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Asymptotic estimates for the distribution of additive functionals of Brownian motion by the Wiener-Hopf factorization method

By

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1. Introduction

In [4], we studied the supremum process of the integral of Brownian motion and obtained the following estimates: Let $b(t)$ be the one dimensional Brownian motion starting at 0. For $r>0$, $A>0$, $\sigma>0$ and $a\in\mathbf{R}$, let

$$P_{ra}(A)=P\left\{\int_0^t b(u)\,du\leqq r+at \text{ for all } 0\leq t\leq A\right\} \tag{1.1}$$

and

$$P_{ra\sigma}=P\left\{\int_0^t b(u)\,du\leq r+at+\sigma t^2 \text{ for all } 0\leq t<\infty\right\}. \tag{1.2}$$

Then it holds

$$P_{ra}(A)\sim C_1(r,a)\ A^{-1/4} \text{ as } A\uparrow\infty \tag{1.3}$$

and

$$P_{ra\sigma}\sim C_2(r,a)\,\sigma^{1/2} \text{ as } \sigma\downarrow 0 \tag{1.4}$$

with positive constants $C_i(r,a)$, $i=1,2$, which can be given explicitly (see [4]). This is a refinement of Sinai's estimates in [9].

These asymptotics follow systematically from the theorem in [4] on a two dimensional process called the *Kolmogorov diffusion* (cf. [5]) :

$$Y(t)=y+b(t),\ X(t)=x+\int_0^t Y(u)\,du=x+yt+\int_0^t b(u)\,du. \tag{1.5}$$

Let T be the first hitting time to the positive y-axis :

$$T=\inf\{t\geq 0\ ;\ X(t)=0,\ Y(t)\geq 0\}. \tag{1.6}$$

We denote by $E_{(x,y)}$ the expectation for the diffusion starting at $(x,y)\in\mathbf{R}^2$.

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Theorem ([4]). *For $a\geq 0$, $b\geq 0$ and $(x, y)\in\mathbf{R}^2$ with $x\leq 0$,*

$$1-E_{(x,y)}(\exp[-a\sigma^2 T-b\sigma Y(T)])\sim C(a, b; x, y)\sqrt{\sigma} \text{ as } \sigma\downarrow 0. \tag{1.7}$$

Our proof in [4] was based on a formula obtained by McKean[6] while Sinai's method was based on an extension of the fluctuation theory of Sparre Andersen for sums of i. i. d. random variables. So it may be a natural question to ask if the above estimates (1.3) and (1.4) (more generally the estimate (1.7)) could be recovered by a fluctuation theory for sums of i. i. d. random variables and Lévy processes. The present paper is an attempt to this problem: The fluctuation theory we use is a version of the Wiener-Hopf decomposition obtained by Rogozin[7] for Lévy processes as the continuous time analogue of Spitzer's identity in [8] for sums of i. i. d. random variables. However we need a generalization of the Spitzer-Rogozin identity to the case of two dimensional Lévy processes (Th.1). Although we could not yet succeed to recover the estimates (1.3) and (1.4) by this method, we can obtain some special cases and furthermore, we can deduce a weaker form of (1.3) for a class of odd additive functionals of Brownian motion including the integral of Brownian motion (Th.3) .

2. The main theorems

Let (τ_t, ξ_t) be a time homogeneous Lévy process, i. e. , a cadlag process with stationary independent increments, with $\tau_0=0$ and $\xi_0=0$. Then the law of this process is determined by the characteristic exponent $\psi(\mu, \eta)$ defined by

$$E\left[e^{i\mu\tau_t+i\eta\xi_t}\right]=e^{-t\psi(\mu,\eta)},\ \mu,\ \eta\in\mathbf{R}. \tag{2.1}$$

Set

$$\bar{\xi}_t=\sup_{0\leq s\leq t}\xi_s,\ \underline{\xi}_t=\inf_{0\leq s\leq t}\xi_s \tag{2.2}$$

and define for $\sigma\geq 0$

$$\phi_+(\sigma, \mu, \eta)=(\sigma+\psi(\mu, 0))\int_0^\infty dtE[e^{-\sigma t+i\mu\tau_t+i\eta\bar{\xi}_t}], \tag{2.3}$$

$$\phi_-(\sigma, \mu, \eta)=(\sigma+\psi(\mu, 0))\int_0^\infty dtE[e^{-\sigma t+i\mu\tau_t+i\eta\underline{\xi}_t}], \tag{2.4}$$

We assume that $\mathrm{Re}\psi(\mu, \eta)>0$ if $|\mu|+|\eta|>0$. Then we have the following generalization of the Wiener-Hopf decomposition theorem which we call the Spitzer-Rogozin identity in continuous time (cf. [7]) :

Theorem 1.

$$\phi_+(\sigma, \mu, \eta) = \exp\left(\int_0^\infty \frac{dt}{t} E[e^{-\sigma t + i\mu\tau_t}\,(e^{i\eta\xi_t - 1})\ ;\ \xi_t > 0]\right), \tag{2.5}$$

$$\phi_-(\sigma, \mu, \eta) = \exp\left(\int_0^\infty \frac{dt}{t} E[e^{-\sigma t + i\mu\tau_t}\,(e^{i\eta\xi_t - 1})\ ;\ \xi_t < 0]\right), \tag{2.6}$$

and

$$\phi_+(\sigma, \mu, \eta)\phi_-(\sigma, \mu, \eta) = \frac{\sigma + \psi(\mu, 0)}{\sigma + \psi(\mu, \eta)}. \tag{2.7}$$

The proof will be given in Section 3.

Let $b(t)$ be the one dimensional Brownian motion with $b(0)=0$. Let L_t be the local time at 0: $L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(b(u))\,du$ and τ_t be the right continuous inverse of L_t: $\tau_t = \inf\{u>0;\ L_u > t\}$. For $\alpha \geq 0$, we set

$$\xi_t = \int_0^{\tau_t} \operatorname{sgn}(b(u)) \cdot |b(u)|^\alpha du \tag{2.8}$$

where

$$\operatorname{sgn} x = \begin{cases} 1 & x>0, \\ -1 & x<0, \\ 0 & x=0. \end{cases}$$

It is well known that (τ_t, ξ_t) is a two dimensional Lévy process with $\tau_0 = 0$ and $\xi_0 = 0$.

As an application of Theorem 1 to this particular case together with the overshoot argument in the fluctuation theory (cf. Bingham[2]), we obtain the following

Theorem 2. *Let $\Theta_r = \Theta(r) = \inf\{t \geq 0\ ;\ \bar{\xi}_t \geq r\}$ for $r>0$. Then we have for $\sigma \geq 0$, $\mu \geq 0$, $\sigma + \mu > 0$, $\lambda \geq 0$*

$$1 - E[e^{-\sigma\Theta(r) - \mu\tau_{\Theta(r)} - \lambda\bar{\xi}_{\Theta(r)}}] \sim \frac{\sqrt{\sigma + \sqrt{2\mu}}\,C(\alpha)}{\phi_+(\sigma, i\mu, i\lambda)}\, r^{\frac{1}{2(\alpha+2)}} \quad \text{as } r \downarrow 0, \tag{2.9}$$

where

$$C(\alpha) = \frac{2^{1+\frac{\alpha+1}{2(\alpha+2)}}(\alpha+2)^{1-\frac{\alpha}{2(\alpha+2)}}\Gamma\left(\frac{1}{\alpha+2}\right)\sqrt{\sin\left(\frac{\pi}{2(\alpha+2)}\right)}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2(\alpha+2)}\right)}. \tag{2.10}$$

The proof will be given in Section 4.

Theorem 2 can be used to obtain some estimates of the law of additive functionals of Brownian motion as stated in Introduction. For this, we introduce the following two dimensional diffusion process $(X(t), Y(t))$:

$$(2.11)\qquad \begin{cases} Y(t) = y + b(t), \\ X(t) = x + \int_0^t \operatorname{sgn}(Y(u)) \cdot |Y(u)|^\alpha du. \end{cases}$$

The law of this diffusion is denoted by $P_{(x,y)}$ as usual. Let

$$(2.12)\qquad \begin{cases} T = \inf\{t \geq 0 \,;\, X(t) = 0\}, \\ T^0 = \inf\{t \geq 0 \,;\, Y(t) = 0\}, \\ X^0 = X(T^0). \end{cases}$$

Then it is easy to see that under $P_{(-r,0)}$ with $r > 0$, $(T + T^0 \circ \theta_T,\ r + X(T + T^0 \circ \theta_T))$ is equally distributed as $(\tau_{\Theta_r},\ \bar{\xi}_{\Theta_r} = \xi_{\Theta_r})$, where θ_t is the usual shift operator on the path space. Hence, by the strong Markov property, we have

$$1 - E[e^{-\mu\tau_{\Theta(r)} - \lambda\bar{\xi}_{\Theta(r)}}] = 1 - E_{(-r,0)}[e^{-\mu(T + T^0 \circ \theta_T) - \lambda r - \lambda X(T + T^0 \circ \theta_T)}]$$
$$= 1 - e^{-\lambda r} E_{(-r,0)}[e^{-\mu T} G(Y(T);\, \lambda, \mu)]$$

where

$$(2.13)\qquad G(y;\, \lambda, \mu) = E_{(0,y)}[e^{-\mu T^0 - \lambda X^0}].$$

It can be proved that $G(y;\, \lambda, \mu)$ is the unique bounded solution of

$$(2.14)\qquad G''(y;\, \lambda, \mu) = 2\,(\mu + \lambda y^\alpha)\, G(y;\, \lambda, \mu)$$

on $[0, \infty)$ with the condition $G(0;\, \lambda, \mu) = 1$. By the scaling property of $b(t)$, we deduce easily for any $c > 0$ the following equivalence in law as 8 dimensional processes:

$$\begin{pmatrix} b_t & L_t \\ \tau_t & \xi_t \\ \psi(\mu, \eta) & \phi_\pm(\sigma, \mu, \eta) \\ \Theta_r & \tau_{\Theta(r)} \end{pmatrix} \overset{\mathrm{d}}{=} \begin{pmatrix} \frac{1}{c} b_{c^2 t} & \frac{1}{c} L_{c^2 t} \\ \frac{1}{c^2} \tau_{ct} & \frac{1}{c^{\alpha+2}} \xi_{ct} \\ c\psi(\mu/c^2,\, \eta/c^{\alpha+2}) & \phi_\pm(\sigma/c,\, \mu/c^2,\, \eta/c^{\alpha+2}) \\ \frac{1}{c} \Theta(c^{\alpha+2} r) & \frac{1}{c^2} \tau_{\Theta(c^{\alpha+2} r)} \end{pmatrix}.$$

Hence, we have

$$(2.15)\qquad E[e^{-\mu\tau_{\Theta(r)} - \lambda\bar{\xi}_{\Theta(r)}}] = E[e^{-r^{\frac{2}{\alpha+2}}\mu\tau_{\Theta(1)} - r\lambda\bar{\xi}_{\Theta(1)}}]$$

and

$$(2.16)\qquad G(cy;\, c^{-(2+\alpha)}\lambda,\, c^{-2}\mu) = G(y;\, \lambda, \mu).$$

Then by Theorem 2. we obtain

Corollary 1. *For each $\mu\geq 0$ and $\lambda\geq 0$*

$$1-E_{(-r,0)}[e^{-\mu T}G(Y(T)\ ;\lambda,\mu)]=1-E_{(-1,0)}[e^{-r^{\frac{2}{\alpha+2}}\mu T}G(r^{\frac{1}{\alpha+2}}Y(T)\ ;\lambda,\mu)]$$
$$\sim\frac{(2\mu)^{1/4}}{\phi_+(0,i\mu,i\lambda)}C(\alpha)\ r^{\frac{1}{2(\alpha+2)}}\ \textit{as}\ r\downarrow 0.$$

In particular, the estimate (1.4) follows if we set $\alpha=1$:

Corollary 2. *If $P_{ra\sigma}$ is defined by* (1.2), *then*

$$P_{r0\sigma}\sim\frac{3^{5/6}2^{4/3}\Gamma\left(\frac{1}{3}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{6}\right)}r^{1/6}\sqrt{\sigma},\ \textit{as}\ \sigma\downarrow 0\ \textit{or}\ r\downarrow 0. \tag{2.17}$$

Remark. This result was obtained in [4] and the general results for $P_{ra\sigma}$ with $a\neq 0$ can be reproduced from $P_{r0\sigma}$ by using the technique developed there.

Proof. The lemma in [4] asserts that

$$P_{r0\sigma}=1-E_{(-r,0)}[e^{-2\sigma^2T-2\sigma Y(T)}]+O(\sigma)\ \text{as}\ \sigma\downarrow 0. \tag{2.18}$$

where these T, $Y(T)$ are defined with $\alpha=1$.

On the other hand if $\alpha=1$, $\lambda=0$, $G(y;0,\mu)$ equals to $e^{-\sqrt{2\mu}y}$. Replacing μ by 2 and r by $r\sigma^3$, we obtain by the scaling property of $(X(t), Y(t))$ as $\sigma\downarrow 0$ or $r\downarrow 0$

$$P_{r0\sigma}\sim 1-E_{(-1,0)}[e^{-2r^{2/3}\sigma^2T-2r^{1/3}\sigma Y(T)}]\sim\sqrt{2}C(1)\ (r\sigma^3)^{1/6}=\frac{3^{5/6}2^{4/3}\Gamma\left(\frac{1}{3}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{6}\right)}r^{1/6}\sqrt{\sigma},$$

because $\phi_+(\sigma,\mu,0)=1$.

As for the estimate (1.3), we obtain the following result which is a generalization of Sinai's estimate in [9] to a class of odd additive functionals of Brownian motion $b(t)$; We could not yet refine it in the precise form as (1. 3), however. Let, for $r>0$, $A>0$ and $\alpha\geq 0$,

$$P_{r0}^{(\alpha)}(A)=P_{(-r,0)}[T>A]$$
$$=P\left[\int_0^t\text{sgn}\ (b(u))\cdot|b(u)|^\alpha du\leq r\ \text{for all}\ 0\leq t\leq A\right]$$

so that $P_{r0}^{(1)}(A)$ equals to $P_{r0}(A)$ defined by (1.1).

Theorem 3.

$$P_{r0}^{(\alpha)}(A)\asymp r^{\frac{1}{2(\alpha+2)}}A^{-\frac{1}{4}}\qquad \textit{as}\ A\cdot r^{-\frac{2}{\alpha+2}}\to\infty,$$

that is, there exist positive constants $c_L^{(\alpha)}<c_U^{(\alpha)}$ such that

$$c_L^{(\alpha)} r^{\frac{1}{2(\alpha+2)}} A^{-1/4} < P_{r0}^{(\alpha)}(A) < c_U^{(\alpha)} r^{\frac{1}{2(\alpha+2)}} A^{-1/4}$$

provided that $A \cdot r^{-\frac{2}{\alpha+2}}$ *is large enough.*

The proof will be given in section 5.

3. Proof of Theorem 1

We prove theorem 1 by an approximation of the Lévy process by random walks. For a random walk, Spitzer's method ([8]) can be easily modified to cover the two dimensional case. The approximation of the Lévy process by random walks can be obtained along the line proposed in [7]. We should note that Theorem 1 can also be proved by the method of Greenwood-Pitman[3].

In [8], Spitzer proved the following

Theorem 4 (Spitzer). *Let* $\mathbf{x} = (x_1, \cdots, x_n)$ *be an arbitrary n-sequence of real numbers, and* $\mathfrak{S}_n$ *be the n-th symmetric group. For* $\sigma \in \mathfrak{S}_n$, *we define*

$$S(\sigma\mathbf{x}) = \max_{0 \leq k \leq n} \left(\sum_{i=1}^{k} x_{\sigma i}\right), \tag{3.1}$$

where the summation over empty set is considered to be 0. *For* $\tau \in \mathfrak{S}_n$ *which is represented as a product of cyclic permutations on mutually disjoint sets:*

$$\tau = (\alpha_1(\tau))(\alpha_2(\tau)) \cdots (\alpha_\nu(\tau)),$$

we define

$$T(\tau\mathbf{x}) = \sum_{j=1}^{\nu} \left(\sum_{k \in \alpha j(\tau)} x_k\right)^+. \tag{3.2}$$

For any fixed $\mathbf{x}$, *the two sets* $[S(\sigma\mathbf{x})]_{\sigma \in \mathfrak{S}_n}$ *and* $[T(\tau\mathbf{x})]_{\tau \in \mathfrak{S}_n}$ *of n! numbers are identical to each other.*

From this theorem it is obvious that for any sequence of pairs of two real numbers

$$(\mathbf{t}, \mathbf{x}) = ((t_1, x_1), (t_2, x_2), \cdots, (t_n, x_n)),$$

the two sets $\left[\left(\sum_{j=1}^{n} t_{\sigma j}, S(\sigma\mathbf{x})\right)\right]_{\tau \in \mathfrak{S}_n}$ and $\left[\left(\sum_{j=1}^{n} t_{\tau j}, T(\tau\mathbf{x})\right)\right]_{\tau \in \mathfrak{S}_n}$ are identical sets.

As a prototype of Theorem 1, we need the following

Theorem 5. *Let* (t_i, x_i), $i = 1, 2, \cdots$ *be two dimensional i. i. d. random variables. If we define*

$$\tilde{\tau}_k = t_1 + t_2 + \cdots + t_k, \quad \tilde{\tau}_0 = 0,$$

$$S_k = x_1 + x_2 + \cdots + x_k, \quad S_0 = 0,$$

$$\phi_k(\mu, \eta) = E[e^{i\mu \tilde{\tau}_k + i\eta S_k^+}],\ S_k^+ = \max(0, S_k),$$

then it holds

$$(3.3)\qquad E\left[e^{i\mu \tilde{\tau}_k + i\eta \max_{0\le k\le n} S_k}\right] = \sum_{\substack{k_1+2k_2+\cdots+jk_j+\cdots+nk_n=n \\ k_j\ge 0}} \prod_{j=1}^{n} \frac{1}{k_j!}\left(\frac{\phi_j(\mu, \eta)}{j}\right)^{k_j}.$$

Furthermore

$$(3.4)\qquad \sum_{n=0}^{\infty} s^n E[e^{i\mu \tilde{\tau}_k + i\eta \max_{0\le k\le n} S_k}] = \exp\left[\sum_{n=1}^{\infty} \frac{s^n}{n} E[e^{i\mu \tilde{\tau}_k + i\eta S_n^+}]\right].$$

Proof. Spitzer proved (3.3) and (3.4) in the case $\mu=0$ by using Theorem 4 and

$$(3.5)\qquad E[e^{i\eta T(\tau x)}] = \prod_{j=1}^{n} (E[e^{i\eta S_j^+}])^{k_j},$$

for $\tau \in \mathfrak{S}_n$ which contains k_j $(0 \leqq k_j \leqq n)$ cyclic permutations of length j, $j=1, 2, \cdots, n$. In our case, we need what we noted after Theorem 4 and

$$(3.6)\qquad E[e^{i\mu \tilde{\tau}_n + i\eta T(\tau x)}] = \prod_{j=1}^{n} (\phi_j(\mu, \eta))^{k_j}.$$

We denote in the following

$$\Phi_+(s, \mu, \eta) := (1 - sE[e^{i\mu \tilde{\tau}_1}]) \sum_{n=0}^{\infty} s^n E[e^{i\mu \tilde{\tau}_n} e^{i\eta \bar{S}_k^n}],\ \bar{S}_n = \max_{0\le k\le n} S_k.$$

$$\Phi_-(s, \mu, \eta) := (1 - sE[e^{i\mu \tilde{\tau}_1}]) \sum_{n=0}^{\infty} s^n E[e^{i\mu \tilde{\tau}_n} e^{i\eta \underline{S}_k^n}],\ \underline{S}_n = \min_{0\le k\le n} S_k.$$

Using (3.4) and the Taylor expansion of $\log\left(\frac{1}{1-x}\right)$ we obtain

$$(3.7)\qquad \Phi_+(s, \mu, \eta) = \exp\left(\sum_{n=1}^{\infty} \frac{s^n}{n} E[e^{i\mu \tilde{\tau}_n}(e^{i\eta S_n} - 1);\ S_n > 0]\right),$$

$$(3.8)\qquad \Phi_-(s, \mu, \eta) = \exp\left(\sum_{n=1}^{\infty} \frac{s^n}{n} E[e^{i\mu \tilde{\tau}_n}(e^{i\eta S_n} - 1);\ S_n < 0]\right)$$

and

$$(3.9)\qquad \Phi_+(s, \mu, \eta)\Phi_-(s, \mu, \eta) = \frac{1 - sE[e^{i\mu \tilde{\tau}_1}]}{1 - sE[e^{i\mu \tilde{\tau}_1} e^{i\eta S_1}]}.$$

To proceed to the continuous time process (τ_t, ξ_t), we set $\tilde{\tau}_n = \tau_{\frac{n}{N}}$, $S_n = \xi_{\frac{n}{N}}$, and $s = e^{-\sigma/N}$ in the above. We will show that the both sides of (3. 7) converge to those of (2. 5) when N tends to infinity. The left hand side converges to $\phi_+(\sigma, \mu, \eta)$ from the fact that (τ_t, ξ_t) has no fixed time of discontinuity :

$$\Phi_+(e^{-\sigma/N}, \mu, \eta)$$

$$=(1-e^{-\sigma/N}E[e^{i\mu\tau_{\frac{1}{N}}}])\sum_{n=0}^{\infty}e^{-\sigma n/N}E\left[e^{i\mu\tau_{\frac{n}{N}}}+i\eta\max_{0\leq k\leq n}\xi_{\frac{k}{N}}\right]$$

$$=N\ (1-e^{-\sigma/N-\frac{1}{N}\phi(\mu,\,0)})\int_0^{\infty}dte^{-\sigma[Nt]/N}E\left[e^{i\mu\tau_{\frac{[Nt]}{n}}}+i\eta\max_{0\leq k\leq[Nt]}\xi_{\frac{k}{N}}\right]$$

$$\rightarrow(\sigma+\phi(\mu,\,0))\int_0^{\infty}e^{-\sigma t}E\left[e^{i\mu\tau_t+i\eta\bar{\xi}_t}\right].$$

This convergence holds for $\mathrm{Im}\eta\geq 0$. For the right hand side of (3.7),

$$\exp\left(\sum_{n=1}^{\infty}\frac{e^{-\sigma n/N}}{n}E\left[e^{i\mu\tau_{\frac{n}{N}}}\left(e^{i\eta\xi_{\frac{n}{N}}-1}\right);\ \xi_{\frac{n}{N}}>0\right]\right)$$

$$=\exp\left(\int_0^{\infty}dt\frac{N}{[Nt]+1}e^{-\sigma([Nt]+1)/N}E\left[e^{i\eta\tau_{\frac{[Nt]+1}{N}}}\left(e^{i\eta\xi_{\frac{[Nt]+1}{N}}}-1\right);\xi_{\frac{[Nt]+1}{N}}>0\right]\right)$$

If $\mu=0$, this is proved by Rogozin([7]) to converge to $\int_0^{\infty}\frac{e^{-\sigma t}dt}{t}E[e^{i\eta\xi_t}-1\ ;\ \xi_t>0]$ for any η, $\mathrm{Im}\eta\geq 0$. In our situation,

$$(3.10)\qquad -\frac{N}{[Nt]+1}e^{-\sigma([Nt]+1)/E}E\left[e^{i\mu\tau_{\frac{[Nt]+1}{N}}}(e^{-\lambda\xi_{\frac{[Nt]+1}{N}}}-1);\ \xi_{\frac{[Nt]+1}{N}}>0\right]$$

for $\lambda\geq 0$ is dominated by

$$-\frac{N}{[Nt]+1}e^{-\sigma([Nt]+1)/N}E\left[e^{-\lambda\xi_{\frac{[Nt]+1}{N}}}-1;\ \xi_{\frac{[Nt]+1}{N}}>0\right],$$

which is positive and converges pointwise to an integrable function as $N\rightarrow\infty$, together with their integral by $\int_0^{\infty}dt$. Moreover (3.10) converges pointwise to

$$-\frac{e^{-\sigma t}}{t}E[e^{i\mu\tau_t}\ (e^{-\lambda\xi_t}-1)\ ;\xi_t>0].$$

Hence it follows that the integral of (3.10) tends to

$$\int_0^{\infty}\frac{e^{-\sigma t}\,dt}{t}E\ \left[e^{i\mu\tau_t}(e^{-\lambda\xi_t}-1);\ \xi_t>0\right],$$

which establishes (2.5) for $\eta=i\lambda$, $\lambda\geq 0$. Continuity and analyticity of both sides complete the proof. (2.7) can be proved either by taking limit in (3.9) or by the formula of the Frullani integral applied to the product of (2.5) and (2.6).

4. Proof of Theorem 2

Let $\tau_t=L_t^{-1}$ and ξ_t be defined by (2. 8). Then $\phi(i\mu,\,0)=\sqrt{2\mu}$. Most of explicit computations involving the properties of Brownian motion is done in Lemma 3.

Lemma 1. *If* $\sigma\geq 0$, $\mu\geq 0$, $\sigma+\mu>0$ *and* $\lambda\geq 0$,

$$1-E[e^{-\sigma\theta(r)-\mu\tau_{\theta(r)}-\lambda\bar{\xi}_{\theta(r)}}]=\frac{\sigma+\sqrt{2\mu}}{\phi_+(\sigma, i\mu, i\lambda)}\int_0^\infty dtE[e^{-\sigma t-\mu\tau_t-\lambda\bar{\xi}_t}\ ;\ \bar{\xi}_t<r]. \tag{4.1}$$

Proof. It is sufficient to prove the following identity:

$$\frac{\phi_+(\sigma, i\mu, i\lambda)}{\sigma+\sqrt{2\mu}}=\int_0^\infty dtE[e^{-\sigma t-\mu\tau_t-\lambda\bar{\xi}_t};\ \bar{\xi}_t<r]$$
$$+\frac{\phi_+(\sigma, i\mu, i\lambda)}{\sigma+\sqrt{2\mu}}E\left[e^{-\sigma\theta(r)-\mu\tau_{\theta(r)}-\lambda\bar{\xi}_{\theta(r)}}\right].$$

By (2.3) and the strong Markov property of (τ_t, ξ_t),

$$\frac{\phi_+(\sigma, i\mu, i\lambda)}{\sigma+\sqrt{2\mu}}=\int_0^\infty duE[e^{-\sigma u-\mu\tau_u-\lambda\bar{\xi}_u}]$$
$$=E\left[\left(\int_0^{\theta(r)}+\int_{\theta(r)}^\infty\right)e^{-\sigma u-\mu\tau_u-\lambda\bar{\xi}_u}du\right]$$
$$=E\left[\int_0^\infty du\ e^{-\sigma u-\mu\tau_u-\lambda\bar{\xi}_u}1_{\{\bar{\xi}_u<r\}}\right]+E\left[\int_0^\infty dt\ e^{-\sigma(\theta(r)+t)-\mu\tau_{(\theta(r)+t)}-\lambda\bar{\xi}_{(\theta(r)+t)}}\right]$$
$$=\int_0^\infty du\ E[e^{-\sigma u-\mu\tau_u-\lambda\bar{\xi}_u};\ \bar{\xi}_u<r]$$
$$+E\left[e^{-\sigma\theta(r)-\mu\tau_{\theta(r)}-\lambda\bar{\xi}_{\theta(r)}}E\left[\int_0^\infty dt\ e^{-\sigma t-\mu\tau_t-\lambda\bar{\xi}_t}\right]\right]$$
$$=\int_0^\infty du\ E\left[e^{-\sigma u-\mu\tau_u-\lambda\bar{\xi}_u};\ \bar{\xi}_u<r\right]+\frac{\phi_+(\sigma, i\mu, i\lambda)}{\sigma+\sqrt{2\mu}}E\left[e^{-\sigma\theta(r)-\mu\tau_{\theta(r)}-\lambda\bar{\xi}_{\theta(r)}}\right].$$

We proceed to obtain the asymptotics for the right hand side of (4.1).

Lemma 2. *If* $\sigma\geq 0$, $\mu\geq 0$, $\sigma+\mu>0$ *and* $\lambda\geq 0$,

$$\int_{-\infty}^\infty d\eta\frac{1-e^{-(\lambda+i\eta)r}}{2\pi(\lambda+i\eta)}\phi_+(\sigma, i\mu, \eta)=(\sigma+\sqrt{2\mu})\int_0^\infty dtE[e^{-\sigma t-\mu\tau_t-\lambda\bar{\xi}_t};\ \bar{\xi}_t<r], \tag{4.2}$$

Proof. We have

$$\int_{-A}^A d\eta\frac{1-e^{-(\lambda+i\eta)r}}{2\pi(\lambda+i\eta)}\int_0^\infty dtE\ [e^{-\sigma t-\mu\tau t+i\eta\bar{\xi}_t}] \tag{4.3}$$
$$=\int_0^\infty dtE\left[e^{-\sigma t-\mu\tau_t}\int_{-A}^A\frac{1-e^{-(\lambda+i\eta)r}}{2\pi(\lambda+i\eta)}e^{i\eta\bar{\xi}_t}d\eta\right].$$

If A tends to infinity, the left hand side converges to

$$\int_{-\infty}^\infty d\eta\frac{1-e^{-(\lambda+i\eta)r}}{2\pi(\lambda+i\eta)}\int_0^\infty dtE[e^{-\sigma t-\mu\tau_t+i\eta\bar{\xi}_t}]=\int_{-\infty}^\infty d\eta\frac{1-e^{-(\lambda+i\eta)r}}{2\pi(\lambda+i\eta)}\cdot\frac{\phi_+(\sigma, i\mu, \eta)}{\sigma+\sqrt{2\mu}}$$

because (4.10) given below assures that the absolute value of the integrand is dominated by

$$\frac{2}{2\pi|\eta|}\left|\frac{\phi_+(\sigma, i\mu, \eta)}{\sigma+\sqrt{2\mu}}\right|\sim\mathrm{const}|\eta|^{-1-\frac{1}{2(\alpha+2)}}\mathrm{as}|\eta|\to\infty,$$

which is integrable outside $[-A, A]$. The right hand side of (4.3) also converges to

$$\int_0^\infty dtE[e^{-\sigma t-\mu\tau_t-\lambda\xi_t};\ \xi_t<r]$$

because the inverse Fourier transform

$$\int_{-A}^{A}\frac{1-e^{-(\lambda+i\eta)r}}{2\pi(\lambda+i\eta)}e^{i\eta x}d\eta$$

converges boundedly to $e^{-\lambda x}\cdot 1_{(0,r)}(x)$.

The next lemma is the key to explicit computations.

Lemma 3. *Let* $\theta=\frac{\pi}{4(\alpha+2)}$. *Then we have as* $|\eta|\to\infty$,

$$\phi_+(\sigma, i\mu, \eta)\sim\frac{\sqrt{\sigma+\sqrt{2\mu}}}{\sqrt{C_0(\alpha)}|\eta|^{\frac{1}{2(\alpha+2)}}}e^{i\theta(\mathrm{sgn}\eta)} \tag{4.4}$$

for $\sigma\geq 0$*and* $\mu\geq 0$ *uniformly on any compact set, where*

$$C_0(\alpha)=\frac{\pi(\alpha+2)^{\frac{\alpha}{\alpha+2}}}{2^{\frac{\alpha+1}{\alpha+2}}\sin\left(\frac{\pi}{2(\alpha+2)}\right)\Gamma\left(\frac{1}{\alpha+2}\right)^2}. \tag{4.5}$$

Proof. First we obtain the asymptotics for the absolute value, and then we prove the existence of the limit of $\arg\phi_+(\sigma, i\mu, \eta)$.

It is obvious that $(\tau_t, \bar{\xi}_t)\overset{d}{=}(\tau_t, -\bar{\xi}_t)$, and hence for any $\eta\in\mathbf{R}$

$$\psi(\mu, \eta)=\psi(\mu, -\eta),\ \mathrm{Im}\mu\geq 0,$$

$$\phi_+(\sigma, i\mu, \eta)=\overline{\phi_-(\sigma, i\mu, \eta)},\ \mu\geq 0. \tag{4.6}$$

Then by (2.7),

$$|\phi_+(\sigma, i\mu, \eta)|=\sqrt{\frac{\sigma+\sqrt{2\mu}}{\sigma+\psi(i\mu, \eta)}},\ \mu\geq 0.$$

Let $M(t)$ be the local martingale

$$F(b_t^+)\ \exp\left(-\frac{1}{2}F'(0)\ L_t-\int_0^t(\mu+i\eta b_s^\alpha)1_{\{b_s>0\}}ds\right)$$

where $F(x)$ is the unique bounded solution of $F''(x)=(2\mu+2i\eta x^\alpha)\ F(x)$ on $[0, \infty)$ with the condition $F(0)=1$. Stopping $M(t)$ at τ_t and at 0, respectively, we obtain

$$E\left[F(b_{\tau_t}^+)\ \exp\left(-\frac{1}{2}F'(0)L_{\tau_t}-\mu\tau_t^+-i\eta\int_0^{\tau_t}(b_u^+)^\alpha du\right)\right]=F(0).$$

Hence

$$E\left[\exp\left(-\mu\tau_t^+ - i\eta\int_0^{\tau_t}(b_u^+)^\alpha du\right)\right] = e^{\frac{t}{2}F'(0)},$$

where τ_t^+ is the time spent by $b(t)$ in $(0, \infty)$ until the stopping time τ_t, and τ_t^- is defined analogously:

$$\tau_t^+ = \int_0^{\tau_t} 1_{\{b_s>0\}} ds, \quad \tau_t^- = \int_0^{\tau_t} 1_{\{b_s<0\}} ds. \tag{4.7}$$

Because the two dimensional processes $(\tau_t^+, \int_0^{\tau_t}|b_u \vee 0|^\alpha du)$ and $(\tau_t^-, \int_0^{\tau_t}|b_u \wedge 0|^\alpha du)$ are independent and identically distributed, it holds

$$E[\exp(-\mu\tau_t - i\eta\int_0^{\tau_t}\mathrm{sgn}(b_u)|b_u|^\alpha du)] = e^{\frac{t}{2}(F'(0)+\overline{F'(0)})} = e^{t\mathrm{Re}F'(0)},$$

and hence

$$\psi(i\mu, \eta) = -\mathrm{Re}F'(0). \tag{4.8}$$

The limiting property of $F'(0)$ can be computed by using some knowledge on the Bessel functions (Abramowitz-Stegun[1]). Let K_ν and I_ν be the usual modified Bessel functions :

$$K_\nu(z) = \frac{\pi}{2} \cdot \frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi}$$

and

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(z^2/4)^k}{k!\Gamma(\nu+k+1)}.$$

Then $w(z) = \sqrt{z}\, K_{\frac{1}{\alpha+2}}\left(\frac{2}{\alpha+2}z^{\frac{\alpha+2}{2}}\right)$ is, up to a multiple constant, a unique bounded solution to $w''(z) = z^\alpha w(z)$ on $[0, \infty)$. Hence if $U(z) = w((2i)^{\frac{1}{\alpha+2}} z)$, then $F(x)$ for $(\mu, \eta) = (0, 1)$ is given by $\frac{U(x)}{U(0)}$. We note the following asymptotic formula:

$$w(z) = \sqrt{z}K_{\frac{1}{\alpha+2}}\left(\frac{2}{\alpha+2}z^{\frac{\alpha+2}{2}}\right) \sim \sqrt{z}\frac{\pi}{2\sin\frac{\pi}{\alpha+2}}\left(\frac{\left(\frac{1}{\alpha+2}z^{\frac{\alpha+2}{2}}\right)^{-\frac{1}{\alpha+2}}}{\Gamma\left(\frac{\alpha+1}{\alpha+2}\right)} - \frac{\left(\frac{1}{\alpha+2}z^{\frac{\alpha+2}{2}}\right)^{\frac{1}{\alpha+2}}}{\Gamma\left(\frac{\alpha+3}{\alpha+2}\right)}\right)$$

$$= \frac{\pi}{2\sin\frac{\pi}{\alpha+2}}\left(\frac{\left(\frac{1}{\alpha+2}\right)^{-\frac{1}{\alpha+2}}}{\Gamma\left(\frac{\alpha+1}{\alpha+2}\right)} - \frac{\left(\frac{1}{\alpha+2}\right)^{\frac{1}{\alpha+2}}z}{\Gamma\left(\frac{\alpha+3}{\alpha+2}\right)}\right).$$

From this $F'(0) = U'(0)/U(0)$ for $(\mu, \eta) = (0, 1)$ can be computed as

$$F'(0)=-2^{\frac{1}{\alpha+2}}e^{\frac{\pi i}{2(\alpha+2)}}\frac{(\alpha+2)^{\frac{\alpha}{\alpha+2}}\Gamma\left(\frac{\alpha+1}{\alpha+2}\right)}{\Gamma\left(\frac{1}{\alpha+2}\right)}.$$

Thus it holds as $|\eta|\to\infty$,

$$\phi(i\mu,\eta)=|\eta|^{\frac{1}{\alpha+2}}\phi(i|\eta|^{-\frac{2}{(\alpha+2)}}\mu,\ \mathrm{sgn}\eta)\sim|\eta|^{\frac{1}{\alpha+2}}\phi(0,\pm1)=C_0(\alpha)|\eta|^{\frac{1}{\alpha+2}}$$

where

$$C_0(\alpha):=\phi(0,\pm1)=-\mathrm{Re}\,F'(0)=\frac{\pi(\alpha+2)^{\frac{\alpha}{\alpha+2}}}{2^{\frac{\alpha+1}{\alpha+2}}\sin\frac{\pi}{2(\alpha+2)}\Gamma\left(\frac{1}{\alpha+2}\right)^2}. \tag{4.9}$$

Hence

$$|\phi_+(\sigma,i\mu,\eta)|\sim\sqrt{\frac{\sigma+\sqrt{2\mu}}{C_0(\alpha)|\eta|^{\frac{1}{\alpha+2}}}}\ \text{as } |\eta|\to\infty. \tag{4.10}$$

arg $\phi_+(\sigma, i\mu, \eta)$ is the imaginary part of the exponent in the right hand of (2.5) :

$$\begin{aligned}\arg\phi_+(\sigma,i\mu,\eta)&=\int_0^\infty\frac{dt}{t}E[e^{-\sigma t-\mu\tau_t}\sin(\eta\xi_t^+)]\\&=\mathrm{sgn}(\eta)\int_0^\infty\frac{dt}{t}E[e^{-\sigma t-\mu t^2\tau_1}\sin(|\eta|t^{\alpha+2}\xi_1^+)]\\&=\frac{\mathrm{sgn}(\eta)}{\alpha+2}\int_0^\infty\frac{du}{u}E[e^{-\sigma(u/|\eta|)^{\frac{1}{\alpha+2}}-\mu(u/|\eta|)^{\frac{2}{\alpha+2}}\tau_1}\sin(u\xi_1^+)]\end{aligned}$$

and, as $|\eta|\to\infty$, this converges to

$$\begin{aligned}&\frac{\mathrm{sgn}(\eta)}{\alpha+2}E\left[\int_0^\infty\frac{dv}{v}\sin v;\ \xi_1>0\right]\\&=\frac{\mathrm{sgn}(\eta)}{\alpha+2}E\left[\frac{\pi}{2};\xi_1>0\right]=\frac{\pi\,\mathrm{sgn}(\eta)}{4\ (\alpha+2)}.\end{aligned}$$

The convergence above follows by the repeated use of the fact that for $x>0$,

$$\left|\int_x^\infty\frac{f(t)\sin t\ dt}{t}\right|<f(0)\left|\int_x^\infty\frac{\sin t\ dt}{t}\right|$$

for any positive decreasing function $f(t)$, and that the integral $\int_x^\infty\frac{\sin t\ dt}{t}$ is continuous and vanishes as $x\to\infty$.

Proof of Theorem 2. From (4.4), it follows that the left hand side of (4.2) behaves as $r\downarrow 0$,

$$\int_{-\infty}^{\infty} d\eta \frac{1-e^{-(\lambda+i\eta)r}}{2\pi(\lambda+i\eta)} \phi_+(\sigma, i\mu, \eta) = \int_{-\infty}^{\infty} d(r\eta) \frac{1-e^{-(\lambda+i\eta)r}}{2\pi r(\lambda+i\eta)} \phi_+(\sigma, i\mu, \frac{1}{r}\cdot r\eta)$$

$$= \int_{-\infty}^{\infty} dx \frac{1-e^{-r\lambda-ix}}{2\pi(r\lambda+ix)} \phi_+(\sigma, i\mu, \frac{1}{r}x)$$

$$\sim \int_{-\infty}^{\infty} dx \frac{1-e^{-ix}}{2\pi ix} \cdot \frac{\sqrt{\sigma+\sqrt{2\mu}}}{\sqrt{C_0(\alpha)}} r^{\frac{1}{2(\alpha+2)}} |x|^{\frac{-1}{2(\alpha+2)}} e^{i\theta(\operatorname{sgn} x)}$$

$$= \frac{\sqrt{\sigma+\sqrt{2\mu}}}{\sqrt{C_0(\alpha)}} r^{\frac{1}{2(\alpha+2)}} \cdot \frac{2(\alpha+2)}{\Gamma\left(\frac{1}{2(\alpha+2)}\right)}.$$

Here the identity $\int_{-\infty}^{\infty} dx \frac{1-e^{-ix}}{2\pi ix} |x|^{\frac{-1}{2(\alpha+2)}} e^{i\theta(\operatorname{sgn} x)} = 2(\alpha+2)/\Gamma\left(\frac{1}{2(\alpha+2)}\right)$ is obtained by using $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ and the following integrals for $\nu > -1$: $\int_0^{\infty} x^{\nu} \sin x dx = \Gamma(1+\nu) \cos(\nu\pi/2)$ and $\int_0^{\infty} x^{\nu} \cos x dx = -\Gamma(1+\nu) \sin(\nu\pi/2)$.

Hence

$$1-E[e^{-\sigma\theta(r)-\mu\tau_{\theta(r)}-\lambda\bar{\xi}_{\theta(r)}}] = \frac{\sigma+\sqrt{2\mu}}{\phi_+(\sigma, i\mu, i\lambda)} \int_0^{\infty} dt E[e^{-\sigma t-\mu\tau_t-\lambda\bar{\xi}_t}; \bar{\xi}_t < r]$$

$$= \frac{1}{\phi_+(\sigma, i\mu, i\lambda)} \int_{-\infty}^{\infty} d\eta \frac{1-e^{-(\lambda+i\eta)r}}{2\pi(\lambda+i\eta)} \phi_+(\sigma, i\mu, \eta)$$

$$\sim \frac{1}{\phi_+(\sigma, i\mu, i\lambda)} \frac{\sqrt{\sigma+\sqrt{2\mu}}}{\sqrt{C_0(\alpha)}} r^{\frac{1}{2(\alpha+2)}} \cdot \frac{2(\alpha+2)}{\Gamma\left(\frac{1}{2(\alpha+2)}\right)}.$$

We have thus obtained (2. 9) with

$$C(\alpha) = \frac{1}{\sqrt{C_0(\alpha)}} \cdot \frac{2(\alpha+2)}{\Gamma\left(\frac{1}{2(\alpha+2)}\right)}$$

$$= \frac{2^{1+\frac{\alpha+1}{2(\alpha+2)}} (\alpha+2)^{1-\frac{\alpha}{2(\alpha+2)}} \sqrt{\sin\left(\frac{\pi}{2(\alpha+2)}\right) \Gamma\left(\frac{1}{\alpha+2}\right)}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2(\alpha+2)}\right)}.$$

5. Proof of Theorem 3

Lemma 4. *Let S and T be non-negative random variables and assume that*

$$P[T+S>A] \sim c_1 A^{-\nu} \text{ and } P[S>A] \sim c_2 A^{-\nu} \text{ as } A \uparrow \infty$$

with $0<\nu<1$ and $0<c_2<c_1$. Then

(5.1) $$(c_1^{1/(\nu+1)}-c_2^{1/(\nu+1)})^{\nu+1}A^{-\nu}<P[T>A]<c_1A^{-\nu} \quad as\ A\uparrow\infty.$$

Proof. We set

$$\liminf_{A\to\infty}\frac{P[T>A]}{A^{-\nu}}=k$$

Then for any $\varepsilon>0$, there exists A_0 such that

$$\frac{P[S>A]}{A^{-\nu}}<c_2+\varepsilon \text{ and } \frac{P[T+S>A]}{[A^{-\nu}}>c_1-\varepsilon \text{ for all } A>A_0$$

and for infinitely many large B,

$$\frac{P[T>B]}{B^{-\nu}}<k+\varepsilon.$$

We choose B large enough to satisfy $\dfrac{B}{\left(\dfrac{c_1-\varepsilon}{c_2+\varepsilon}-1\right)^{1/(\nu+1)}}>A_0$. Then

$$\begin{aligned}(c_1-\varepsilon)(A+B)^{-\nu}&<P[T+S>A+B]\\ &<P[S>A]+P[T>B]\\ &<(c_2+\varepsilon)A^{-\nu}+(k+\varepsilon)B^{-\nu}.\end{aligned}$$

Hence

$$(k+\varepsilon)B^{-\nu}>(c_1-\varepsilon)(A+B)^{-\nu}-(c_2+\varepsilon)A^{-\nu}.$$

The last term is maximized when we set $A=B\left(\dfrac{c_1-\varepsilon}{c_2+\varepsilon}-1\right)^{-1/(\nu+1)}$ to yield

$$k+\varepsilon>\left\{(c_1-\varepsilon)^{1/(\nu+1)}-(c_2+\varepsilon)^{1/(\nu+1)}\right\}^{\nu+1}.$$

We intend to apply this lemma with $(T, S)=(T, T^0 \circ \theta_T)$. If we set $\sigma=0$, $\lambda=0$ in Theorem 2, it follows

$$1-E_{(-1,0)}\left[e^{-r^{\frac{2}{\alpha+2}}\mu(T+T^0\circ\theta_T)}\right]\sim(2\mu)^{1/4}r^{\frac{1}{2(\alpha+2)}}C(\alpha) \qquad \text{as } r\downarrow 0.$$

We rewrite this using the Tauberian theorem,

(5.2) $$P_{(-1,0)}[(T+T^0\circ\theta_T)>A]\sim\frac{2^{1/4}A^{-1/4}}{\Gamma\left(\frac{3}{4}\right)}C(\alpha) \qquad as\ A\uparrow\infty.$$

On the other hand, we can extract $T^0 \circ \theta_T$:

(5.3) $$\begin{aligned}1-E_{(-1,0)}[e^{-r^{\frac{2}{\alpha+2}}\mu(T^0\circ\theta_T)}]&=1-E_{(-1,0)}[G(Y(T):0,\mu r^{\frac{2}{\alpha+2}})]\\ &=1-E_{(-1,0)}[e^{-\sqrt{2\mu}r^{\frac{1}{\alpha+2}}Y(T)}].\end{aligned}$$

The asymptotic behavior of the last term can be obtained by the Tauberian theorem and Corollary 1:

Lemma 5. *As $r \downarrow 0$, it holds*

$$1-E_{(-1,0)}[G(r^{\frac{1}{\alpha+2}}Y(T);\lambda,0)] \sim \frac{2(\alpha+2)}{\Gamma\left(\frac{1}{2(\alpha+2)}\right)}(|\lambda|r)^{\frac{1}{2(\alpha+2)}} \tag{5.4}$$

and hence

$$1-E_{(-1,0)}\left[e^{-\sqrt{2\mu}\, r^{\frac{1}{\alpha+2}}Y(T)}\right] \sim (2\mu)^{1/4} r^{1/2(\alpha+2)} \frac{2^{1-\frac{1}{2(\alpha+2)}}(\alpha+2)^{1+\frac{1}{\alpha+2}}\Gamma\left(\frac{1}{\alpha+2}\right)\sin\frac{\pi}{2(\alpha+2)}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2(\alpha+2)}\right)}. \tag{5.5}$$

The proof is given at the end of this section. From (5.5) and (5.3) we know

$$P_{(-1,0)}[(T^0 \circ \theta_T)>A] \sim \frac{2^{1/4}A^{-1/4}}{\Gamma\left(\frac{3}{4}\right)} \cdot \frac{2^{1-\frac{1}{2(\alpha+2)}}(\alpha+2)^{1+\frac{1}{\alpha+2}}\Gamma\left(\frac{1}{\alpha+2}\right)\sin\frac{\pi}{2(\alpha+2)}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2(\alpha+2)}\right)}$$

$$= \frac{2^{1/4}A^{-1/4}}{\Gamma\left(\frac{3}{4}\right)} C(\alpha)\sqrt{\frac{\alpha+2}{2}\sin\frac{\pi}{2(\alpha+2)}}. \tag{5.6}$$

Comparing (5.2) and (5.6) explicitly, we can apply Lemma 4 and thereby complete the proof of Theorem 3 with

$$c_L^{(\alpha)} = (c_1^{4/5}-c_2^{4/5})^{5/4} = \frac{2^{1/4}}{\Gamma\left(\frac{3}{4}\right)}C(\alpha)\left(1-\left(\frac{\alpha+2}{2}\sin\frac{\pi}{2(\alpha+2)}\right)^{\frac{2}{5}}\right)^{\frac{5}{4}},\quad c_U^{(\alpha)} = \frac{2^{1/4}}{\Gamma\left(\frac{3}{4}\right)}C(\alpha).$$

Remark. For $\alpha=1$, $P_{ra}(A) \sim C(r,a)A^{-1/4}$ has been obtained in [4]. It amounts

$$P_{10}(A) \sim 0.7182\cdots \cdot A^{-1/4},$$

while our Theorem 3 asserts only

$$0.1231\cdots \cdot A^{-1/4} < P_{10}(A) < 1.972\cdots \cdot A^{-1/4} \text{ for all large } A.$$

Proof of Lemma 5. From (4.4),

$$\phi_+(\sigma, i\mu, i\lambda) \sim \frac{\sqrt{\sigma+\sqrt{2\mu}}}{\sqrt{C_0(\alpha)}}\lambda^{-\frac{1}{2(\alpha+2)}} \qquad \text{as } \lambda\to\infty.$$

Hence if we make $\mu \downarrow 0$ in Corollary 1, we get the asymptotic (5.4). For (5.5), we must compute

$$\lim_{A\to\infty}\frac{P_{(-1,0)}[Y(T)>A]}{A^{-1/2}} = C. \tag{5.7}$$

Once C is explicitly obtained, we readily obtain

$$(5.8)\qquad 1-E_{(-1,0)}\left[e^{-\sqrt{2\mu}\,r^{1/(\alpha+2)}Y(T)}\right]\sim(2\mu)^{\frac{1}{4}}r^{\frac{1}{2(\alpha+2)}}\Gamma\Big(\frac{1}{2}\Big)C.$$

To compute C explicitly, we note that $G(y;\lambda,0)$ is a bounded solution to $G''(y)=2\lambda y^{\alpha}G(y)$ on $[0,\infty)$ and hence $G(y;\frac{1}{2},0)=w(y)/w(0)$ where $w(z)=\sqrt{z}K_{\frac{1}{\alpha+2}}\Big(\frac{2}{\alpha+2}z^{\frac{\alpha+2}{2}}\Big)$. By (5.4) we deduce that

$$\int_0^{\infty}\frac{-w'(y)}{w(0)}P[r^{\frac{1}{\alpha+2}}Y(T)>y]dy\sim\frac{2(\alpha+2)}{\Gamma\Big(\frac{1}{2(\alpha+2)}\Big)}\Big(\frac{r}{2}\Big)^{\frac{1}{2(\alpha+2)}},$$

and hence

$$(5.9)\qquad C\int_0^{\infty}\frac{-w'(y)}{w(0)}y^{-1/2}dy=\frac{2(\alpha+2)}{\Gamma\Big(\frac{1}{2(\alpha+2)}\Big)}2^{-\frac{1}{2(\alpha+2)}}.$$

As we obtained in Section 4,

$$w(0)=\frac{\pi(\alpha+2)^{\frac{1}{\alpha+2}}}{2\sin\frac{\pi}{\alpha+2}\Gamma\Big(\frac{\alpha+1}{\alpha+2}\Big)}=\frac{(\alpha+2)^{\frac{1}{\alpha+2}}\Gamma\Big(\frac{1}{\alpha+2}\Big)}{2}.$$

Finally we integrate $-w'(y)\,y^{-1/2}$:

$$-\int_0^{\infty}w'(y)\,y^{-1/2}dy$$
$$=-\int_0^{\infty}y^{-1/2}dy\left\{\frac{1}{2\sqrt{y}}K_{\frac{1}{\alpha+2}}\Big(\frac{2}{\alpha+2}y^{\frac{\alpha+2}{2}}\Big)+\sqrt{y}\cdot y^{\frac{\alpha}{2}}K'_{\frac{1}{\alpha+2}}\Big(\frac{2}{\alpha+2}y^{\frac{\alpha+2}{2}}\Big)\right\}$$

and by the change of variable $\zeta=\frac{2}{\alpha+2}y^{\frac{\alpha+2}{2}}$ so that $d\zeta=y^{\frac{\alpha}{2}}dy$, this is equal to

$$-\int_0^{\infty}d\zeta\left\{\frac{1}{(\alpha+2)\zeta}K_{\frac{1}{\alpha+2}}(\zeta)+K'_{\frac{1}{\alpha+2}}(\zeta)\right\}.$$

Note that $\frac{\nu}{z}K_{\nu}(z)+K'_{\nu}(z)=-K_{\nu-1}(z)$ and $K_{\nu}(z)=\frac{1}{2}\int_0^{\infty}d\lambda\cdot\lambda^{\pm\nu-1}e^{-\frac{z}{2}(\lambda+1/\lambda)}$, the above further equals to

$$\int_0^{\infty}d\zeta K_{\frac{1}{\alpha+2}-1}(\zeta)$$
$$=\int_0^{\infty}d\lambda\cdot\lambda^{\frac{1}{\alpha+2}-2}\int_0^{\infty}\frac{d\zeta}{2}e^{-\frac{\zeta}{2}(\lambda+1/\lambda)}$$
$$=\int_0^{\infty}d\lambda\frac{\lambda^{\frac{1}{\alpha+2}-1}}{\lambda^2+1}=\frac{\pi}{2\sin\frac{\pi}{2(\alpha+2)}}.$$

From (5.9) we obtain

$$\frac{\pi}{2\sin\frac{\pi}{2(\alpha+2)}}C=\frac{2(\alpha+2)}{\Gamma\left(\frac{1}{2(\alpha+2)}\right)}2^{-\frac{1}{2(\alpha+2)}}\cdot\frac{(\alpha+2)^{\frac{1}{\alpha+2}}\Gamma\left(\frac{1}{\alpha+2}\right)}{2}$$

and hence

$$(2\mu)^{\frac{1}{4}r^{\frac{1}{2(\alpha+2)}}}\Gamma\left(\frac{1}{2}\right)C=(2\mu)^{\frac{1}{4}r^{\frac{1}{2(\alpha+2)}}}\frac{2^{1-\frac{1}{2(\alpha+2)}}(\alpha+2)^{1+\frac{1}{\alpha+2}}\Gamma\left(\frac{1}{\alpha+2}\right)\sin\frac{\pi}{2(\alpha+2)}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2(\alpha+2)}\right)}.$$

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References

[1] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, new edition, National Bureau of Standards, 1964.

[2] N. H. Bingham, Fluctuation theory in continuous time, Adv. Appl. Prob., **7** (1975), 705-766.

[3] P. Greenwood and J. Pitman, Fluctuaton identities for Lévy process and splitting at the maximum, Adv. Appl. Prob., **12** (1980), 893-902.

[4] Y. Isozaki and S. Watanabe, An asymptotic formula for the Kolmogorov diffusion and a refinement of Sinai's estimates for the integral of Brownian motion, Proc. Japan Acad., **70A**-9 (1994), 271-276.

[5] A. N. Kolmogorov, Zuffälige Bewegungen, Ann. Math. II., **35** (1934), 116-117.

[6] H. P. McKean, Jr., A winding problem for a resonator driven by a white noise, J. Math. Kyoto Univ., **2** (1963), 227-235.

[7] B. A. Rogozin, On distributions of functionals related to boundary problems for processes with independent increments, Theory of Probability and its Applications, **11**-4 (1966), 580-591.

[8] F. Spitzer, A combinatorial lemma and its application to probability theory, Trans. of A. M. S., **82** (1956), 323-339.

[9] Ya. G. Sinai, Distribution of some functionals of the integral of a random walk, Theor. Math. Phys., **90** (1992), 219-241.