J. Math. Kyoto Univ. (JMKYAZ)
36-1 (1996) 31-43

Normal subgroups and heights of characters

By

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Introduction

Let G be a finite group and p a prime. Suppose we are given an irreducible character χ of G such that χ_N is irreducible for a normal subgroup N of G. Then every irreducible character ζ of G lying over χ_N is written as $\zeta=\chi\theta$ for a unique irreducible character θ of G/N. Let B (resp. $\bar{B}$) be the block of G (resp. G/N) to which ζ (resp. θ) belongs. It is natural to ask how B and $\bar{B}$ are related. If χ is the trivial character then B is just a block which dominates $\bar{B}$ and basic facts, including the relations between defect groups of these blocks, are known (cf. [11, Chapter 5, Sections 8.2 and 8.3]). (We note that we have shown, with no restrictions on N, that there exists a block of G/N dominated by B with defect group DN/N for a defect group D of B, cf. [10, Remark 4.7].) We shall show in Section 1 that, for an arbitrary χ, the situation is quite analogous to that of the usual "domination" above. The same is true when χ is an irreducible Brauer character. Actually the results are obtained in a more general setting, that is, we consider "V-domination" for suitable indecomposable G-modules V.

To explain the results in Section 2 we need to introduce some notation. Let B be a block of G which covers a block b of a normal subgroup N of G. Let ξ be an irreducible character in b. Let $T_G(b)$ be the inertial group of b in G. As in [10] we call a defect group D of B an *inertial defect group* of B if D is a defect group of the Fong-Reynolds correspondent of B in $T_G(b)$. Fix an inertial defect group D of B. Let $\mathrm{Irr}(B|\xi)$ be the set of irreducible characters in B lying over ξ. In Section 2 we shall show that

$$\min\{\mathrm{ht}(\chi)-\mathrm{ht}(\xi)\,|\,\chi\in\mathrm{Irr}(B|\xi)\}$$

is determined by information on DN and the $T_G(b)$-conjugates of ξ. This extends some results in [10]. As an application we shall obtain a result related to the Dade conjecture [3]. We shall also obtain a slight extension of the Gluck-Wolf theorem [5].

In Section 3 we shall give the modular version of the above.

Throughout this paper let (K, R, k) be a p-modular system. We assume that K is sufficiently large with respect to G and that k is algebraically closed.

Communicated by Prof. T. Hirai, October 27, 1994

The maximal ideal of R is denoted by (π). Let ν be the valuation of K normalized so that $\nu(p)=1$.

The author would like to express his thanks to Professor Y. Tsushima for his interest in [10] and to the referee for valuable suggestions.

1. Domination of blocks of a factor group

Throughout this section, N is a normal subgroup of a finite group G and V is an indecomposable $\mathfrak{o}G$-module such that V_N is indecomposable, where $\mathfrak{o}$ denotes R or k. The block of N to which V_N belongs is denoted by b.

We say that a block B of G *dominates a block* $\bar{B}$ *of* G/N *through* V (simply B *V-dominates* $\bar{B}$) if there exists an $\mathfrak{o}[G/N]$-module X in $\bar{B}$ such that $V\otimes \mathrm{Inf}(X)$ $(=V \otimes_{\mathfrak{o}} \mathrm{Inf}(X))$ lies in B, where $\mathrm{Inf}(X)$ denotes the inflation of X to G. So when V is the trivial module, "V-domination" coincides with the usual "domination".

In the following we understand $\pi=0$ when $\mathfrak{o}=k$. All $\mathfrak{o}G$-modules are assumed to be $\mathfrak{o}$-free of finite rank.

The following extends [6, VII. 9.12 (i), (iii)] slightly.

Lemma 1.1 (i) *Let W be an indecomposable $\mathfrak{o}[G/N]$-module. If $\mathfrak{o}=R$, assume that $W/\pi W$ is indecomposable. Then $V\otimes \mathrm{Inf}(W)$ is indecomposable. In particular, $V\otimes \mathrm{Inf}(W)$ is indecomposable for every projective indecomposable module W.*

(ii) *Let W and W' be $\mathfrak{o}[G/N]$-modules. If $V\otimes\mathrm{Inf}(W) \mid V\otimes \mathrm{Inf}(W')$, then $W/\pi W \mid W'/\pi W'$.*

Proof. We shall prove the assertion by mimicking the proof of [6, VII 9.12]. Put $E=\mathrm{End}_{\mathfrak{o}N}(V)$.

(i) Let $\phi\in \mathrm{End}_{\mathfrak{o}G}(V\otimes \mathrm{Inf}(W))$ be an idempotent. Put $m=\mathrm{rank}_{\mathfrak{o}}W$. Let $\{w_i\}$ be an $\mathfrak{o}$-basis of W. Let

$$w_i g=\sum_j a_{ij}(g)w_j,\ a_{ij}\in\mathfrak{o},\ \text{for every } g\in G.$$

Put

$$(v\otimes w_i)\phi=\sum_j v\phi_{ij}\otimes w_j,\ \phi_{ij}\in E.$$

As in [6], we get

$$\sum_j a_{ij}(g)\phi_{jk}=\sum_j \phi_{ij}^g a_{jk}(g),\ \text{for } 1\leqq i,k\leqq m.$$

Since $E=\mathfrak{o}1_V+J(E)$, we may take $\lambda_{ij}\in\mathfrak{o}$, $\rho_{ij}\in J(E)$ so that $\phi_{ij}=\lambda_{ij}1_V+\rho_{ij}$ for $1\leqq i,j\leqq m$. Then we get

$$\sum_j a_{ij}(g)\lambda_{jk}\equiv\sum_j \lambda_{ij}a_{jk}(g) \pmod{\pi},\ \text{for } 1\leqq i,k\leqq m,$$

since $\pi\mathfrak{o}1_V=\mathfrak{o}1_V\cap J(E)$.

Let $\Lambda \in \mathrm{Mat}_m(k)$ be the matrix whose (i, j)-th entry is $\lambda_{ij}+\pi\mathfrak{o}$. Since $W/\pi W$ is indecomposable, the above shows that Λ is the identity matrix or 0. We may assume $\Lambda=0$. So $(\psi_{ij}) \in \mathrm{Mat}_m(J(E))=J(\mathrm{Mat}_m(E))$. Since (ψ_{ij}) is an idempotent, it follows that $(\psi_{ij})=0$ and hence $\psi=0$. This completes the proof.

(ii) Let ϕ: $V\otimes \mathrm{Inf}(W) \to V\otimes \mathrm{Inf}(W')$ and ψ: $V\otimes \mathrm{Inf}(W') \to V\otimes \mathrm{Inf}(W)$ be $\mathfrak{o}G$-homomorphisms such that $\phi\psi$ is the identity map of $V\otimes \mathrm{Inf}(W)$. Let $\{w_i\}$ (resp. $\{w'_s\}$) be an $\mathfrak{o}$-basis of W(resp. W'). We may write

$$(v\otimes w_i)\phi=\sum_s v\phi_{is}\otimes w'_s,\ v\in V,$$

where $\phi_{is}\in E$. Also

$$(v\otimes w'_s)\psi=\sum_i v\psi_{si}\otimes w_i,\ v\in V,$$

where $\psi_{si}\in E$. Then we get

$$\sum_s \phi_{is}\psi_{sj}=\delta_{ij}1_V,$$

where δ_{ij} is the Kronecker delta. Put $\phi_{is}=\lambda_{is}\ 1_V+\rho_{is}$, $\psi_{si}=\mu_{si}\ 1_V+\sigma_{si}$, where λ_{is}, $\mu_{si}\in\mathfrak{o}$, ρ_{is}, $\sigma_{si}\in J(E)$. We get

$$\sum_s \lambda_{is}\mu_{sj}\equiv\delta_{ij}\ (\mathrm{mod}\ \pi),$$

as above. Now define the k-linear map $\bar{\phi}$: $W/\pi W \to W'/\pi W'$ by

$$\bar{w}_i\bar{\phi}=\sum_s \bar{\lambda}_{is}\ \bar{w}'_s,$$

where $\bar{w}_i=w_i+\pi W$, $\bar{w}'_s=w'_s+\pi W'$, and $\bar{\lambda}_{is}=\lambda_{is}+\pi\mathfrak{o}$. Similarly define the k-linear map $\bar{\psi}$: $W'/\pi W' \to W/\pi W$ by

$$\bar{w}'_s\bar{\psi}=\sum_i \bar{\mu}_{si}\ \bar{w}_i.$$

Then clearly $\bar{\phi}\,\bar{\psi}$ is the identity map of $W/\pi W$. On the other hand, if we let

$$w_i g=\sum_j a_{ij}(g)w_j,\ a_{ij}(g)\in\mathfrak{o},\ \text{and}$$

$$w'_s g=\sum_t b_{st}(g)w'_t,\ b_{st}(g)\in\mathfrak{o},\ \text{for every } g\in G,$$

then we get

$$\sum_j a_{ij}(g)\,\phi_{jt}=\sum_s \phi_{is}{}^g\ b_{st}(g).$$

From this we get as above,

$$\sum_j a_{ij}(g)\,\lambda_{jt}\equiv\sum_s \lambda_{is}\ b_{st}(g)\ (\mathrm{mod}\ \pi).$$

This implies that $\bar{\phi}$ is a kG-homomorphism. Similarly $\bar{\psi}$ is a kG-homomorphism. Thus the result follows.

Theorem 1.2. (i) *A block B of G V-dominates a block of G/N if and only if B covers b.*

(ii) *Every block $\overline{B}$ of G/N is V-dominated by a unique block, say B, of G. In that case, for every $\mathfrak{o}[G/N]$-module W in $\overline{B}$, $V \otimes \mathrm{Inf}(W)$ lies in B.*

Proof. (i) if part: If B covers b, then $(V_N)^G$ has an indecomposable summand U lying in B. Since $(V_N)^G \cong V \otimes \mathfrak{o}[G/N]$, we have, by Lemma 1.1 (i), $U \cong V \otimes \mathrm{Inf}(P)$ for some projective indecomposable $\mathfrak{o}[G/N]$-module P. So B V-dominates the block of G/N containing P.

only if part: This is easy to see.

(ii) Let $\overline{B}$ be a block of G/N. Choose a projective indecomposable $\mathfrak{o}[G/N]$-module P in $\overline{B}$, then $V \otimes \mathrm{Inf}(P)$ is indecomposable. Let B be the block of G to which $V \otimes \mathrm{Inf}(P)$ belongs. So $\overline{B}$ is V-dominated by B. To prove the assertion, it suffices to show that for every $\mathfrak{o}[G/N]$-module W in $\overline{B}$, $V \otimes \mathrm{Inf}(W)$ lies in B. Suppose that we are given projective indecomposable $\mathfrak{o}[G/N]$-modules P_1 and P_2 in $\overline{B}$ such that $V \otimes \mathrm{Inf}(P_1)$ lies in B and that there is a non-zero $\mathfrak{o}[G/N]$-homomorphism $f: P_1 \to P_2$. Then $1_V \otimes f: V \otimes \mathrm{Inf}(P_1) \to V \otimes \mathrm{Inf}(P_2)$ is non-zero. Since $V \otimes \mathrm{Inf}(P_2)$ is indecomposable, it follows that $V \otimes \mathrm{Inf}(P_2)$ lies in B. So, since $V \otimes \mathrm{Inf}(P)$ lies in B, the indecomposability of the Cartan matrix of $\overline{B}$ yields that $V \otimes \mathrm{Inf}(Q)$ lies in B for every projective indecomposable module Q in $\overline{B}$. For every $\mathfrak{o}[G/N]$-module W in $\overline{B}$, there is a surjection: $V \otimes \mathrm{Inf}(P_W) \to V \otimes \mathrm{Inf}(W) \to 0$, where P_W is the projective cover of W. Since $V \otimes \mathrm{Inf}(P_W)$ lies in B by the above, so does $V \otimes \mathrm{Inf}(W)$. This completes the proof.

We need the following.

Lemma 1.3 *Let N_1 be a normal subgroup of a group G_1 and let H be a subgroup of G_1 such that $H \geqq N_1$. Let b_1 be a G_1-invariant block of N_1. If B_1 is a block of H for which $B_1^{G_1}$ is defined, then B_1 covers b_1 if and only if $B_1^{G_1}$ covers b_1.*

Proof. There are a kG_1-module X in $B_1^{G_1}$ and a kH-module Y in B_1 such that Y is a direct summand of X_H by [11, Theorem 5.3.10] (see also [10, Corollary 1.7 (i)]). This yields the assertion.

Theorem 1.4. *Let B be a block of G covering b and let D be a defect group of B. Then:*

(i) *For every block $\overline{B}$ of G/N which is V-dominated by B, a defect group of $\overline{B}$ is contained in DN/N.*

(ii) *Furthermore for some block $\overline{B}$ of G/N which is V-dominated by B, DN/N is a defect group of $\overline{B}$.*

Proof. (i) If $\mathfrak{o}=k$, let W be an irreducible module in $\overline{B}$ of height 0. If $\mathfrak{o}=R$, let W be an R-form of an irreducible $K[G/N]$-module in $\overline{B}$ of height 0 such that $W/\pi W$ is indecomposable, cf. [4, I. 17.12] for the existence of such a W. Then $V \otimes \mathrm{Inf}(W)$ is indecomposable by Lemma 1.1 (i) and lies in B by Theorem 1.2 (ii). Let Q be a vertex of $V \otimes \mathrm{Inf}(W)$. Since $V \otimes \mathrm{Inf}(W)$ is QN-projective,

$$V \otimes \mathrm{Inf}(W) \mid ((V \otimes \mathrm{Inf}(W))_{QN})^G \cong V \otimes ((\mathrm{Inf}(W))_{QN})^G.$$

Clearly $((\mathrm{Inf}(W))_{QN})^G \cong \mathrm{Inf}\{(W_{QN/N})^{G/N}\}$. Hence $W/\pi W$ is a summand of $(W_{QN/N})^{G/N}/\pi(W_{QN/N})^{G/N}$ by Lemma 1.1. By the choice of W and Green's theorem, $W/\pi W$ is an indecomposable module whose vertex is a defect group of $\bar{B}$. Since $(W_{QN/N})^{G/N}/\pi(W_{QN/N})^{G/N}$ is QN/N-projective and Q is contained in a defect group of B, the result follows.

(ii) Put $H = N_G(D)N$ and let $\tilde{B}$ be the unique block of H with defect group D such that $\tilde{B}^G = B$. Since V_N lies in b, b is G-invariant. So $\tilde{B}$ covers b by Lemma 1.3. Hence by Theorem 1.2 (i) there is a block B_1 of H/N which is V_H-dominated by $\tilde{B}$. Since DN/N is normal in H/N, it follows from (i) that DN/N is a defect group of B_1. Here we note that $H = N_G(DN)$, i.e. $H/N = N_{G/N}(DN/N)$. In fact, since b is G-invariant, if $\hat{b}$ is a unique block of DN that covers b, then D is a defect group of $\hat{b}$ by [10, Lemma 2.2] and $\hat{b}$ is $N_G(DN)$-invariant. Hence the "Frattini argument" shows that $H = N_G(DN)$. Thus if we put $\bar{B} = B_1{}^{G/N}$, then $\bar{B}$ has defect group DN/N by the First Main Theorem. So it suffices to prove that $\bar{B}$ is V-dominated by B.

Let W be a module chosen as in the proof of (i) for B_1. Then $V_H \otimes \mathrm{Inf}(W)$ is an indecomposable module in $\tilde{B}$ as above. (Here $\mathrm{Inf}(W)$ is the inflation of W to H.) By the proof of (i) we see there is a vertex Q of $V_H \otimes \mathrm{Inf}(W)$ such that $QN = DN$. Now let U be the Green correspondent of W with respect to $(G/N, DN/N, H/N)$. (Note that DN/N is a vertex of W.) Then U lies in $\bar{B}$ by the Nagao-Green theorem [11, Theorem 5.3.12]. Clearly $V_H \otimes \mathrm{Inf}(W) \mid (V \otimes \mathrm{Inf}(U))_H$, so there is an indecomposable summand X of $V \otimes \mathrm{Inf}(U)$ such that $V_H \otimes \mathrm{Inf}(W) \mid X_H$. Since $C_G(Q) \leqq N_G(QN) = N_G(DN) = H$, X belongs to $\tilde{B}^G = B$ by the Nagao-Green theorem again. Then $V \otimes \mathrm{Inf}(U)$ lies in B by Theorem 1.2 (ii). So $\bar{B}$ is V-dominated by B. This completes the proof.

Let χ (resp. ϕ) be an irreducible character (resp. irreducible Brauer character) of G such that χ_N (resp. ϕ_N) is irreducible. We say that a block of B of G χ-dominates (resp. ϕ-dominates) a block $\bar{B}$ of G/N, if $\chi \otimes \zeta$ (resp. $\phi \otimes \psi$) lies in B for an irreducible character ζ (resp. an irreducible Brauer character ψ) in $\bar{B}$. (In this paper we write $\chi \otimes \zeta$ (or $\phi \otimes \psi$) instead of $\chi\zeta$ (or $\phi\psi$) to avoid unnecessary confusions.)

Corollary 1.5 *Let χ and ϕ be as above. For every block B of G, let* $\mathrm{Bl}(B, \chi)$ (*resp.* $\mathrm{Bl}(B, \phi)$) *be the set of blocks of G/N which are χ-dominated (resp. ϕ-dominated) by B.*

(i) (i. a) $\mathrm{Bl}(B, \chi) \neq \emptyset$ *if and only if B covers the block of N to which χ_N*

belongs.

(i. b) *Assume* $\mathrm{Bl}(B, \chi) \neq \emptyset$. *Let D be a defect group of B. Then for every block* $\bar{B} \in \mathrm{Bl}(B, \chi)$, *a defect group of* $\bar{B}$ *is contained in* DN/N. *Furthermore there is a block* $\bar{B} \in \mathrm{Bl}(B, \chi)$ *such that* DN/N *is a defect group of* $\bar{B}$.

(i. c) *Every block* $\bar{B}$ *of* G/N *is* χ-*dominated by a unique block, say B, of G. In that case, for every* $\theta \in \mathrm{Irr}(\bar{B})$, $\chi \otimes \theta \in \mathrm{Irr}(B)$.

(ii) (ii. a) $\mathrm{Bl}(B, \phi) \neq \emptyset$ *if and only if B covers the block of N to which* ϕ_N *belongs.*

(ii. b) *Assume* $\mathrm{Bl}(B, \phi) \neq \emptyset$. *Let D be a defect group of B. Then for every block* $\bar{B} \in \mathrm{Bl}(B, \phi)$, *a defect group of* $\bar{B}$ *is contained in* DN/N. *Furthermore there is a block* $\bar{B} \in \mathrm{Bl}(B, \phi)$ *such that* DN/N *is a defect group of* $\bar{B}$.

(ii. c) *Every block* $\bar{B}$ *of* G/N *is* ϕ-*dominated by a unique block, say B, of G. In that case, for every* $\theta \in \mathrm{IBr}(\bar{B})$, $\phi \otimes \theta \in \mathrm{IBr}(B)$.

Proof. (i) Let V be an R-form of a KG-module affording χ. Then clearly a block $\bar{B}$ of G/N is χ-dominated by B if and only if $\bar{B}$ is V-dominated by B.

(i. a) By the above, the assertion follows from Theorem 1.2(i).

(i. b) Similarly this follows from Theorem 1.4.

(i. c) As is well-known, $\chi \otimes \theta$ is irreducible for every $\theta \in \mathrm{Irr}(G/N)$. Theorem 1.2 (ii) yields that $\{\chi \otimes \theta | \theta \in \mathrm{Irr}(\bar{B})\}$ is contained in a single block of G.

The proof of (ii) is similar.

2. Normal subgroups and heights of irreducible characters

For an irreducible character χ lying in a (p-) block B of a group G, let θ_χ be the class function on G defined by

$$\begin{aligned} \theta_\chi(x) &= p^{\mathrm{d}(B)}\chi(x) && \text{if } x \text{ is } p\text{-regular,} \\ &= 0 && \text{otherwise,} \end{aligned}$$

where $\mathrm{d}(B)$ is the defect of B.

Lemma 2.1 *Let B be a block of G. Let b be a block of a subgroup H of G such that* $b^G = B$ *and that* $\mathrm{d}(b) = \mathrm{d}(B)$. *Let* ζ *be an irreducible character of height 0 in b. Then for every* $\chi \in \mathrm{Irr}(B)$, *we have*:

(i) $\nu((\chi_H, \theta_\zeta)_H) = \mathrm{ht}(\chi)$.

(ii) *There is a constituent* $\eta \in \mathrm{Irr}(b)$ *of* χ_H *with* $\mathrm{ht}(\eta) \leqq \mathrm{ht}(\chi)$.

Proof. (i) By Frobenius reciprocity $(\chi_H, \theta_\zeta)_H = (\theta_\chi, \zeta^G)_G$. As in [10, Section 1], let $(\zeta^G)^* = \sum \zeta^G(x^{-1})x$, where x runs through the p-regular elements of G. Then $(\theta_\chi, \zeta^G)_G |G| / (p^{\mathrm{d}(B)}\chi(1)) = \omega_\chi((\zeta^G)^*)$, where ω_χ is the central character corresponding to χ. Since B-component of ζ^G is of height 0 [10, Proposition 1.8(ii)], the result follows from [10, Theorem 1.3].

(ii) This follows from (i), cf. [1].

In the rest of this section we use the following notation:

N is a normal subgroup of a group G, ξ is an irreducible character of N, b is the block of N to which ξ belongs, and B is a block of G covering b.

Let $T_G(\xi)$ be the inertial group of ξ in G. Let $\mathrm{Irr}(B|\xi)$ be the set of irreducible characters in B lying over ξ, that is,

$$\mathrm{Irr}(B|\xi)=\{\chi\in\mathrm{Irr}(B)\,|\,(\chi_N,\ \xi)_N\neq 0\}\,.$$

Let $T_G(b)$ be the inertial group of b in G.

The following generalizes Corollary 4.2 (i) in [10].

Lemma 2.2. *For every* $\chi\in\mathrm{Irr}(B|\xi)$, *we have* $\mathrm{ht}(\chi)\geqq\mathrm{ht}(\xi)$.

Proof. Let $\chi\in\mathrm{Irr}(B|\xi)$. Let $\chi'\in\mathrm{Irr}(T_G(\xi)|\xi)$ be such that $\chi'^G=\chi$ and let B' be the block of $T_G(\xi)$ to which χ' belongs. Then it follows that $\mathrm{ht}(\chi)=\mathrm{ht}(\chi')+\mathrm{d}(B)-\mathrm{d}(B')\geqq\mathrm{ht}(\chi')$, since $B'^G=B$. So we may assume ξ is G-invariant. Take a central extension of G,

$$1\rightarrow Z\rightarrow\widehat{G}\xrightarrow{f}G\rightarrow 1,$$

such that $f^{-1}(N)=N_1\times Z$, $N_1\lhd\widehat{G}$ and that ξ extends to a character of $\widehat{G}$, say $\widehat{\xi}$, under the identification of N_1 with N through f, and that Z is a finite cyclic group. Let $\widehat{B}$ (resp. $\widehat{\chi}$) be the inflation of B (resp. χ) to $\widehat{G}$. Then there is an irreducible character θ of $\widehat{G}/N$ such that $\widehat{\chi}=\widehat{\xi}\otimes\theta$. Let $\overline{B}$ be the block of $\widehat{G}/N$ to which θ belongs. Then we get $\mathrm{ht}(\widehat{\chi})=\mathrm{ht}(\xi)+\mathrm{ht}(\theta)+\mathrm{d}(\widehat{B})-\mathrm{d}(b)-\mathrm{d}(\overline{B})$. Let $\widehat{D}$ be a defect group of $\widehat{B}$. Then $\widehat{D}N/N$ contains a defect group of $\overline{B}$ by Corollary 1.5 (i), so we get $\mathrm{d}(\widehat{B})-\mathrm{d}(b)-\mathrm{d}(\overline{B})\geqq 0$. (Note that $\widehat{D}\cap N$ is a defect group of b [8, Proposition 4.2].) On the other hand, since $\widehat{G}$ is a central extension of G, $\widehat{D}Z/Z$ is a defect group of B. This implies $\mathrm{ht}(\widehat{\chi})=\mathrm{ht}(\chi)$, since a Sylow p-subgroup of Z is contained in $\widehat{D}$. Hence $\mathrm{ht}(\chi)\geqq\mathrm{ht}(\xi)$.

Fix an inertial defect group D of B and let $\widehat{b}$ be a unique block of DN covering b. Put

$$\alpha(\xi,B)=\min\{\mathrm{ht}(\chi)-\mathrm{ht}(\xi)\,|\,\chi\in\mathrm{Irr}(B|\xi)\}\,,$$

$$\alpha'(\xi,B)=\min\{\mathrm{ht}(\zeta)-\mathrm{ht}(\xi)\,|\,\zeta\in\mathrm{Irr}(\widehat{b}\,|\{\xi^{T_G(b)}\})\},\ \text{and}$$

$$\beta(\xi,B)=\min\left\{\mathrm{d}(B)-\nu(|Q|)\,\middle|\,\begin{array}{l}Q\text{ is a subgroup of }D\text{ such that}\\ \xi^t\text{ extends to }QN\text{ for some}\\ t\in T_G(b)\end{array}\right\},$$

where $\mathrm{Irr}(\widehat{b}\,|\{\xi^{T_G(b)}\})$ denotes the set of irreducible characters in $\widehat{b}$ lying over a $T_G(b)$-conjugate of ξ.

We note that the quantities $\alpha'(\xi,B)$ and $\beta(\xi,B)$ do not depend on a particular choice of D, since D is determined up to $T_G(b)$-conjugacy.

We have shown in [10, Theorem 4.4 (i)] that if $\mathrm{ht}(\xi)=0$, then $\alpha(\xi, B)=0$ if and only if $\beta(\xi, B)=0$. Now we extend this as follows:

Theorem 2.3 *With the notation above, we have* $\alpha(\xi, B)=\alpha'(\xi, B)=\beta(\xi, B)$.

Proof. $\alpha(\xi, B)=\alpha'(\xi, B)$: We may assume that $G=T_G(b)$ by the Fong-Reynolds theorem. First we show that for any $\chi\in\mathrm{Irr}(B|\xi)$ there is a character $\zeta\in\mathrm{Irr}(\hat{b}\,|\{\xi^{T_G(b)}\})$ such that $\mathrm{ht}(\chi)\geqq\mathrm{ht}(\zeta)$. Let $\tilde{B}$ be the unique block of $N_G(D)N$ with defect group D such that $\tilde{B}^G=B$. By Lemma 2.1 there is a constituent $\tilde{\chi}\in\mathrm{Irr}(\tilde{B})$ of $\chi_{N_G(D)N}$ with $\mathrm{ht}(\chi)\geqq\mathrm{ht}(\tilde{\chi})$. Since $\tilde{B}$ covers b by Lemma 1.3, $\tilde{B}$ covers $\hat{b}$ (note that $DN\triangleleft N_G(D)N$). Furthermore, since b is G-invariant, $\hat{b}$ is $N_G(D)N$-invariant. So every irreducible constituent ζ of $\tilde{\chi}_{DN}$ lies in $\hat{b}$ and by Lemma 2.2 $\mathrm{ht}(\tilde{\chi})\geqq\mathrm{ht}(\zeta)$. Thus any such ζ is a required character.

Next we show that for any $\zeta\in\mathrm{Irr}(\hat{b}\,|\{\xi^{T_G(b)}\})$, there is a character $\chi\in\mathrm{Irr}(B|\xi)$ such that $\mathrm{ht}(\chi)\leqq\mathrm{ht}(\zeta)$. This is proved as in the proof of Lemma 4.3 in [10]. In fact, let $\tilde{B}$ be the block of $N_G(D)N$ as above. Since $\tilde{B}$ covers $\hat{b}$, there is a character $\tilde{\chi}\in\mathrm{Irr}(\tilde{B}|\zeta)$. Then, as is well-known, $\nu(\tilde{\chi}(1))\leqq\nu(|N_G(D)N/DN|)+\nu(\zeta(1))$. Since $\tilde{B}^G=B$, we have $\nu(\tilde{\chi}^B(1))=\nu(\tilde{\chi}^G(1))$, where $\tilde{\chi}^B$ denotes the B-component of $\tilde{\chi}^G$ ([4, V. 1.3]). So there is an irreducible constituent χ of $\tilde{\chi}^B$ such that $\nu(\chi(1))\leqq(\tilde{\chi}^G(1))$. Then easy computations show that $\mathrm{ht}(\chi)\leqq\mathrm{ht}(\zeta)$ and, by Frobenius reciprocity, $\chi\in\mathrm{Irr}(B|\xi)$. This completes the proof.

$\alpha'(\xi, B)=\beta(\xi, B)$: For every $t\in T_G(b)$, put

$$\alpha'_t=\min\{\mathrm{ht}(\zeta)-\mathrm{ht}(\xi)\,|\,\zeta\in\mathrm{Irr}(\hat{b}\,|\xi^t)\}, \text{ and}$$

$$\beta_t=\min\left\{\mathrm{d}(B)-\nu(|Q|)\;\middle|\;\begin{array}{l}Q \text{ is a subgroup of } D \text{ such that}\\ \xi^t \text{ extends to } QN\end{array}\right\}.$$

Since $\alpha'(\xi, B)=\min\{\alpha'_t|t\in T_G(b)\}$ and $\beta(\xi, B)=\min\{\beta_t|t\in T_G(b)\}$, it suffices to show that $\alpha'_t=\beta_t$ for every $t\in T_G(b)$. Fix $t\in T_G(b)$ and put $\xi_1=\xi^t$. Let Q be a subgroup of D such that QN has a character η with $\eta_N=\xi_1$. There is an irreducible constituent ζ of η^{DN} with

$$\nu(\zeta(1))\leqq\nu(\eta(1))+\nu(|DN:QN|)\leqq\nu(\xi(1))+\nu(|D:Q|).$$

Since $\nu(|DN|)-\mathrm{d}(B)=\nu(|N|)-\mathrm{d}(b)$, we get $\mathrm{ht}(\zeta)\leqq\mathrm{ht}(\xi)+\mathrm{d}(B)-\nu(|Q|)$. Since ζ lies in $\hat{b}$, it follows that $\alpha'_t\leqq\beta_t$. Conversely let $\zeta\in\mathrm{Irr}(\hat{b}\,|\xi_1)$. Since DN/N is a p-group, there are a subgroup H with $N\leqq H\leqq DN$ and a character $\eta\in\mathrm{Irr}(H)$ such that $\eta_N=\xi_1$ and that $\eta^{DN}=\zeta$ by [7, Theorem 6.22]. We have $H=QN$ with $Q=D\cap H$. Then $\mathrm{ht}(\zeta)=\mathrm{ht}(\xi)+\mathrm{d}(B)-\nu(|Q|)$. Hence $\beta_t\leqq\alpha'_t$. Thus $\alpha'_t=\beta_t$. This completes the proof.

In [3], E. C. Dade conjectures the following. Assume that $\mathrm{O}_p(G)$ is central in G and that $O_p(G)$ is not a defect group of a block B of G. Then for every irreducible character ϕ of $O_p(G)$ and for all integers h,

$$(*) \quad k(B, h|\phi) = \sum_C (-1)^{|C|+1} \sum_{B'} k(B', d(B') - d(B) + h|\phi),$$

where C runs through a certain set of "p-chains" with $|C| > 0$ and B' runs through the blocks of $N_G(C)$ with $B'^G = B$. Here $k(B, h|\phi)$ denotes the number of irreducible characters in B of height h which lie over ϕ.

Put $h_1 = \min\{\nu(\zeta(1)) \mid \zeta \in \mathrm{Irr}(D|\phi)\}$.

Corollary 2.4 *The equality* (*) *is true for every* $h < h_1$.

Proof. Let $h < h_1$. We shall show that all the terms appearing in (*) are 0. By applying Theorem 2.3 with $O_p(G)$ and ϕ in place of N and ξ, we get $\min\{\mathrm{ht}(\chi) \mid \chi \in \mathrm{Irr}(B|\phi)\} = h_1$. Hence $k(B, h|\phi) = 0$.

If $k(B', d(B') - d(B) + h|\phi) \neq 0$ for some B', then, by Theorem 2.3, there is a subgroup $Q \geqq O_p(G)$ of a defect group D' of B' such that ϕ extends to Q and that $\nu(|D':Q|) \leqq d(B') - d(B) + h$. Let D_1 be a defect group of B containing D'. Then $\nu(|D_1 : Q|) \leqq h$, so $k(B, h'|\phi) \neq 0$ for some $h' \leqq h$ by Theorem 2.3. This contradicts the above. Thus the result follows.

Now put

$$\gamma(\xi, B) = \max\{\mathrm{ht}(\chi) - \mathrm{ht}(\xi) \mid \chi \in \mathrm{Irr}(B|\xi)\}.$$

For a solvable group X, let $\mathrm{dl}(X)$ be the derived length of X. Define the commutator subgroups of X by $X^{(0)} = X$, $X^{(i)} = [X^{(i-1)}, X^{(i-1)}]$ $(i \geqq 1)$. The following is a slight extension of a theorem of Gluck-Wolf [5]. (In fact, letting $N = 1$, we recover Theorem D in [5].)

Theorem 2.5. *Let D be a defect group of B. If G/N is p-solvable, then* $\mathrm{dl}(DN/N) \leqq 2\gamma(\xi, B) + 1$.

Proof. First we assume $\gamma(\xi, B) = 0$ and show that DN/N is abelian. We argue by induction on $|G/N|$.

We may assume ξ is G-invariant. In fact, let $\chi \in \mathrm{Irr}(B|\xi)$ and let $\chi' \in \mathrm{Irr}(T_G(\xi)|\xi)$ be such that $\chi'^G = \chi$, and let B' be the block of $T_G(\xi)$ to which χ' belongs. Then $\mathrm{ht}(\chi) = \mathrm{ht}(\chi') + d(B) - d(B') \geqq \mathrm{ht}(\chi') \geqq \mathrm{ht}(\xi)$ by Lemma 2.2. Hence equality holds throughout by assumption. Thus B' and B have a common defect group. For any $\eta \in \mathrm{Irr}(B'|\xi)$, we have $\eta^G \in \mathrm{Irr}(B|\xi)$ and $\mathrm{ht}(\eta) = \mathrm{ht}(\eta^G) = \mathrm{ht}(\xi)$. Thus $\gamma(\xi, B') = 0$. So, if $T_G(\xi) \neq G$, then the result follows by induction.

We may assume $O_{p'}(G/N) = 1$. In fact, let $L/N = O_{p'}(G/N) \neq 1$. Choose $\eta \in \mathrm{Irr}(L|\xi)$ so that the block of L containing η is covered by B. Clearly $\mathrm{ht}(\eta) = \mathrm{ht}(\xi)$. This and $\mathrm{Irr}(B|\eta) \subseteq \mathrm{Irr}(B|\xi)$ show $\gamma(\eta, B) = 0$. By induction DL/L is abelian and then so is DN/N, since L/N is a p'-group.

Now let

$$1 \to Z \to \widehat{G} \xrightarrow{f} G \to 1,$$

be a central extension of G as in the proof of Lemma 2.2. Choose any $\chi \in \mathrm{Irr}(B|\xi)$. Let $\widehat{B}$ (resp. $\widehat{\chi}$) be the inflation of B (resp. χ) to $\widehat{G}$. Put $\overline{G}=\widehat{G}/N$. There is an irreducible character θ of $\overline{G}$ such that $\widehat{\chi}=\widehat{\xi}\otimes\theta$. Let $\overline{B}$ be the block of $\overline{G}$ to which θ belongs. Let $\widehat{D}$ be a defect group of $\widehat{B}$ and put $\overline{D}=\widehat{D}N/N$. Then, since $\mathrm{ht}(\chi)=\mathrm{ht}(\xi)$, we get that $\overline{D}$ is a defect group of $\overline{B}$ and that $\mathrm{ht}(\theta)=0$, cf. the proof of Lemma 2.2. Now put $\overline{Z}=ZN/N$. Let $\mu\in\mathrm{Irr}(Z)$ be a constituent of $\widehat{\xi}_Z$. We may regard μ as a character of $\overline{Z}$ in a natural way. Since $\overline{G}/\overline{Z}\cong G/N$, we see $\mathrm{O}_{p'}(\overline{G})=\mathrm{O}_{p'}(\overline{Z})$. Then, since $\overline{G}$ is p-solvable, it follows from Fong's theorem (cf. for example [9, Theorem 0.28]) that all irreducible characters of $\overline{G}$ lying over the character μ^{-1} of $\overline{Z}$ lie in $\overline{B}$ and that $\overline{D}$ is a Sylow p-subgroup of $\overline{G}$. So for every $\theta'\in\mathrm{Irr}(\overline{G}|\mu^{-1})$, $\widehat{\xi}\otimes\theta'\in\mathrm{Irr}(\widehat{B}|\xi)$ by Corollary 1.5 (i) and then $\widehat{\xi}\otimes\theta'$ is inflated from a character in $\mathrm{Irr}(B|\xi)$, which implies (as above) $\mathrm{ht}(\theta')=0$ and $\theta'(1)$ is prime to p. Thus by Gluck-Wolf [5, Theorem A], $\overline{D}\,\overline{Z}/\overline{Z}$ is abelian. Since $DN/N\cong\widehat{D}NZ/NZ\cong\overline{D}\,\overline{Z}/\overline{Z}$, the result follows.

For the general case we argue by induction on $|G/N|$ along the line of the proof of Corollary 14.7 (a) in [9]. By the above, we may assume that $\gamma(\xi,B)\geqq 1$ and that DN/N is nonabelian. Let $N=L_0\lhd L_1\lhd\cdots\lhd L_n=G$ be a chief series (of G/N). Take blocks b_i of L_i so that $b_0=b$, $b_n=B$, and b_i covers b_{i-1} for $1\leqq i\leqq n$. Let Q_i be a defect group of b_i for $0\leqq i\leqq n$. Since DN/N is nonabelian, we can choose $j\geqq 1$ so that Q_jN/N is nonabelian and $Q_{j-1}N/N$ is abelian. (Note that then L_j/L_{j-1} is an abelian p-group.) By the above, there is $\zeta\in\mathrm{Irr}(b_j|\xi)$ such that $\mathrm{ht}(\zeta)-\mathrm{ht}(\xi)\geqq 1$. Then $\gamma(\zeta,B)\leqq\gamma(\xi,B)-1$ and by induction $\mathrm{dl}(DL_j/L_j)\leqq 2\gamma(\zeta,B)+1$. Put $d=\mathrm{dl}(DL_j/L_j)$. So $D^{(d)}\leqq D\cap L_j$. On the other hand, $Q_j^{(1)}\leqq Q_j\cap L_{j-1}$, since L_j/L_{j-1} is abelian. Since $D\cap L_j$ is G-conjugate to Q_j and $Q_j\cap L_{j-1}$ is L_j-conjugate to Q_{j-1} by [8, Proposition 4.2], the fact that $Q_{j-1}^{(1)}\leqq N$ now implies $\mathrm{dl}(DN/N)\leqq d+2$. Thus $\mathrm{dl}(DN/N)\leqq 2(\gamma(\xi,B)-1)+3=2\gamma(\xi,B)+1$. This completes the proof.

3. Normal subgroups and heights of irreducible Brauer characters

In this section we shall show the modular version of Theorem 2.3.

Throughout this section we use the following notation:

N is a normal subgroup of a group G, ϕ is an irreducible Brauer character of N, b is the block of N to which ϕ belongs, and B is a block of G covering b.

Let $T_G(\phi)$ be the inertial group of ϕ in G. Let $\mathrm{IBr}(B|\phi)$ be the set of irreducible Brauer characters in B lying over ϕ.

The following is well-known in the case of (ordinary) irreducible characters, cf. [11, Lemma 5.3.1 (ii)].

Lemma 3.1. *Let the notation be as above. Let $\phi \in \mathrm{IBr}(B|\psi)$ and let $\phi' \in \mathrm{IBr}(T_G(\psi)|\psi)$ be such that $\phi'^G = \phi$. If B' is the block of $T_G(\psi)$ containing ϕ', then B'^G is defined and equals B.*

Proof. By the Fong-Reynolds theorem, we may assume that b is G-invariant. Let $T' = \bigcap T_G(\xi)$, where ξ runs through $\mathrm{Irr}(b)$. Clearly $T' \lhd T_G(b) = G$. Also $T' \lhd T_G(\psi)$, since ψ is an integral linear combination of the irreducible characters in b (on the set of p-regular elements of N). Let B_1 be a block of T' covered by B'. Then by [10, Lemma 4.14 (i)], $B' = B_1^{T_G(\psi)}$. Since B also covers B_1, $B = B_1^G$ by the same reason. Hence B'^G is defined and equals B ([11, Lemma 5.3.1]).

Lemma 3.2. *Let the notation be as above. Then*

(i) $\mathrm{ht}(\phi) \geqq \mathrm{ht}(\psi)$ *for every* $\phi \in \mathrm{IBr}(B|\psi)$.

(ii) *If ψ is G-invariant, then there is $\phi \in \mathrm{IBr}(B|\psi)$ with* $\mathrm{ht}(\phi) = \mathrm{ht}(\psi)$.

Proof. (i) The proof is much the same as that of Lemma 2.2. But we repeat it here, since it is necessary for the proof of (ii).

Let $\phi \in \mathrm{IBr}(B|\psi)$. Let $\phi' \in \mathrm{IBr}(T_G(\psi)|\psi)$ be such that $\phi'^G = \phi$ and let B' be the block of $T_G(\psi)$ to which ϕ' belongs. Then it follows that $\mathrm{ht}(\phi) = \mathrm{ht}(\phi') + \mathrm{d}(B) - \mathrm{d}(B') \geqq \mathrm{ht}(\phi')$, since $B'^G = B$ by Lemma 3.1. So we may assume ψ is G-invariant. Take a central extension of G,

$$1 \to Z \to \widehat{G} \overset{f}{\to} G \to 1,$$

such that $f^{-1}(N) = N_1 \times Z$, $N_1 \lhd \widehat{G}$ and that ψ extends to a Brauer character of $\widehat{G}$, say $\widehat{\psi}$, under the identification of N_1 with N through f, and that Z is a finite cyclic group. Let $\widehat{B}$ (resp. $\widehat{\phi}$) be the inflation of B (resp. ϕ) to $\widehat{G}$. There is an irreducible Brauer character θ of $\widehat{G}/N$ such that $\widehat{\phi} = \widehat{\psi} \otimes \theta$. If $\bar{B}$ is the block of $\widehat{G}/N$ to which θ belongs, $\mathrm{d}(\widehat{B}) - \mathrm{d}(b) - \mathrm{d}(\bar{B}) \geqq 0$ by Corollary 1.5 (ii). Since $\mathrm{ht}(\widehat{\phi}) = \mathrm{ht}(\phi)$, we get $\mathrm{ht}(\phi) \geqq \mathrm{ht}(\psi)$.

(ii) Let $\widehat{G}$, $\widehat{\psi}$, $\widehat{B}$ be as above. Clearly $\widehat{B}$ covers b. So by Corollary 1.5 (ii), we can choose a block $\bar{B}$ of $\widehat{G}/N$ which is $\widehat{\psi}$-dominated by $\widehat{B}$ and for which $\mathrm{d}(\widehat{B}) - \mathrm{d}(b) - \mathrm{d}(\bar{B}) = 0$. Let θ be an irreducible Brauer character lying in $\bar{B}$ of height 0. Then $\widehat{\psi} \otimes \theta$ is an irreducible Brauer character lying in $\widehat{B}$ by Corollary 1.5 (ii) and $\mathrm{ht}(\widehat{\psi} \otimes \theta) = \mathrm{ht}(\psi)$. Since $\widehat{B}$ covers the principal block $B_0(Z)$ of Z and $\mathrm{IBr}(B_0(Z))$ consists of only the trivial character, $\widehat{\psi} \otimes \theta$ is trivial on Z. Thus $\widehat{\psi} \otimes \theta$ is inflated from some $\phi \in \mathrm{IBr}(B|\psi)$ and then $\mathrm{ht}(\widehat{\psi} \otimes \theta) = \mathrm{ht}(\phi)$ as above. So $\mathrm{ht}(\phi) = \mathrm{ht}(\psi)$. This completes the proof.

Fix an inertial defect group D of B and let $\widehat{b}$ be a unique block of DN covering b. Let $T_G(b)$ be the inertial group of b in G.

Put

$$\alpha(\psi, B) = \min\{\mathrm{ht}(\phi) - \mathrm{ht}(\psi) \mid \phi \in \mathrm{IBr}(B|\psi)\},$$
$$\alpha'(\psi, B) = \min\{\mathrm{ht}(\theta) - \mathrm{ht}(\psi) \mid \theta \in \mathrm{IBr}(\widehat{b} \mid \{\psi^{T_G(b)}\})\}, \text{ and}$$
$$\beta(\psi, B) = \min\left\{ \mathrm{d}(B) - \nu(|Q|) \;\middle|\; \begin{array}{l} Q \text{ is a subgroup of } D \text{ such} \\ \text{that } \psi^t \text{ extends to } QN \text{ for} \\ \text{some } t \in T_G(b) \end{array} \right\},$$

where $\mathrm{IBr}(\widehat{b} \mid \{\psi^{T_G(b)}\})$ denotes the set of irreducible Brauer characters in $\widehat{b}$ lying over a $T_G(b)$-conjugate of ψ.

As in Section 2, the quantities $\alpha'(\psi, B)$ and $\beta(\psi, B)$ do not depend on a particular choice of D. Also we have shown in [10, Theorem 4.4 (ii)] that if $\mathrm{ht}(\psi) = 0$, then $\alpha(\psi, B) = 0$ if and only if $\beta(\psi, B) = 0$. We extend this as follows:

Theorem 3.3. *With the notation above, we have* $\alpha(\psi, B) = \alpha'(\psi, B) = \beta(\psi, B)$.

Proof. We may rewrite $\beta(\psi, B)$ as follows:

$$\beta(\psi, B) = \min\{\nu(|D: D \cap T_G(\psi^t)|) \mid t \in T_G(b)\}.$$

In fact, if ψ^t, $t \in T_G(b)$, is Q-invariant for a subgroup $Q \leqq D$, then ψ^t necessarily extends to QN. From this the above follows.

$\alpha(\psi, B) = \beta(\psi, B)$: By the Fong-Reynolds theorem, we may assume that b is G-invariant. First we show $\alpha(\psi, B) \leqq \beta(\psi, B)$. Let $t \in G$ and put $Q = D \cap T_G(\psi^t)$. We shall show there is $\phi \in \mathrm{IBr}(B|\psi)$ with $\mathrm{ht}(\phi) - \mathrm{ht}(\psi) \leqq \mathrm{d}(B) - \nu(|Q|)$. We claim there is a block B' of $T = T_G(\psi^t)$ such that:

B' covers b, $B'^G = B$ and Q is contained in a defect group of B'.

Since $Q \leqq D$, there is a block B_1 of $N_G(Q)N$ with $B_1{}^G = B$. Then B_1 covers b by Lemma 1.3. Choose $\phi_1 \in \mathrm{IBr}(B_1|\psi^t)$ and let $\phi_2 \in \mathrm{IBr}(N_G(Q)N \cap T|\psi^t)$ be such that $\phi_2{}^G = \phi_1$. Let B_2 be the block of $N_G(Q)N \cap T$ to which ϕ_2 belongs. Then B_2 covers b and, by Lemma 3.1. $B_2{}^{N_G(Q)N} = B_1$. Clearly B_2 covers a unique block $\widetilde{b}$ of QN that covers b. (Note that $QN \triangleleft N_G(Q)N \cap T$.) Since $Q \geqq D \cap N$, Q is a defect group of $\widetilde{b}$ by [10, Lemma 4.13]. So a defect group D_2 of B_2 contains Q by [8, Proposition 4.2] and then, since $C_T(D_2) \leqq C_T(Q) \leqq N_G(Q)N \cap T$, $B_2{}^T$ is defined. Put $B' = B_2{}^T$. Then B' covers b by Lemma 1.3 and, since $B_2{}^G = (B_2{}^{N_G(Q)N})^G = B$, $B'^G = B$ by [11, Lemma 5.3.1]. Since a defect group of B' contains D_2, B' is a required block.

By Lemma 3.2 (ii), there is $\phi' \in \mathrm{IBr}(B'|\psi^t)$ with $\mathrm{ht}(\phi') = \mathrm{ht}(\psi^t)$. Now let $\phi = \phi'^G$. Then $\phi \in \mathrm{IBr}(B|\psi^t) = \mathrm{IBr}(B|\psi)$ by Lemma 3.1, and $\mathrm{ht}(\phi) - \mathrm{ht}(\psi) = \mathrm{ht}(\phi) - \mathrm{ht}(\phi') = \mathrm{d}(B) - \mathrm{d}(B') \leqq \mathrm{d}(B) - \nu(|Q|)$. Thus $\alpha(\psi, B) \leqq \beta(\psi, B)$.

Now we show the reverse inequality. Let $\phi \in \mathrm{IBr}(B|\psi)$. Let $\phi' \in \mathrm{IBr}(T_G(\psi)|\psi)$ be such that $\phi'^G = \phi$ and B' the block of $T_G(\psi)$ to which ϕ' belongs. Let D' be a defect group of B'. Since $B'^G = B$ by Lemma 3.1, we get that $\mathrm{ht}(\phi) - \mathrm{ht}(\phi') = \mathrm{d}(B) - \mathrm{d}(B')$ and that $D'^t \leqq D$ for some $t \in G$. Then D'^t

$\leqq D\cap T_G(\psi^t)$ and $\nu(|D : D\cap T_G(\psi^t)|) \leqq \mathrm{d}(B) - \mathrm{d}(B') = \mathrm{ht}(\phi) - \mathrm{ht}(\phi') \leqq \mathrm{ht}(\phi) - \mathrm{ht}(\psi)$ by Lemma 3.2 (i). Thus the reverse inequality is also true.

$\alpha'(\psi, B) = \underset{\sim}{\beta}(\psi, B)$: Let $t \in T_G(b)$. Since DN/N is a p-group and ψ^t belongs to b, $\mathrm{IBr}(b\,|\psi^t)$ consists of a single character, say θ. Since the ramification index of θ relative to N equals 1, we get $\mathrm{ht}(\theta) - \mathrm{ht}(\psi) = \nu(|DN : DN\cap T_G(\psi^t)|)$. So

$$\alpha'(\psi, B) = \min\{\nu(|DN : DN\cap T_G(\psi^t)|) \mid t \in T_G(b)\}.$$

Since $\nu(|DN : DN\cap T_G(\psi^t)|) = \nu(|D : D\cap T_G(\psi^t)|)$, the result follows.

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Note added on August 30, 1995.
For shorter (module-theoretical) proofs of Lemma 2.2 and Lemma 3.2 (i), and related results, see A. Watanabe: Normal subgroups and multiplicities of indecomposable modules, preprint.