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Minimization of the embeddings of the curves into the affine plane

By

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0. Introduction

Let C be a smooth affine algebraic curve with only one place at infinity defined over an algebraically closed field k of characteristic zero; we also call C a *once punctured* smooth algebraic curve. Assume that C is embedded into the affine plane $\mathbf{A}^2$ as a closed curve. The image of C by an algebraic automorphism of $\mathbf{A}^2$ is again a curve of the same nature as C. One may then ask what is the smallest among the degrees of $\varphi(C)$ when φ ranges over automorphisms of $\mathbf{A}^2$. We say that $\varphi(C)$ is a *minimal embedding* of C if the degree of $\varphi(C)$ is the smallest.

The question was first treated by Abhyankar-Moh [1] and Suzuki [12] in the case of genus g of C is zero. Namely, *a minimal embedding of the affine line is a coordinate line.* The cases $g = 2, 3, 4, \ldots$ were treated by Neumann [8] by topological methods and by A'Campo-Oka [3] depending on Tschirnhausen resolution tower.

We shall here propose a different algebro-geometric approach based on the classification of degenerations of curves, which enables us to describe an automorphism φ of $\mathbf{A}^2$ minimizing the degree of $\varphi(C)$.

Our theorem is the following:

Theorem. *Let C be a once punctured smooth algebraic curve of genus g, which is embedded into the affine plane $\mathbf{A}^2 = \operatorname{Spec} k[x, y]$ as a closed curve defined by $f(x, y) = 0$. Then there exists new coordinates u, v of $\mathbf{A}^2$ such that*

(1) *$k[x, y] = k[u, v]$, and*

(2) *$h(u, v) := f(x(u, v), y(u, v))$ and $e = \deg h(u, v)$ are given as follows if $g \leq 4$;*

Case $g = 0$: *$e = 1$ and $h = u$.*

Case $g = 1$: *$e = 3$ and $h = v^2 - (u^3 + au + b)$ with a, $b \in k$.*

Case $g = 2$: *$e = 5$ and $h = v^2 - (u^5 + au^3 + bu^2 + cu + d)$ with a, b, c, $d \in k$.*

Case $g = 3$: *There are three types:*

(1) *$e = 4$ and $h = v^3 + g_1(u)v - (u^4 + g_2(u))$ with $g_i(u) \in k[u]$ and $\deg g_i(u) \leq 2$ for $i = 1, 2$.*

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(2) $e=7$ *and* $h=v^2-(u^7+g(u))$ *with* $g(u)\in k[u]$ *and* $\deg g(u)\leq 5$.

(3) $e=6$ *and the multiplicity sequence of singularities at the point at infinity* P_0 *is* (2^7), *where* (2^7) *implies that there are 7 double points centered at* P_0.

Case $g=4$: *There are four types.*

(1) $e=5$ *and* $h=v^3+g_1(u)v-(u^5+g_2(u))$ *with* $g_i(u)\in k[u]$ *and* $\deg g_i(u)\leq 3$ *for* $i=1, 2$.

(2) $e=9$ *and* $h=v^2-(u^9+g(u))$ *with* $g(u)\in k[u]$ *and* $\deg g(u)\leq 7$.

(3) $e=6$ *and the multiplicity sequence of singularities at the point at infinity is* (2^6).

(4) $e=9$ *and the multiplicity sequence of singularities is* (3^8).

(3) *The automorphism* φ *of* $\mathbf{A}^2$ *induced by* ${}^a\varphi(x)=x(u, v)$ *and* ${}^a\varphi(y)=y(u, v)$ *is described explicitly as a Cremona transformation of* $\mathbf{P}^2$ *induced by* φ (*cf.* *Lemma* 8).

For the arguments using Lemma 9 below, we are indebted to A. Sathaye who instructed us how to use Lemma 9. We are very grateful to him.

1. Minimal degenerations

Embed $\mathbf{A}^2$ into the projective plane $\mathbf{P}^2$ with the line at infinity l_∞. Let C be as above and let $\bar{C}$ be the closure of C in $\mathbf{P}^2$, which is a curve of degree, say d, having a one-place point P_0 on l_∞.

Consider a linear pencil Λ on $\mathbf{P}^2$ generated by $\bar{C}$ and dl_∞. The point P_0 is a base point of Λ. Then, by blowing up P_∞ and its infinitely near points which are base points of Λ and by taking the proper transform of Λ, we can eliminate the base points of Λ. We assume that we need the last blowing-up to make the linear system free from base points. When the base points are eliminated after finitely many blowing-ups, we obtain a birational morphism $\sigma: V\to\mathbf{P}^2$ and a surjective morphism $\rho: V\to\mathbf{P}^1$ such that V is a nonsingular projective surface, that σ is a composite of the above blowing-ups and that the fibers of ρ correspond bijectively to the members of the proper transform $\sigma'\Lambda$ of Λ. Let E be the exceptional curve arising from the last blowing-up.

Now the following result is proved in [4, 5].

Lemma 1. *With the notations and assumptions as above, we have*:

(1) E *is a cross-section of the fibration* $\rho: V\to\mathbf{P}^1$, *and every fiber of* ρ *is smooth at the point of intersection with* E.

(2) *A general fiber of* ρ *is a smooth projective curve of genus* g.

(3) *The proper transform* $F_0:=\sigma'\bar{C}$ *of* $\bar{C}$ *is a fiber of* ρ.

(4) *Let* F_∞ *be the fiber of* ρ *comprising the proper transform* $L:=\sigma'(l_\infty)$. *Then* $(F_\infty)_{\mathrm{red}}$ *consists of* L *and all* (*irreducible*) *exceptional curves but* E *which arise from the blowing-ups effected to make the pencil* Λ *free from base points. All other fibers of* ρ *are irreducible.*

(5) *Denote F_∞ by Γ. Then Γ consists of nonsingular rational curves with simple normal crossings. Only the component L is possibly a (-1) curve among the irreducible components of Γ. Furthermore, the dual graph of Γ as shown below in Figure* 1 *is a tree such that the branching number at each vertex is at most* 3;

If L is not a (-1) curve the fibration $\rho: V \to \mathbf{P}^1$ is minimal, i.e., there are no (-1) curves contained in the fibers of ρ. If ρ is not minimal, we contract the component L and all subsequently contractible components of Γ to obtain a minimal fibration. Thus, we have a birational morphism $\tau: V \to \bar{V}$ and a minimal fibration $\bar{\rho}: \bar{V} \to \mathbf{P}^1$ of curves of genus g such that $\rho = \bar{\rho} \cdot \tau$. Let $\bar{\Gamma} := \tau_* \Gamma$ be the direct image of Γ. We shall show that:

Lemma 2. *$\bar{\Gamma}$ consists of nonsingular rational curves with simple normal crossings, and the dual graph of $\bar{\Gamma}$ is a tree as shown in Figure* 2.

Proof. Suppose the assertion does not hold. Then, in the course of blowing down contractible components of Γ, we have a (-1) component M with branching number 3 whose location in the dual graph of the direct image of Γ is shown as follows (see Figure 2):

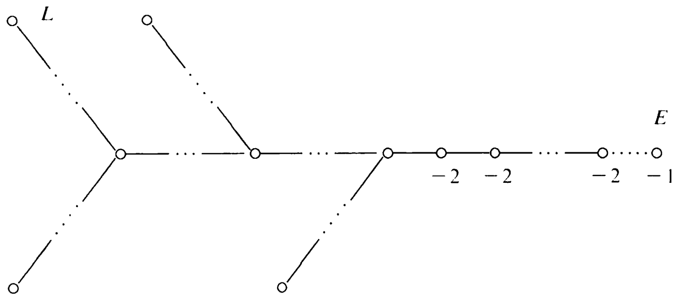


Figure 1

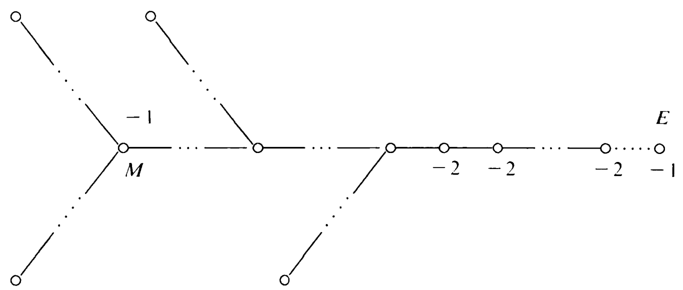
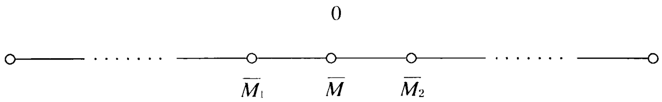


Figure 2

Then, starting with E, all the components lying on the right side of M can be contracted as a part of the reverse process of elimination of base points of the pencil Λ. The image $\overline{M}$ of M together with two remaining branches forms a linear chain of nonsingular rational curves

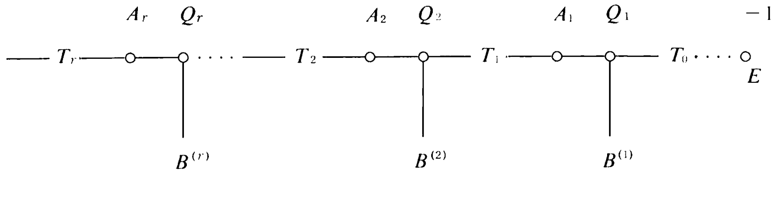


where $(\overline{M}^2)=0$ and two adjacent components $\overline{M}_1$, $\overline{M}_2$ of $\overline{M}$ have self-intersection number ≤ -2. However, this linear chain is the dual graph of the boundary divisor of a minimal normal completion of the affine plane. Then one of the adjacent components $\overline{M}_1$ and $\overline{M}_2$ must have positive self-intersection number by Morrow [6]. So, we are led to a contradiction. Q.E.D.

Thus the dual graph of $\overline{\Gamma}$ looks the same as the one in Figure 2 with the self-intersection number of M is replaced by -2. We regard it as a tree, and call the horizontal linear chain and the slanted branches the *trunk* and *branches* of the tree, respectively. We number the branches the first, the second, ... from the right, i.e., from the branches close to E.

We shall evaluate the contributions of irreducible components of $\overline{\Gamma}$ in the intersection number $(\overline{\Gamma}\cdot K_{\overline{V}})$.

We consider the case where the dual graph of $\overline{\Gamma}$ has more than r branches and the first r branches consist only of (-2) components. A part of $\overline{\Gamma}$ containing the first r branches then looks like the following:



where

(1) $Q_i - B^{(i)}$ $(1 \leq i \leq n)$ is a linear chain of (-2) components of length u_i with multiplicities (in $\overline{\Gamma}$) as shown below:

(2) $(A_i^2) = -(u_i + 1)$ and the multiplicity of A_i in $\bar{\Gamma}$ is $s_i + u_1 u_2 \ldots u_{i-1}$ for $1 \le i \le r$, where we set $u_0 = 1$,

(3) T_i $(0 \le i \le \mathrm{r})$ is a (possibly empty) linear chain of (-2) components, and one of the end components of T_0 meets E.

Lemma 3. *With the notations and assumptions as above, we have*

$$\sum_{i=1}^{n} (s_i + u_1 \ldots u_{i-1})(u_i - 1) \le 2g - 2\,,$$

where g is the genus. In particular, we have

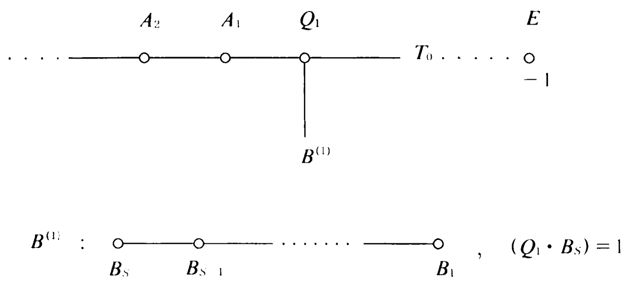
$$r + 2^r \le 2g - 1\,.$$

Proof. Note that $(\bar{\Gamma}^2) = 0$ and $(\bar{\Gamma} \cdot K_{\bar{V}}) = 2g - 2$. Furthermore, $(K_{\bar{V}} \cdot Z) = -2 - (Z^2) \ge 0$ for every component Z of $\bar{\Gamma}$. This implies that $\sum_{i=1}^{r} A_i \cdot K_{\bar{V}} \le 2g - 2$, where $A_i \cdot K_{\bar{V}} = u_i - 1$. Since the multiplicity of A_i in $\bar{\Gamma}$ is $s_i + u_1 \ldots u_{i-1}$, we obtain the first inequality by computing the contribution δ of the components A_i $(1 \le i \le r)$. Note that $u_i \ge 2$ and $s_i \ge 1$ for $1 \le i \le r$. Hence we have

$$\delta \ge \sum_{i=1}^{r} (1 + 2^{i-1}) = r + 2^r - 1\,.$$

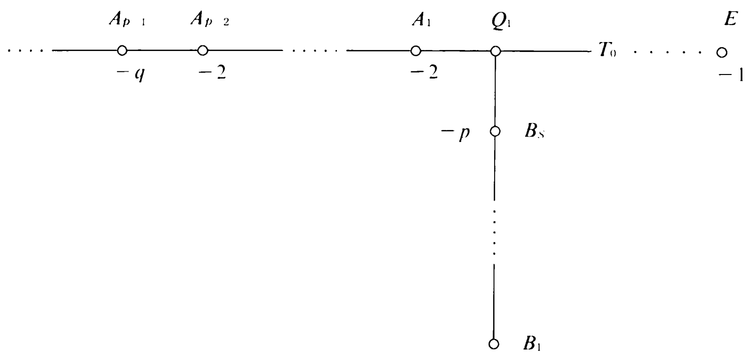
Since $2g - 2 \ge \delta$, we have $r + 2^r \le 2g - 1$. Q.E.D.

We assume, for a while, that the dual graph of $\bar{\Gamma}$ has at least two branching points, i.e., vertices where the branching number is at least (in fact, exactly) 3, and that the first branch contains a component with self-intersection ≤ -3. The dual graph looks like the following near the first branch;



Then, it is not possible that $(A_1^2) = (B_s^2) = -2$. In fact, since $(Q_1^2) = -2$, Q_1 becomes a (-1) curve after the contraction of the components of T_0. If $(A_1^2) = (B_s^2) = -2$, the proper transform of A_1 has self-intersection ≥ 0 after contracting Q_1, B_s and all contractible components in the branch $B^{(1)}$. Thus we would have the graph of a minimal normal completion of $\mathbf{A}^2$ which is not a linear chain by the hypothesis that the dual graph of $\bar{\Gamma}$ has at least two branching points. This contradicts a theorem of Ramanujam [10]. So, $(B_s^2) = -p \le -3$ or $(A_1^2) = -q \le -3$.

We consider the case $(B_s^2) = -p \leq -3$ first. Then the dual graph of $\bar{\Gamma}$ looks like:



where the components of T_0, Q_1, A_1, ..., A_{p-2} are (-2) curves and $q \geq 3$. Furthermore, any component of A_1, ..., A_{p-2} does not represent a branching point of the dual graph of $\bar{\Gamma}$. In fact, if A_i does, the proper transform of A_i after the contraction of the components of T_0, Q_1, A_1, ..., A_{i-1} is a (-1) curve meeting three other components of the image of $\bar{\Gamma}$ which is the boundary graph of a normal completion of $\mathbf{A}^2$. This is impossible.

Let m, α_i and β_j be the multiplicities of Q_1, A_i $(1 \leq i \leq p-1)$ and B_j $(1 \leq j \leq s)$ in the fiber $\bar{\Gamma}$, respectively. Note that T_0 consists of $m-1$ components and its component meeting E has multiplicity 1. Since $(\bar{\Gamma} \cdot B_s) = 0$, we have $m - p\beta_s + \beta_{s-1} = 0$. By Lemma 5 below, $\beta_s > \beta_{s-1} \geq 0$, where we set $\beta_{s-1} = 0$ if $s = 1$. So, $m > (p-1)\beta_s$. Furthermore, since $(\bar{\Gamma} \cdot Q_1) = (\bar{\Gamma} \cdot A_i) = 0$ $(1 \leq i \leq p-2)$, we compute

$$\alpha_1 = 2m - (m-1) - \beta_s = m - (\beta_s - 1)$$

$$\alpha_2 = 2\alpha_1 - m = m - 2(\beta_s - 1)$$

$$\cdots\cdots\cdots$$

$$\alpha_{p-1} = m - (p-1)(\beta_s - 1) \geq 1 + (p-1)\beta_s - (p-1)(\beta_s - 1) = p$$

Let δ be the contribution of the components B_s and A_{p-1} to $(\bar{\Gamma} \cdot K_{\bar{V}})$. Then we have

$$\delta = (q-2)\alpha_{p-1} + (p-2)\beta_s \geq p(q-2) + (p-2)\beta_s\,.$$

By evaluating the value of β_s (cf. the proof of Lemma 5), we have

Lemma 4. *With the notations and assumptions as above, the following assertions hold*:

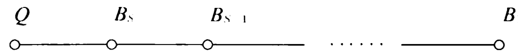
(1) $\delta \geq pq - p - 2$ *if* $s = 1$, $\delta \geq pq - 4$ *if* $s = 2$ *and* $(B_{s-1}^2) = -2$, $\delta = pq + q - 6$ *if* $s = 1$ *and* $\beta_s = 2$, *and* $\delta \geq pq + p - 6$ *otherwise.*

(2) $\delta = pq - p - 2$ *if and only if* $s = 1$ *and* $\beta_s = 1$.
(3) $\delta = pq - 4$ *if and only if* $s = 2$, $\beta_1 = 1$ *and* $(B_1^2) = -2$.
(4) *If* $s = 1$ *and* $p = 3$ *then* $q = 3$.

Proof. We note that $\delta \geq pq - p - 2$ if $\beta_s = 1$, $\delta \geq pq - 4$ if $\beta_s = 2$ and $\delta \geq pq + p - 6$ if $\beta_s \geq 3$, that $\beta_s = 1$ only if $s = 1$ and that $\beta_s > \beta_{s-1} > 0$ if $s \geq 2$ (cf. Lemma 5). The assertions (1) and (2) follow from these remarks. Suppose $\delta = pq - 4$. Then $\beta_s \leq 2$. If $\beta_s = 1$ then $s = 1$ and $p = 2$, which contradicts the hypothesis $p \geq 3$. So, $\beta_s = 2$. If $s = 1$ then $m = 2p$, $\alpha_{p-1} = p + 1$ and $\delta = pq + q - 6$. Hence $q = 2$, which is again a contradiction. Thus $\beta_s = 2$ and $s = 2$. Then $(B_{s-1}^2) = -2$ and $\beta_{s-1} = 1$. As for the assertion (4), if $s = 1$ and $p = 3$, the component B_s is contractible after the contraction of the components of T_0, Q_1 and A_1, and the component A_{p-1} must become a (-1) curve after the contraction of B_s. So, $q = 3$. Q.E.D.

In the above argument we have used the following:

Lemma 5. *Consider a branch in the fiber* $\bar{\Gamma}$



where Q *is a component on the trunk and* B_i $(1 \leq i \leq s)$ *is a component in the branch with* $(B_i^2) = -b_i \leq -2$ *and multiplicity* β_i *in* $\bar{\Gamma}$. *Then we have*:

(1) *For* $1 \leq i \leq s$,

$$\beta_{i+1} = \beta_1 \det \begin{pmatrix} b_i & -1 & 0 & \cdots & 0 \\ -1 & b_{i-1} & -1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & -1 & b_2 & -1 \\ 0 & \cdots & 0 & -1 & b_1 \end{pmatrix}$$

where β_{s+1} *is the multiplicity of* Q *in* $\bar{\Gamma}$.
(2) *For* $1 \leq i \leq s$, $\beta_{i+1} > \beta_i$. *In particular,* $s = 1$ *if* $\beta_s = 1$.

Proof. Since $(\bar{\Gamma} \cdot B_i) = 0$ for $1 \leq i \leq s$, we have

$$\beta_{i+1} - b_i \beta_i + \beta_{i-1} = 0\,,$$

where $\beta_0 = 0$. To show the assertion (1), we proceed by induction on i. Note that $(\bar{\Gamma} \cdot B_1) = \beta_2 - b_1 \beta_1 = 0$. The cofactor expansion along the first row of

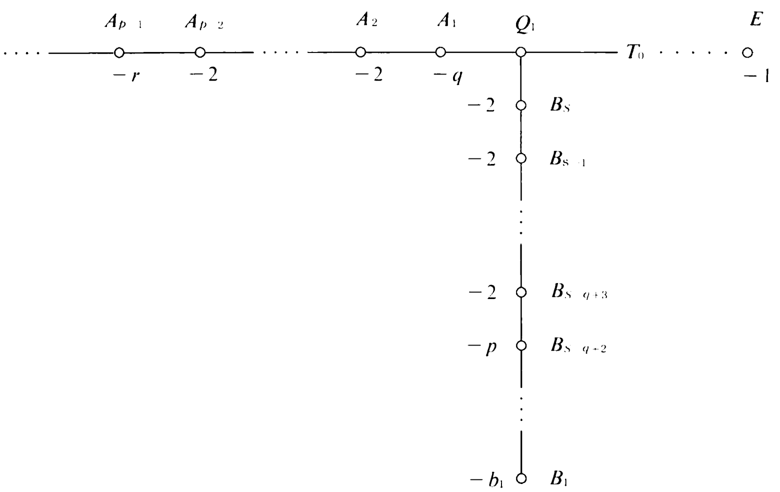
$$\beta_1 \det \begin{bmatrix} b_i & -1 & 0 & \cdots & 0 \\ -1 & b_{i-1} & -1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & \cdots & -1 & b_2 & -1 \\ 0 & \cdots & 0 & -1 & b_1 \end{bmatrix}$$

is $b_i\beta_i - \beta_{i-1}$ by the induction hypothesis. Hence β_{i+1} is given as stated. The assertion (2) is also shown by induction if one notes that

$$\beta_{i+1} - \beta_i = (b_i - 1)\beta_i - \beta_{i-1} > (b_i - 2)\beta_i \geq 0 ,$$

where $\beta_i > \beta_{i-1}$ by the induction hypothesis and $b_i \geq 2$. Q.E.D.

We next consider the case $(A_1^2) = -q \leq -3$. Then the dual graph $\bar{\Gamma}$ looks like:



where $(Q_1^2) = -2$, T_0 is a chain of (-2) curves with its end component meeting E, $(B_{s-q+2}^2) = -p \leq -3$ and $(A_{p-1}^2) = -r \leq -3$. Let m, β_i and α_j be the multiplicities of Q_1, B_i $(1 \leq i \leq s)$ and A_j $(1 \leq j \leq p-1)$ in $\bar{\Gamma}$, respectively. Set $\beta = \beta_{s-q+2}$ and $\beta' = \beta_{s-q+3}$. We note that note of $A_1, \ldots, A_{p-2}$ represents a branching point of the dual graph of $\bar{\Gamma}$.

A straightforward computation shows that

$$\beta' = p\beta - \beta_{s-q+1} \geq (p-1)\beta + 1 \qquad \text{as } \beta > \beta_{s-q+1} ,$$

$$\cdots\cdots$$

$$\beta_s = (q-2)\beta' - (q-3)\beta ,$$

$$m = (q-1)\beta' - (q-2)\beta_s\,,$$

$$\alpha_1 = 2m - (m-1) - \beta_s = \beta' - \beta + 1 \geq (p-2)\beta + 2\,,$$

$$\alpha_2 = q\alpha_1 - m = \beta' - 2\beta + q\,,$$

$$\cdots\cdots$$

$$\alpha_{p-1} = \beta' - (p-1)\beta + (p-2)q - (p-3) \geq (p-2)q - p + 4\,.$$

Let δ be the contribution of the components B_{s-q+2}, A_1 and A_{p-1} in $(\bar{\Gamma}\cdot K_{\bar{V}})$. Then we have

$$\delta \geq (p-2)\beta + (q-2)\{(p-2)\beta + 2\} + (r-2)\{(p-2)q - p + 4\}$$

$$= (p-2)(q-1)(r+\beta-2) + 2(q+r) - 8\,.$$

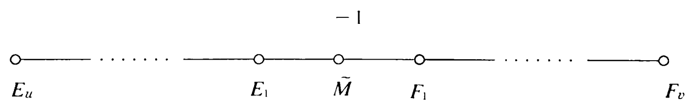
Hence we easily obtain the following:

Lemma 6. *With the notations and assumptions as above, the following assertions hold*:

(1) $\delta \geq 8$ *if* $\beta = 1$ *and* $\delta \geq 10$ *if* $\beta \geq 2$.

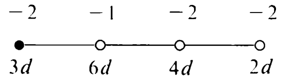
(2) $\delta = 8$ *if and only if* $s = q-1$, $\beta = 1$ *and* $p = q = r = 3$.

If we contract E and all components of the trunk and all branches except for the leftmost component of the trunk and those of two branches at the base, the dual graph G of the image of $\bar{\Gamma}$ looks like:

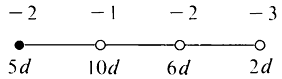


where $(E_i^2) \leq -2$ and $(F_j^2) \leq -2$. If we contract $\tilde{M}$ and all contractible components of the graph G, we will obtain the boundary dual graph of a minimal normal completion of $\mathbf{A}^2$. Note that the proper transform M of $\tilde{M}$ in $\bar{\Gamma}$ has self-intersection number -2. Let δ be the contribution of the components E_i $(1 \leq i \leq u)$ and F_j $(1 \leq j \leq v)$ to $(\bar{\Gamma}\cdot K_{\bar{V}})$. With the notations of Lemma 1, we suppose that the image $\tilde{C}$ of the smooth fiber F_0 meets $\tilde{M}$ in a one-place point P with $(\tilde{C}\cdot\tilde{M}) = d > 0$. We have the following result:

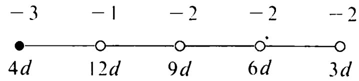
Lemma 7. *Assume that* $\delta \leq 8$. *Then the following list exhausts all possible graphs of* G, *where the positive multiples of* d *attached to the vertices indicate the multiplicities of the corresponding components in* $\bar{\Gamma}$:

(1) $\delta=0$ 

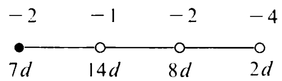
$(d\geq 1)$

(2) $\delta=2d$, 

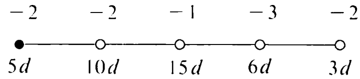
$(1\leq d\leq 4)$

(3) $\delta=4d$, 

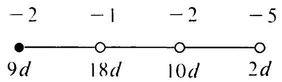
$(d=1,2)$

(4) $\delta=4d$, 

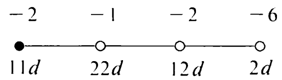
$(d=1,2)$

(5) $\delta=6d$, 

$(d=1)$

(6) $\delta=6d$, 

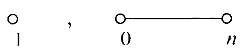
$(d=1)$

(7) $\delta=8d$, 

$(d=1)$

A black circle in the list indicates that the corresponding component or an exceptional curve obtained by blowing up points of the component can be brought to the line at infinity of $\mathbf{P}^2$.

Proof. Starting with $\tilde{M}$, contract all possible components of the graph G. In the course of contractions, when a component with self-intersection 0, say A, is produced for the first time, other components have negative self-intersection. Furthermore, if there are two components B_1, B_2 meeting A then $(B_1^2)\leq -2$ or $(B_2^2)\leq -2$. Then, by Morrow's list [6], the boundary dual graph of a minimal normal completion of $\mathbf{A}^2$ which is obtained from the graph G is either one of the following:



$(n\leq -2)$

We then retrieve the dual graph G by the reverse process of blowing-ups. Note that we have to choose a center of blowing-up on components with self-intersection ≥ -1 whenever such components exist; if there are two such components, the center is necessarily the intersection point of the two components. The resulting graphs appearing in the course of the blowing-ups must be linear chains.

On the other hand, as $\tilde{C}$ meets the component $\tilde{M}$ in a one-place point P

with $(\tilde{C} \cdot \tilde{M}) = d > 0$, the multiplicity of $\tilde{C}$ at P is d as well because the blowing-up with center P separates the proper transforms of $\tilde{C}$ and $\tilde{M}$. If we obtain a minimal normal completion of $\mathbf{A}^2$ by the above-mentioned contraction and transform it to $\mathbf{P}^2$ by a birational morphism with $\mathbf{A}^2$ kept intact, we obtain an irreducible curve $\bar{C}_1$ with a one-place point P_1 on the line at infinity, which might differ from the curve $\bar{C}$ we started with. We can express the degree of $\bar{C}_1$ and the multiplicity at P_1 in terms of d. Then we can determine the multiplicities of the components of G in $\bar{\Gamma}$ and hence their contributions to $(\bar{\Gamma} \cdot K_{\bar{V}})$. The rest is a straightforward computation. Q.E.D.

2. Minimal embeddings and Abhyankar-Moh theory

Let C be as in the sections 1 and 2. Suppose that $C^{(1)} := \varphi(C)$ has the smallest degree for an automorphism φ of $\mathbf{A}^2$. Let φ^* be the associated ring automorphism of $k[x, y]$. We may view φ^* as defining a (non-linear) change of corrdinates $x' = \varphi^*(x)$, $y' = \varphi^*(y)$. Namely, we consider φ as a Cremona transformation of $\mathbf{P}^2$ keeping $\mathbf{A}^2$ intact. As we defined $\sigma\colon V \to \mathbf{P}^2$ and $\tau\colon V \to \bar{V}$ for C and its closure $\bar{C}$, we similarly define $\sigma^{(1)}\colon V^{(1)} \to \mathbf{P}^2$ and $\tau^{(1)}\colon V^{(1)} \to \bar{V}^{(1)}$ for $C^{(1)}$ and its closure $\bar{C}^{(1)}$. Then $V^{(1)}$ has a minimal fibration $\bar{\rho}^{(1)}\colon \bar{V}^{(1)} \to \mathbf{P}^1$ such that the proper transform $F_0^{(1)}$ of the closure $\bar{C}^{(1)}$ is a fiber. Let $\bar{\Gamma}^{(1)}$ be the fiber of $\bar{\rho}^{(1)}$ containing the proper transform of the line at infinity of $\mathbf{P}^2$.

Let $\psi\colon \bar{V} \to \bar{V}^{(1)}$ be the birational mapping induced by the Cremona transformation $\varphi\colon \mathbf{P}^2 \to \mathbf{P}^2$. Since φ maps the curves C_λ defined by $f = \lambda$ isomorphically to the curves $C_\lambda^{(1)}$ defined by $f^{(1)} = \lambda$, where $\lambda \in k$ and C, $C^{(1)}$ are defined by $f = 0$, $f^{(1)} = 0$, respectively, ψ is a fiber-preserving isomorphism between $\bar{V} - \bar{\Gamma}$ and $\bar{V}^{(1)} - \bar{\Gamma}^{(1)}$, and ψ is decomposed as $\psi = \theta^{(1)} \cdot \theta^{-1}$, where $\theta\colon W \to \bar{V}$ and $\theta^{(1)}\colon W \to \bar{V}^{(1)}$ are the composites of blowing-ups with centers on $\bar{\Gamma}$ and $\bar{\Gamma}^{(1)}$, respectively. Meanwhile, since $\bar{\Gamma}$ and $\bar{\Gamma}^{(1)}$ have no (-1) components, $\theta^{(1)}$ must coincide with θ upto an isomorphism near $\bar{\Gamma}$ and $\bar{\Gamma}^{(1)}$. Namely, the mapping $\psi\colon \bar{V} \to \bar{V}^{(1)}$ is an isomorphism such that $\bar{\rho} = \bar{\rho}^{(1)} \cdot \psi$. Hence we know the following:

Lemma 8. *With the notations and assumptions as above, the curve $\bar{C}^{(1)}$ with its embedding into $\mathbf{P}^2$ is obtained from $\bar{V}$ by first contracting E and all contractible components of $\bar{\Gamma}$ to obtain a minimal normal completion of $\mathbf{A}^2$ and then transforming it to $\mathbf{P}^2$ by such a birational transformation that the image $\bar{C}^{(1)}$ of the fiber F_0 has the smallest degree.*

In the next section we shall consider the cases of low genus $g = 1, 2, 3, 4$. Our approach is to classify all possible types of the dual graphs of $\bar{\Gamma}$ and to verify then the existence or non-existence of curves with given types of the dual graphs. The following result of Sathaye-Stenerson [11] based on Abhyankar-Moh's theory of approximate roots is a crucial criterion for the existence of such curves.

A sequence of positive integers $(\delta_0, \ldots, \delta_h)$ is said to be a *characteristic δ-sequence* if it satisfies the following three conditions:

1. Set $d_i = \gcd(\delta_0, \ldots, \delta_{i-1})$ for $1 \leq i \leq h+1$. Set $n_i = d_i/d_{i+1}$ for $1 \leq i \leq h$. Then $d_{h+1} = 1$ and $n_i > 1$ for all $2 \leq i \leq h$.
2. $\delta_i n_i \in \langle \delta_0, \ldots, \delta_{i-1} \rangle =$ the semigroup generated by $\{\delta_0, \ldots, \delta_{i-1}\}$.
3. $\delta_i < \delta_{i-1} n_{i-1}$ for $i \geq 2$.

The semigroup $\Delta = \langle \delta_0, \delta_1, \ldots, \delta_n \rangle$ is called the *planer semigroup* generated by the δ-sequence $(\delta_0, \delta_1, \ldots, \delta_h)$. We define the conductor of Δ as

$$c(\Delta) = 1 - \delta_0 + \sum_{i=1}^{h} (n_i - 1)\delta_i .$$

Let C be an irreducible curve on $\mathbf{A}^2$ defined by $f(x, y) = 0$ such that the closure $\bar{C}$ of C on $\mathbf{P}^2$ has only one place at P_0 on the line at infinity. Let v be the (normalized) valuation of the function field $K = k(\bar{C})$ with center at P_0. We may (and shall) assume by a suitable change of coordinates x, y on $\mathbf{A}^2$ that $f = f(x, y)$ is monic in y with coefficients in $k[x]$ and that $\deg_x(f - ax^m) < m$ for $0 \neq a \in k$, where $m = \deg_x(f)$ and $n = deg_y(f)$ (cf. Abhyankar-Singh [2, Lemma (1.11)]). Furthermore, we assume that C is not rational. Then $n = -v(x)$ and $m = -v(y)$ (cf. *ibid.*). Let $g_i \in k[x, y]$ be the i-th approximate root of $f(x, y)$ for $1 \leq i \leq h$, which is monic in y and unique, and let $\delta_i = -v(g_i(x, y))$. Let $\delta_0 = -v(x) = n$ and $\delta_1 = -v(y) = m$. Then $(\delta_0, \delta_1, \ldots, \delta_h)$ is a characteristic δ-sequence. It is known that g_i is monic of degree d_1/d_i in y and that

$$\deg_y(f(x, y) - g_i(x, y)^{d_i}) < d_1 - (d_1/d_i) .$$

Furthermore, if $c(\Delta)$ is the conductor of the planar semigroup Δ generated by $(\delta_0, \delta_1, \ldots, \delta_h)$, the following genus formula is known (cf. Abhyankar-Singh [2]):

$$c(\Delta) = 2p_a(C) ,$$

where $p_a(C)$ is the arithmetic genus of the curve $\bar{C}$ with its singularities at the point P_0 and its infinitely near points all resolved.

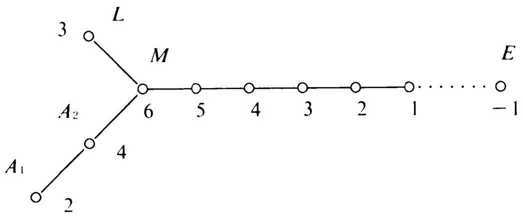
What we shall make use of is the following result [11, Theorem]:

Lemma 9. *Let $(\delta_0, \delta_1, \ldots, \delta_h)$ be a chracteristic δ-sequence. Then there exists an irreducible curve C with one place at infinity such that C is defined by $f(x, y) = 0$ with $f \in k[x, y]$ monic in y and that the degree semigroup of C is the planar semigroup generated by $(\delta_0, \delta_1, \ldots, \delta_h)$.*

3. Minimal embeddings in low genus

Let C be a smooth affine algebraic curve on $\mathbf{A}^2$ with only one place at infinity and let g be the geometric genus of C. We retain here all the notations and assumptions in the previous sections.

3.1. Case of $g = 1$. Since $(\bar{\Gamma} \cdot K_{\bar{V}}) = 0$, $\bar{\Gamma}$ consists of (-2) curves and the dual graph of $\bar{\Gamma}$ has therefore one branching point. The dual graph consists of three linear chains meeting in one vertex, and one of three branches meets the exceptional curve E. So, with the notations of Lemma 7, $\tilde{C}$ is smooth and $d=1$. By Lemma 7, the dual graph of $\bar{\Gamma}$ is:



where the attached integers indicate the multiplicities in $\bar{\Gamma}$. Then the contraction of all components of the trunk, M, A_2 and A_1 maps $\bar{V}$ to $\mathbf{P}^2$ with the image of L as the line at infinity l_∞. The image $\bar{C}$ of F_0 under the above contraction is a smooth curve of degree 3 and l_∞ is the inflectional tangent of $\bar{C}$. It is not hard to show that C is expressed as

$$y^2 = x^3 + ax + b$$

where $(x, y, 1)$ is a system of inhomogeneous coordinates of $\mathbf{P}^2$ and $a, b \in k$.

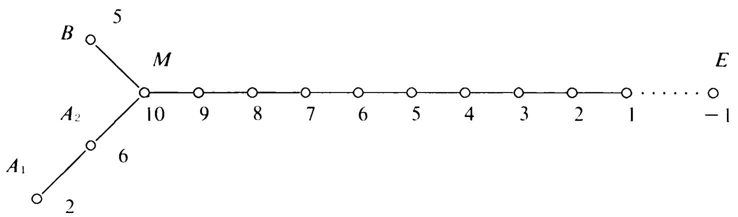
3.2. Case of $g=2$. Since $(\bar{\Gamma}\cdot K_{\bar{V}})=2$, $\bar{\Gamma}$ consists of (-2) curves except for

(i) two (-3) curves with multiplicity 1, or

(ii) one (-3) curve with multiplicity 2, or

(iii) one (-4) curve with multiplicity 1.

Consider, first, the case where the dual graph of $\bar{\Gamma}$ has only one branching point. Then the graph consists of three linear chains L_1, L_2 and L_3 meeting in one vertex, and one of the linear chains, say L_3, meets E. Then L_3 consists only of (-2) curves and the curve $\tilde{C}$ is smooth with $d=1$. By Lemma 7, the dual graph must be:



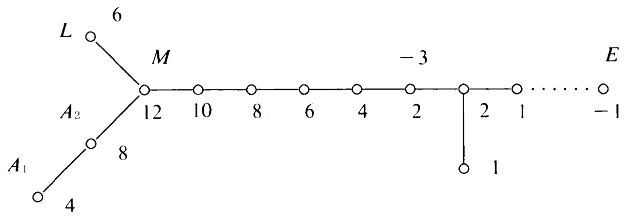
where $(A_1^2) = -3$. Blow up a point on B which is not the intersection point $B \cap M$ and let L be the exceptional curve. Then all components of $\bar{\Gamma}$ are contracted together with E, and the contraction brings $\bar{V}$ to $\mathbf{P}^2$ with the image of L as the line at infinity. The image $\bar{C}$ of F_0 is a curve of degree 5 meeting l_∞

in a one-place point P_0, where the multipliticy sequence of singularities is (3, 2). Then it is not hard to show that C is expressed as

$$y^2 = x^5 + ax^3 + bx^2 + cx + d\,,$$

where $(x, y, 1)$ is a system of inhomogeneous coordinates of $\mathbf{P}^2$ and $a, b, c, d \in k$.

Now suppose that the dual graph of $\bar{\Gamma}$ has more than one branching point. Lemmas 4 and 6 imply that the first branch of the graph consists of (-2) curves. Then the argument in Lemma 3 implies that the graph consists of (-2) curves and one (-3) curve located next to the first branching point. Then $d = (\tilde{C} \cdot \tilde{M}) = 2$ and the dual graph is given as follows:



The contraction of E and all components of $\bar{\Gamma}$ except for L brings $\bar{V}$ to $\mathbf{P}^2$ with the image of L as the line at infinity l_∞. Let $\bar{C}$ be the image of F_0 under the contraction. Then $\bar{C}$ is a curve of degree 6 with a one-place point P_0 on l_∞, and the multiplicity sequence of singularities at P_0 is (2^8), where 2^8 signifies that 2 is iterated 8 times. Now we apply Lemma 9. We have:

$$\delta_0 = d_1 = 4\,, \quad \delta_1 = 6\,, \quad d_2 = 2\,, \quad h = 2\,, \quad n_1 = n_2 = 2\,,$$

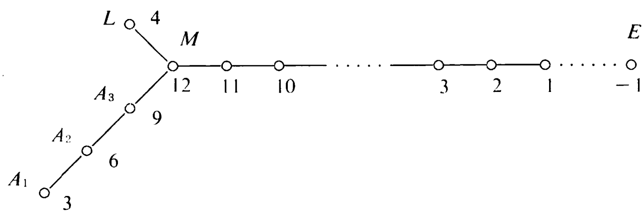
$$\delta_2 < 2\delta_1 = 12\,, \qquad 2\delta_2 \in \langle 4, 6 \rangle\,.$$

Then δ_2 is one of 3, 5, 7, 9, 11, while $c(\Delta) = \delta_2 + 3 = 2p_a(C)$ and $p_a(C) = 2$. This is a contradiction. Thus this case is impossible.

We can use a complete list of classification of degenerate fibers in a pencil of genus two (cf. Namikawa-Ueno [7] and Ogg [9]).

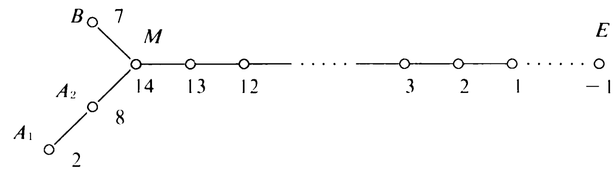
3.3. Case of $g = 3$. Since $(\bar{\Gamma} \cdot K_{\bar{V}}) = 4$, $\bar{\Gamma}$ may contain $(-a)$ curves with $a = 2, 3, 4, 5$. Consider first the case where the dual graph of $\bar{\Gamma}$ has only one branching point. Then, in view of Lemma 7, the following two cases are possible:

(I)



where $(L^2) = -3$ and all other components are (-2) curves.

(II)



where $(A_1^2) = -4$ and all other components are (-2) curves.

In the case (I) we obtain $\mathbf{P}^2$ from $\bar{V}$ by contracting E and all components of $\bar{\Gamma}$ except for L. The image of L is the line at infinity l_∞ and the image of $\bar{C}$ of F_0 is a smooth curve of degree 4. It is not hard to show that C is defined by an equation

$$y^3 + g_1(x)y = x^4 + g_2(x),$$

where $g_i(x) \in k[x]$ and $\deg g_i(x) \leq 2$ for $i = 1, 2$.

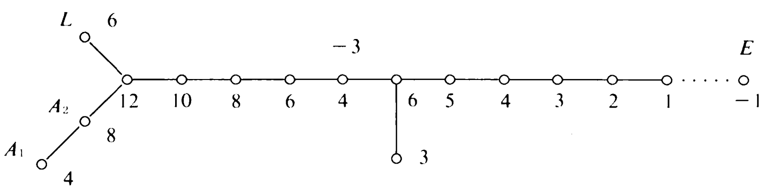
In the case (II), let σ_1 be the blowing-up of a point Q_1 on B such that $Q_1 \neq B \cap M$ and let σ_2 be the blowing-up of a point Q_2 on $\sigma_1^{-1}(Q_1)$ such that $Q_2 \neq \sigma_1^{-1}(Q_1) \cap \sigma_1'(B)$. Then contract E and all components of $\bar{\Gamma}$ as well as the proper transform $\sigma_2'(\sigma_1^{-1}(Q_1))$ to obtain $\mathbf{P}^2$ with the line at infinity l_∞, which is the image of $\sigma_2^{-1}(Q_2)$. Let $\bar{C}$ be the image of F_0 under the above birational mapping. Then $\bar{C}$ is a curve of degree 7 with a one-place point P_0 at infinity, and the multiplicity sequence of singularities at P_0 is $(5, 2^2)$. It is not hard to show that C is then defined by an equation

$$y^2 = x^7 + g(x),$$

where $g(x) \in k[x]$ and $\deg g(x) \leq 5$.

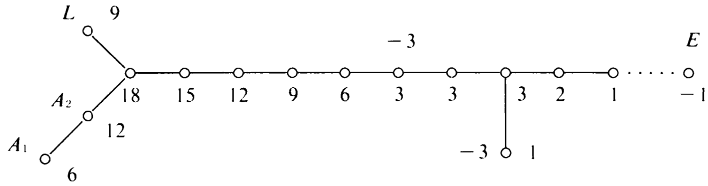
Consider next the case where the dual graph of $\bar{\Gamma}$ has more than one branching point. We then make use of Lemmas 3, 4, 6 and 7 to deduce the following two possibilities:

(III)



where all components are (-2) curves except for one (-3) component.

(*IV*)



where all components are (-2) curves except for two (-3) components.

In the case (III), contract E and all components of $\bar{\Gamma}$ except for L to bring $\bar{V}$ to $\mathbf{P}^2$. Then L gives rise to the line at infinity l_∞. The image $\bar{C}$ of F_0 is a curve of degree 6 with a one-place point P_0 on l_∞, and the multiplicity of singularities at P_0 is (2^7). Furthermore, we have

$$\delta_0 = d_1 = 4\,, \quad \delta_1 = 6\,, \quad d_2 = 2\,, \quad d_3 = 1\,, \quad n_1 = 2\,, \quad n_2 = 2\,,$$

$$\delta_2 \in \langle 2, 3 \rangle\,, \qquad \delta_2 \text{ is odd}\,, \quad \delta_2 < 12\,, \quad \text{and}$$

$$c(\Delta) = \delta_2 + 3 = 6\,.$$

So, $\delta_2 = 3$, and such a curve C exists by virtue of Lemma 9. A defining equation of C is given by

$$(x^3 + y^2)^2 + ax^4 + bx^3 + \frac{1}{4}a^2x^2 + cx + ax^2y^2 + by^2 + dy = 0\,,$$

with $d \neq 0$ (cf. [8]).

In the case (IV), we consider a similar contraction of $\bar{V}$ to $\mathbf{P}^2$ and obtain a curve of degree 9 with a one-place point P_0 on the line at infinity. The multiplicity sequence of singularities is $(3^8, 2)$. Furthermore, we have

$$\delta_0 = d_1 = 6\,, \quad \delta_1 = 9\,, \quad d_2 = 3\,, \quad d_3 = 1\,, \quad n_1 = 2\,, \quad n_2 = 3\,,$$

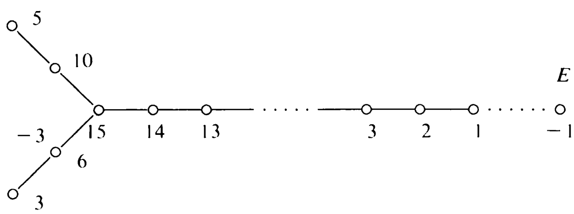
$$\delta_2 \in \langle 2, 3 \rangle\,, \quad 3 \nmid \delta_2\,, \quad \delta_2 < 18\,, \quad \text{and}$$

$$c(\Delta) = 2(\delta_2 + 2) = 6\,.$$

So, $\delta_2 = 1 \notin \langle 2, 3 \rangle$, a contradiction. Hence this case is impossible.

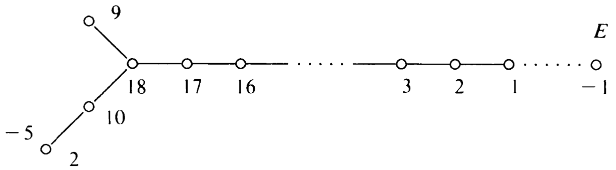
3.4. Case of genus 4. Note that $(\bar{\Gamma} \cdot K_{\bar{V}}) = 2g - 2 = 6$. The following list exhaust all possible dual graphs of $\bar{\Gamma}$.

(*I*)



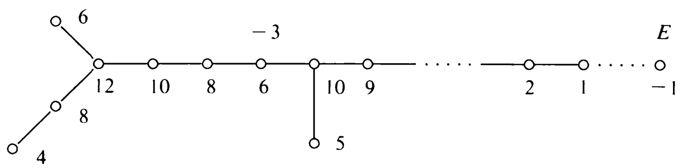
where $\deg \overline{C} = 5$ and the multiplicity sequence of singularities at the point at infinity is (2^2).

(*II*)



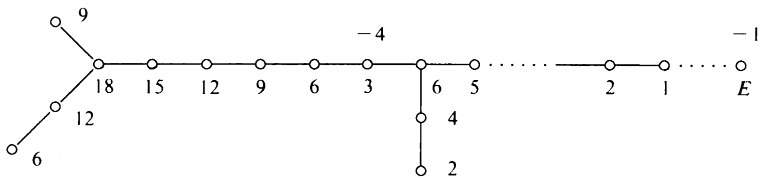
where $\deg \overline{C} = 9$ and the multiplicity sequence is $(7, 2^3)$.

(*III*)



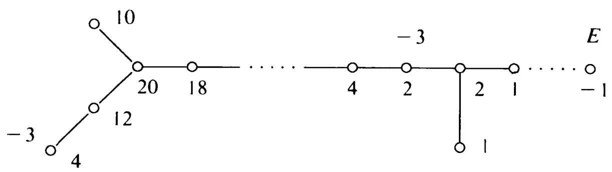
where $\deg \overline{C} = 6$ and the multiplicity sequence is (2^6).

(*IV*)



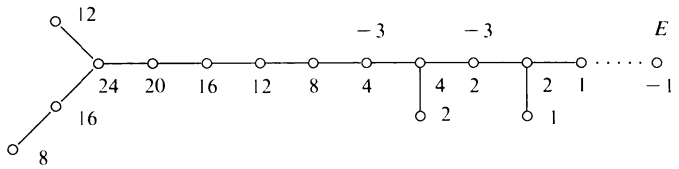
where $\deg \overline{C} = 9$ and the multiplicity sequence is (3^8).

(*V*)



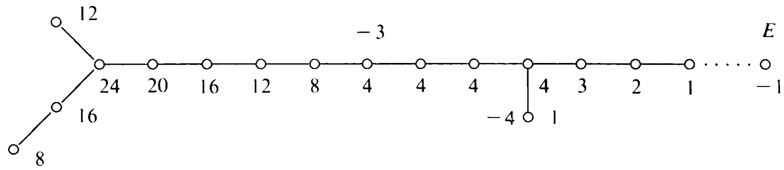
where $\deg \overline{C} = 10$ and the multiplicity sequence is $(6, 4, 2^{11})$.

(VI)



where $\deg \bar{C} = 12$ and the multiplicity sequence is $(4^8, 2^3)$.

(VII)



where $\deg \bar{C} = 12$ and the multiplicity sequence is $(4^8, 3)$.

The cases (I) ~ (IV) do exist. In the case (I), $\bar{C}$ is defined by an equation

$$y^3 + g_1(x)y = x^5 + g_2(x),$$

where $g_i(x) \in k[x]$ and $\deg g_i(x) \leq 3$ for $i = 1, 2$. In the case (II), $\bar{C}$ is defined by an equation

$$y^2 = x^9 + g(x), \qquad \deg g(x) \leq 7.$$

In the cases (III) and (IV), we know the existence of $\bar{C}$ by Lemma 9. According to [8], C is defined respectively in the cases (III) and (IV) by

$$(x^3 + y^2)^2 + ax^4 + bx^3 + cx^2 + dx + axy^2 + by^2 + exy + fy = 0, \qquad (e \neq 0)$$

$$(x^3 + y^2)^3 + 3ax^7 + bx^6 + 3a^2x^5 + 2abx^4 + cx^3 + a^2bx^2 + dx$$
$$+ 6ax^4y^2 + 2bx^3y^2 + 3a^2x^2y^2 + 2abxy^2 + (c - a^3)y^2$$
$$- 6axy^4 + by^4 = 0, \qquad (d \neq a^4 - 8ac).$$

By the same lemma, we can show by the same reasoning as in the cases of $g = 2, 3$ that the remaining cases (V) ~ (VII) are impossible.

Thus we have completed a proof of our theorem stated in the introduction. We finally note that the above argument can be applied to the case of higher genus, though more complicated classification of possible types of the dual graph of $\bar{\Gamma}$ will be involved.

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