J. Math. Kyoto Univ. (JMKYAZ)
36-1 (1996) 61-72

Nonexistence of twisted Hecke algebras

By

Akihiko GYOJA

0. Introduction.

Let (W, S) be a Coxeter system with finite $S \neq \phi$ (cf. [1]). The group ring $\mathbf{C}[W]$ can be deformed in two ways.

(0.1) The twisted group ring. Let $\mathbf{C}[W]'$ be the vector space with the basis $\{e'_w\}_{w\in W}$. Assume that $\mathbf{C}[W]'$ has an associative $\mathbf{C}$-algebra structure, and that

$$e'_x e'_y \in \mathbf{C}^\times e'_{xy}$$

$$e'_x e'_1 = e'_1 e'_x = e'_x$$

for all $x, y \in W$. If we express $e'_x e'_y = c_{x,y} e'_{xy}$ with $c_{x,y} \in \mathbf{C}^\times$, then $c := \{c_{x,y}\}_{x,y\in W}$ becomes a 2-cocycle in $H^2(W, \mathbf{C}^\times)$. We say that the $\mathbf{C}$-algebra $\mathbf{C}[W]'$ is *obtained by twisting the group ring* $\mathbf{C}[W]$ *by the cocycle c.*

(0.2) The q-deformation of the group ring. (The Iwahori-Hecke algebra.) Let $q = \{q_w\}_{w\in W}$ be a family of non-zero complex numbers such that

(0.2.1) $$q_x q_y = q_{xy} \text{ if } l(x) + l(y) = l(xy),$$

where $l(w)$ is the length of $w \in W$. Let $H(q, W)$ be the vector space with the basis $\{T_w\}_{w\in W}$. Then there is a unique associative $\mathbf{C}$-algebra structure in $H(q, W)$ such that

(0.2.2) $$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w \\ q_s T_{sw} + (q_s - 1) T_w & \text{if } sw < w, \end{cases}$$

where $\leq$ is the Bruhat order. This $\mathbf{C}$-algebra $H(q, W)$ is called the Iwahori-Hecke algebra (cf. [1, Chap.4, §2, Ex. 23]), which we shall regard as a *q-deformation of the group ring* $\mathbf{C}[W]$.

The purpose of this note is to show that, in a sense, 'the q-deformation of the twisted group ring' does not exist.

Let us explain our result more precisely. Let notation be as above, and H a vector space over $\mathbf{C}$ with the basis $\{e_w\}_{w\in W}$.

Received February 3, 1995

Theorem. *Assume that H has an associative $\mathbf{C}$-algebra structure such that*

$$e_s e_w \in \mathbf{C}^\times e_{sw} \ (sw > w),$$

$$e_s e_w \in \mathbf{C}^\times e_{sw} + \mathbf{C}^\times e_w \ (sw < w),$$

$$e_1 e_w = e_w e_1 = e_w,$$

for $s \in S$ and $w \in W$. Then there exists $\{q_w\}_{w \in W}$ satisfying (0.2.1) *and such that, if we put $T_w := b_w e_w$ with suitable $\{b_w\}_w \in (\mathbf{C}^\times)^W$, then $\{q_w\}_w$ and $\{T_w\}_w$ satisfy* (0.2.2).

Let us explain our motivation. In [4, p.109, *l*.5], T. A. Springer conjectured that the algebra of self-intertwining operators (= the Hecke algebra) of a representation of a finite Chevalley group with connected center induced from a cuspidal representation of a parabolic subgroup is isomorphic to the twisted group ring of a certain subgroup $W(\omega)$ of the Weyl group with a 2-cocycle $\gamma(\omega)$ in $H^2(W(\omega), \mathbf{C}^\times)$. This conjecture was proved affirmatively by R. Howlett and G. Lehrer [2] (cf. [3]) and $\gamma(\omega)$ turns out to be always the trivial cohomology class. The motivation of the present work is to understand why a twisted version of the Hecke algebra does not appear.

Before concluding the introduction, it would be worth mentioning the recent work of S. Ariki [6], where he considers for an irreducible finite Coxeter group W, *when* $\overline{W} \longrightarrow W/$(center of W) $=: \overline{W}$ *is the representation group of* $\overline{W}$, *i.e., the universal central extension of* $\overline{W}$, and he shows that this is the case if and only if W is of type E_8, H_4 or $I_2^{(4m)}$. Then he constructs 'a q-deformation of the twisted group ring of $\overline{W}$'.

1. Let H be a vector space over $\mathbf{C}$ with a basis $\{e_w\}_{w \in W}$. Define linear endomorphisms p_s, q_t $(s, t \in S)$ of H as follows:

$$p_s(e_w) = \begin{cases} c_{s,w} e_{sw} & (sw > w) \\ c_{s,w} e_{sw} + d_s e_w & (sw < w), \end{cases} \tag{1.1}$$

$$q_t(e_w) = \begin{cases} c_{w,t} e_{wt} & (wt > w) \\ c_{w,t} e_{wt} + d_t e_w & (wt < w), \end{cases} \tag{1.2}$$

where $s, t \in S$, $w \in W$, $c_{s,w}$, $c_{w,t} \in \mathbf{C}^\times$, $d_s \in \mathbf{C}^\times$. (The same $c_{s,t}$ $(s, t \in S)$ appear both in (1.1) and (1.2).)

Lemma. The following three conditions are equivalent:

(1.3)
$$p_s q_t = q_t p_s,$$
$$p_s(e_1) = q_s(e_1) = e_s \qquad (s, t \in S).$$

(1.4)
$$c_{w,t} c_{s,wt} = c_{sw,t} c_{s,w},$$
$$c_{1,s} = c_{s,1} = 1 \qquad (s, t \in S, w \in W),$$
$$d_s c_{w,t} = c_{s,w} d_t, \text{ if } sw = wt.$$

(1.5) There is an associative algebra structure in H such that

$$p_s(e_w) = e_s e_w,\ q_t(e_w) = e_w e_t,$$
$$e_1 e_w = e_w e_1 = e_w \quad (s,\ t \in S,\ w \in W).$$

Proof. The proof of (1.3)$\Leftrightarrow$(1.4) can be easily done by using the following fact: *If $s,\ t \in S,\ w \in W,\ l(sw) = l(wt)$ and $l(swt) = l(w)$, then $sw = wt$.* Since (1.5) $\Rightarrow$ (1.3) is trivial, let us prove (1.3) $\Rightarrow$ (1.5). Let $w = s_l \cdots s_2 s_1$ ($s_i \in S$) be a reduced decomposition of w ($\in W$). Let

$$p_w = c_{s_2,s_1}^{-1}\, c_{s_3,s_2s_1}^{-1} \cdots c_{s_l,s_{l-1}\cdots s_1}^{-1}\, p_{s_l} \cdots p_{s_1}$$

and

$$q_w = c_{s_l,s_{l-1}}^{-1}\, c_{s_ls_{l-1},s_{l-2}}^{-1} \cdots c_{s_l\cdots s_2,s_1}^{-1}\, q_{s_1} \cdots q_{s_l}.$$

Then, by (1.3), we have

$$p_x(e_1) = q_x(e_1) = e_x,$$
$$p_x(e_y) = p_x q_y(e_1) = q_y p_x(e_1) = q_y(e_x) \qquad (x,\ y \in W).$$

Since $p_x(e_y)$ does not depend on the choice of the reduced decomposition of y, q_y is well defined. In the same way, we can show that p_x is well-defined. Let us define a multiplication by $e_x e_y = p_x(e_y)$. Then, as we have shown above,

$$q_y(e_x) = p_x(e_y) = e_x e_y.$$

Hence, noting $p_x q_z = q_z p_x$, we get

$$e_x(e_y e_z) = p_x q_z(e_y) = q_z p_x(e_y) = (e_x e_y) e_z.$$

Thus we have defined a desired $\mathbf{C}$-algebra structure.

2. Let H be as in §1 and $\{c_{s,w}\} \in (\mathbf{C}^\times)^{S\times W}$, $\{d_{s,w}\} \in \mathbf{C}^{S\times W}$.

Lemma. *The following conditions are equivalent:*

(2.1) *There is an associative $\mathbf{C}$-algebra structure in H such that*

$$e_s e_w = \begin{cases} c_{s,w}\, e_{sw} & (sw > w) \\ c_{s,w}\, e_{sw} + d_{s,w} e_w & (sw < w), \end{cases}$$
$$e_1 e_w = e_w e_1 = e_w.$$

(2.2) (*a*) $d_{s,w} = d_{s,s}$, *which we shall denote by* d_s.

(*b*) $\{c_{s,w}\}$ *can be extended to a 2-cocycle* $\{c_{x,y}\}_{x,y\in W}$ *of* $H^2(W, \mathbf{C}^\times)$, *i.e.*, $c_{y,z}\, c_{xy,z}^{-1} c_{x,yz}\, c_{x,y}^{-1} = 1$ $(x,\ y,\ z \in W)$.
(*Such an extension is unique. See* [5; Chap. 2, §§7-9] *for group cohomologies.*)

(*c*) $d_s c_{w,t} = c_{s,w} d_t$, *if* $sw = wt$.

(*d*) $c_{1,s}=c_{s,1}=1$.

Here $s, t \in S$, $w \in W$.

Proof. First, let us prove (2.1) $\Rightarrow$ (2.2). If $sw > w$, then

$$\begin{aligned} e_s(e_s e_w) &= c_{s,w} e_s e_{sw} \\ &= c_{s,w} c_{s,sw} e_w + c_{s,w} d_{s,sw} e_{sw}, \\ (e_s e_s) e_w &= (c_{s,s} e_1 + d_{s,s} e_s) e_w \\ &= c_{s,s} e_w + d_{s,s} c_{s,w} e_{sw}. \end{aligned}$$

Hence $d_{s,sw}=d_{s,s}$ and we get (a). Let $w=s_l \cdots s_2 s_1$ ($s_i \in S$) be a reduced decomposition of $w (\in W)$. Then

(2.3) $$e_w e_t = (c^{-1}_{s_2,s_1} \cdots c^{-1}_{s_l,s_{l-1}\cdots s_1})(c_{s_1,t} c_{s_2,s_1 t} \cdots c_{s_l,s_{l-1}\cdots s_1 t}) e_{wt}, \quad \text{if } wt > w$$

(2.4) $$e_w e_t = (c^{-1}_{s_2,s_1} \cdots c^{-1}_{s_l,s_{l-1}\cdots s_1} c_{t,t} c_{s_3,s_2} \cdots c_{s_l,s_{l-1}\cdots s_2}) e_{wt} + d_t e_w, \quad \text{if } t = s_1.$$

Write (2.3) and (2.4) as

$$e_w e_t = \begin{cases} c_{w,t} e_{wt} & (wt > w) \\ c_{w,t} e_{wt} + d_t e_w & (wt < w) \end{cases}$$

with suitable $c_{w,t} \in \mathbf{C}^\times$. Define linear endomorphisms p_s and q_t by $p_s(e_w) = e_s e_w$ and $q_t(e_w) = e_w e_t$. Then p_s and q_t are of the forms (1.1) and (1.2), and satisfy the condition (1.3). Hence $\{c_{s,w}\}$, $\{c_{w,t}\}$ and $\{d_s\}$ satisfy the condition (1.4). In particular, we get (c) and (d). Note that, even if we replace d_s by 0, (1.4) remains valid. Hence, by the implication (1.4) $\Rightarrow$ (1.5) for $d_s=0$, we get an associative $\mathbf{C}$-algebra structure in H such that

$$e_s * e_w = c_{s,w} e_{sw}, \quad e_1 * e_w = e_w * e_1 = e_w.$$

Then

$$e_x * e_y = c_{x,y} e_{xy} \quad (x, y \in W)$$

with some $\{c_{x,y}\} \in (\mathbf{C}^\times)^{W\times W}$. As is easily seen $\{c_{x,y}\} \in H^2(W, \mathbf{C}^\times)$. The uniqueness of $c_{x,y}$ can be proved by an induction on $l(x)$, using the cocycle condition on $\{c_{x,y}\}$. Thus we get (b). The implication (2.2) $\Rightarrow$ (2.1) is a direct consequence of Lemma in §1.

3. Let $\{c_{x,y}\}$ be a cocycle in $H^2(W, \mathbf{C}^\times)$ normalizd so that $c_{w,1}=c_{1,w}=1$. (The cocycle appeared in (2.2) satisfies this condition.)

Lemma. *The following conditions are equivalent:*

(3.1) *There is* $\{d_s\} \in (\mathbf{C}^\times)^S$ *such that* $d_s c_{w,t} = c_{s,w} d_t$, *if* $sw = wt$.

(3.2) $c_{s,w}=c_{w,s}$, *if* $sw=ws$.

Here $s, t \in S$ *and* $w \in W$.
A 2-cocycle which is normalized as above and satisfies these equivalent conditions is said to be *admissible*, provisionally in this paper. The admissibility depends only on the cohomology classes.

Proof. Since (3.1) ⇒ (3.2) is trivial, let us prove (3.2) ⇒ (3.1).

First, assume that $sz=zt$, $sw=ws$, $s, t \in S$ and $w, z \in W$. Then

$$c_{w,z}c_{sw,z}^{-1}c_{s,wz}c_{s,w}^{-1}=1,$$

$$c_{s,z}^{-1}c_{ws,z}c_{w,sz}^{-1}c_{w,s}=1,$$

$$c_{z,t}c_{wz,t}^{-1}c_{w,zt}c_{w,z}^{-1}=1.$$

Multiplying these three equalities, we get

$$c_{s,z}/c_{z,t}=c_{s,wz}/c_{wz,t}.$$

This implies that, if $sw=wt$, $c_{s,w}/c_{w,t}$ does not depend on w, which we shall denote by $a_{s,t}$. Note that $a_{s,t}$ $(s, t \in S)$ is defined if and only if s and t are conjugate in W.

Next, assume that $sw=wt$, $tz=zu$, $s, t, u \in S$ and $w, z \in W$. Then

$$c_{w,z}c_{sw,z}^{-1}c_{s,wz}c_{s,w}^{-1}=1,$$

$$c_{t,z}^{-1}c_{wt,z}c_{w,tz}^{-1}c_{w,t}=1,$$

$$c_{z,u}c_{wz,u}^{-1}c_{w,zu}c_{w,z}^{-1}=1.$$

Multiplying these three equalities, we get

$$(c_{s,w}/c_{w,t})\,(c_{t,z}/c_{z,u})=c_{s,wz}/c_{wz,u}.$$

Hence $a_{s,t}a_{t,u}=a_{s,u}$. Let $\{S_i\}$ be the W-conjugacy classes of S and fix a representative u_i for each S_i. Let $d_s=a_{s,u_i}$ if $s \in S_i$. Then $a_{s,t}=d_s/d_t$. Hence

$$c_{s,w}/c_{w,t}=d_s/d_t \quad \text{if} \quad sw=wt.$$

4. Now, our task is to prove that an admissible cocycle is a coboundary, i.e., there exists $\{b_x\}_{x\in W} \in (\mathbf{C}^\times)^W$ such that

(4.1) $$c_{x,y}=b_y b_{xy}^{-1} b_x.$$

In fact, if (4.1) holds, the multiplication law as in (2.1) can be written as

$$e_s e_w=\begin{cases} b_w b_{sw}^{-1} b_s e_{sw} & (sw>w) \\ b_w b_{sw}^{-1} b_s e_{sw}+d_s e_w & (sw<w). \end{cases}$$

If we put $f_w=b_w^{-1}e_w$, then

(4.2) $$f_s f_w=\begin{cases} f_{sw} & (sw>w) \\ f_{sw}+b_s^{-1}d_s f_w & (sw<w), \end{cases}$$

and $f_1f_w = f_wf_1 = f_w$ ($s \in S$, $w \in W$). (Note that an admissible cocycle is assumed to be normalized as in §3, and that $b_1 = 1$.) Let $\{r_s\}_{s\in S}$ be an element of $(\mathbf{C}^\times)^S$ such that

(4.3) $$r_s = r_t, \text{ if } s \text{ and } t \text{ are conjugate in } W,$$

(4.4) $$b_s^{-1}d_s\, r_s = r_s^2 - 1.$$

(Note that from (3.1) follows $b_s^{-1}d_s = b_t^{-1}d_t$ if $sw = wt$.) Let $\{r_w\}_{w\in W}$ be an element of $(\mathbf{C}^\times)^W$ which is an extension of $\{r_s\}_{s\in S}$ and satisfies

(4.5) $$r_xr_y = r_{xy}, \text{ if } l(xy) = l(x) + l(y).$$

By (4.3), such an extension (uniquely) exists. Let $T_w = r_wf_w$ and $q_w = r_w^2$. Then by (4.2), (4.4) and (4.5), we get

$$T_sT_w = \begin{cases} T_{sw} & (sw > w) \\ q_sT_{sw} + (q_s - 1)\, T_w & (sw < w), \end{cases}$$

and

$$q_xq_y = q_{xy}, \text{ if } l(xy) = l(x) + l(y).$$

The remainder of this paper is devoted to prove that an admissible cocycle is a coboundary.

5. In this section, we give some preliminaries on group cohomologies. See [5; Chap. 2, §9].

Let W be a group and present it as a quotient F/K of a free group F. Let $\bar{K} = K/[K, F]$ and $\bar{F} = F/[K, F]$, where $[K, F]$ is the group generated by the commutators $[k, f] = kfk^{-1}f^{-1}$ ($k \in K$, $f \in F$). Then

$$1 \longrightarrow \bar{K} \longrightarrow \bar{F} \longrightarrow W \longrightarrow 1,$$
$$\bar{K} \subset Z(\bar{F}),$$

where $Z(\bar{F})$ is the center of $\bar{F}$. For each $x \in W$, let us fix an element $\bar{f}_x$ of $\bar{F}$ such that $\bar{f}_x \longmapsto x$. Then

$$\bar{f}_x\bar{f}_y = \bar{k}_{x,y}\bar{f}_{xy} \quad (x, y \in W)$$

with some $\bar{k}_{x,y} \in \bar{K}$, and

$$\bar{k}_{y,w}\,\bar{k}_{xy,w}^{-1}\,\bar{k}_{x,yw}\,\bar{k}_{x,y}^{-1} = 1 \qquad (x, y, z \in W).$$

We have the following exact sequence:

(5.1) $$0 \to \mathrm{Hom}(W, \mathbf{C}^\times) \to \mathrm{Hom}(\bar{F}, \mathbf{C}^\times) \to \mathrm{Hom}(\bar{K}, \mathbf{C}^\times) \xrightarrow{\gamma} H^2(W, \mathbf{C}^\times) \to 0,$$

where γ is defined by

(5.2) $$\gamma(\phi) = \{\phi(\bar{k}_{x,y})\}.$$

See [5; Chap. 2, (9.5)] for the proof.

For a group G, denote Hom $(G, \mathbf{C}^{\times})$ by $G^{\vee}$. Then (5.1) can be written as follows:

(5.3) $0 \to (W/[W, W])^{\vee} \to (\bar{F}/[\bar{F}, \bar{F}])^{\vee} \xrightarrow{\rho} \bar{K}^{\vee} \xrightarrow{\tau} H^2(W, \mathbf{C}^{\times}) \to 0.$

Since

$$0 \to \bar{K} \cap [\bar{F}, \bar{F}] \to \bar{K} \to \bar{F}/[\bar{F}, \bar{F}]$$

is exact,

$$\text{image } \rho = \{\phi \in \bar{K}^{\vee}; \phi \mid_{\bar{K} \cap [\bar{F}, \bar{F}]} = 1\}.$$

Hence

(5.4) $$H^2(W, \mathbf{C}^{\times}) = \bar{K}^{\vee}/\text{image } \rho = (\bar{K} \cap [\bar{F}, \bar{F}])^{\vee}.$$

Here $\phi \in (\bar{K} \cap [\bar{F}, \bar{F}])^{\vee}$ is identified with $\{\tilde{\phi}(\bar{k}_{x,y})\} \in H^2(W, \mathbf{C}^{\times})$, where $\tilde{\phi} \in \bar{K}^{\vee}$ is any extension of ϕ. Since

$$\bar{K}/(\bar{K} \cap [\bar{F}, \bar{F}]) = \bar{K}[\bar{F}, \bar{F}]/[\bar{F}, \bar{F}] \subset \bar{F}/[\bar{F}, \bar{F}] = F/[F, F],$$

$\bar{K}/(\bar{K} \cap [\bar{F}, \bar{F}])$ is torsion free. Hence

(5.5) $$\text{torsion } (\bar{K}) \subset \bar{K} \cap [\bar{F}, \bar{F}].$$

6. From now on, we assume that W is a Coxeter group, i.e., W is defined by the following presentation: W is generated by a finite set S which satisfies the relations

$$s^2 = 1 \qquad (s \in S)$$
$$(st)^{m(s,t)} = 1 \quad (s, t \in S),$$

where $m(s, t)$ $(s, t \in S)$ are given integers. We assume that $S = \{s_1, \cdots, s_l\}$. In our case, we may take the free group generated by the set $\{\tilde{s} \mid s \in S\}$ as F. (See the preceding section.) Then K is the minimal normal subgroup of F containing

$$\{\tilde{s}^2, (\tilde{s}\tilde{t})^{m(s,t)} \mid s, t \in S\}.$$

Let $\bar{s}$ be the image of $\tilde{s}$ by $F \to \bar{F}$, and take $\bar{f}_x$ so that $\bar{f}_s = \bar{s}$ $(s \in S)$. Then $\bar{K}$ is the subgroup of $\bar{F}$ generated by

$$\{\bar{s}^2, (\bar{s}\bar{t})^{m(s,t)}\}.$$

Remember that $\bar{K}$ is contained in the center of $\bar{F}$. Assume $s, t \in S$ given. First, assume that $m(s, t) = m = 2k$ and let $\bar{w} = \bar{s}\bar{t}\bar{s}\bar{t} \cdots \bar{t}\bar{s}$ ($2k$-1 factors). Then

$$\begin{aligned} \bar{t}^2 &= (\bar{w}\bar{t}\bar{w}^{-1})^2 = (\bar{w}\bar{t}\bar{w}\bar{t}\bar{t}^{-1}\bar{w}^{-2})^2 \\ &= ((\bar{s}\bar{t})^m \bar{t}^{-1} \underbrace{\bar{s}^{-2}\bar{t}^{-2}\bar{s}^{-2} \cdots \bar{t}^{-2}\bar{s}^{-2}}_{2k\text{-}1 \text{ factors}})^2 \\ &= (\bar{s}\bar{t})^{2m} \bar{t}^{-2}((\bar{s}^{-2})^k (\bar{t}^{-2})^{k-1})^2. \end{aligned}$$

Hence

(6.1) $$((\bar{s}\,\bar{t})^{m})^{2}=(\bar{s}^{2})^{m}(\bar{t}^{2})^{m}.$$

Next, assume that $m(s, t)=m=2k+1$ and let $\bar{w}=\bar{t}\,\bar{s}\cdots\bar{s}\,\bar{t}\,\bar{s}$ ($2k$ factors) and $\bar{w}'=\bar{s}\,\bar{t}\cdots\bar{t}\,\bar{s}\,\bar{t}$ ($2k$ factors). Then

$$\begin{aligned}\bar{s}^{2}&=(\bar{w}'\bar{s}\,\bar{w}'^{-1})^{2}=(\bar{w}'\bar{s}\,\bar{w}\,\bar{t}.\ \bar{t}^{-1}\bar{w}^{-1}\bar{w}'^{-1})^{2}\\&=((\bar{s}\,\bar{t})^{m}\bar{t}^{-1}(\bar{s}^{-1}\bar{t}^{-1}\cdots\bar{t}^{-1})\ (\bar{t}^{-1}\cdots\bar{s}^{1}))^{2}\\&=((\bar{s}\,\bar{t})^{m}\bar{t}^{-1}\underbrace{(\bar{s}^{-2}\bar{t}^{-2}\cdots\bar{t}^{-2})}_{2k\text{ factors}})^{2}\\&=(\bar{s}\,\bar{t})^{2m}\bar{t}^{-2}(\bar{s}^{-2})^{2k}(\bar{t}^{-2})^{2k}.\end{aligned}$$

Hence (6.1) holds also in this case. Since $\bar{K}$ is a commutative group generated by

$$\bar{s}^{2}\qquad(s\in S)$$

and

$$(\bar{s}\,\bar{t})^{m(s,t)}\quad(s,t\in S),$$

we have

$$\operatorname{rank}\bar{K}\leqq l$$

by (6.1). On the other hand,

$$\begin{aligned}\bar{K}/(\bar{K}\cap[\bar{F},\bar{F}])&\cong\text{image }(\bar{K}\longrightarrow\bar{F}/[\bar{F},\bar{F}])\\&\cong\text{image }(K\longrightarrow F/[F,F])\\&=<\tilde{\tilde{s}}^{2};\ \tilde{\tilde{s}}^{\,m(s,t)}\tilde{\tilde{t}}^{\,m(s,t)}>\\&=<\tilde{\tilde{s}}^{2};\ \tilde{\tilde{s}}\,\tilde{\tilde{t}}\ (m(s,t)\ odd)>,\end{aligned}$$

where $\langle x\rangle$ is the group generated by X, and $\tilde{\tilde{s}}$ denotes the image of $\tilde{s}$ by $F\rightarrow F/[F,F]$. Hence $\bar{K}/(\bar{K}\cap[\bar{F},\bar{F}])$ is a free $\mathbf{Z}$-module of rank l. Hence

(6.2) $$\text{torsion }\bar{K}=\bar{K}\cap[\bar{F},\bar{F}]$$

and

(6.3) $$\operatorname{rank}\bar{K}=l.$$

(See (5.5). Although the following fact is not used in the sequel, it would be worth noting here: Since $\bar{K}$ is finitely generated, torsion $\bar{K}$ is a finite group. Hence, by (5.4) and (6.2), we have

$$H^{2}(W,\mathbf{C}^{\times})=(\text{torsion }\bar{K})^{\vee}\cong\text{torsion }\bar{K}.)$$

Thus, what we should prove is the following fact:

(6.4) *If* $\{\phi(\bar{k}_{x,y})\}$ *with* $\phi\in\bar{K}^{\vee}$ *is admissible, then*

$$\phi\,|_{\text{torsion }\bar{K}}\equiv 1.$$

(See the end of §4. See also (5.1) and (5.2).)

7. In this section, we shall write the admissibility condition on $\{\phi(\bar{k}_{x,y})\}$ in terms of $\phi(\in \bar{K}^{\vee})$.

Let us apply the argument of §5 to our situation. If $s \in S$, $w \in W$, $sw=ws$, then

$$\bar{f}_s \bar{f}_w = \bar{k}_{s,w} \bar{f}_{sw},$$
$$\bar{f}_w \bar{f}_s = \bar{k}_{w,s} \bar{f}_{ws},$$
$$[\bar{f}_s, \bar{f}_w] = \bar{k}_{s,w} / \bar{k}_{w,s}.$$

Hence $\{\phi(\bar{k}_{x,y})\}$ is admissible if and only if

$$(7.1) \qquad \phi([\bar{f}_s, \bar{f}_w]) = 1 \quad (s \in S,\ w \in W,\ sw=ws).$$

Let

$$\bar{M} = \langle [\bar{f}_s, \bar{f}_w] \mid [s, w]=1 \rangle \qquad (\subset \bar{K}).$$

The condition (7.1) is equivalent to

$$(7.2) \qquad \phi \mid _{\bar{M}} \equiv 1.$$

Let $\bar{\bar{K}} = \bar{K}/\bar{M}$ and $\bar{\bar{F}} = \bar{F}/\bar{M}$. It suffices to prove that

$$(7.3) \qquad \bar{\bar{K}} \cong \mathbf{Z}^l.$$

In fact, by (6.3), (7.3) is equivalent to say that

$$(7.4) \qquad (\text{torsion } \bar{K}) = \bar{M}.$$

Hence (7.2) implies (6.4), which is what we should prove.

8. Denote the image of $\bar{f}_x$, $\bar{k}_{x,y}$ in $\bar{F}/\bar{M}$ by $\bar{\bar{f}}_x$, $\bar{\bar{k}}_{x,y}$, respectively. Note that $\bar{\bar{k}}_{s,w} = \bar{\bar{k}}_{w,s}$ in $\bar{\bar{F}}$, if sw=ws. Hence, by the same argument as in the proof of Lemma in §3 we can find an element $\{\bar{\bar{d}}_s\} \in \bar{\bar{K}}^S$ such that

$$\bar{\bar{d}}_s \bar{\bar{k}}_{w,t} = \bar{\bar{k}}_{s,w} \bar{\bar{d}}_t, \text{ if sw=wt.}$$

Lemma. We have

$$(8.1) \qquad (\bar{\bar{f}}_s \bar{\bar{f}}_t)^{m(s,t)} = (\bar{\bar{f}}_s^2 \bar{\bar{f}}_t^2)^n, \quad \text{if } m(s,t) = 2n,$$

$$(8.2) \qquad (\bar{\bar{f}}_s \bar{\bar{f}}_t)^{m(s,t)} = (\bar{\bar{d}}_s / \bar{\bar{d}}_t)\ (\bar{\bar{f}}_s^2)^n\ (\bar{\bar{f}}_t^2)^{n+1}, \quad \text{if } m(s,t) = 2n+1.$$

Proof. For the sake of brevity, we shall prove (8.1) (resp. (8.2)) assuming $n=2$ (resp. $n=1$). First, let us prove (8.1). Since

$$\bar{\bar{f}}_s \bar{\bar{f}}_t \bar{\bar{f}}_s \bar{\bar{f}}_t = \bar{\bar{k}}_{s,t} \bar{\bar{k}}_{st,s} \bar{\bar{k}}_{sts,t} \bar{\bar{f}}_{stst}$$

$$= \bar{k}_{s,t}\bar{k}_{st,s}\bar{k}_{t,sts}\bar{f}_{tsts}$$
$$= \bar{f}_t\bar{f}_s\bar{f}_t\bar{f}_s,$$

we have

$$(\bar{f}_s\bar{f}_t)^4 = (\bar{f}_s\bar{f}_t\bar{f}_s\bar{f}_t)(\bar{f}_t\bar{f}_s\bar{f}_t\bar{f}_s) = (\bar{f}_s^2\bar{f}_t^2)^2.$$

Next, let us prove (8.2). Since

$$\bar{d}_t\bar{f}_s\bar{f}_t\bar{f}_s = \bar{d}_t\bar{k}_{s,t}\bar{k}_{st,s}\bar{f}_{sts} \tag{8.3}$$
$$= \bar{k}_{s,t}\bar{k}_{t,st}\bar{d}_s\bar{f}_{tst}$$
$$= \bar{d}_s\bar{f}_t\bar{f}_s\bar{f}_t,$$

we have

$$(\bar{f}_s\bar{f}_t)^3 = (\bar{f}_t\bar{f}_s\bar{f}_t\cdot\bar{d}_s/\bar{d}_t)(\bar{f}_t\bar{f}_s\bar{f}_t)$$
$$= (\bar{d}_s/\bar{d}_t)\bar{f}_s^2(\bar{f}_t^2)^2.$$

Lemma. *If $m(s, t)$ is odd, then*

$$\bar{d}_s^2\bar{f}_s^{-2} = \bar{d}_t^2\bar{f}_t^{-2}. \tag{8.4}$$

Proof. For the sake of brevity, we shall prove (8.4) assuming that $m(s, t)=3$. By (8.3), we have

$$\bar{d}_t^2(\bar{f}_s\bar{f}_t\bar{f}_s)(\bar{f}_s\bar{f}_t\bar{f}_s) = \bar{d}_s^2(\bar{f}_t\bar{f}_s\bar{f}_t)(\bar{f}_t\bar{f}_s\bar{f}_t),$$
$$\bar{d}_t^2\bar{f}_s^2\bar{f}_t^2\bar{f}_s^2 = \bar{d}_s^2\bar{f}_t^2\bar{f}_s^2\bar{f}_t^2.$$

Hence we get (8.4).

9. Let $S=\{s_1,\cdots, s_l\}$, $\bar{f}_i=\bar{f}_{s_i}$, $\bar{f}_i=\bar{f}_{s_i}$, $\bar{d}_i=\bar{d}_{s_i}$, $m_{ij}=m(s_i, s_j)$ and $n_{ij}=[m_{ij}/2]$. Define a character $\phi_i \in F^\vee$ by $\phi_i(\tilde{s}_i)=2$ and $\phi_i(\tilde{s}_j)=1$ $(j\neq i)$. Since $F^\vee$, $\bar{F}^\vee$ and $\bar{\bar{F}}^\vee$ are naturally identified with each other, ϕ^i can be regarded as a character of $\bar{F}$. Remember that we are assuming that $\bar{f}_i=\bar{s}_i$.

Lemma. *The following two conditions for $x_i, x_{ij} \in \mathbf{Z}$ are equivalent:*

$$\prod_{i=1}^{l}(\bar{f}_i^2)^{x_i}\prod_{i<j}((\bar{f}_i\bar{f}_j)^{m_{ij}})^{x_{ij}}=1. \tag{9.1}$$

$$2x_i+\sum_{i<j} m_{ij}x_{ij}+\sum_{i>j} m_{ji}x_{ji}=0 \quad (1\leqq i\leqq l). \tag{9.2}$$

(*In* (9.2), *the summations are taken for* j.)

Note that this Lemma implies:

(9.3) $$\prod_{i=1}^{l} (\bar{f}_i^2)^{x_i}=1 \text{ if and only if } x_1=\cdots=x_l=0.$$

(9.4) *If* $\bar{f} :=\prod_{i=1}^{l} (\bar{f}_i^2)^{x_i}\prod_{i<j} ((\bar{f}_i\bar{f}_j)^{m_{ij}})^{x_{ij}} \neq 1$, *then*

$$\bar{f}^2,\ \bar{f}^3,\ \cdots\neq 1.$$

Since (9.3) and (9.4) together with (6.3) implies $\bar{K}\cong \mathbf{Z}^l$, this lemma concludes our proof. By applying ϕ_i to (9.1), we get (9.2). Hence our task is to prove implication (9.2)⇒(9.1).

Proof of (9.2) ⇒ (9.1). Assume that (9.2) holds. By using (8.1) and (8.2), the left hand side of (9.1) can be written as follows:

(9.5)
$$\begin{aligned}
&\prod_{i=1}^{l} (\bar{f}_i^2)^{x_i} \prod_{\substack{i<j\\ m_{ij}\ even}} (\bar{f}_i^2\bar{f}_j^2)^{n_{ij}x_{ij}}\\
&\times \prod_{\substack{i<j\\ m_{ij}\ odd}} ((\bar{d}_i/\bar{d}_j)\ (\bar{f}_i^2)^{n_{ij}}(\bar{f}_j^2)^{n_{ij}+1})^{x_{ij}}\\
&=\prod_{\substack{i<j\\ m_{ij}\ odd}} (\bar{d}_i/\bar{d}_j)^{x_{ij}}\prod_{i=1}^{l} (\bar{f}_i^2)^{y_i},
\end{aligned}$$

where

(9.6)
$$\begin{aligned}
y_i&=x_i+\sum_{\substack{i<j\\ m_{ij}\ even}} n_{ij}x_{ij}+\sum_{\substack{j<i\\ m_{ij}\ even}} n_{ji}x_{ji}\\
&\quad+\sum_{\substack{i<j\\ m_{ij}\ odd}} n_{ij}x_{ij}+\sum_{\substack{j<i\\ m_{ij}\ odd}} (n_{ji}+1)x_{ji}\\
&=\frac{1}{2}(2x_i+\sum_{i<j} m_{ij}x_{ij}+\sum_{i>j} m_{ji}x_{ji})\\
&\quad-\frac{1}{2}(\sum_{\substack{i<j\\ m_{ij}\ odd}} x_{ij}-\sum_{\substack{i>j\\ m_{ij}\ odd}} x_{ji})\\
&=-\frac{1}{2}(\sum_{\substack{i<j\\ m_{ij}\ odd}} x_{ij}-\sum_{\substack{i>j\\ m_{ij}\ odd}} x_{ji}).
\end{aligned}$$

(Here the summations are taken for j.) Hence (9.5) is equal to

(9.7) $$\prod_{i=1}^{l} (\bar{d}_i^2\bar{f}_i^{-2})^{-y_i},$$

where y_i is given by (9.6), and our task is to prove that (9.7) equals 1. For this purpose, it suffices to prove that

(9.8) $$\prod_{s_i\in S'} (\bar{d}_i^2\bar{f}_i^{-2})^{y_i}=1$$

for each W-conjugacy class S' of S. By (8.4), $\bar{d}_i^2 \bar{f}_i^{-2}$ is constant on S'. Hence, by (9.6), we can prove (9.8). Thus we conclude the implication (9.2) $\Rightarrow$ (9.1) and the proof of our main theorem is over.

DEPARTMENT OF FUNDAMENTAL SCIENCES,
FACULTY OF INTEGRATED HUMAN STUDIES,
KYOTO UNIVERSITY

References

[1] N. Bourbaki, Groupes et algèbres de Lie, Chap. IV, V, VI, Masson, 1981.
[2] R. Howlett and G. Lehrer, Induced cuspidal representations and generalized Hecke rings, Invent. Math., **58** (1980), 37-64.
[3] G. Lusztig, Characters of reductive groups over a finite field, Annals of Math. Studies **107** (1984), Princeton.
[4] T. A. Springer, Cusp forms for finite groups ; Seminar on algebraic groups and related topics, Lecture Notes in Math., **131** (1970), Springer.
[5] M. Suzuki, Group theory I, Springer, Berlin-Heidelberg-New York, 1982.
[6] S. Ariki, q-Analogue of a twisted group ring, preprint.