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Failure of analytic hypoellipticity for some operators of $X^2 + Y^2$ type

Dedicated to Professor Tosinobu Muramatu on his 60th birthday

By

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0. Introduction and result

A differential operator P is said to be hypoelliptic, if for any C^∞ function f in some open set U all solutions u to $Pu = f$ belong to $C^\infty(U)$. Also P is said to be analytic hypoelliptic, if $f \in C^\omega(U)$ implies $u \in C^\omega(U)$. Let Ω be an open set in $\mathbf{R}^n$ and $X_1, \dots, X_r$ be real vector fields with analytic coefficients. It is well known that, if $X_1, \dots, X_r$ and their commutators $[X_{j_1}, X_{j_2}], \dots, [X_{j_1}[X_{j_2}, \dots, [X_{j_{k-1}}, X_{j_k}] \cdots] \dots$ generate the tangent space $T_x\mathbf{R}^n$ for all $x \in \Omega$ then the operator

$$(1) \qquad P = \sum_{j=1}^{r} X_j^2$$

is hypoelliptic in Ω (L. Hörmander [7]).

Note that such an assumption as above is not sufficient for analytic hypoellipticity (*cf.* F. Treves [12], D. S. Tartakoff [11] and A. Grigis-J. Sjöstrand [4]). Indeed, there are some negative results. Some hypoelliptic operators of type (1) were shown to be not analytic hypoelliptic. Such operators can be seen, for example, in the following papers: M. S. Baouendi- C. Goulaouic [1], G. Métivier [8], B. Helffer [6], Pham The Lai-D. Robert [9], N. Hanges-A. A. Himonas [5] and M. Christ [2]. The purpose of the present paper is to give new examples of hypoelliptic operators which fail to be analytic hypoelliptic.

Here, we consider the operator

$$(2) \qquad P = \frac{\partial^2}{\partial x^2} + \left(x^k \frac{\partial}{\partial y} - x^l \frac{\partial}{\partial t} \right)^2$$

in $\mathbf{R}^3$. If the non negative integers k, l satisfy $k < l$, then Hörmander's theorem can be applied, hence the operator P is hypoelliptic. With this hypothesis the result of the present paper is following

Theorem. *The operator P in* (2) *is not analytic hypoelliptic, if either of the following assumptions is satisfied*:

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(i) $\dfrac{l+1}{l-k}$ *is not a positive integer.*

(ii) *The both* $l-k$ *and* $\dfrac{l+1}{l-k}$ *are odd integers.*

The operator P in (2) with $k=0$ was considered by M. Christ [2] (see also N. Hanges-A. A. Himonas [5]). He proved that such operators are not analytic hypoelliptic except the case $l=1$ (see the assumption (i) in Theorem). Our proof of Theorem is based on his method, so Theorem is an extension of his result.

The present paper is organized as follows: First, we shall explain the outline of a proof of Theorem in Section 1. A lemma, which is essential to a proof of our Theorem, will be proven in Section 2. Section 3 will be devoted to a proof of Proposition 2 which is also necessary for a proof of the lemma.

1. Outline of Proof

Theorem will be proved, if we construct a non real analytic solution u to $Pu \equiv 0$. To do so, for $\zeta \in \mathbf{C}$, set

$$P_\zeta = -\frac{d^2}{dx^2} + (x^k\zeta - x^l)^2. \tag{3}$$

The argument as in the following lemma is standard (*cf.* [1], [2], [5] and [8]).

Lemma 1. *If there exist* $\zeta \in \mathbf{C}$ *and* $f \in L^\infty(\mathbf{R})$ *not identically equal to zero satisfying* $P_\zeta f \equiv 0$, *then the operator* P *in* (2) *is not analytic hypoelliptic.*

Proof. Let F be defined by

$$F(x, y, t) = \int_0^\infty e^{it\tau + i\tau^{\frac{k+1}{l+1}}\zeta y - \tau^{\frac{k+1}{l+1}}} f(\tau^{\frac{1}{l+1}}x)\,d\tau.$$

Since the function f is of class $L^\infty(\mathbf{R})$, the above integral converges in a region $|y| < \varepsilon$ for some $\varepsilon > 0$. Also it is easy to see that $PF \equiv 0$, since we have $P_\zeta f \equiv 0$. On the other hand, one can show that F is not real analytic at $(x, y, z) = (0, 0, 0)$. In fact, observe that

$$\begin{aligned}\left|\frac{\partial^m F}{\partial t^m}(0, 0, 0)\right| &= |f(0)| \int_0^\infty \tau^m e^{-\tau^{\frac{k+1}{l+1}}}\,d\tau \\ &= |f(0)| \cdot \frac{l+1}{k+1} \cdot \Gamma\left(\frac{l+1}{k+1}(m+1)\right).\end{aligned}$$

The last relation obviously yields that, if $f(0) \neq 0$, then there is no constant C such that $\left|\dfrac{\partial^m F}{\partial t^m}(0, 0, 0)\right| \leq C^{m+1} m!$, because $\dfrac{l+1}{k+1} > 1$. In case $f(0) = 0$, $f \not\equiv 0$

and the uniqueness of the solution imply that $f'(0) \neq 0$. In this case, we have

$$\left|\frac{\partial^{m+1} F}{\partial t^m \partial x}(0,0,0)\right| = |f'(0)| \cdot \frac{l+1}{k+1} \cdot \Gamma\left(\frac{l+1}{k+1}(m+1) + \frac{1}{k+1}\right).$$

Thus F is not real analytic also in this case.

By virtue of Lemma 1, it suffices to show the existence of $\zeta \in \mathbf{C}$ and $f \in L^\infty(\mathbf{R})$ as in Lemma 1 from one of the hypotheses in our Theorem. To show this, we use M. Christ's procedure.

The next result is well known in much greater generality (*cf.* E. A. Coddington-N. Levinson [3] or Y. Sibuya [10]).

Proposition 1. *For each $\zeta \in \mathbf{C}$, there exist unique solutions f_ζ^+, f_ζ^- to $P_\zeta f_\zeta^+ \equiv 0$, $P_\zeta f_\zeta^- \equiv 0$, respectively, having the following asymptotic behaviors:*

(i) $$f_\zeta^+(x) = |x|^{-\frac{l}{2}} e^{-\phi_\zeta(x)}(1 + O(|x|^{-1})) \qquad \textit{as } x \to \infty,$$

(ii) $$f_\zeta^-(x) = |x|^{-\frac{l}{2}} e^{(-1)^l \phi_\zeta(x)}(1 + O(|x|^{-1})) \qquad \textit{as } x \to -\infty,$$

where $\phi_\zeta(x) = \dfrac{x^{l+1}}{l+1} - \zeta \dfrac{x^{k+1}}{k+1}$. *Moreover, for each $x \in \mathbf{R}$, these functions are holomorphic with respect to $\zeta \in \mathbf{C}$ and also real valued for $\zeta \in \mathbf{R}$.*

Y. Sibuya called f_ζ^+ (resp. f_ζ^-) a subdominant solution on positive (resp. negative) real axis in his book [10]. For our purpose, it suffices to show the existence of $\zeta \in \mathbf{C}$ where f_ζ^+ and f_ζ^- are linearly dependent. Because, for such $\zeta \in \mathbf{C}$, the both subdominant solutions decay exponentially as $x \to \pm\infty$, hence remain bounded. Next consider the wronskian

$$W(\zeta) = f_\zeta^+(x)(f_\zeta^-)'(x) - (f_\zeta^+)'(x) f_\zeta^-(x).$$

Then it is also obvious that the existence of $\zeta \in \mathbf{C}$ mentioned above is equivalent to that of $\zeta \in \mathbf{C}$ with $W(\zeta) = 0$.

The following lemma is essential to the proof of our Theorem. It gives some information concerning zeros of $W(\zeta)$.

Lemma 2. *The Wronskian $W(\zeta)$ is an entire function of order $\dfrac{l+1}{l-k}$. More precisely, there exist positive constants C and ε such that*

(i) $$|W(\zeta)| \leq C \exp(C|\zeta|^{\frac{l+1}{l-k}}), \qquad \textit{for all } \zeta \in \mathbf{C},$$

(ii) $$|W(\zeta)| \geq \varepsilon \exp(\varepsilon|\zeta|^{\frac{l+1}{l-k}}), \qquad \textit{for all } \zeta \in \mathbf{R}_+.$$

Moreover, if $l - k$ is an odd integer, then the inequality (ii) *holds for all $\zeta \in \mathbf{R}$.*

A proof of Lemma 2 will be given in the next section. Taking Lemma 2 for granted, let us prove our Theorem.

First we consider the case (i), i.e., the case in which $\frac{l+1}{l-k}$ is not a positive integer. Suppose that $W(\zeta) \neq 0$ for arbitrary $\zeta \in \mathbf{C}$. Then, the function

$$w(\zeta) = \log W(\zeta) = \log |W(\zeta)| + i \arg W(\zeta)$$

is defined as an entire function with respect to ζ. By Lemma 2, it satisfies

$$|\mathrm{Re}\, w(\zeta)| \leq \log C + C|\zeta|^{\frac{l+1}{l-k}}, \qquad \zeta \in \mathbf{C}.$$

Hence, $w(\zeta)$ must be a polynomial of at most degree $\left[\frac{l+1}{l-k}\right]$. On the other hand, it follows also from Lemma 2 that

$$|\mathrm{Re}\, w(\zeta)| \geq \log \varepsilon + \varepsilon|\zeta|^{\frac{l+1}{l-k}}, \qquad \zeta \in \mathbf{R}_+.$$

However, note that, if $\frac{l+1}{l-k} \notin \mathbf{N}$ then $\left[\frac{l+1}{l-k}\right]$ is smaller than $\frac{l+1}{l-k}$. This contradicts to Lemma 2, so the function $W(\zeta)$ has zeros.

An argument of a proof in the case (ii) is similar. The above argument gives that, if $W(\zeta) \neq 0$ for any $\zeta \in \mathbf{C}$, then $W(\zeta) = e^{P(\zeta)}$ with polynomial $P(\zeta)$ of degree $N = \frac{l+1}{l-k}$ (which is a positive odd integer in this case). Put

$$P(\zeta) = A_N \zeta^N + A_{N-1}\zeta^{N-1} + \cdots + A_0.$$

Since $W(\zeta)$ is real valued for $\zeta \in \mathbf{R}$, all A_j are real. Let us now observe that

$$|W(\zeta)| \geq \varepsilon \exp(\varepsilon|\zeta|^N), \qquad \zeta \in \mathbf{R}_+$$

implies $A_N > 0$. On the other hand,

$$|W(\zeta)| \geq \varepsilon \exp(\varepsilon|\zeta|^N), \qquad \zeta \in \mathbf{R}_-$$

and the hypothesis that N is an odd integer imply $A_N < 0$. Thus, there is a contradiction, therefore $W(\zeta)$ has zeros. This completes the proof.

2. The estimates of the wronskian

In this section, we show the estimates of the wronskian in Lemma 2. First we give a proposition, which is related to Proposition 1 in the preceding section. From the proposition below, we can get an information concerning dependence of $f_\zeta^{\pm}(x)$ on $\zeta \in \mathbf{C}$. Its proof will be given in the next section.

Proposition 2. *Put* $A = C_0(1+|\zeta|)^{\frac{1}{l-k}}$. *If we take the positive constant* C_0 *sufficiently large, then the following inequalities hold with a constant* C *(independent of* ζ*) satisfying* $0 < C < 1$.

(i) $$|f_\zeta^+(x) - |x|^{-\frac{l}{2}} e^{-\phi_\zeta(x)}| \leq C|x|^{-\frac{l}{2}}|e^{-\phi_\zeta(x)}|,$$

$$|(f_\zeta^+)'(x) + |x|^{\frac{l}{2}} e^{-\phi_\zeta(x)}| \le C|x|^{\frac{l}{2}} |e^{-\phi_\zeta(x)}|,$$

$$\text{for } x \ge A.$$

(ii)
$$|f_\zeta^-(x) - |x|^{-\frac{l}{2}} e^{(-1)^l \phi_\zeta(x)}| \le C|x|^{-\frac{l}{2}} |e^{(-1)^l \phi_\zeta(x)}|,$$

$$|(f_\zeta^-)'(x) - |x|^{\frac{l}{2}} e^{(-1)^l \phi_\zeta(x)}| \le C|x|^{\frac{l}{2}} |e^{(-1)^l \phi_\zeta(x)}|,$$

$$\text{for } x \le -A.$$

Let us start the proof of Lemma 2. Suppose $\zeta \in \mathbf{R}$. Then $f_\zeta^+(x)$ and $f_\zeta^-(x)$ are real valued. Moreover, it follows from Proposition 2 that $f_\zeta^+(x) > 0$, $(f_\zeta^+)'(x) < 0$ for $x \ge A$ and $f_\zeta^-(x) > 0$, $(f_\zeta^-)'(x) > 0$ for $x \le -A$. Note that the equation $(f_\zeta^+)''(x) = (x^k\zeta - x^l)^2 f_\zeta^+(x)$ forces $f_\zeta^+(x)(f_\zeta^+)''(x) \ge 0$ for all $x \in \mathbf{R}$. Furthermore, observe that $f_\zeta^+(x)(f_\zeta^+)''(x) \ge 0$ and the boundary condition $f_\zeta^+(A) > 0$, $(f_\zeta^+)'(A) < 0$ imply $f_\zeta^+(x) > 0$, $(f_\zeta^+)'(x) < 0$ for $x \le A$. Thus one can see that $f_\zeta^+(x) > 0$, $(f_\zeta^+)'(x) < 0$ for all $x \in \mathbf{R}$, and similarly that $f_\zeta^-(x) > 0$, $(f_\zeta^-)'(x) > 0$ for all $x \in \mathbf{R}$.

We are going to estimate a lower bound on the wronskian $W(\zeta)$ for $\zeta \in \mathbf{R}$. Firstly, the preceding observation yields that

$$\begin{aligned} W(\zeta) &= f_\zeta^+(0)(f_\zeta^-)'(0) - (f_\zeta^+)'(0) f_\zeta^-(0) \\ &\ge f_\zeta^+(0)(f_\zeta^-)'(0). \end{aligned}$$

So our remaining task is to estimate $f_\zeta^+(0)$ and $(f_\zeta^-)'(0)$ from below.

Set

$$\begin{cases} g_\zeta(x) = \varepsilon A^{-\frac{l}{2}} e^{-\phi_\zeta(x)} \\ u(x) \; = f_\zeta(x) - g_\zeta(x). \end{cases}$$

Notice that Proposition 2 implies $f_\zeta^+(A) \ge \delta A^{-\frac{l}{2}} e^{-\phi_\zeta(A)}$ and $-(f_\zeta^+)'(A) \ge \delta A^{\frac{l}{2}} e^{-\phi_\zeta(A)}$ for some positive constant δ. Hence, if we take $\varepsilon > 0$ sufficiently small, it holds that $u(A) > 0$ and $u'(A) < 0$.

Observe next

(4)
$$P_\zeta g_\zeta = \{x^{k-1}(l x^{l-k} - k\zeta)\} g_\zeta,$$

whence $P_\zeta g_\zeta(x) > 0$ for $x \ge |\zeta|^{\frac{1}{l-k}}$. Thus it holds that

$$\begin{aligned} \frac{d^2u}{dx^2} &= -P_\zeta u + (x^k\zeta - x^l)^2 u \\ &= P_\zeta g_\zeta + (x^k\zeta - x^l)^2 u \\ &\ge (x^k\zeta - x^l)^2 u, \qquad \text{for } |\zeta|^{\frac{1}{l-k}} \le x \le A. \end{aligned}$$

Therefore, $u(x)$ satisfies $u''(x) \ge 0$ for $|\zeta|^{\frac{1}{l-k}} \le x \le A$ and $u(A) > 0$, $u'(A) < 0$, whence $u(x)>0$ and $u'(x)<0$ for $|\zeta|^{\frac{1}{l-k}}\le x\le A$. As a consequence, by setting $x_\zeta=|\zeta|^{\frac{1}{l-k}}$, $y_\zeta=\left|\frac{l+1}{k+1}\zeta\right|^{\frac{1}{l-k}}$, the following inequalities are valid for $\zeta>0$:

$$\begin{aligned} f_\zeta^+(x_\zeta) &\ge g_\zeta(x_\zeta) \\ &= \varepsilon A^{-\frac{l}{2}}\exp\left\{\left(\frac{1}{k+1}-\frac{1}{l+1}\right)|\zeta|^{\frac{l+1}{l-k}}\right\}, \\ -(f_\zeta^+)'(y_\zeta) &\ge -g_\zeta'(y_\zeta) \\ &= \varepsilon\cdot\left(\frac{l+1}{k+1}\right)^{k/(l-k)}\cdot\frac{l-k}{k+1}\cdot|\zeta|^{\frac{l}{l-k}}\cdot A^{-\frac{l}{2}}. \end{aligned}$$

Observe that the uniqueness of f_ζ^+, f_ζ^- implies

$$f_\zeta^-(x)=\begin{cases} f_{-\zeta}^+(-x) & ,l-k=\text{odd}, \\ f_\zeta^+(-x) & ,l-k=\text{even}. \end{cases}$$

Moreover, the argument at the begining of the present section gives that $f_\zeta^+(0)\ge f_\zeta^+(x_\zeta)$ and $-(f_\zeta^+)'(0)\ge -(f_\zeta^+)'(y_\zeta)$. Thus, if $l-k$ is an even integer and $\zeta>0$, it holds that

$$\begin{aligned} W(\zeta) &\ge -f_\zeta^+(0)(f_\zeta^+)'(0) \\ &\ge -f_\zeta^+(x_\zeta)(f_\zeta^+)'(y_\zeta) \\ &\ge -g_\zeta(x_\zeta)g_\zeta'(y_\zeta) \\ &\ge C\exp\left\{\left(\frac{1}{k+1}-\frac{1}{l+1}\right)|\zeta|^{\frac{l+1}{l-k}}\right\}, \end{aligned}$$

where C is a positive constant independent of ζ. Since we have $\frac{1}{k+1}-\frac{1}{l+1}>0$, we have established the estimate (ii) of Lemma 2 in case $l-k$ is an even integer.

Next let us consider the case in which $l-k$ is an odd integer. If $\zeta>0$, the relation (4) implies that $P_{-\zeta}g_{-\zeta}(x)>0$ for $1\le x\le A$. Hence the above argument also gives

$$\begin{aligned} -(f_{-\zeta}^+)'(1) &\ge -g_{-\zeta}'(1) \\ &= \varepsilon A^{-\frac{l}{2}}(1+\zeta)e^{-\phi_{-\zeta}(1)}. \end{aligned}$$

Thus if $l-k$ is an odd integer and $\zeta>0$, we have

$$\begin{aligned} W(\zeta) &\ge f_\zeta^+(0)(f_\zeta^-)'(0) \\ &= -f_\zeta^+(0)(f_{-\zeta}^+)'(0) \\ &\ge -f_\zeta^+(x_\zeta)(f_{-\zeta}^+)'(1) \\ &\ge -g_\zeta(x_\zeta)(g_{-\zeta})'(1) \end{aligned}$$

$$= \varepsilon^2 A^{-l}(1+\zeta)e^{-\phi_\zeta(x_\zeta)-\phi_{-\zeta}(1)}.$$

Observe that

$$-\phi_\zeta(x_\zeta)-\phi_{-\zeta}(1)=\left(\frac{1}{k+1}-\frac{1}{l+1}\right)\zeta^{\frac{l+1}{l-k}}-\left(\frac{1}{l+1}+\frac{\zeta}{k+1}\right),$$

where $\frac{1}{k+1}-\frac{1}{l+1}>0$ and $\frac{l+1}{l-k}>1$. So, the estimate (ii) of Lemma 2 holds also in the case $l-k$ is odd and $\zeta>0$.

The case in which $l-k$ is an odd integer and $\zeta<0$ can be treated in a similar way. Indeed, in this case, it holds that

$$\begin{aligned} W(\zeta) &\geq -(f_\zeta^+)'(0)f_\zeta^-(0) \\ &= -(f_\zeta^+)'(0)f_{-\zeta}^+(0) \\ &\geq -g_\zeta'(1)g_{-\zeta}(x_{-\zeta}) \\ &= \varepsilon^2 A^{-l}(1-\zeta)e^{-\phi_{-\zeta}(x_{-\zeta})-\phi_\zeta(1)}. \end{aligned}$$

Hence, observing

$$-\phi_{-\zeta}(x_{-\zeta})-\phi_\zeta(1)=\left(\frac{1}{k+1}-\frac{1}{l+1}\right)|\zeta|^{\frac{l+1}{l-k}}-\left(\frac{1}{l+1}-\frac{\zeta}{k+1}\right),$$

one can see that the estimate (ii) in Lemma 2 holds in this case. This finishes the proof of the lower bound.

Now we turn to estimate an upper bound on $W(\zeta)$ for arbitrary $\zeta\in\mathbf{C}$. First notice that Proposition 2 yields

$$\text{(5)} \qquad \begin{cases} |f_\zeta^+(A)| \leq CA^{-\frac{l}{2}}|e^{-\phi_\zeta(A)}|, \\ |(f_\zeta^+)'(A)| \leq CA^{\frac{l}{2}}|e^{-\phi_\zeta(A)}|, \end{cases}$$

where C is a positive constant independent of $\zeta\in\mathbf{C}$. On the other hand, observe that

$$\begin{aligned} \left|\frac{d^2}{dx^2}f_\zeta^+(x)\right| &= |x^k\zeta - x^l|^2|f_\zeta^+(x)| \\ &\leq C|\zeta|^{\frac{2l}{l-k}}|f_\zeta^+(x)|, \end{aligned}$$

for $0\leq x\leq A$, whence

$$\begin{aligned} &\frac{d}{dx}\{|\zeta|^{\frac{2l}{l-k}}|f_\zeta^+(x)|^2+|(f_\zeta^+)'(x)|^2\} \\ &\leq C|\zeta|^{\frac{l}{l-k}}\{|\zeta|^{\frac{2l}{l-k}}|f_\zeta^+(x)|^2+|(f_\zeta^+)'(x)|^2\}, \end{aligned}$$

for $0\leq x\leq A$. Furthermore, by Gronwall's inequality, it follows

$$(6)\qquad |\zeta|^{\frac{2l}{l-k}}|f_\zeta^+(0)|^2+|(f_\zeta^+)'(0)|^2 \le C\{|\zeta|^{\frac{2l}{l-k}}|f_\zeta^+(A)|^2+|(f_\zeta^+)'(A)|^2\}e^{C|\zeta|^{\frac{l}{l-k}}A}.$$

As a consequence of (5) and (6), one can see that, for large $|\zeta|$,

$$\begin{cases} |f_\zeta^+(0)|\le C|\zeta|^{-\frac{l}{2(l-k)}}e^{C|\zeta|^{\frac{l+1}{l-k}}}, \\ |(f_\zeta^+)'(0)|\le C|\zeta|^{\frac{l}{2(l-k)}}e^{C|\zeta|^{\frac{l+1}{l-k}}}. \end{cases}$$

Since the above argument can also be applied to the estimates for $f_\zeta^-(0)$ and $(f_\zeta^-)'(0)$, we finally see that the wronskian have the upper bound

$$|W(\zeta)|\le C\exp(C|\zeta|^{\frac{l+1}{l-k}}).$$

This completes the proof of Lemma 2.

3. Proof of Proposition 2

Proposition 2 asserts that, in some sense, $|x|^{-\frac{l}{2}}e^{-\phi_\zeta(x)}$ approximates to $f_\zeta^+(x)$ in the region $x\ge A$, and $|x|^{-\frac{l}{2}}e^{(-1)^l\phi_\zeta(x)}$ approximates to $f_\zeta^-(x)$ in the region $x\le -A$. In the present section, we show this for $f_\zeta^+(x)$, since the proof for $f_\zeta^-(x)$ will be completely parallel.

Set $A=C_0(1+|\zeta|)^{\frac{1}{l-k}}$,

$$(7)\qquad G(x)=\frac{1}{\sqrt{\phi'(x)}}e^{-\phi(x)}$$

and

$$(8)\qquad w(x)=G(x)-f_\zeta^+(x).$$

Here, we abbreviate $\phi_\zeta(x)$ as $\phi(x)$. It will be shown that, if we take C_0 sufficiently large, then we have

$$(9)\qquad \sup_{x\ge A}\frac{|w(x)|}{|x|^{-l-1}|G(x)|}\le C_1<\infty$$

with a positive constant C_1 independent of $\zeta\in\mathbf{C}$. Observe now that, by taking C_0 sufficiently large, the inequality

$$\left|x^{-\frac{l}{2}}-\frac{1}{\sqrt{\phi'(x)}}\right|\le C_2x^{-\frac{l}{2}}\qquad \text{for } x\ge A$$

holds with a constant C_2 satisfying $0<C_2<1$. Hence, from (7), (8) and (9) one can get

$$|f_\zeta^+(x) - |x|^{-\frac{l}{2}} e^{-\phi(x)}| \le C|x|^{-\frac{l}{2}}|e^{-\phi(x)}| \qquad \text{for } x \ge A$$

with a constant C satisfying $0 < C < 1$.

Now, denote by $u_y = u_y(x)$ the solution in the interval $[A, y]$ of the initial value problem:

$$\begin{cases} P_\zeta u_y(x) = P_\zeta G(x), & A < x < y, \\ u_y(y) = u_y'(y) = 0. \end{cases}$$

We shall prove that the inequality

$$(10) \qquad |u_y(x)| \le C|x|^{-l-1}|G(x)|, \qquad A \le x \le y$$

holds with a constant C independent of y and ζ. Also we shall prove that, for y_1, y_2 satisfying $A \le x \le y_1 \le y_2$, the inequality

$$(11) \qquad |u_{y_1}(x) - u_{y_2}(x)| \le C|y_1|^{-l-1}|G(x)|$$

holds with a constant C independent of y_1, y_2 and ζ. This granted, u_y converges uniformly on arbitrary compact set of $[A, \infty)$ as y tends to ∞. It is also clear that the limit function $w(x)$ satisfies $P_\zeta w \equiv P_\zeta G$ and

$$|w(x)| \le C|x|^{-l-1}|G(x)| \qquad \text{for } A \le x < \infty.$$

Thus, one can see that the function $G(x) - w(x)$ is a solution to $P_\zeta(G - w) \equiv 0$ having the same asymptotic behavior as $f_\zeta^+(x)$. Hence, we have $f_\zeta^+ \equiv G - w$ and w satisfies (9).

Now let us prove (10). Set

$$\psi = \log G = -\phi(x) - \frac{1}{2}\log \phi'(x)$$

and

$$D = \frac{d}{dx} + \psi', \qquad \tilde{D} = -\frac{d}{dx} + \psi'.$$

Then, we have

$$\begin{aligned} D \circ \tilde{D} &= -\frac{d^2}{dx^2} + (\psi')^2 + \psi'' \\ &= P_\zeta + E, \end{aligned}$$

where

$$E = -\frac{1}{2}\left(\frac{\phi''}{\phi'}\right)' + \frac{1}{4}\left(\frac{\phi''}{\phi'}\right)^2.$$

Notice that the inequality

$$|E(x)| \le C|x|^{-2}, \qquad x \ge A$$

holds with a constant C independent of ζ. Also observe that $P_\zeta G(x) = -\left(\frac{1}{\sqrt{\phi'(x)}}\right)'' e^{-\phi(x)}$, whence

$$|P_\zeta G(x)| \le C|x|^{-2}|G(x)|, \qquad x \ge A$$

holds with a constant C independent of ζ.

With y fixed and abbreviating u_y as u, put

$$B = \sup_{A \le x \le y} \frac{|u(x)|}{|x|^{-l-1}|G(x)|} < \infty.$$

We shall show that B has a bound independent of y and ζ. Put $v = \tilde{D}u$. Then, we have

$$\begin{aligned} e^{-\psi}\frac{d}{dx}e^{\psi}v &= Dv = D \circ \tilde{D}u \\ &= P_\zeta u + Eu = P_\zeta G + Eu. \end{aligned}$$

Since we have $v(y) = 0$, one has for $A \le s \le y$

$$e^{\psi(s)}v(s) = \int_y^s e^{\psi(t)}[P_\zeta G(t) + (Eu)(t)]\,dt,$$

whence we have

$$\begin{aligned} |v(s)| &\le |e^{-\psi(s)}| \int_s^y |e^{\psi(t)}|[C|t|^{-2}|G(t)| + C|t|^{-2}|u(t)|]\,dt \\ &\le C|s|^{\frac{l}{2}}|e^{\phi(s)}| \int_s^y |t|^{-\frac{l}{2}}|e^{-\phi(t)}|[|t|^{-\frac{l}{2}-2}|e^{-\phi(t)}| + |t|^{-2}|u(t)|]\,dt \\ &\le C|s|^{\frac{l}{2}}|e^{\phi(s)}| \cdot |s|^{-2l-2}|e^{-2\phi(s)}| + C|s|^{\frac{l}{2}}|e^{\phi(s)}| \int_s^y |t|^{-\frac{l}{2}-2}|e^{-\phi(t)}u(t)|\,dt \end{aligned}$$

$$(12) \qquad = C|s|^{-\frac{3}{2}l-2}|e^{-\phi(s)}| + C|s|^{\frac{l}{2}}|e^{\phi(s)}| \int_s^y |t|^{-\frac{l}{2}-2}|e^{-\phi(t)}u(t)|\,dt.$$

Furthermore, from the equalities, $v = \tilde{D}u = -e^{\psi}\frac{d}{dx}e^{-\psi}u$ it follows that, for $A \le x \le y$, we have

$$\begin{aligned} |u(x)| &\le |e^{\psi(x)}| \int_x^y |e^{-\psi(s)}v(s)|\,ds \\ &\le C|x|^{-\frac{l}{2}}|e^{-\phi(x)}| \int_x^y |s|^{-l-2}\,ds \end{aligned}$$

$$+ C|x|^{-\frac{l}{2}}|e^{-\phi(x)}| \int_{s=x}^{y} |s|^l |e^{2\phi(s)}| \int_{t=s}^{y} |t|^{-\frac{l}{2}-2} |e^{-\phi(t)} u(t)| \, dt ds$$

$$\leq C|x|^{-l-1}|G(x)| + C|G(x)| \int_{t=x}^{y} |t|^{-\frac{l}{2}-2} |e^{-\phi(t)} u(t)| \int_{s=x}^{t} |s|^l |e^{2\phi(s)}| \, ds dt$$

$$\leq C|x|^{-l-1}|G(x)| + C|G(x)| \int_{x}^{y} |t|^{-\frac{l}{2}-2} |e^{-\phi(t)} u(t)| \cdot |e^{2\phi(t)}| \, dt$$

$$\leq C|x|^{-l-1}|G(x)| + C|G(x)| \int_{x}^{y} |t|^{-2l-3} B \, dt$$

$$\leq C|x|^{-l-1}|G(x)| [1 + B|x|^{-l-1}].$$

Take the positive constant C_0 large enough such that $C|x|^{-l-1} \leq \frac{1}{2}$ holds for $x \geq C_0$. Then, from the last inequality, we obtain $B \leq 2C$. All constants denoted by C are independent of y and ζ, so the inequality (10) is established.

We turn to show (11) for $A \leq x \leq y_1 \leq y_2$. By the above argument, already we have

$$u_{y_2}(y_1) \leq C|y_1|^{-l-1}|G(y_1)|.$$

Also it follows from (12) that, for $A \leq s \leq y_2$, we have

$$\tilde{D} u_{y_2}(s) \leq C|s|^{-\frac{3}{2}l-2}|e^{-\phi(x)}| + C|s|^{\frac{l}{2}}|e^{\phi(s)}| \int_{s}^{y_2} |t|^{-\frac{l}{2}-2} |e^{-\phi(t)} u_{y_2}(t)| \, dt$$

$$\leq C|s|^{-l-2}|G(s)| + C|s|^{\frac{l}{2}}|e^{\phi(s)}| \int_{s}^{y_2} |t|^{-2l-3} |e^{-2\phi(t)}| \, dt$$

$$\leq C|s|^{-l-2}|G(s)| + C|s|^{-\frac{5}{2}l-3}|e^{-\phi(x)}|$$

$$\leq C|s|^{-l-2}|G(s)|.$$

Set $w_1 = u_{y_1} - u_{y_2}$ and $v = \tilde{D} w_1$. Recall that $u_{y_1}(y_1) = u'_{y_1}(y_1) = 0$. Thus by the above observation, we have

$$|w_1(y_1)| \leq C|y_1|^{-l-1}|G(y_1)| \leq C|y_1|^{-\frac{3}{2}l-1}|e^{-\phi(y_1)}|$$

and

$$|v(y_1)| \leq C|y_1|^{-l-2}|G(y_1)| \leq C|y_1|^{-\frac{3}{2}l-2}|e^{-\phi(y_1)}|.$$

Let us estimate $w_1(x)$ for $A \leq x \leq y_1$. Since we have

$$e^{-\psi} \frac{d}{ds} e^{\psi} v = Dv = D \circ \tilde{D} w_1$$
$$= P_\zeta w_1 + E w_1 = E w_1,$$

it holds that for $A \leq s \leq y_1$

$$|e^{\psi(s)}v(s)| \leq |e^{\psi(y_1)}v(y_1)| + \int_s^{y_1} |e^{\psi(t)}(Ew_1)(t)|\,dt$$
$$\leq C|y_1|^{-2l-2}|e^{-2\phi(y_1)}| + \int_s^{y_1} |t|^{-\frac{l}{2}-2}|w_1(t)|\,dt.$$

Moreover, since we have $-e^{\psi}\frac{d}{dx}e^{-\psi}w_1 = v$, the argument to get (10) also gives that, for $A \leq x \leq y_1$, we have

$$|e^{-\psi(x)}w_1(x)| \leq |e^{\psi(y_1)}w_1(y_1)| + \int_x^{y_1} |e^{-\psi(s)}v(s)|\,ds$$
$$\leq C|y_1|^{-l-1} + C|y_1|^{-2l-2}|e^{-2\phi(y_1)}|\cdot|e^{2\phi(y_1)}|$$
$$+ \int_x^{y_1} |t|^{-\frac{l}{2}-2}|e^{-\phi(t)}w_1(t)| \int_x^t |e^{-2\psi(s)}|\,ds dt$$
$$\leq C|y_1|^{-l-1} + C\int_x^{y_1} |t|^{-\frac{l}{2}-2}|e^{-\phi(t)}w_1(t)|\cdot|e^{2\phi(t)}|\,dt.$$

Setting now

$$B_1 = \sup_{A \leq x \leq y_1} \frac{|w_1(x)|}{|G(x)|} < \infty,$$

we consequently obtain that

$$B_1 \leq C|y_1|^{-l-1} + C\int_x^{y_1} B_1|t|^{-l-2}dt$$
$$\leq C|y_1|^{-l-1} + B_1\cdot C\cdot|x|^{-l-1}$$
$$\leq C|y_1|^{-l-1} + \frac{B_1}{2}.$$

Thus, the inequality $B_1 \leq 2C|y_1|^{-l-1}$ holds, so (11) is established.

Finally we give a proof of the inequality:

$$(13) \qquad |(f_\zeta^+)'(x) + |x|^{\frac{l}{2}}e^{-\phi(x)}| \leq C|x|^{\frac{l}{2}}|e^{-\phi(x)}|, \qquad \text{for } x \geq A,$$

with a constant C satisfying $0 < C < 1$. First observe that

$$(f_\zeta^+)' = (G - w)' = G' + \tilde{D}w - \psi'w.$$

Concerning the first term on the right hand side, by taking the constant C_0 sufficiently large, the inequality

$$|G'(x) + |x|^{\frac{l}{2}} e^{-\phi(x)}| \leq C|x|^{\frac{l}{2}} |e^{-\phi(x)}|, \qquad x \geq A$$

holds with a constant C with $0 < C < 1$. Also observe that

$$|\tilde{D}w(x)| \leq C|x|^{-\frac{3}{2}l-2} |e^{-\phi(x)}|, \qquad x \geq A$$

and

$$|\psi'(x)w(x)| \leq C|x|^{-\frac{l}{2}-1} |e^{-\phi(x)}|, \qquad x \geq A.$$

They are consequences of (12) and (10) respectively, by taking the limits as y tends to ∞. Thus, combining these inequalities, we obtain (13). The proof of Proposition 2 is now complete.

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