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Commutant algebra of Cartan-type Lie superalgebra $W(n)$

By

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Introduction

Undoubtedly, Weyl's classical reciprocity law is very important in Lie theory ([4]). It tells us that there is a correspondence between irreducible representations of general linear group $GL(V)$ in m-fold tensor space $V \otimes \cdots \otimes V$ and those of $\mathfrak{S}_m$ (symmetric group of degree m). The correspondence is one-to-one and so an irreducible representation of $\mathfrak{S}_m$ determines that of $GL(V)$ and vice versa. If we consider various m, then all the irreducible polynomial representations of $GL(V)$ appear in the decomposition, so one gets a classification of irreducible representations via those of $\mathfrak{S}_m$.

In the article [2], the first author studied an analogous phenomenon for a Cartan-type Lie algebra of vector fields. In the present paper, we want to do it for a Cartan-type Lie *superalgebra* $W(n)$. By definition $W(n)$ is a Lie superalgebra of all the superderivations on a Grassmann algebra $\Lambda(n)$ of n-variables (see [1]), so $W(n)$ acts on $\Lambda(n)$ naturally. We call it the natural representation of $W(n)$ and denote it by ψ. To study an analogue of Weyl's reciprocity, the first important thing is to calculate the commutant algebra of the natural representation of $W(n)$ in m-fold tensor Grassmann algebra $\otimes^m \Lambda(n)$. Let End $[m]$ be the set of all the maps from $[m]$ to $[m]$, where $[m]=\{1, 2, \cdots, m\}$, and denote the semigroup ring of End $[m]$ by $\mathfrak{E}_m$. There is the natural represntation of $\mathfrak{E}_m$ on $\otimes^m \Lambda(n)$ (see Section 1.3) and denote the image algebra of this representation by $\mathscr{E}_m$. Also we denote the commutant algebra of $\psi^{\otimes m}(W(n))$ in End $\otimes^m \Lambda(n)$ by $\mathscr{C}_m$ (see Section 1.2). One of the main results is the identification of the commutant algebra $\mathscr{C}_m$ and the semigroup ring $\mathscr{E}_m$ (Theorems 2.3 and 3.2). However, in Theorem 2.3 we assume $m \leq n$ (the rank n is larger then the power of tensor product m), and in Theorm 3.2 we restrict ourselves to the case $n=1$. In the future studies, we want to clarify the relationship between $\mathscr{C}_m$ and $\mathscr{E}_m$ fo general m and n.

The another main result is Theorem 3.3, which says that the bicommutant algebra of $W(1)$ coincides with the image of its universal enveloping algebra $U(W(1))$. We consider it an analogy of Weyl's reciprocity for $W(1)$

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$\times$ End$[m]$. However, we did not get any decomposition of the (non-semisimple) representation of $W(1) \times$ End $[m]$ on $\otimes^m \Lambda(1) \simeq \Lambda(m)$, and it is a future subject of ours.

We divide this article into three sections. In Section 1, we give the basic definitions. In Section 2, we calculate the commutant algebra $\mathscr{C}_m$ for the case $m \leq n$. In Section 3, we get the commutant algebra and bicommutant algebra for the case $n=1$.

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1. Lie superalgebra $W(n)$ and its natural representation

1.1. Tensor product of a natural representation of $W(n)$. Let $\Lambda(n)$ be a Grassmann algebra over $\mathbf{C}$ in n variables $\xi_1, \xi_2, \cdots, \xi_n$. If we set deg $\xi_i = 1$, then $\Lambda(n)$ becomes a $\mathbf{Z}$-graded algebra. Let Λ_k be the space of k-homogeneous elements of $\Lambda(n)$:

$$\Lambda_k = \langle \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_k} | 1 \leq i_1 < \cdots < i_k \leq n \rangle,$$

where $\langle B \rangle$ denotes the vector space over $\mathbf{C}$ spanned by B. Then $\Lambda(n) = \otimes_{k=0}^{n} \Lambda_k$, a direct sum, where $\Lambda_0 = \mathbf{C}$. We can naturally condider $\Lambda(n)$ as a superalgebra, where even elements are $\Lambda_{\bar{0}} = \bigoplus_{k:even} \Lambda_k$, and odd elements are $\Lambda_{\bar{1}} = \bigoplus_{k:odd} \Lambda_k$.

Let A be a superalgebra and End$_s$ A the set of linear endomorphisms on A of degree s, that is $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and End$_s$ $A = \{X \in$ End $A | X(A_i) \subseteq A_{i+s}, (i \in \mathbf{Z}_2)\}$. A superderivation of degree s $(s \in \mathbf{Z}_2)$ of a superalgebra A is an endomorphism $D \in$ End$_s$ A with the property

$$D(ab) = D(a)b + (-1)^{s \deg a} a D(b)$$

for any homogeneous $a, b \in A$. The space of all the superderivations of degree s is denoted by Der$_s$ A.

Let $W(n)$ be the set of all superderivations over $\Lambda(n)$, then it becomes naturally a Lie superalgebra. According to results in [1], every derivation $D \in W(n)$ can be written in the form

$$D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i}$$

with $P_i \in \Lambda(n)$ $(1 \leq i \leq n)$, where $\frac{\partial}{\partial \xi_i}$ is a superderivation of degree 1 defined by

$$\frac{\partial}{\partial \xi_i} \xi_j = \delta_{ij}.$$

By definition, the Lie superalgebra $W(n)$ acts on Grassmann superalgebra $\Lambda(n)$ as follows: for $\forall D \in W(n)$ and $\forall \xi_{i1}\wedge\cdots\wedge\xi_{ir}$

$$D(\xi_{i1}\wedge\cdots\wedge\xi_{ir})=\sum_{s=1}^{r}(-1)^{(s-1)\deg D}\xi_{i1}\wedge\cdots\wedge D(\xi_{is})\wedge\cdots\wedge\xi_{ir}.$$

We call it the natural representation of $W(n)$, and denote it by ψ.

Let us consider m-fold tensor product $\otimes^m\Lambda(n)$. Then we have a natural isomorphism as $W(n)$-modules

$$\otimes^m\Lambda(n)\cong\Lambda[\xi_{ij}|1\leq i\leq n,\ 1\leq j\leq m]=:\Lambda(n,m),$$

where $\Lambda[\xi_{ij}|1\leq i\leq n,\ 1\leq j\leq m]$ is a Grassmann algebra generated by ξ_{ij} $(1\leq i \leq n,\ 1\leq j\leq m)$. In the following, we identify $\otimes^m\Lambda(n)$ with $\Lambda(n,m)$. By means of a tensor prodct, $W(n)$ is imbedded into End $(\otimes^m\Lambda(n))\cong$ End $\Lambda(n,m)$. More precisely, an element

$$D=\sum_{i=1}^{n}P_i(\xi_1,\cdots,\xi_n)\frac{\partial}{\partial\xi_i}\in W(n)$$

corresponds to an element

$$\psi^{\otimes m}(D)=\sum_{i=1}^{n}\sum_{\alpha=1}^{m}P_i(\xi_{1\alpha},\cdots,\xi_{n\alpha})\ \frac{\partial}{\partial\xi_{i\alpha}}\in \text{Der}\ \Lambda(n,m)$$

via m-fold tensor product $\psi^{\otimes m}$ of ψ.

1.2. Definition of commutant algebra $\mathscr{C}_m$. By definition of commutativity in Lie superalgebra, we say that a and b are commutative if $[a,b]=0$, whence $ab=(-1)^{\deg a\cdot\deg b}\,ba$ if a and b are homogeneous. Let $\mathscr{C}_m$ denote the commutant algebra of $\psi^{\otimes m}(W(n))$ in End $\Lambda(n,m)$:

$$\mathscr{C}_m=\{E\in \text{End}\ (\otimes^m\Lambda(n))\,|\,[E,D]=0,\ \forall D\in\psi^{\otimes m}(W(n))\}.$$

Then $\mathscr{C}_m$ has a natural $\mathbf{Z}_2$-graded structure, $\mathscr{C}_m=\mathscr{C}_{m,\bar{0}}\oplus\mathscr{C}_{m,\bar{1}}$. However, $\mathscr{C}_{m,\bar{1}}$ vanishes as we see in the following lemma.

Lemma 1.1. *The odd subspace $\mathscr{C}_{m,\bar{1}}$ of $\mathscr{C}_m$ vanishes:*

$$\mathscr{C}_{m,\bar{1}}=\{0\}.$$

Proof. Let $\Lambda(n,m)_k$ be a space of homogeneous Grassmannian polynomials of degree k and put

$$D_0=\sum_{i=1}^{n}\sum_{j=1}^{m}\xi_{ij}\frac{\partial}{\partial\xi_{ij}}\in\psi^{\otimes m}(W(n)).$$

Since D_0 is an Euler operator and its eigenvalues are precisely $\mathbf{Z}$-degrees of Grassmannian polynomials, $E\in\mathscr{C}_m$ must preserve homogeneous spaces $\Lambda(n,m)_k$. Therefore $\deg E$ is zero. Q. E. D.

By the above lemma, we have $\mathscr{C}_m = \mathscr{C}_{m,\bar{0}}$, hence the commutant algebra $\mathscr{C}_m$ becomes

$$\mathscr{C}_m = \{E \in \text{End } (\otimes^m \Lambda(n)) | ED = DE, \ \forall D \in \psi^{\otimes m}(W(n))\}.$$

1.3. Action of the permutation semigroup. Denote by $[m]$ the set $\{1, 2, \cdots, m\}$ of integers, and put $\text{End}[m] = \{\varphi: [m] \to [m]\}$ the set of all the maps from $[m]$ to itself. By composition of maps, $\text{End}[m]$ becomes a semigroup with unit, whose group elements form a permutation group $\mathfrak{S}_m$ of degree m. Denote the semigroup ring of $\text{End}[m]$ by $\mathfrak{E}_m$. An element $\varphi \in \text{End}[m]$ acts on $\Lambda(n, m)$ as $(\varphi P)(\xi_{ij}) = P(\xi_{i\varphi(j)})$ $(P \in \Lambda(n, m))$ and we extend it to $\mathfrak{E}_m$ by linearity (see[2]), thus, we have a representation of $\mathfrak{E}_m$ on $\Lambda(n, m)$. Denote the image algebra of this representation by $\mathscr{E}_m \subset \text{End } \Lambda(n, m)$. Using the above notations, we can state

Lemma 1.2. *For arbitrary n and m, we have*

$$\mathscr{E}_m \subseteq \mathscr{C}_m.$$

Proof. Let $D = \sum_{i=1}^{n} P_i(\xi_1, \cdots, \xi_n) \frac{\partial}{\partial \xi_i} \in W(n)$, where $P_i \in \Lambda(n)$. Then

$$\psi^{\otimes m}(D) = \sum_{i=1}^{n} \sum_{\alpha=1}^{m} P_i(\xi_{1\alpha}, \cdots, \xi_{n\alpha}) \frac{\partial}{\partial \xi_{i\alpha}}.$$

For any $\varphi \in \text{End}[m]$, we have

$$\begin{aligned}
&\varphi\psi^{\otimes m}(D)(\xi_{i_1 j_1} \wedge \cdots \wedge \xi_{i_r j_r})) \\
&= \varphi\left(\sum_{i=1}^{n} \sum_{\alpha=1}^{m} P_i\left(\xi_{1\alpha}, \cdots, \xi_{n\alpha}\right) \frac{\partial}{\partial \xi_{i\alpha}} (\xi_{i_1 j_1} \wedge \cdots \wedge \xi_{i_r j_r})\right) \\
&= \sum_{i=1}^{n} \sum_{\alpha=1}^{m} P_i\left(\xi_{1\varphi(\alpha)}, \cdots, \xi_{n\varphi(\alpha)}\right) \frac{\partial}{\partial \xi_{i\varphi(\alpha)}} (\xi_{i_1\varphi(j_1)} \wedge \cdots \wedge \xi_{i_r\varphi(j_r)}) \\
&= P_{i_1}(\xi_{1\varphi(j_1)}, \cdots, \xi_{n\varphi(j_1)}) \widehat{\xi_{i\varphi(j_1)}} \wedge \xi_{i_2\varphi(j_2)} \wedge \xi_{i_r\varphi(j_r)} \\
&\quad + (-1) P_{i_2}(\xi_{1\varphi(j_2)}, \cdots, \xi_{n\varphi(j_2)}) \xi_{i_1\varphi(j_1)} \wedge \widehat{\xi_{i_2\varphi(j_2)}} \wedge \cdots \wedge \xi_{i_r\varphi(j_r)} \\
&\quad + \cdots \\
&\quad + (-1)^{r-1} P_{i_r}(\xi_{1\varphi(j_r)}, \cdots, \xi_{n\varphi(j_r)}) \xi_{j_1\varphi(j_1)} \wedge \cdots \wedge \xi_{i_{r-1}\varphi(j_{r-1})} \wedge \widehat{\xi_{i_r\varphi(j_r)}},
\end{aligned}$$

where $\wedge$ means an elimination. On the other hand,

$$\psi^{\otimes m}(D)\varphi(\xi_{i_1 j_1} \wedge \cdots \wedge \xi_{i_r j_r})$$

$$= \psi^{\otimes m}(D)\,(\xi_{i1\varphi(j1)} \wedge \cdots \wedge \xi_{ir\varphi(jr)})$$

$$= \sum_{i=1}^{n}\sum_{\alpha=1}^{m} P_i(\xi_{1\alpha}\cdots\xi_{n\alpha})\,(\delta_{ii1}\delta_{\alpha\varphi(j1)}\widehat{\xi_{i1\varphi(j1)}} \wedge \xi_{i2\varphi(j2)} \wedge \cdots \wedge \xi_{ir\varphi(jr)}$$

$$+ (-1)\,\delta_{ii_2}\delta_{\alpha\varphi(j2)}\xi_{i1\varphi(j1)} \wedge \widehat{\xi_{i2\varphi(j2)}} \wedge \cdots \wedge \xi_{ir\varphi(jr)}$$

$$+ \cdots$$

$$+ (-1)^{r-1}\delta_{ii_r}\delta_{\alpha\varphi(jr)}\xi_{i1\varphi(j1)} \wedge \xi_{i2\varphi(j2)} \wedge \cdots \wedge \widehat{\xi_{ir\varphi(jr)}}).$$

So we obtain

$$\varphi\psi^{\otimes m}(D) = \psi^{\otimes m}(D)\,\varphi \quad (\varphi \in \mathrm{End}[m]).$$

By the definition of $\mathscr{E}_m$, we complete the proof of the lemma. Q. E. D.

2. Commutant algebra of $\psi^{\otimes m}(W(n))$ (the case $m \leq n$)

2.1. Semigroup ring $\mathscr{E}_m$ and commutant algebra $\mathscr{C}_m$. Let $(\psi, \Lambda(n))$ be the natural representation of $W(n)$ and $(\psi^{\otimes m}, \Lambda(n, m))$ its m-fold tensor product. Denote by $U(W(n))$ the universal enveloping algebra of $W(n)$, then we have

Lemma 2.1. *Put $\xi(\alpha) = (\xi_{1\alpha}, \cdots, \xi_{n\alpha})$. Then the subalgebra $\psi^{\otimes m}(U(W(n)))$ in* End $\Lambda(n, m)$ *is generated by*

$$\sum_{1 \leq \alpha_1, \cdots, \alpha_k \leq m} P_1(\xi(\alpha_1)) \cdots P_k(\xi(\alpha_k)) \frac{\partial^k}{\partial\xi_{b_1\alpha_1}\cdots\partial\xi_{b_k\alpha_k}},$$

where $1 \leq b_1, \cdots, b_k \leq n$ are indices, P_i is a Grassmannian polynomial in n-variables, and $P_i(\xi(\alpha)) = P_i(\xi_{1\alpha}, \cdots, \xi_{n\alpha})$.

Proof. Let $P_i \in \Lambda(n)$ be homogeneous and fix $1 \leq b_1, b_2 \leq n$. Then it holds that

$$\sum_{\alpha_1=1}^{m} P_1(\xi(\alpha_1)) \frac{\partial}{\partial\xi_{b_1,\alpha_1}} \sum_{\alpha_2=1}^{m} P_2(\xi(\alpha_2)) \frac{\partial}{\partial\xi_{b_2,\alpha_2}}$$

$$= \sum_{\alpha_1,\alpha_2} \Big\{ P_1(\xi(\alpha_1)) \frac{\partial}{\partial\xi_{b_1,\alpha_1}} P_2(\xi(\alpha_2)) \frac{\partial}{\partial\xi_{b_2,\alpha_2}}$$

$$+ (-1)^{\deg P_2} P_1(\xi(\alpha_1)) P_2(\xi(\alpha_2)) \frac{\partial^2}{\partial\xi_{b_1\alpha_1}\partial\xi_{b_2\alpha_2}} \Big\}.$$

By definition of $\psi^{\otimes m}$, we know that the left hand side and the first term in the right hand side of the above formula is in $\psi^{\otimes m}(U(W(n)))$, whence

$$\sum_{\alpha_1,\alpha_2} P_1(\xi(\alpha_1))P_2(\xi(\alpha_2))\frac{\partial^2}{\partial\xi_{b_1\alpha_1}\partial\xi_{b_2\alpha_2}}\in\phi^{\otimes m}(U(W(n))).$$

We can complete the proof of the lemma by using indinduction on k. Q. E. D.

Lemma 2.2. *If $m\leq n$, then the representation of $\mathfrak{E}_m$ on $\Lambda(n, m)$ is faithful, hence we have*

$$\dim\mathscr{E}_m=\dim\mathfrak{E}_m=m^m.$$

Proof. Put

$$\mathcal{N}_k:=\{\varphi|\varphi\in\mathrm{End}[m],\ \#\mathrm{Im}(\varphi)=k\}=\{\varphi_h^k|h=1,2,\cdots,n_k\},$$

where $n_k=\#\mathcal{N}_k$. Then the semigroup ring $\mathfrak{E}_m$ is generated by $\mathcal{N}_k$'s, hence $\mathscr{E}_m=\sum_{k=1}^{m}\langle\mathcal{N}_k\rangle$. Suppose

$$\sum_{k=1}^{m}\sum_{h=1}^{nk}c_h^k\varphi_h^k|_{\Lambda(n,m)}=0\quad(c_h^k\in\mathbf{C},\ \varphi_h^k\in\mathcal{N}_k).$$

We want to prove $c_h^k=0$. By the assumption and the condition $m\leq n$, we have

$$\sum_{k=1}^{m}\sum_{h=1}^{nk}c_h^k\varphi_h^k(\xi_{11}\wedge\xi_{12}\wedge\cdots\wedge\xi_{1m}\wedge\prod_{i=2}^{m}\xi_{ii})=0.$$

Note that $\varphi_h^k\,(1\leq k\leq m-1)$ kills the vector in the left hand side. So we obtain $c_h^m=0$ from the formula

$$\sum_h c_h^m\xi_{1\varphi_h^m(1)}\wedge\xi_{1\varphi_h^m(2)}\wedge\cdots\wedge\xi_{1\varphi_h^m(m)}\wedge\prod_{i=2}^{m}\xi_{i\varphi_h^m(i)}=0.$$

We prove the result by induction on k. Assume that $c_h^t=0\,(t\geq k+1)$. Let us consider the case $t=k$. We have

$$\sum_t\sum_h c_h^t\varphi_h^t(\xi_{11}\wedge\xi_{1i_2}\wedge\cdots\wedge\xi_{1i_k}\wedge\prod_{i=2}^{m}\xi_{ii})=0,$$

where $1<i_2<i_3<\cdots<i_k$. So by the induction hypothesis, we get

$$\sum_h c_h^k\varphi_h^k(\xi_{11}\wedge\xi_{1i_2}\wedge\cdots\wedge\xi_{1i_k}\wedge\prod_{i=2}^{m}\xi_{ii})$$

$$=\sum_h c_h^k\xi_{1\varphi_h^k(1)}\wedge\xi_{1\varphi_h^k(i_2)}\wedge\cdots\xi_{1\varphi_h^k(i_k)}\wedge\prod_{i=2}^{m}\xi_{i\varphi_h^k(i)}=0.$$

For any two elements $\varphi_{h_1}^k$, $\varphi_{h_2}^k$ of $\mathcal{N}_k$, obviously,

$$\xi_{1\varphi_{h_1}^k(1)}\wedge\xi_{1\varphi_{h_1}^k(i_2)}\wedge\cdots\wedge\xi_{1\varphi_{h_1}^k(i_k)}\wedge\prod_{i=2}^{m}\xi_{i\varphi_{h_1}^k(i)}$$

$$=\xi_{1\varphi^k_{h_2}(1)}\wedge\xi_{1\varphi^k_{h_2}(i_2)}\wedge\cdots\wedge\xi_{1\varphi^k_{h_2}(i_k)}\wedge\prod_{i=2}^{m}\xi_{i\varphi^k_{h_2}(2)}\neq 0,$$

if and only if

$$(\varphi^k_{h_1}(1),\ \varphi^k_{h_1}(2),\ \cdots,\ \varphi^k_{h_1}(m))=(\varphi^k_{h_2}(1),\ \varphi^k_{h_2}(2),\ \cdots,\ \varphi^k_{h_2}(m)),\ \text{i.e.,}\ \varphi^k_{h_1}=\varphi^k_{h_2}.$$

By the linear independence of elements

$$\left\{\xi_{1\varphi^k_h(1)}\wedge\xi_{1\varphi^k_h(i_2)}\wedge\cdots\wedge\xi_{1\varphi^k_h(i_k)}\wedge\prod_{i=2}^{m}\xi_{i\varphi^k_h(i)}\ (\neq 0)\,|\,\varphi^k_h\in\mathcal{N}_k\right\},$$

we obtain $c^k_h=0$. By induction, we complete the proof. Q. E. D.

Now we can prove the following

Theorem 2.3. *If $m\leq n$, then the commutant algebra $\mathscr{C}_m$ of $\psi^{\otimes m}(W(n))$ coincides with the representation image $\mathscr{E}_m$ of the semigroup ring $\mathfrak{E}_m$ of the permutation semigroup* End$[m]$*:*

$$\mathscr{C}_m=\mathscr{E}_m.$$

Proof. Take an $E\in\mathscr{C}_m$. For Grassmannian polynomials $P_1,\ \cdots,\ P_m$ in n-variables, put

$$X(P_1,P_2,\cdots,P_m)=\sum_{1\leq\alpha_1,\cdots,\alpha_m\leq m}P_1(\xi(\alpha_1))\cdots P_m(\xi(\alpha_m))\frac{\partial^m}{\partial\xi_{1\alpha_1}\cdots\partial\xi_{m\alpha_m}},$$

which is in $\psi^{\otimes m}(U(W(n)))$ by Lemma 2.1. Then we have

$$E(P_1(\xi(1))P_2(\xi(2))\cdots P_m(\xi(m)))=EX(P_1,P_2,\cdots,P_m)(\xi_{11}\wedge\xi_{22}\wedge\cdots\wedge\xi_{mm})$$
$$=X(P_1,P_2,\cdots,P_m)E(\xi_{11}\wedge\xi_{22}\wedge\cdots\xi_{mm}).$$

Since $\Lambda(n,m)$ is generated by $\{P_1(\xi(1))\cdots P_m(\xi(m))\,|P_i\in\Lambda(n)\}$, E is completely determined by $E(\xi_{11}\wedge\xi_{22}\wedge\cdots\wedge\xi_{mm})$. On the other hand, Euler operators

$$\sum_{\alpha=1}^{m}\xi_{j\alpha}\frac{\partial}{\partial\xi_{j\alpha}}\quad(1\leq j\leq n)$$

are in $\psi^{\otimes m}(W(n))$, and

$$\sum_{\alpha=1}^{m}\xi_{j\alpha}\frac{\partial}{\partial\xi_{j\alpha}}E(\xi_{11}\wedge\cdots\wedge\xi_{mm})=E\Big(\sum_{\alpha=1}^{m}\xi_{j\alpha}\frac{\partial}{\partial\xi_{j\alpha}}(\xi_{11}\wedge\cdots\wedge\xi_{mm})\Big)$$

$$=\begin{cases}E(\xi_{11}\wedge\cdots\wedge\xi_{mm}) & \text{if } 1\leq j\leq m,\\ 0 & \text{if } m<j\leq n.\end{cases}\qquad(2.1)$$

This means that if $1 \leq j \leq m$, then $E(\xi_{11} \wedge \cdots \wedge \xi_{mm})$ is the eigenvector of the Euler operator with eigenvalue 1 and if $m+1 \leq j \leq n$, then $E(\xi_{11} \wedge \cdots \wedge \xi_{mm})$ is in the kernel of the Euler operator. So $E(\xi_{11} \wedge \cdots \wedge \xi_{mm})$ is degree 1 in $(\xi_{j_1}, \cdots, \xi_{j_m})$ if $1 \leq j \leq m$ and degree 0 in $(\xi_{j_1}, \cdots, \xi_{j_m})$ if $m+1 \leq j \leq n$. Hence we obtain

$$E(\xi_{11} \wedge \cdots \wedge \xi_{mm}) = \sum_{1 \leq j_1, \cdots, j_m \leq m} a_{j_1 \cdots j_m} (\xi_{1 j_1} \wedge \cdots \wedge \xi_{m j_m}).$$

So dim $\mathscr{C}_m$ is less than or equal to m^m.

On the other hand, by Lemma 1.2, $\mathscr{C}_m$ contains the subalgebra $\mathscr{E}_m$ and by Lemma 2.2, its dimension is equal to m^m if $m \leq n$. Therefore we conclude the theorem. Q. E. D.

3. A kind of Weyl reciprocity for $W(1) \times \mathrm{End}[m]$

3.1. Commutant algebra of $\psi^{\otimes m}(W(1))$. In this section, we consider the case $n=1$. In this case, we get a stronger result which is independent of m. For $n=1$, it becomes

$$W(1) = \left\langle \frac{\partial}{\partial \xi}, \xi \frac{\partial}{\partial \xi} \right\rangle, \ \deg\left(\frac{\partial}{\partial \xi}\right) = 1, \ \deg\left(\xi \frac{\partial}{\partial \xi}\right) = 0.$$

For convenience, we use isomorphism

$$\Lambda(1, m) := < \xi_1, \xi_2, \cdots, \xi_m > \cong \Lambda(m).$$

So we have

$$D_{-1} := \psi^{\otimes m}\left(\frac{\partial}{\partial \xi}\right) = \sum_{i=1}^{m} \frac{\partial}{\partial \xi_i}, \ D_0 := \psi^{\otimes m}\left(\xi \frac{\partial}{\partial \xi}\right) = \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial \xi_i}.$$

Obviously, $D_{-1}(\Lambda_k) \subseteq \Lambda_{k-1}$, $D_0(\Lambda_k) \subseteq \Lambda_k$ for any k. Further, we have

Lemma 3.1. *The operator* D_{-1} *is an exact derivation, i.e.,* $(\mathrm{D}_{-1})^2 = 0$ *and the chain complex*

$$0 \longrightarrow \Lambda_m \xrightarrow{D_{-1}} \Lambda_{m-1} \xrightarrow{D_{-1}} \cdots \xrightarrow{D_{-1}} \Lambda_2 \xrightarrow{D_{-1}} \Lambda_1 \xrightarrow{D_{-1}} \Lambda_0 \longrightarrow 0$$

is exact.

Proof. First of all, note that $(D_{-1})^2 = 0$. So D_{-1} is a boundary operator. To prove the exactness, we define linear operators P_k: $\Lambda_k \longrightarrow \Lambda_{k+1}$ by

$$P_k(\xi_{i_1} \wedge \cdots \wedge \xi_{i_k}) = \xi_1 \wedge \xi_{i_1} \wedge \cdots \wedge \xi_{i_k},$$

where $0 \leq k \leq m$. We put $P_{-1} = 0$ for convenience. Then

$$\begin{aligned}&(D_{-1}P_k+P_{k-1}D_{-1})(\xi_{i_1}\wedge\cdots\wedge\xi_{i_k})\\&=D_{-1}(\xi_1\wedge\xi_{i_1}\wedge\cdots\wedge\xi_{i_k})+P_{k-1}D_{-1}(\xi_{i_1}\wedge\cdots\wedge\xi_{i_k})\\&=\xi_{i_1}\wedge\cdots\wedge\xi_{i_k}+\sum_{t=1}^{k}(-1)^t\xi_1\wedge\xi_{i_1}\wedge\cdots\wedge\widehat{\xi_{i_t}}\wedge\cdots\wedge\xi_{i_k}\\&+P_{k-1}\Big(\sum_{t=1}^{k}(-1)^{t-1}\xi_{i_1}\wedge\cdots\wedge\widehat{\xi_{i_t}}\wedge\cdots\wedge\xi_{i_k}\Big)\\&=\xi_{i_1}\wedge\cdots\wedge\xi_{i_k}.\end{aligned}$$

So we obtain

$$D_{-1}P_k+P_{k-1}D_{-1}=I\quad(k=0,1,2,\cdots,m).$$

By general theory of algebraic topology, the lemma follows from the above formula (e.g., see [3, p.163]). Q. E. D.

By the above lemmas, we can prove the following

Theorem 3.2. *Let $n=1$ and the notations be as above. Then the commutant algebra $\mathscr{C}_m$ of $\phi^{\otimes m}(W(1))$ coincides with the representation image $\mathscr{E}_m$ of semigroup ring $\mathfrak{S}_m$ of the permutation semigroup* End$[m]$:

$$\mathscr{E}_m=\mathscr{C}_m.$$

Proof. By Lemma 1.2. we have $\mathscr{E}_m\subseteq\mathscr{C}_m$, so it is enough to prove $\mathscr{C}_m\subseteq\mathscr{E}_m$. For this purpose, we introduce some notations. For any $E\in\mathscr{C}_m$, put

$$E_k:=E\,|_{\Lambda_k},$$

$$\mathfrak{J}_k:=\{E\in\mathscr{C}_m|E_l=0(\forall\, l>k)\},$$

$$D_{-1,k}:=D_{-1}|_{\Lambda_k}.$$

Clearly,

$$\mathscr{C}_m=\mathfrak{J}_m\supseteq\cdots\supseteq\mathfrak{J}_1\supseteq\mathfrak{J}_0=(0),$$

and

$$\mathscr{C}_m\simeq(\mathfrak{J}_m/\mathfrak{J}_{m-1})\oplus(\mathfrak{J}_{m-1}/\mathfrak{J}_{m-2})\oplus\cdots\oplus\mathfrak{J}_1\ \text{(as a vector space)}.$$

We divide the proof into 5 steps.

STEP 1. Taking a complementary subspace Λ'_k of $\mathfrak{R}(D_{-1,k+1})$ (here $\mathfrak{R}(D_{-1,k+1})$ is the image space of $D_{-1,k+1}$), we have

$$\Lambda_k=\mathfrak{R}(D_{-1,k+1})\oplus\Lambda'_k. \tag{3.1}$$

We define a linear operator P_k: $\mathfrak{J}_k/\mathfrak{J}_{k-1}\to\mathrm{Hom}_{\mathbb{C}}(\Lambda'_k,\Lambda_k)$ by

$$P_k(\overline{E}):=E_k|_{\Lambda'_k},\ \text{for any}\ \overline{E}\in\mathfrak{J}_k/\mathfrak{J}_{k-1},$$

where $E\in\mathfrak{J}_k$ is a representative of $\overline{E}\in\mathfrak{J}_k/\mathfrak{J}_{k-1}$. This map is well-defined, because if two elements $E, E'\in\mathscr{C}_m$ satisfy $E-E'\in\mathfrak{J}_{k-1}$, then by definition, $(E-E')_k=E_k-E'_k=0$.

STEP 2. We show that P_k is an injection. Assume that $P_k(\overline{E})=0$ for an $\overline{E}\in\mathfrak{J}_k/\mathfrak{J}_{k-1}$, then there exists a representative $\overline{E}\in\mathfrak{J}_k$ such that $E|_{\Lambda'_k}=0$. On the other hand, for any $x\in\mathfrak{R}(D_{-1,k+1})$, there exists $y\in\Lambda_{k+1}$ such that $x=D_{-1}y$, so

$$E_kx=E_kD_{-1}y=D_{-1}E_{k+1}y=0,$$

whence $E_k|_{\Lambda_{k+1}}=0$. Then we see $E_k=0$ in total and $E\in\mathfrak{J}_{k-1}$. Thus we obtain $\overline{E}=0$.

STEP 3. We show P_k is surjective. For any $H\in\mathrm{Homc}(\Lambda'_k,\Lambda_k)$, we define $E\in\mathrm{Homc}(\Lambda'(n),\Lambda(n))$ as follows:

$$\begin{cases} E|_{\Lambda_l}=0 \quad (l\neq k-1, k), \\ E|_{\mathfrak{R}(D_{-1,k+1})}=0,\ E|_{\Lambda'_k}=H, \\ E|_{\mathfrak{R}(D_{-1,k})}=D_{-1,k}H, \quad E|_{\Lambda'_{k-1}}=0 \end{cases}$$

By Lemma 3.1 and the definiton of E, we have $D_0E=ED_0$, $D_{-1}E=ED_{-1}$ and $E\in\mathfrak{J}_k$. Furthermore, let $\overline{E}$ be the image of E in $\mathfrak{J}_k/\mathfrak{J}_{k-1}$, Then

$$P_k(\overline{E})=E_k|_{\Lambda'_k}=H.$$

So P_k is surjective.

So far, we have the following result.

$$\begin{aligned}\mathscr{C}_m &\simeq (\mathfrak{J}_m/\mathfrak{J}_{m-1})\oplus(\mathfrak{J}_{m-1}/\mathfrak{J}_{m-2})\oplus\cdots\oplus\mathfrak{J}_1 \\ &\simeq\mathrm{Homc}(\Lambda'_m,\Lambda_m)\oplus\mathrm{Homc}(\Lambda'_{m-1},\Lambda_{m-1})\oplus\cdots\oplus\mathrm{Homc}(\Lambda'_1,\Lambda_1).\end{aligned}$$

STEP 4. Let us prove that above Λ'_k can be replaced by $\Lambda_{k-1}\wedge\xi_m$, that is,

$$\Lambda_k=\mathfrak{R}(D_{-1,k+1})\oplus(\Lambda_{k-1}\wedge\xi_m), \tag{3.2}$$

where $k=1, 2, \cdots, m$ and $D_{1,m+1}=0$. If $x\wedge\xi_m=D_{-1,k+1}y$ with $x\in\Lambda_{k-1}$, $y\in\Lambda_{k+1}$, then

$$D_{-1,k}(x\wedge\xi_m)=D_{-1,k}D_{-1,k+1}y=0,$$

and

$$D_{-1,k}(x\wedge\xi_m)=(D_{-1,k}\,x)\wedge\xi_m-(-1)^{k-1}x.$$

So it holds that

$$x\wedge\xi_m=((-1)^{k-1}(D_{-1,k}\,x)\wedge\xi_m)\wedge\xi_m=0.$$

This means that

$$\mathfrak{R}(D_{-1,k+1}) \cap (\Lambda_{k-1} \wedge \xi_m) = (0). \tag{3.3}$$

On the other hand, we have

$$\dim \Lambda'_k = \binom{m-1}{m-k}$$

which is proved by induction on k. By a direct calculation, we get

$$\dim\ (\Lambda_{k-1} \wedge \xi_m) = \binom{m-1}{k-1} = \binom{m-1}{m-k}.$$

Thus, from (3.1) and (3.3), we obtain the direct sum relation (3.2).

STEP 5. A basis of the space $\Lambda'_k = \Lambda_{k-1} \wedge \xi_m$ is given by

$$\{\xi_{j_1} \wedge \cdots \wedge \xi_{j_{k-1}} \wedge \xi_m | 1 \leq j_1 < j_2 < \cdots < j_{k-1} < m\}.$$

For any basis element $\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_k}$ of Λ_k, we define $\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k} \in$ End $[m]$ as follows:

$$\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k} = \begin{pmatrix} 1 & 2 & \cdots & j_1 & \cdots & j_{k-1} & \cdots & m \\ i_k & i_k & \cdots & i_1 & \cdots & i_{k-1} & \cdots & i_k \end{pmatrix},$$

i.e., $\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k}(j_l) = i_l\ (1 \leq l \leq k-1)$, $\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k}(j) = i_k\ (j \notin \{j_1, \cdots, j_{k-1}\})$. Then we have

$$\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k}(\xi_{j_1} \wedge \cdots \wedge \xi_{j_{k-1}} \wedge \xi_m) = \xi_{i_1} \wedge \cdots \wedge \xi_{i_{k-1}} \wedge \xi_{i_k},$$

and

$$\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k}(\xi_{s_1} \wedge \cdots \wedge \xi_{s_{k-1}} \wedge \xi_m) = 0 \text{ for } (s_1, \cdots, s_{k-1}) \neq (j_1, \cdots, j_{k-1}).$$

Therefore the set

$$\left\{\varphi^{j_1 \cdots j_{k-1} m}_{i_1 \cdots i_{k-1} i_k} | 1 \leq j_1 < j_2 < \cdots < j_{k-1} \leq m-1,\ 1 \leq i_1 < i_2 < \cdots < i_k \leq m\right\}$$

is a basis of $\mathrm{Hom}_{\mathbf{C}}(\Lambda'_k, \Lambda_k)$. So we get a surjection

$$\mathscr{E}_m \longrightarrow \bigoplus_{k=1}^{m} \mathrm{Hom}_{\mathbf{C}}(\Lambda'_k, \Lambda_k) \simeq \mathbf{C}^m$$

and $\dim \mathscr{C}_m \leq \dim \mathscr{E}_m$. By Lemma 1.2, we have finally $\mathscr{E}_m = \mathscr{C}_m$. Q. E. D.

3.2. The bicommutant algebra.

In the special case where $n=1$, we also get the bicommutant algebra of $\psi^{\otimes m}(W(1))$. The next Theorem 3.3 states that it is the image of the enveloping algebra $\psi^{\otimes m}(U(W(1)))$. Therefore, in this case, we have an analogue of Schur duality for $W(1)\times \mathrm{End}[m]$.

Theorem 3.3. *The bicommutant algebra of m-fold tensor product $\psi^{\otimes m}$ of the natural representation ψ of $W(1)$ is equal to the image $\psi^{\otimes m}(U(W(1)))$ of the enveloping algebra.*

Proof. Let $U(W(1))$ be the universal enveloping algebra of $W(1)$. Then by Poincaré-Birkhoff-Witt theorem, $U(W(1))$ is spanned over $\mathbf{C}$ as

$$U(W(1))=\left\langle\left(\xi\frac{\partial}{\partial\xi}\right)^k,\left(\xi\frac{\partial}{\partial\xi}\right)^k\frac{\partial}{\partial\xi}\middle|k=0,1,\cdots\right\rangle/\mathbf{C}.$$

Denote the bicommutant algebra by $\mathscr{C}'_m$. Obviously $\psi^{\otimes m}(U(W(1)))\subseteq\mathscr{C}'_m$ holds. Put

$$D_0=\psi^{\otimes m}\left(\xi\frac{\partial}{\partial\xi}\right),\ D_{-1}=\psi^{\otimes m}\left(\frac{\partial}{\partial\xi}\right),$$

then, under the representation $\psi^{\otimes m}$,

$$\psi^{\otimes m}(U(W(1)))=\langle I,\ D_0^k,\ D_0^{k-1}D_{-1}|k=1,2,\cdots\rangle/\mathbf{C}.$$

We prove that $\{I, D_0, \cdots, D_0^m, D_{-1}, D_0D_{-1}, \cdots, D_0^{m-1}D_{-1}\}$ are linearly independent, hence $\dim\psi^{\otimes m}(U(W(n)))\geq 2m+1$.

In fact, assume that

$$\sum_{i=0}^{m}k_iD_0^i+\sum_{i=1}^{m}k_{m+i}D_0^{i-1}\mathrm{D}_{-1}=0.$$

then

$$\left(\sum_{i=0}^{m}k_iD_0^i+\sum_{i=1}^{m}k_{m+i}D_0^{i-1}D_{-1}\right)(\xi_1\wedge\cdots\wedge\xi_r)=0.$$

Take $r=0, 1, \cdots, m$, then we get $k_0=0$ and

$$\begin{pmatrix}1&1&\cdots&1\\2&2^2&\cdots&2^m\\\vdots&\vdots&\ddots&\vdots\\m&m^2&\cdots&m^m\end{pmatrix}\begin{pmatrix}k_1\\k_2\\\vdots\\k_m\end{pmatrix}=0.$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (m-1) & \cdots & (m-1)^{m-1} \end{pmatrix} \begin{pmatrix} k_{m+1} \\ k_{m+2} \\ k_{m+3} \\ \vdots \\ k_{2m} \end{pmatrix} = 0.$$

So we have $k_1 = k_2 = \cdots = k_{2m} = 0$.

Second, we prove that

$$\dim \mathscr{C}'_m \leq 2m+1.$$

For this purpose, let Q_k be the projection from $\Lambda(m)$ onto Λ_k, then

$$\mathscr{C}'_m = \bigoplus_{i,j=0}^{m} Q_i \mathscr{C}'_m Q_j.$$

For any $\bar{E} \in \mathscr{C}'_m$, we have $E = \sum_{i,j=1}^{m} E_{ij}$ where $E_{ij} = Q_i E Q_j$, and E_{ij} is essentially a linear operator from Λ_j to Λ_i. For any multi-index $(j_1, \cdots, j_k)$, take a $\varphi \in \mathfrak{S}_m$ such that

$$\varphi(j_t) = t \quad (1 \leq t \leq k).$$

By $E\varphi = \varphi E$, we have

$$E(\xi_{j_1} \wedge \cdots \wedge \xi_{j_k}) = \varphi^{-1} E(\xi_1 \wedge \cdots \wedge \xi_k).$$

So the value $E(\xi_{j_1} \wedge \cdots \wedge \xi_{j_k})$ is determained uniquely by $E(\xi_1 \wedge \cdots \wedge \xi_k)$.

Let us now consider E_{ij}.

1°. THE CASE E_{ij} $(j<i)$. Put

$$E_{ij}(\xi_1 \wedge \cdots \wedge \xi_j) = \sum_{1 \leq t_1 < \cdots < t_i \leq m} c_{t_1 \cdots t_i} \xi_{t_1} \wedge \cdots \wedge \xi_{t_i}.$$

Take a $\varphi \in$ End$[m]$such that $\varphi(k) = k$ $(1 \leq k \leq j)$, $\varphi(k) = j$ $(j < k \leq m)$. Then, because of $E_{ij}\varphi = \varphi E_{ij}$,

$$E_{ij}(\xi_1 \wedge \cdots \wedge \xi_j) = \sum_{1 \leq t_1 < \cdots < t_i \leq m} c_{t_1 \cdots t_i} \xi_{\varphi(t_1)} \wedge \cdots \wedge \xi_{\varphi(t_i)} = 0,$$

whence $E_{ij} = 0$.

2°. THE CASE E_{ij} $(j \geq i+2)$. Put

$$E_{ij}(\xi_1 \wedge \cdots \wedge \xi_j) = \sum_{1 \leq t_1 < \cdots < t_i \leq m} c_{t_1 \cdots t_i} \xi_{t_1} \wedge \cdots \wedge \xi_{t_i}.$$

For any $1 \leq k_1 < k_2 < \cdots < k_i \leq m$, $1 \leq k \leq m$, and $k \notin \{k_1, \cdots, k_i\}$, take $\varphi \in$ End$[m]$ such that $\varphi(k_l) = k_l$ $(1 \leq l \leq i)$, $\varphi(t) = k$ $(t \notin \{k_1, \cdots, k_i\})$. Then $\varphi(\xi_1 \wedge \cdots \wedge \xi_j) = 0$, and it holds that

$$\begin{aligned} 0 = E_{ij}\varphi(\xi_1 \wedge \cdots \wedge \xi_j) &= \varphi E_{ij}(\xi_1 \wedge \cdots \wedge \xi_j) \\ &= \sum_{1 \leq t_1 < \cdots < t_i \leq m} c_{t_1 \cdots t_i} \xi_{\varphi(t_1)} \wedge \cdots \wedge \xi_{\varphi(t_i)}. \end{aligned}$$

Therefore we obtain $c_{k_1\cdots k_i}=0$, whence $E_{ij}=0$.

According to the above facts, an operator $E\in \mathscr{C}'_m$ is determined completely by the values

$$E_{0,0}(1),\ E_{1,1}(\xi_1),\ \cdots,\ E_{m,m}(\xi_1\wedge\cdots\wedge\xi_m),$$

and

$$E_{1,0}(\xi_1),\ E_{2,1}(\xi_1\wedge\xi_2),\ \cdots,\ E_{m,m-1}(\xi_1\wedge\cdots\wedge\xi_m).$$

Further, if we take

$$\varphi=\begin{pmatrix}1 & 2 & \cdots & r & r+1 & \cdots & j & \cdots & m\\ 1 & 2 & \cdots & r & r & \cdots & r & \cdots & r\end{pmatrix},$$

then φ maps $\Lambda_r(m)$ into the one-dimensional subspace $\mathbf{C}\ \xi_1\wedge\cdots\wedge\xi_r$. So by $E\varphi=\varphi E$, we get

$$E_{r,r}(\xi_1\wedge\cdots\wedge\xi_r)\in\mathbf{C}\xi_1\wedge\cdots\wedge\xi_r,$$

$$E_{r,r-1}(\xi_1\wedge\cdots\wedge\xi_{r-1})\in\mathbf{C}\xi_1\wedge\cdots\wedge\xi_r.$$

Therefore, it holds that

$$\dim \mathscr{C}'_m\leq 2m+1.$$

In conclusion, we have $\dim \mathscr{C}'_m=2m+1$ and $\mathscr{C}'_m=\psi^{\otimes m}(U(W(1)))$. This completes the proof of the theorem. Q. E. D.

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