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A remark on Brauer's height zero conjecture

By

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Introduction

Let G be a finite group and p a prime. Let B be a p-block of G with defect group D. A well-known conjecture of R. Brauer asserts:

(HZ) Every irreducible character in B is of height 0 if and only if D is abelian.

This assertion is naturally divided into two parts:

(AHZ) If D is abelian, then every irreducible character in B is of height 0.

(HZA) If every irreducible character in B is of height 0, then D is abelian.

The assertion (AHZ) has been reduced to the case of quasi-simple groups by Berger and Knörr [2]. See also [10]. The assertion (HZA) has been shown to be true for p-solvable groups by Gluck and Wolf [6].

Let $k_0(B)$ be the number of irreducible characters of height 0 in B. Let $\tilde{B}$ be the Brauer correspondent of B in $N_G(D)$. Then the Alperin-McKay conjecture asserts:

(AM) $\quad k_0(B) = k_0(\tilde{B})$.

For p-solvable groups the assertion (AM) has been shown to be true, cf. Dade [3], Okuyama and Wajima [11].

On the other hand, R. Knörr and G. R. Robinson have shown that if the conjecture (AM) and Alperin's weight conjecture [1] are true, then so is the conjecture (AHZ), cf. [8, Proposition 5.6] for a more precise statement. Moreover E. C. Dade has given a conjecture which implies (AM) and Alperin's weight conjecture (and hence (AHZ)) are true, cf. [4, p. 188].

Here we prove the following

Theorem. *Assume that* (AM) *is true for the principal blocks of all finite groups and that* (HZA) *is true for the principal blocks of all simple groups. Then* (HZA) *is true for the principal blocks of all finite groups.*

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Proof of Theorem

Let G be a minimal counterexample to the assertion (HZA) (for the case of principal blocks). Let $N_0 \neq G$ be a normal subgroup of G. Let b_0 be the principal block of N_0 and let δ_0 be a Sylow p-subgroup of N_0. Clearly we have that every $\xi \in \mathrm{Irr}(b_0)$ has p'-degree, and hence δ_0 is abelian by the choice of G. In particular,

(*) G has no normal subgroup ($\neq G$) of p'-index.

Let N be a maximal normal subgroup of G. Let B (resp. b) be the principal block of G (resp. N). Let D be a Sylow p-subgroup of G and put $\delta = D \cap N$. Then δ is a Sylow p-subgroup of N and δ is abelian by the above. Also $D \leqq N_G(\delta)$ and D is nonabelian.

If $N = 1$, then G is simple and D is abelian by our assumption of Theorem. Hence $N \neq 1$. Let $T = \cap T_G(\xi)$, where ξ runs through $\mathrm{Irr}(b)$. (Here $T_G(\xi)$ is the inertial group of ξ in G.) Then $N \leqq T \lhd G$. Since G/N is simple, either of the following holds:

(I) $T = G$.

(II) $T = N$.

We first consider the case (I). Every irreducible character χ in the principal block of G/N belongs to B, when considered as a character of G, so χ has p'-degree. Hence DN/N is abelian by the choice of G. Let b^* be the principal block of DN. We claim that every irreducible character in b^* has p'-degree. Let $\zeta \in \mathrm{Irr}(b^*)$ and $\xi \in \mathrm{Irr}(b)$ be a constituent of ζ_N. There is a character $\chi \in \mathrm{Irr}(B)$ such that ξ is a constituent of χ_N. Since χ has p'-degree, so does ξ. Hence it follows from [10, Theorem 4.4 (i)] that ξ extends to $D^x N$ for some $x \in G$. So there is an extension η of ξ to DN, since ξ is G-invariant. Then $\zeta = \eta\psi$ for some $\psi \in \mathrm{Irr}(DN/N)$. Since DN/N is abelian, $\psi(1) = 1$ and $\zeta(1) = \eta(1) = \xi(1)$ is prime to p. So the claim is proved. If $DN \neq G$, then D is abelian by the choice of G, a contradiction. Hence $DN = G$. Since G/N is simple, G/N is of order p.

Let B_1 (resp. b_1) be the principal block of $N_G(\delta)$ (resp. $N_N(\delta)$). Since $G = N_G(\delta)N$ by the Frattini argument, $N_G(\delta)/N_N(\delta) \cong G/N$ is of order p. In particular, B_1 is a unique block of $N_G(\delta)$ covering b_1. This implies that

$$k(B_1) = ip + \frac{k(b_1) - i}{p},$$

where i denotes the number of $N_G(\delta)$-invariant irreducible characters in b_1 and $k(B_1)$ (resp. $k(b_1)$) denotes the number of irreducible characters in B_1 (resp. b_1). Since δ is a normal abelian Sylow p-subgroup of $N_N(\delta)$, $k(b_1) = k_0(b_1)$ by Ito's theorem. By (AM), $k_0(b_1) = k_0(b)$ and $k_0(B_1) = k_0(B)$. Now, since every irreducible character in B has p'-degree and every irreducible character in b is G-invariant, we get $k_0(B) = pk_0(b)$ as above. Combining the above equalities with the trivial inequality $k(B_1) \geqq k_0(B_1)$, we get that $k_0(b_1) \leqq i \leqq k(b_1) = k_0(b_1)$. Thus equality holds throughout, so $k(B_1) = k_0(B_1)$. Thus every irreducible character in B_1 has p'-degree. So if $N_G(\delta) \neq G$, then D is abelian by the choice of G, a contradiction. Hence $N_G(\delta) = G$.

Now b is a unique block of N covering the principal block of $C_N(\delta)$. So it follows that B is a unique block of G covering the principal block of $C_N(\delta)$. Thus every irreducible character χ of $G/C_N(\delta)$ lies in B, when regarded as a character of G, and χ has p'-degree. Since $G/C_N(\delta)$ has a normal p-complement $N/C_N(\delta)$, it follows that $G/C_N(\delta)$ has a normal Sylow p-subgroup, cf. (the proof of) Theorem 12.33 in [7]. Thus, by (*), $N = C_N(\delta)$. So N has a normal p-complement by Burnside's theorem. Hence G has a normal p-complement and then $\mathrm{Irr}(B)$ is identified with $\mathrm{Irr}(D)$. So every irreducible character of D is linear and D is abelian, a contradiction. (Of course, we could have used Fong [5, Theorem (3E)].)

Hence (II) occurs. By the Frattini argument, $G = N_G(\delta)N$. So $N \leqq C_G(\delta)N \lhd G$. Thus either of the following holds, since G/N is simple.

(a) $G = C_G(\delta)N$.

(b) $C_G(\delta) \leqq N$.

Consider the case (a). Let Q be a Sylow p-subgroup of $C_G(\delta)$. Then by (a), $|G: C_G(\delta)| = |N: C_N(\delta)|$ is prime to p, so Q is a Sylow p-subgroup of G. Since $\delta \leqq Q$, $Q = C_Q(\delta) \leqq T = N$ by [10, Lemma 4.14 (ii)]. Hence G/N is a p'-group, which contradicts (*). Hence the case (b) occurs. Now (b) implies that B is a unique block of G which covers b. So, when regarded as a character of G, every $\chi \in \mathrm{Irr}(G/N)$ belongs to B and hence χ has p'-degree. Then G/N has a normal abelian Sylow p-subgroup by Michler [9, Theorem 2.3]. Since G/N is simple, G/N is either a p'-group or of order p. The former is impossible by (*). In the latter case, there is $\xi \in \mathrm{Irr}(b)$ with $T_G(\xi) = N$, by (II). Then $\xi^G \in \mathrm{Irr}(B)$ and $\xi^G(1)$ is divisible by p, which is a final contradiction. This completes the proof.

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