_{t \geq 0}$ be associated generator and semigroup. In addition, $\{T_t\}$ is a strongly continuous contraction semigroup on $L^p(\rho^2\mu)$ for all $p \in [1, \infty)$.

We can define Sobolev spaces associated with $\{T_t\}$ similarly:

$$V_r = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} T_t dt,$$

$$\mathscr{F}_{r,p} := V_r(L^p(\rho^2\mu)).$$

If there is a fear of confusion, we denote $\mathfrak{G}^{(p)}$, $\{T_t^{(p)}\}_{t\geq 0}$ and $V_r^{(p)}$ to specify the space $L^p(\rho^2\mu)$. We will later use the following fact substantially: for $p > q \geq 1$, $\mathfrak{G}^{(q)}$, $\{T_t^{(q)}\}$ and $V_r^{(q)}$ are extensions of $\mathfrak{G}^{(p)}$, $\{T_t^{(p)}\}$ and $V_r^{(p)}$, respectively, e.g., $\mathrm{Dom}(\mathfrak{G}^{(p)}) \subseteq \mathrm{Dom}(\mathfrak{G}^{(q)})$ and $\mathfrak{G}^{(q)} = \mathfrak{G}^{(p)}$ on $\mathrm{Dom}(\mathfrak{G}^{(p)})$. We can define the (r, p)-capacity using $\mathscr{F}_{r,p}$. The following condition (R) is important to develop the capacity theory.

(R) $\mathscr{F}_{r,p} \cap C_b(B)$ is dense in $\mathscr{F}_{r,p}$.

Here $C_b(B)$ denotes the set of all bounded continuous functions on B. For example, Fukushima-Kaneko [8] showed the continuity from below of the capacity under the condition (R). We shall establish that condition (R) is satisfied in our situation.

Theorem 2.1. *Assume $\rho, \rho^{-1} \in W^{\infty,\infty-}$. Then $\mathscr{F}C_b^\infty$ is dense in $\mathscr{F}_{r,p}$ for $p \in [1, \infty)$.*

Before giving a proof, we prepare the following:

Proposition 2.2. *For any $p \geq 1$,*

$$\mathscr{F}C_b^\infty \subseteq W^{\infty,p+} \subseteq \mathscr{F}_{\infty,p} := \bigcap_{r\geq 0} \mathscr{F}_{r,p}. \tag{2.9}$$

Proof. First, recall that $\mathscr{F}C_b^\infty \subseteq \mathscr{F}_{2,2}$ and

$$\mathfrak{G}^{(2)}f(x) = Lf(x) + 2(D\log\rho(x), Df(x))_{H^*}, \qquad f \in \mathscr{F}C_b^\infty.$$

By the assumption, $\mathfrak{G}^{(2)}f \in L^p(\rho^2\mu)$ for any $p \geq 1$. Hence

$$f = V_2^{(2)}(I - \mathfrak{G}^{(2)})f = V_2^{(p)}(I - \mathfrak{G}^{(2)})f.$$

This implies $f \in \mathrm{Ran}(V_2^{(p)}) = \mathscr{F}_{2,p}$ and

$$\mathfrak{G}^{(p)}f = \mathfrak{G}^{(2)}f = Lf + 2(D\log\rho, Df)_{H^*}.$$

Take any q with $q > p$. Then for any $f \in W^{2,q}$, there exists a sequence $\{f_n\} \subseteq \mathscr{F}C_b^\infty$ such that

$$f_n \to f \text{ in } W^{2,q} \text{ as } n \to \infty.$$

Hence

$$\begin{aligned} &\|\mathfrak{G}^{(p)}f_n - Lf - 2(D\log\rho, Df)_{H^*}\|_{L^p(\rho^2\mu)} \\ &\leq \left\{\int_B |Lf_n(x) - Lf(x)|^p \rho^2(x)\mu(dx)\right\}^{1/p} \\ &\quad + \left\{\int_B |D\log\rho|_{H^*}^p |Df_n(x) - Df(x)|_{H^*}^p \rho^2(x)\mu(dx)\right\}^{1/p} \end{aligned}$$

$$\leq \left\{ \int_B |Lf_n(x) - Lf(x)|^q \mu(dx) \right\}^{1/q} \left\{ \int_B \rho^{2q/(q-p)} \mu(dx) \right\}^{(q-p)/qp}$$

$$+ 2 \left\{ \int_B |D \log \rho|_{H^*}^{pq/(q-p)} \rho^{2q/(q-p)}(x) \mu(dx) \right\}^{(q-p)/qp}$$

$$\times \left\{ \int_B |Df_n(x) - Df(x)|_{H^*}^q \mu(dx) \right\}^{1/q}$$

$$\leq \| Lf_n - Lf \|_q \| \rho \|_{2q/(q-p)}^{2/p} + 2 \| Df_n - Df \|_q \| |D \log \rho|_{H^*}^p \rho^2 \|_{q/(q-p)}^{1/p}$$

$$\to 0 \text{ as } n \to \infty$$

which yields $f \in \mathscr{F}_{2,p}$ and

$$\mathfrak{G}^{(p)} f = Lf + 2(D \log \rho, Df)_{H^*}. \tag{2.10}$$

We have therefore $W^{2,p+} \subseteq \mathscr{F}_{2,p}$. Moreover, by the expression (2.10), we have $\mathfrak{G}^{(p)} f \in W^{\infty,p+}$ for $f \in W^{\infty,p+}$. This implies that $W^{\infty,p+} \subseteq \mathrm{Dom}\,(I - \mathfrak{G}^{(p)})$ and

$$(I - \mathfrak{G}^{(p)}) W^{\infty,p+} \subseteq W^{\infty,p+}.$$

Hence, by the iteration, we have

$$W^{\infty,p+} \subseteq \bigcap_{k=1}^{\infty} \mathrm{Dom}\,((\mathfrak{G}^{(p)})^k) = \mathscr{F}_{\infty,p}.$$

This completes the proof.

The following lemma is well-known.

Lemma 2.3. *Let X be a subspace of $\mathscr{F}_{r,p}$, $r \geq 0$, $p \geq 1$. Then the following conditions are equivalent to each other.*

(i) *X is dense in $\mathscr{F}_{r,p}$.*

(ii) $\mathrm{Ker}\,(((I - \mathfrak{G}^{(p)})^{r/2} \upharpoonright X)^*) = \{0\}$, *i.e., if $u \in L^{p'}(\rho^2 \mu)$ satisfies*

$${}_{L^{p'}(\rho^2\mu)}\langle u, (I - \mathfrak{G}^{(p)})^{r/2} v \rangle_{L^p(\rho^2\mu)} = 0 \ \ \forall v \in X,$$

then $u = 0$.

Now we can give a proof of Theorem 2.1.

Proof of Theorem 2.1. For $r > s$, $\mathscr{F}_{r,p}$ is dense in $\mathscr{F}_{s,p}$, (see, e.g., [10, Proposition 2.5]), it is enough to prove the assertion in the case that r is an even integer, say $r = 2k$. By Lemma 2.3, we shall show $\mathrm{Ker}\,(((I - \mathfrak{G}^{(p)})^k \upharpoonright \mathscr{F}C_b^\infty)^*) = \{0\}$.

To show this, take any $u \in \mathrm{Ker}\,(((I - \mathfrak{G}^{(p)})^k \upharpoonright \mathscr{F}C_b^\infty)^*)$. Note that $\rho^2 u \in L^{p-}(\mu)$ since $u \in L^p(\rho^2 \mu)$ and $\rho \in L^{\infty-}(\mu)$. By the assumption,

$$\langle \rho^2 u, (I - \mathfrak{G})^k f \rangle_\mu = 0, \ \forall f \in \mathscr{F}C_b^\infty. \tag{2.11}$$

Here $\langle \cdot, \cdot \rangle_\mu$ denotes the inner product in $L^2(\mu)$. We use this notation extensively to denote any kind of pairing, e.g., $L^p(\mu)$ and $L^{p'}(\mu)$ or $W^{r,p}$ and $W^{-r,p'}$, etc. Since $\mathfrak{G}$ is expressed as (2.10), (2.11) holds for any $f \in W^{\infty,\infty-}$. On the other hand, $(I - \mathfrak{G})^k$ can be written as

$$(I - \mathfrak{G})^k = (I - L)^k - R_k \tag{2.12}$$

where R_k is a linear combination of derivatives with degree $\leq 2k - 1$. Moreover all coefficients of R_k belong to $W^{\infty,\infty-}$. Hence we have that R_k: $W^{s,q} \to W^{s-2k+1,\alpha}$ is bounded for $q > \alpha > 1$, $s \in \mathbf{R}$. By using $(I - L)^{-k}f \in W^{\infty,\infty-}$ for $f \in W^{\infty,\infty-}$, we have

$$\langle \rho^2 u, (I - \mathfrak{G})^k (I - L)^{-k} f \rangle_\mu = 0.$$

Hence for any α, q with $\alpha < q < p$,

$$\begin{aligned} |\langle \rho^2 u, f \rangle_\mu| &= |\langle \rho^2 u, R_k (I - L)^{-k} f \rangle_\mu| \\ &\leq \|\rho^2 u\|_q \|R_k (I - L)^{-k} f\|_{q'} \\ &\leq \|\rho^2 u\|_q \|R_k\|_{\mathscr{L}(W^{2k-1,\alpha'}, L^{q'}(\mu))} \|(I - L)^{-k} f\|_{2k-1,\alpha'} \\ &= \|\rho^2 u\|_q \|R_k\|_{\mathscr{L}(W^{2k-1,\alpha'}, L^{q'}(\mu))} \|f\|_{-1,\alpha'} \end{aligned}$$

which yields $\|\rho^2 u\|_{1,\alpha} \leq \|\rho^2 u\|_q \|R_k\|_{\mathscr{L}(W^{2k-1,\alpha'}, L^{q'}(\mu))} < \infty$. Since α, q is arbitrary, we have $\rho^2 u \in W^{1,p-}$. Using this, we repeat the above argument:

$$\begin{aligned} |\langle \rho^2 u, f \rangle_\mu| &\leq \|\rho^2 u\|_{1,q} \|R_k (I - L)^{-k} f\|_{-1,q'} \\ &\leq \|\rho^2 u\|_{1,q} \|R_k\|_{\mathscr{L}(W^{2k-2,\alpha'}, W^{-1,q'})} \|(I - L)^{-k} f\|_{2k-2,\alpha'} \\ &\leq \|\rho^2 u\|_{1,q} \|R_k\|_{\mathscr{L}(W^{2k-2,\alpha'}, W^{-1,q'})} \|f\|_{-2,\alpha'}. \end{aligned}$$

We have therefore $\rho^2 u \in W^{2,p-}$. Repeating above procedure, we eventually arrive at $\rho^2 u \in W^{\infty,p-}$. But $\rho^{-1} \in W^{\infty,\infty-}$, we easily get $u \in W^{\infty,p-}$. By Proposition 2.2, $u \in \mathscr{F}_{k,q}$ for any $q < p$. Hence for $f \in W^{\infty,\infty-}$,

$$\begin{aligned} 0 &= \langle \rho^2 u, (I - \mathfrak{G})^k f \rangle_\mu \\ &= {}_{L^q(\rho^2\mu)}\langle u, (I - \mathfrak{G}^{(q')})^k f \rangle_{L^{q'}(\rho^2\mu)} \\ &= {}_{L^q(\rho^2\mu)}\langle (I - \mathfrak{G}^{(q)})^k u, f \rangle_{L^{q'}(\rho^2\mu)} \end{aligned}$$

which yields $(I - \mathfrak{G}^{(q)})u = 0$. Since $I - \mathfrak{G}^{(q)}$ is injective, we have $u = 0$.

3. Essential self-adjointness

In the previous section, we have proved that $\mathscr{F}C_b^\infty$ is dense in $\mathscr{F}_{r,p}$. In the case $r = 2$, $p = 2$, this asserts that $\mathfrak{G}^{(2)}$ is essentially self-adjoint on $\mathscr{F}C_b^\infty$. Here the basic Hilbert space is $L^2(\rho^2\mu)$.

In this section, we restrict ourselves to the case $r = 2$, $p = 2$ and discuss the essential self-adjointness. Here we take the Ornstein-Uhlenbeck process with a general drift as a basic process though in the previous section we have taken the standard Ornstein-Uhlenbeck process. Precisely speaking, the associated

Dirichlet form is given by

$$\mathscr{E}^{\mathrm{O-U}}(u, v) = \int_B (\sqrt{A^*} Du(x), \sqrt{A^*} Dv(x))_{H^*}\mu(dx). \tag{3.1}$$

Here (B, H, μ) is an abstract Wiener space as before and A^* is a strictly positive self-adjoint operator in H^*. We simply denote $\sqrt{A^*}D$ by D_A.

We assume that $C^\infty(A^*)\cap B^*$ is dense in H^*. Here $C^\infty(A^*) = \bigcap_{k=1}^{\infty} \mathrm{Dom}((A^*)^k)$. In this case, the space $\mathscr{F}C_b^\infty$ needs be modified as follows: $\mathscr{F}C_b^\infty$ is the set of all functions f that can be expressed as

$$f(x) = F(\langle x, l_1\rangle, \cdots, \langle x, l_n\rangle),\ l_1, \dots, l_n \in C^\infty(A^*)\cap B^*$$

where $F\in C_b^\infty(\mathbf{R}^n)$. Then the bilinear form $\mathscr{E}^{\mathrm{O-U}}$ of (3.1) is well-defined on $\mathscr{F}C_b^\infty$ and closable. By taking closure, we get a Dirichlet form, which we want. In addition, we *assume* that there exists a diffusion process, which we call the Ornstein-Uhlenbeck process with the drift A, associated with the Dirichlet form (3.1). We can define the Sobolev space $W^{r,p}$ associated with this diffusion process.

Take a function $\rho\in W^{2,\infty-}$ with $\rho > 0$ a.e. Let us consider the following Dirichlet form:

$$\mathscr{E}(u, v) = \int_B (D_A u(x), D_A v(x))_{H^*}\rho(x)^2\mu(dx). \tag{3.2}$$

As in Section 2, we can see that $\mathscr{E}$ on $\mathscr{F}C_b^\infty$ is closable. Moreover, the associated generator can be seen by

$$\mathscr{E}(u, v) = -\int_B \{L_A u + 2(D_A \log\rho, D_A u)_{H^*}\} v\rho^2\mu(dx). \tag{3.3}$$

We also denote the closure of $(\mathscr{E}, \mathscr{F}C_b^\infty)$ by $\mathscr{E}$. We use the same notations as in Section 2, e.g., $\{T_t\}_{t\geq 0}$ and $\mathfrak{G}$ denote associated semigroup and generator, respectively. The following is the main theorem in this section.

Theorem 3.1. *Assume* $\rho\in W^{2,\infty-}$. *Further we assume that* $|D_A\rho|_{H^*}/\rho\in L^{2+}(\mu)$ *and* $L_A\rho/\rho\in L^{1+}(\mu)$. *Then* $\mathfrak{G}^{(2)}$ *is essentially self-adjoint on* $\mathscr{F}C_b^\infty$.

Before proving this theorem, we prepare the following:

Proposition 3.2. *Under the same condition as in Theorem* 3.1, *it holds that* $D_A^*(D_A\rho/\rho)\in L^{1+}(\mu)$.

Proof. Formally we have

$$D_A^*\left(\frac{D_A\rho}{\rho}\right) = -\frac{L_A\rho}{\rho} - \frac{|D_A\rho|^2}{\rho^2}\in L^{1+}(\mu). \tag{3.4}$$

To show this, take any $\varepsilon > 0$. Then

$$D_A^*\left(\frac{D_A\rho}{\rho+\varepsilon}\right)=-\frac{L_A\rho}{\rho+\varepsilon}-\frac{|D_A\rho|_{H^*}^2}{(\rho+\varepsilon)^2}.$$

By using $L_A\rho/\rho\in L^{1+}(\mu)$ and $|D_A\rho|/\rho\in L^{2+}(\mu)$, it is easy to see that the right hand side converges as $\varepsilon\to 0$. Consequently, we get (3.4).

For notational simplicity, we denote $D_A\log\rho$ in place of $D_A\rho/\rho$. $D_A\log\rho$ can not be defined in the Malliavin calculus because we do not assume the integrability of $\log\rho$. We also denote $L_A\log\rho$ in place of $-D_A^*(D_A\rho/\rho)$.

Proposition 3.3. *Assume the same condition as in Theorem* 3.1. *Then for any* $p\in[1,2]$, $W^{2,p+}\subseteq\mathscr{F}_{2,p}$. *More precisely, for any* $q>p$, *there exists a constant* c *such that*

$$\|(I-\mathfrak{G}^{(p)})f\|_{L^p(\rho^2\mu)}\le c\,\|f\|_{2,q},\qquad f\in W^{2,q}. \tag{3.5}$$

Proof. For any $f\in\mathscr{F}C_b^\infty$, we have

$$\begin{aligned}
&\|(I-L_A)f-2(D_A\log\rho,\,D_Af)_{H^*}\|_{L^p(\rho^2\mu)}\\
&\le\left\{\int_B|(I-L_A)f|^p\rho^2\mu(dx)\right\}^{1/p}+2\left\{\int_B\frac{|D_A\rho|_{H^*}^p}{\rho^p}|D_Af|_{H^*}^p\rho^2\mu(dx)\right\}^{1/p}\\
&\le\left\{\int_B|(I-L_A)f|^q\mu(dx)\right\}^{1/q}\left\{\int_B\rho^{2q/(q-p)}\mu(dx)\right\}^{(q-p)/qp}\\
&\quad+2\left\{\int_B|D_Af|_{H^*}^q\mu(dx)\right\}^{1/q}\left\{\int_B|D_A\rho(x)|_{H^*}^{pq/(q-p)}\rho^{(2-p)q/(q-p)}\mu(dx)\right\}^{(q-p)/qp}\\
&\le c\,\|f\|_{2,q}.
\end{aligned}$$

Here we used the fact $\|D_Af\|_q\le M\|f\|_{1,q}$ for some constant $M>0$. By noting that $\mathscr{F}C_b^\infty$ is dense in $W^{2,q}$, we get (3.5).

We introduce the following function. Let χ be a C^∞-function on $\mathbf{R}$ such that $0\le\chi\le 1$ and

$$\chi(t)=\begin{cases}1 & \text{if } t\ge 1,\\ 0 & \text{if } t\le\frac{1}{2}.\end{cases}$$

For $n\in\mathbf{N}$, we set

$$\chi_n(t)=\chi(2^nt).$$

Then it is easy to see that $\operatorname{supp}[\chi_n']\subseteq[2^{-n-1},2^{-n}]$ and there exists a constant C such that

$$|\chi_n'(t)|\le Ct^{-1}. \tag{3.6}$$

In fact, we may take $C=\|\chi'\|_\infty$. Further, we set $\varphi_n=\chi_n(\rho)$. Notice that

$$|D_A\varphi_n(x)|_{H^*}\le C\frac{|D_A\rho(x)|_{H^*}}{\rho(x)}. \tag{3.7}$$

Now let us turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. We shall prove that $\mathrm{Ker}((I-\mathfrak{G}^{(2)}\uparrow\mathscr{F}C_b^\infty)^*)=\{0\}$. Take any $v\in\mathrm{Ker}((I-\mathfrak{G}^{(2)}\uparrow\mathscr{F}C_b^\infty)^*)$. Then for any $f\in\mathscr{F}C_b^\infty$,

$$(v,(I-\mathfrak{G}^{(2)})f)_{L^2(\rho^2\mu)}=\langle\rho^2 v,(I-L_A)f-2(D_A\log\rho,D_Af)_{H^*}\rangle_\mu=0. \tag{3.8}$$

By Proposition 3.3, we can see that (3.8) holds for $f\in W^{\infty,\infty-}$.

If $f\in W^{\infty,\infty-}$, then $(I-L_A)^{-1}f\in W^{\infty,\infty-}$ and therefore we have

$$\langle\rho^2 v,(I-\mathfrak{G}^{(2)})(I-L_A)^{-1}f\rangle_\mu=\langle\rho^2 v,f-2(D_A\log\rho,D_A(I-L_A)^{-1}f)_{H^*}\rangle_\mu=0.$$

Hence

$$\langle\rho^2 v,f\rangle_\mu=\langle\rho^2 v,2(D_A\log\rho,D_A(I-L_A)^{-1}f)_{H^*}\rangle_\mu.$$

Then for any $1<q<2$,

$$\begin{aligned}|\langle\rho^2 v,f\rangle_\mu|&=\left|\left\langle\rho^2 v,2\left(\frac{D_A\rho}{\rho},D_A(I-L_A)^{-1}f\right)_{H^*}\right\rangle_\mu\right|\\&=|\langle\rho v,2(D_A\rho,D_A(I-L_A)^{-1}f)_{H^*}\rangle_\mu|\\&\le 2\,\|\rho vD_A\rho\|_q\,\|D_A(I-L_A)^{-1}f\|_p\\&\le\left\{\int_B\rho^q v^q|D_A\rho|^q\mu(dx)\right\}^{1/q}\|f\|_{-1,p}\\&\le\left\{\int_B\rho^2v^2\mu(dx)\right\}^{1/2}\left\{\int_B|D_A\rho|^{2q/(2-q)}\mu(dx)\right\}^{(2-q)/2q}\|f\|_{-1,p}\end{aligned}$$

which yields

$$\|\rho^2 v\|_{1,q}\le\|v\|_{L^2(\rho^2\mu)}\,\|\rho\|_{1,2q/(2-q)}<\infty. \tag{3.9}$$

Thus we have

$$\rho^2 v\in W^{1,2-}. \tag{3.10}$$

Using (3.10), we have for $\alpha>1$,

$$\begin{aligned}|\langle\rho^2 v,f\rangle_\mu|&\le 2|\langle\rho^2 v,(D_A\log\rho,D_A(I-L_A)^{-1}f)_{H^*}\rangle_\mu|\\&\le 2|\langle D_A^*(\rho^2 vD_A\log\rho),(I-L_A)^{-1}f\rangle_\mu|\\&\le 2|\langle(D_A(\rho^2v),D_A\log\rho)_{H^*},(I-L_A)^{-1}f\rangle_\mu|\\&\quad+2|\langle\rho^2vL_A\log\rho,(I-L_A)^{-1}f\rangle_\mu|\\&\le 2\,\|(D_A(\rho^2v),D_A\log\rho)_{H^*}\|_\alpha\,\|(I-L_A)^{-1}f\|_{\alpha'}\\&\quad+2\,\|\rho^2vL_A\log\rho\|_\alpha\,\|(I-L_A)^{-1}f\|_{\alpha'}\\&\le 2\{\|(D_A(\rho^2v),D_A\log\rho)_{H^*}\|_\alpha+\|\rho^2vL_A\log\rho\|_\alpha\}\,\|f\|_{2,\alpha'}.\end{aligned}$$

By noting $\rho^2v\in W^{1,2-}$ and $D_A\log\rho\in L^{2+}(\mu)$, we can choose $\alpha>1$ so that

$\|(D_A(\rho^2 v), D_A \log \rho)_{H^*}\|_\alpha < \infty$. Moreover, note that

$$\|\rho^2 v L_A \log \rho\|_r = \left\|\rho^2 v\left(\frac{L_A\rho}{\rho} + \frac{|D_A\rho|_{H^*}^2}{\rho^2}\right)\right\|_r$$
$$= \|\rho v L_A \rho\|_r + \|\rho v (D_A \log \rho, D_A\rho)_{H^*}\|_r.$$

Now by using $v \in L^2(\rho^2\mu)$, $\rho \in W^{2,\infty-}$ and $|D_A \log \rho|_{H^*} \in L^{2+}(\mu)$ we can choose $r > 1$ so that $\|\rho^2 v L_A \log \rho\|_r < \infty$ at the same time. Thus we have

$$\rho^2 v \in W^{2,1+}. \tag{3.11}$$

Since $\varphi_n/\rho^2 \in W^{2,\infty-}$, we have $\varphi_n v = \rho^2 v \cdot (\varphi_n/\rho^2) \in W^{2,1+}$. Now by Proposition 3.3, we obtain $\varphi_n v \in \mathscr{F}_{2,1}$. We shall show that $(I - \mathfrak{G}^{(1)})(\varphi_n v)$ converges in $L^1(\rho^2\mu)$.

To show this, we note that for $f \in \mathscr{F}C_b^\infty$,

$$\begin{aligned}
&{}_{L^1(\rho^2\mu)}\langle \mathfrak{G}(\varphi_n v) - \varphi_n v, f\rangle_{L^\infty(\rho^2\mu)} \\
&= (\varphi_n v, \mathfrak{G} f)_{L^2(\rho^2\mu)} - (\varphi_n v, f)_{L^2(\rho^2\mu)} \\
&= (v, \mathfrak{G}(f\varphi_n))_{L^2(\rho^2\mu)} - (v, 2(D_A f, D_A\varphi_n)_{H^*})_{L^2(\rho^2\mu)} \\
&\quad - (v, f\mathfrak{G}\varphi_n)_{L^2(\rho^2\mu)} - (\varphi_n v, f)_{L^2(\rho^2\mu)} \\
&= -\langle v\rho^2, 2(D_A f, D_A\varphi_n)_{H^*}\rangle_\mu - \langle v\rho^2, f\mathfrak{G}\varphi_n\rangle_\mu \\
&= 2\langle (D_A(v\rho^2), D_A\varphi_n)_{H^*}, f\rangle_\mu + 2\langle v\rho^2 L_A\varphi_n, f\rangle_\mu - \langle v\rho^2, f\mathfrak{G}\varphi_n\rangle_\mu \\
&= 2\langle (D_A(v\rho^2), D_A\varphi_n)_{H^*}, f\rangle_\mu + \langle v\rho^2 L_A\varphi_n, f\rangle_\mu \\
&\quad - 2\langle v\rho^2, f(D_A \log \rho, D_A\varphi_n)_{H^*}\rangle_\mu.
\end{aligned}$$

By the monotone class theorem, the above equation holds for any $f \in L^\infty(\rho^2\mu)$, i.e.,

$$\begin{aligned}
{}_{L^1(\rho^2\mu)}\langle \mathfrak{G}(\varphi_n v) - \varphi_n v, f\rangle_{L^\infty(\rho^2\mu)} &= 2\langle (D_A(v\rho^2), D_A\varphi_n)_{H^*}, f\rangle_\mu + \langle v\rho^2 L_A\varphi_n, f\rangle_\mu \\
&\quad - 2\langle v\rho^2, f(D_A \log \rho, D_A\varphi_n)_{H^*}\rangle_\mu.
\end{aligned}$$

Let us estimate each term of the right hand side. To show this we take q' so that $D_A \log \rho \in L^{q'}(\mu)$. First,

$$\begin{aligned}
&|\langle (D_A(v\rho^2), D_A\varphi_n)_{H^*}, f\rangle_\mu| \\
&\le \int_B |D_A(v\rho^2)|_{H^*} |D_A\varphi_n|_{H^*} |f(x)|\,\mu(dx) \\
&\le \left\{\int_B |D_A(v\rho^2)|_{H^*}^q \mu(dx)\right\}^{1/q} \left\{\int_B |D_A\varphi_n|_{H^*}^{q'} \mu(dx)\right\}^{1/q'} \|f\|_{L^\infty(\rho^2\mu)} \\
&\le \|v\rho^2\|_{1,q} \|f\|_{L^\infty(\rho^2\mu)} \left\{\int_B |\chi_n'(\rho)|^{q'} |D_A\rho|_{H^*}^{q'} \mu(dx)\right\}^{1/q'} \\
&\le \|v\rho^2\|_{1,q} \|f\|_{L^\infty(\rho^2\mu)} \left\{\int_{\{\rho \le 2^{-n}\}} \left\{\frac{|D_A\rho|_{H^*}}{\rho}\right\}^{q'} \mu(dx)\right\}^{1/q'}
\end{aligned}$$

$$\leq \|v\|_{L^2(\rho^2\mu)} \|\rho\|_{1,2q/(2-q)} \|f\|_{L^\infty(\rho^2\mu)} \left\{ \int_{\{\rho \leq 2^{-n}\}} |D_A \log \rho|_{H^*}^{q'} \mu(dx) \right\}^{1/q'}.$$

Here, in the last line we used (3.9).

Secondly,

$$|\langle v\rho^2 L_A \varphi_n, f\rangle_\mu|$$
$$\leq \left\{ \int_B |v(x)|^2 \rho(x)^2 \mu(dx) \right\}^{1/2} \left\{ \int_B |L_A \varphi_n|^2 \rho^2 \mu(dx) \right\}^{1/2} \|f\|_{L^\infty(\rho^2\mu)}.$$

Further, by noting that

$$L_A \varphi_n = -D_A^* D_A \varphi_n = -D_A^*(\chi_n'(\rho) D_A \rho) = \chi_n''(\rho)(D_A\rho, D_A\rho)_{H^*} + \chi_n'(\rho) L_A \rho$$

we have

$$\left\{ \int_B |L_A \varphi_n|^2 \rho^2 \mu(dx) \right\}^{1/2}$$
$$\leq \left\{ \int_\infty \{|\chi_n''(\rho)| |D_A \rho|_{H^*}^2 + |\chi_n'(\rho)| |L_A \rho|\}^2 \rho^2 \mu(dx) \right\}^{1/2}$$
$$\leq \left\{ \int_{\{\rho \leq 2^{-n}\}} \left\{ C \frac{|D_A\rho|_{H^*}^2}{\rho^2} + C \frac{|D_A \rho|_{H^*}}{\rho} |L_A \rho| \right\}^2 \rho^2 \mu(dx) \right\}^{1/2}$$
$$\leq C \left\{ \int_{\{\rho \leq 2^{-n}\}} \frac{|D_A \rho|_{H^*}^4}{\rho^4} \rho^2 \mu(dx) \right\}^{1/2}$$
$$+ C \left\{ \int_{\{\rho \leq 2^{-n}\}} \frac{|D_A \rho|_{H^*}^2}{\rho^2} |L_A \rho|^2 \rho^2 \mu(dx) \right\}^{1/2}$$
$$\leq C \left\{ \int_{\{\rho \leq 2^{-n}\}} \frac{|D_A \rho|_{H^*}^2}{\rho^2} |D_A \rho|_{H^*}^2 \mu(dx) \right\}^{1/2}$$
$$+ C \left\{ \int_{\{\rho \leq 2^{-n}\}} |D_A \rho|_{H^*}^2 |L_A \rho|^2 \mu(dx) \right\}^{1/2}.$$

Hence

$$|\langle v\rho^2 L_A \varphi_n, f\rangle_\mu| \leq C \left\{ \int_B |v(x)|^2 \rho(x)^2 \mu(dx) \right\}^{1/2} \|f\|_{L^\infty(\rho^2\mu)}$$
$$\times \left[\left\{ \int_{\{\rho \leq 2^{-n}\}} |D_A \log \rho|_{H^*}^2 |D_A \rho|_{H^*}^2 \mu(dx) \right\}^{1/2} + \left\{ \int_{\{\rho \leq 2^{-n}\}} |D_A \rho|_{H^*}^2 |L_A \rho|^2 \mu(dx) \right\}^{1/2} \right].$$

Thirdly,

$$|\langle v\rho^2, f(D_A \log \rho, D_A \varphi_n)_{H^*}\rangle_\mu|$$

$$\leq \left\{\int_B |v(x)|^2 \rho(x)^2 \mu(dx)\right\}^{1/2} \|f\|_{L^\infty(\rho^2\mu)}$$

$$\times \left\{\int_B |D_A \log \rho|_{H^*}^2 |D_A \varphi_n|^2 \rho^2 \mu(dx)\right\}^{1/2}$$

$$\leq \|v\|_{L^2(\rho^2\mu)} \|f\|_{L^\infty(\rho^2\mu)} \left\{\int_B |D_A \log \rho|_{H^*}^2 |\chi_n'(\rho)|^2 |D_A \rho|_{H^*}^2 \rho^2 \mu(dx)\right\}^{1/2}$$

$$\leq C \|v\|_{L^2(\rho^2\mu)} \|f\|_{L^\infty(\rho^2\mu)} \left\{\int_{\{\rho \leq 2^{-n}\}} |D_A \log \rho|_{H^*}^2 \rho^{-2} |D_A \rho|_{H^*}^2 \rho^2 \mu(dx)\right\}^{1/2}$$

$$\leq C \|v\|_{L^2(\rho^2\mu)} \|f\|_{L^\infty(\rho^2\mu)} \left\{\int_{\{\rho \leq 2^{-n}\}} |D_A \log \rho|_{H^*}^2 |D_A \rho|_{H^*}^2 \mu(dx)\right\}^{1/2}.$$

Thus we have

$$\begin{aligned}\|(\mathfrak{G}^{(1)} - I)(\varphi_n v)\|_{L^1(\rho^2\mu)} \leq \|v\|_{L^2(\rho^2\mu)} \Bigg[&\left\{\int_{\{\rho \leq 2^{-n}\}} |D_A \log \rho|_{H^*}^{q'} \mu(dx)\right\}^{1/q'} \\ &+ 2C\left\{\int_{\{\rho \leq 2^{-n}\}} |D_A \log \rho|_{H^*}^2 |D_A \rho|_{H^*}^2 \mu(dx)\right\}^{1/2} \\ &+ C\left\{\int_{\{\rho \leq 2^{-n}\}} |D_A \rho|_{H^*}^2 |L_A \rho|^2 \mu(dx)\right\}^{1/2}\Bigg] \\ &\to 0 \text{ as } n \to \infty\end{aligned}$$

since $\rho \in W^{2,\infty-}$ and we have taken q' so that $|D_A \log \rho|_{H^*} \in L^{q'}(\mu)$.

Thus we have $(\mathfrak{G}^{(1)} - I)(\varphi_n v) \to 0$ in $L^1(\rho^2\mu)$. Since $\varphi_n v \to v$ in $L^1(\rho^2\mu)$ as $n \to \infty$, we have $v \in \mathscr{F}_{2,1}$ and $(\mathfrak{G}^{(1)} - I)v = 0$. Since $\mathfrak{G}^{(1)} - I$ is injective, we have $v = 0$. This completes the proof.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] S. Albeverio and M. Röckner, Classical Dirichlet forms on topological vector spaces—the construction of the associated diffusion process, Probab. Th. Rel. Fields, **83** (1989), 405–434.

[2] S. Albeverio, Ju. G. Kondratiev and M. Röckner, An approximate criteriom of essential self-adjointness of Dirichlet operators, Potential Anal., **1** (1992), 307–317.

[3] S. Albeverio, M. Fukushima, W. Hansen, Z. M. Ma and M. Röckner, An invariance result for capacities on Wiener space, J. Funct. Anal., **106** (1992), 35–49.

[4] D. Feyel and A. de La Pradelle, Espaces de Sobolev gaussiens, Ann. Inst. Fourier, Grenoble, **39** (1989), 875–908.

[5] D. Feyel and A. de La Pradelle, Capacités gaussiennes, Ann. Inst. Fourier, Grenoble, **41** (1991), 49–76.

[6] D. Feyel and A. de La Pradelle, Opérateurs linéaires et espaces de Sobolev sur l'espace de

Wiener, C. R. Acad. Sci. Paris, **313**, Serie I, (1991) 727–729.

[7] D. Feyel and A. de La Pradelle, Opérateurs linéaires gaussiens, Poential Anal., **3** (1994), 89–105.

[8] M. Fukushima and H. Kaneko, On (r, p)-capacities for general Markovian semigroups, in "Infinite dimensional analysis and stochastic processes," pp. 41–47, ed. by S. Albeverio, Longman Scientific & Technical, Harlow, 1985.

[9] H. Kaneko, On (r, p)-capacities for Markov rocesses, Osaka J. Math., **23** (1986), 325–336.

[10] T. Kazumi and I. Shigekawa, Measures of finite (r, p)-energy and potentials on a separable metric space, Séminaire de Probabilités, XXVI, ed. by J. Azéma, P. A. Meyer and M. Yor, Lecture Notes in Math., vol. 1526, pp. 415–444, Berlin-Heidelberg-New York, Springer-Verlag, 1992.

[11] Ju. G. Kondratiev and T. V. Tsycalenko, Infinite-dimensional Dirichlet operators I: Essential selfadjointness and associated elliptic equations, Potential Anal., **2** (1993), 1–21.

[12] S. Kusuoka, Dirichlet forms and diffusion rocesses on Banach space, J. Fac. Science Univ. Tokyo, Sec. 1A **29** (1982), 79–95.

[13] V. G. Maz'ya and V. P. Khavin, Non-linear potential theory, Russian Math. Surveys, **27** (1983), 71–148.

[14] M. Röckner and T.-S. Zhang, Uniqueness of generalized Schrödinger operators and applications, J. Funct. Anal., **105** (1992), 187–231.

[15] B. Schmuland, An alternative compactification for classical Dirichlet forms on topological vector spaces, Stochastics, **33** (1990), 75–90.

[16] I. Shigekawa, Sobolev spaces over the Wiener space based on an Ornstein-Uhlenbeck operator, J. Math. Kyoto Univ., **32** (1992), 731–748.

[17] B. Simon, The $P(\phi)_2$ Euclidean (quantum) field theory, Princeton University Press, Princeton, New Jersy, 1974.

[18] H. Sugita, Sobolev spaces of Wiener functionals and Malliavin's calculus, J. Math. Kyoto Univ., **25** (1985), 31–48.

[19] H. Sugita, Positive generalized Wiener functions and potential theory over abstract Wiener spaces, Osaka J. Math., **25** (1988), 665–696.

[20] M. Takeda, (r, p)-capacity on the Wiener space and properties of Brownian motion, Z. Waht. verw. Gebiete, **68** (1984), 149–162.

[21] M. Takeda, On the uniqueness of Markovian self-adjoint extension of diffusion operators on infinite dimensional spaces, Osaka J. Math., **22** (1985), 733–742.

[22] M. Takeda, On the uniqueness of Markovian self-adjoint extension, Lecture Notes in Math., vol. 1250 (Stochastic processes — Mathematics and Physics), 319–325, Springer, Berlin, 1985.

[23] N. Wielens, On the essential self-adjointness of generalized Schrödinger operators, J. Funct. Anal., **61** (1985), 98–115.