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A counter-example to the q-Levi Problem in $\mathbf{P}^n$

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§0. Introduction

Let $D \subset \mathbf{P}^n$ be an open set which is locally Stein. It follows then, from the characterization of the pseudoconvexity of D by the plurisubharmonicity of $-\log \delta_D$ [10], that D is itself Stein (if $D \neq \mathbf{P}^n$). A generalization of the above statement in the q-convex case would be the following:

*) Let $D \subset \mathbf{P}^n$ be an open subset which is locally q-complete. Then D is q-convex.

We consider here the classical definitions of q-convexity as introduced by Andreotti and Grauert in [1].

The statement *) could be called the q-Levi Problem in $\mathbf{P}^n$. It is known [8] that *) has an affirmative answer if the boundary ∂D of D is smooth. In this particular case the boundary distance δ_D (with respect to the Fubini metric on $\mathbf{P}^n$) is also smooth near ∂D and $-\log \delta_D$ is a q-convex function at the points of D which are sufficiently close to ∂D.

In this paper we consider domains $D \subset \mathbf{P}^n$ with non-smooth boundary, therefore the distance δ_D is only continuous. Under the assumption that $D \subset \mathbf{P}^n$ is locally q-complete it follows then that D has certain global q-convexity properties, but with respect to some other classes of functions: D is a pseudoconvex domain of order $(n-q)$ [4], [5], D is q-complete with corners [6].

The aim of this paper is to give a counter-example to *), therefore to show that the q-Levi Problem in $\mathbf{P}^n$ does not hold.

More precisely we prove:

Theorem 1. *There exists a domain $D \subset \mathbf{P}^3$ which is locally 2-complete but D is not 2-convex.*

§1. The construction of the counter-example proving Theorem 1

Let us recall first some basic definitions and results which will be needed in this paper.

If U is an open subset in $\mathbf{C}^n$, a function $\varphi \in C^\infty(U, \mathbf{R})$ is called q-convex iff the Levi form $L(\varphi)$ has at least $(n-q+1)$ positive (>0) eigenvalues at any

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point of U. Using local coordinates this notion can be easily extended to complex manifolds.

A complex manifold X is called q-convex [1] iff there exists a C^∞ function $\varphi: X \to \mathbf{R}$ which is q-convex outside a compact subset K of X and such that φ is an exhaustion function on X, i.e. $X_c = \{\varphi < c\} \Subset X$ for every $c \in \mathbf{R}$. If K may be taken to be the empty set then X is said to be q-complete.

A complex manifold X is called cohomologically q-convex if $\dim_{\mathbf{C}} H^i(X, \mathscr{F}) < \infty$ for every $i \geq q$ and for every $\mathscr{F} \in \mathrm{Coh}(X)$. If $H^i(X, \mathscr{F}) = 0$ for every $i \geq q$ and every $\mathscr{F} \in \mathrm{Coh}(X)$ then X is said to be cohomologically q-complete. By a main result in [1] it follows that:

a) q-convex $\Rightarrow$ cohomologically q-convex

b) q-complete $\Rightarrow$ cohomologically q-complete

An open subset D of a complex manifold X is said to be locally q-complete if for every point $x \in \partial D$ there is an open neighbourhood U of x such that $U \cap D$ is q-complete.

In [7] the following result is proved:

Proposition 1. *Let X be a complex manifold and $V_1, \ldots, V_q$ Stein open subsets. Then $V = V_1 \cup \cdots \cup V_q$ is q-complete.*

We shall need also the following special case of a result due to Siu [9] (see [2] for a generalization to the q-complete case):

Proposition 2. *Let X be a complex manifold and $A \subset X$ a closed Stein submanifold. Then A has a fundamental system of Stein open neighbourhoods.*

We can now begin the construction of our example. We consider the Segre embedding $\tau: \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ given in homogeneous coordinates by $\tau([x:y], [z:t]) = [xz : xt : yz : yt]$.

We fix a point $p \in \mathbf{P}^1$ and we choose a sequence of points $p_\nu \to p$, $p_\nu \in \mathbf{P}^1$, $p_\nu \neq p$. Let $A = \bigcup_{\nu \geq 1} \{p_\nu \times \mathbf{P}^1\} \cup \{p \times \mathbf{P}^1\} \subset \mathbf{P}^1 \times \mathbf{P}^1$, $B = \tau(A)$ and define $D = \mathbf{P}^3 \backslash B$.

We shall prove that:

i) for every Stein domain $U \subset \mathbf{P}^3$ the intersection $U \cap D$ is 2-complete

ii) D is not cohomologically 2-convex

This of course will end the proof of Theorem 1.

Proof of Claim i). Let $Y = \tau(\mathbf{P}^1 \times \mathbf{P}^1)$ which is a quadric in $\mathbf{P}^3$. Then $V_1 = U \backslash Y$ is Stein being the complement of a hypersurface in the Stein domain U. On the other hand $U \cap Y \backslash B$ is Stein being the interior of an intersection of Stein domains in the Stein manifold $U \cap Y$. Indeed, for every $k \in \mathbf{N}$ $M_k = U \cap Y \backslash \tau(p_1 \times \mathbf{P}^1 \cup \cdots \cup p_k \times \mathbf{P}^1)$ is Stein being the complement of a divisor in the Stein manifold $U \cap Y$ and clearly $U \cap Y \backslash B = (\cap M_k)^\circ$ (the interior being taken in $U \cap Y$). By Proposition 2 there is Stein open subset V_2 of $\mathbf{P}^3$ such that $V_2 \subset U$ and $V_2 \cap Y = U \cap Y \backslash B$. Since clearly $U \cap D = V_1 \cup V_2$ it follows from Proposition 1 that $U \cap D$ is 2-complete, which proves Claim i).

Proof of Claim ii). Let Ω^3 denote the canonical sheaf of $\mathbf{P}^3$ and $\mathcal{O}$ the structure sheaf of $\mathbf{P}^3$. If D would be cohomologically 2-convex then in particular it would follow that $\dim_C H^2(D, \Omega^3) < \infty$. By Serre duality this implies that $\dim_C H_c^1(D, \mathcal{O}) < \infty$.

From the exact sequence:

$$\cdots \to H^0(\mathbf{P}^3, \mathcal{O}) \to H^0(B, \mathcal{O}|_B) \to H_c^1(D, \mathcal{O}) \to \cdots$$

it follows that $\dim_C H^0(B, \mathcal{O}|_B) < \infty$ where $\mathcal{O}|_B$ means sheaf theoretic restriction. But this is impossible since B has infinitely many connected components. Thus the proof of Theorem 1 is complete.

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