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Ropes in projective space

By

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Let C be a degree d non-degenerate integral curve in $\mathbf{P}^r$. In 1983, a remarkable paper of L. Gruson, R. Lazarsfeld, and C. Peskine showed, among other results, that C must be $(d+2-r)$-regular [18]. Such a theorem bounding the regularity in terms of d and r alone is not possible for non-reduced schemes. By considering the genus as well as the degree, Gotzmann was able to obtain bounds for the regularity of an arbitrary nonreduced one-dimensional scheme [17]. If no conditions are placed on the genus, one can construct non-degenerate locally Cohen-Macaulay schemes of degree two with arbitrarily high regularity. In general, one can construct multiplicity two structures on any curve such that the homogeneous ideal has generators in arbitrarily high degree. Multiplicity two structures on the line are called double lines and they provide us with our first example of a ribbon.

In 1986, the first author showed that double lines and their deficiency modules exhibit a form of extremal behavior with respect to liaison [27]. In 1993, M. Martin-Deschamps and D. Perrin obtained several nice bounds on the Hartshorne-Rao (or deficiency) module for an arbitrary 1-dimensional locally Cohen-Macaulay scheme [24]. Double lines exhibit extremal behavior with respect to these bounds as well. Multiplicity two structures arise naturally in questions concerning self-linkage. Rao was able to utilize this fact to obtain restrictions on the cohomology of rank two vector bundles on $\mathbf{P}^4$. Here we see that for questions concerning both regularity and liaison, relatively simple non-reduced schemes can provide us with quite interesting behavior.

Let C be a smooth and irreducible curve in $\mathbf{P}^n$ with homogeneous ideal I, and let Y be a subscheme of $\mathbf{P}^n$ with ideal J. We will call Y an *α-rope* on C if the ideal J satisfies $I^2 \subset J \subset I$ and Y is a locally Cohen-Macaulay multiplicity α structure on C. (A general definition of a rope can be found for instance in Chandler's thesis, c.f. [5], but it is straightforward to check that this general definition coincides with the one given above in the case of a smooth and irreducible curve, when everything is embedded in $\mathbf{P}^n$.) A 2-rope is called a *ribbon.*

In this paper we are interested in certain aspects of the study of ribbons and ropes. Ferrand showed that on any smooth integral curve, there exists

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a ribbon which is subcanonical [13]. As mentioned before, Rao was able to apply complete intersection ribbons to questions concerning the cohomology of rank 2 vector bundles in $\mathbf{P}^4$ [33]. Fundamental work on the scheme structure of a ribbon was carried out by Bayer and Eisenbud in [1]. Recent work on ribbons by Eisenbud and Harris has led to improve bounds on the degree of sub-linebundles of the normal bundle of non-degenerate integral curves in $\mathbf{P}^r$ [9]. Similar questions were studied for ropes by Chandler [5]. Fong was able to give some results relating ribbons to the degeneration of smooth curves [14]. Ribbons and ropes certainly merit further study! In this paper, we focus on questions concerning the regularity and deficiency modules of these non-reduced objects.

In the first section we include background information and first results. Beginning with a smooth irreducible curve, C, in $\mathbf{P}^n$ with homogeneous ideal I, we choose a regular sequence, $F_1, \ldots, F_k$, in such a way that the ideal $T = (F_1, \ldots, F_k)$ satisfies $T \subset I$ and the scheme defined by T is smooth at the general point of C. This allows for the construction of a particular exact sequence. Using a result of Chandler, one can compute the genus of the scheme defined by I^2 in terms of the degree and genus of C. By combining the aforementioned genus calculation and exact sequence with two other exact sequences we compute the Hilbert polynomial of (I^2, T). Using this result we determine

Corollary 0.1. *Let C be a smooth irreducible curve with homogeneous ideal I in $\mathbf{P}^n$. Every $(n-k)$-rope supported on C is defined, up to embedded points, by an ideal of the form $(I^2, F_1, \ldots, F_k)$ where $F_1, \ldots, F_k$ form a regular sequence.*

This leaves open the possibility of two separate families of ropes. The first family would consist of ropes that can be defined by ideals of the form $(I^2, F_1, \ldots, F_k)$ without having to remove embedded points. The second family would consist of ropes which can not be defined in this way. We show that each family is non-empty. A natural path of investigation is to find other ways to distinguish between the two families. By showing that the embedded points of the scheme defined by $(I^2, F_1, \ldots, F_k)$ occur precisely at the singularities of the scheme defined by $(F_1, \ldots, F_k)$ along C we determine

Corollary 0.2. *Let C be a smooth irreducible curve with homogeneous ideal I in $\mathbf{P}^n$. Let J be the homogeneous ideal of an $(n-k)$-rope supported on C. J is the saturation of an ideal of the form $(I^2, F_1, \ldots, F_k)$ if and only if there exists a regular sequence $(G_1, \ldots, G_k)$ in J such that the corresponding scheme is smooth along C.*

Associated to any locally Cohen-Macaulay curve, Y, in $\mathbf{P}^n$ is a finite length graded module $M(Y) = \bigoplus_{k \in \mathbf{Z}} H^1(\mathbf{P}^n, \mathcal{I}_Y(k))$ over the polynomial ring $S = K[X_0, \ldots, X_n]$. In the context of liaison, $M(Y)$ is often called the deficiency or Hartshorne-Rao module of Y. The second section is concerned with the study of the deficiency module of a ribbon in $\mathbf{P}^n$. Let J be the homogeneous ideal of a ribbon supported on an arithmetically Cohen-Macaulay curve C in $\mathbf{P}^3$ where

C is not a complete intersection. If J contains an element which is smooth along C then a short argument shows that J has non-trivial deficiency module. With more work we arrive at the main theorem of section two which describes the deficiency module of a ribbon supported on an arithmetically Cohen-Macaulay curve, C, in $\mathbf{P}^n$. In the special case where C is a complete intersection in $\mathbf{P}^3$, we get a very clean and precise description.

Theorem 0.3. *Let C be a complete intersection in $\mathbf{P}^3$ with ideal $I = (G_1, G_2)$. Let $F = AG_1 + BG_2$ be smooth along C. Let Y be the ribbon defined by (I^2, F). If* $\deg(A)\cdot\deg(B) = 0$ *then $M(Y) = 0$. Otherwise*

$$M(Y) \cong \left(\frac{S}{(A, B, G_1, G_2)}\right)(\deg(F) - \deg(G_1G_2)).$$

This provides a natural generalization to the case of the deficiency module of a double line in $\mathbf{P}^3$ [27]. Natural questions arise as to when such ribbons are linked to one another or to other ribbons; this second question appears to be rather hard in general.

Given a coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n$, the regularity of $\mathscr{F}$, $\operatorname{reg}(\mathscr{F})$, is the smallest integer r with $H^i(\mathbf{P}^n, \mathscr{F}(k-i)) = 0$ for all $k \geq r$ and $i > 0$. We denote by $e(C)$ the index of speciality of C; that is, $e(C) = \max\{t \in \mathbf{Z} \mid h^2(\mathscr{I}_C(t)) \neq 0\}$. In the third section we bound the regularity of the ideal sheaf of ropes in $\mathbf{P}^n$. We first show that if $\mathscr{I}$ is the ideal sheaf of a one-dimensional scheme in $\mathbf{P}^n$ then $\operatorname{reg}(\mathscr{I}^n) \leq n\cdot\operatorname{reg}(\mathscr{I})$. If Y' is a one-dimensional subscheme of $\mathbf{P}^n$ then we let Y denote the union of the top dimensional components of Y'. We show that $\operatorname{reg}(Y) \leq \operatorname{reg}(Y')$. In the first section, every rope on a curve C is shown to be obtained by taking the union of the top dimensional components of an ideal of the form $(I^2, F_1, \ldots, F_k)$. Combining all of this, we obtain the main result of section three.

Theorem 0.4. *Let C be a smooth irreducible curve in $\mathbf{P}^n$ with homogeneous ideal I_C and ideal sheaf $\mathscr{I}_C$. Let Y be an $(n-k)$-rope on C with homogeneous ideal I_Y and ideal sheaf $\mathscr{I}_Y$. Let $F_1, \ldots, F_k$ be a regular sequence in I_Y such that the scheme defined by $(F_1, \ldots, F_k)$ is smooth at the general point of C. Let d_i denote the degree of F_i. Then we have an inequality* $\operatorname{reg}(\mathscr{I}_Y) \leq \max\{2\operatorname{reg}(\mathscr{I}_C), e(C) + d_i + 2, d_1 + \cdots + d_k - (k-1)\}$.

For ribbons in $\mathbf{P}^3$ we get a simpler statement.

Corollary 0.5. *Let C be a smooth irreducible curve in $\mathbf{P}^3$ with homogeneous ideal I_C and ideal sheaf $\mathscr{I}_C$. Let Y be a ribbon on C with homogeneous ideal I_Y and ideal sheaf $\mathscr{I}_Y$. Let d be the smallest degree such that there exists an $F \in I_Y$ smooth at the general point of C. Then* $\operatorname{reg}(\mathscr{I}_Y) \leq \max\{2\operatorname{reg}(\mathscr{I}_C), e(C) + d + 2\}$.

In the special case where the ribbon lies on a smooth surface, this bound on the regularity can be written in terms of the regularity of C, the genus of C, and the degree and genus of the ribbon. We further note that by the result of [18], we can then write this bound purely in terms of the genus of C and the degree and genus of the ribbon.

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1. Background and first results

Throughout the paper, S will denote the polynomial ring $K[X_0, \ldots, X_n]$, where K is an algebraically closed field of characteristic zero. If we start with a $\wp$-primary ideal I then the ideal I^2 might fail to be $\wp$-primary. However, the highest dimensional component in a primary decomposition of I^2 will be $\wp$-primary. In fact, the highest dimensional components in any two primary decompositions of I^2 will agree. This well-defined component is known as the symbolic square of I and is given the special symbol $I^{(2)}$. More generally we can define the symbolic n^{th} power, $I^{(n)}$, of I as the unique $\wp$-primary component in a primary decomposition of I^n. If I is the ideal of a reduced curve, C, in $\mathbf{P}^n$ then $I^{(n)}$ defines a locally Cohen-Macaulay scheme supported on C. It is a somewhat surprising fact that for such an I, $I^{(n)}$ can be readily computed via the formula $I^{(n)} = \text{annihilator}\,(\text{Ext}^{n-1}(I^n, S))$ [11]. This has been implemented as a script in the computer algebra system Macaulay [3]. Unless otherwise stated, in what follows we will assume I is a prime ideal defining a smooth curve C in $\mathbf{P}^n$. For such an ideal, $I^{(n)} = \bar{I}^n$ (where $\bar{I}^n$ denotes the saturation of I^n). This means essentially that I^n picks up no non-irrelevant embedded primes. According to Zariski's main lemma on holomorphic functions (cf. [10], [37]), $I^{(n)}$ is the largest ideal all of whose elements vanish to order at least n on C.

We are interested in finding an expression for the Hilbert polynomial of an ideal of the form $(I^2, F_1, F_2, \ldots, F_k)$. We will assume that $F_1, F_2, \ldots, F_k$ form a regular sequence in I and that the scheme defined by $(F_1, \ldots, F_k)$ is smooth at the general point of C. We proceed via several lemmata. Our first lemma enables us to give several formulations to the same problem. This will prove useful both for explicit computations and for future insight. The second lemma sets the stage for a key exact sequence. The third lemma provides us with the final ingredient necessary for our calculation.

Lemma 1.1. *Let $\wp$ be the homogeneous ideal of an irreducible variety X in $\mathbf{P}^n$. Let $F_1, \ldots, F_k$ be a regular sequence in $\wp$ and denote by Y the complete intersection defined by the F_i. The following are equivalent:*

a) *Y is smooth at a general point of X.*

b) *the local ring $S_\wp/(F_1, \ldots, F_k)S_\wp$ is regular.*

c) $F_1 \notin \wp^{(2)}$, $F_2 \notin (\wp^2, F_1)^{(1)}$, $\ldots$, $F_k \notin (\wp^2, F_1, \ldots, F_{k-1})^{(1)}$.

Where the "$^{(1)}$" denotes the $\wp$-primary component of the corresponding ideal.

Proof. We have that Y is smooth at the general point of X if and only if the $k \times k$-minors of the Jacobian matrix $(\partial/\partial x_i F_j)$ do not all vanish identically on X. This is the case if and only if these minors are not all contained in the ideal $\wp$. By the Jacobian criterion for smoothness (cf. [25] Theorem 30.3), this occurs if and only if the ring $S_\wp/(F_1, \dots, F_k)S_\wp$ is regular (as the ground field is algebraically closed). This establishes the equivalence between a) and b).

The ring $S_\wp$ is a regular local ring, hence condition b) is equivalent to the fact that the $F_1, \dots, F_k$ are part of a regular system of parameters of the maximal ideal $\wp S_\wp$ of $S_\wp$ [25] (we write F_i for the image of F_i in $S_\wp$). This, in turn, is equivalent to the following condition:

$$F_1 \notin \wp^2 S_\wp,\ F_2 \notin (\wp^2, F_2)S_\wp,\ \dots,\ F_k \notin (\wp^2, F_1, \dots, F_{k-1})S_\wp\,.$$

Now to conclude the proof, use the fact that given an ideal J with a primary decomposition $J = Q \cap Q_1 \cap \cdots \cap Q_l$, where Q is $\wp$-primary and $\wp$ is not an embedded prime, one has the equality $Q = JS_\wp \cap S$.

Lemma 1.2. *Let I be the ideal of a smooth irreducible curve in $\mathbf{P}^n$. Let $F_1, \dots, F_k$ be a regular sequence in I, such that the complete intersection defined by the F_i is smooth at the general point of C. Let $d_i = \deg(F_i)$. Then the map*

$$\bigoplus I(-d_i) \to I^2 \cap (F_1, \dots, F_k)$$

that sends $(G_1, \dots, G_k)$ to $\sum G_i F_i$ is a surjection.

Proof. The given map is certainly well defined. We will show that if $G_1, \dots, G_k$ are forms such that $\sum G_i F_i \in I^2$, then each of the G_i has to be in I, and this will establish Lemma 1.2. Let P be a general point of C. We can differentiate the relation

$$\sum G_i F_i \in I^2$$

and evaluate the differentiated expression at P. For each j, we obtain

$$0 = \sum_i \partial/\partial x_j G_i(P) F_i(P) + G_i(P) \partial/\partial x_j F_i(P) = \sum_i G_i(P) \partial/\partial x_j F_i(P)\,.$$

The first equality follows from the expression being an element of I^2. The second equality follows from the F_i being elements of I. Now, as the complete intersection defined by the F_i is smooth at P, we have that the rank of the matrix $(\partial/\partial x_j F_i(P))$ is k. This implies that the only solution to the system $(\sum_i \partial/\partial x_j F_i(P) G_i(P) = 0,\ j = 0, \dots, n)$ is the zero solution. In other words, we have $G_i(P) = 0$ for each i. As this is true for a general point of C, this implies that $G_i \in I$ for each i. This concludes the proof of Lemma 1.2.

Corollary 1.3. *Let $T = (F_1, F_2, \dots, F_k)$. We have an exact sequence*

$$0 \to \textstyle\bigwedge^k (\bigoplus_i S(-d_i)) \to \bigwedge^{k-1} (\bigoplus_i S(-d_i)) \to \cdots \to \bigwedge^2 (\bigoplus_i S(-d_i))$$
$$\xrightarrow{\varphi} \textstyle\bigoplus_i I(-d_i) \to I^2 \cap T \to 0\,.$$

Proof. Note that the image of ϕ is in $\bigoplus_i I(-d_i)$; then look at the Koszul relations on the map from $\bigoplus_i I(-d_i)$ to $I^2 \cap T$.

Let Y be a nonsingular subvariety of a nonsingular variety X. Let $\mathscr{I}$ denote the ideal sheaf of Y in X. Recall that the normal sheaf $\mathscr{N}_{Y/X}$ of Y in X is defined by $\mathscr{N}_{Y/X} = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_X)$. $\mathscr{N}_{Y/X}$ is a locally free sheaf on Y. If $Y = C$ is a smooth irreducible curve and $X = \mathbf{P}^n$ then $\mathscr{N}_{C/\mathbf{P}^n}$ will be locally free of rank $n-1$. Let J be the homogeneous ideal of a locally Cohen-Macaulay scheme, D, supported on C. If J satisfies $I^2 \subseteq J \subseteq I$ then D is the embedding of a *rope* on C. If the degree of D is α times the degree of C then D is the embedding of an *α-rope* on C. Associated with any α-rope on C embedded in $\mathbf{P}^n$ is a rank $\alpha - 1$ locally free subsheaf, E, of $\mathscr{N}_{C/\mathbf{P}^n}$. In the special case $\alpha = 2$ we call D a *ribbon* on C. In the thesis of Chandler [5] we find an expression relating the genus of this α-rope to the genus of the underlying curve and the degree and rank of E. The scheme defined by I^2 is an n-rope on C corresponding to $E = \mathscr{N}_{C/\mathbf{P}^n}$. This formula allows us to determine the genus of the scheme defined by I^2.

Lemma 1.4. *Let C be a smooth curve in $\mathbf{P}^n$ with homongeneous ideal I, degree $\deg(C)$ and genus g. The genus of the scheme defined by I^2 is $G = (n+1)\cdot(\deg(C) + g - 1) + g$.*

Proof. From [5] the genus of the scheme defined by I^2 is $G = \text{degree}(\mathscr{N}_{C/\mathbf{P}^n}) - n\cdot(1-g) + 1$. Letting ω_C denote the canonical sheaf of C, we have $\omega_C \cong \omega_{\mathbf{P}^n} \otimes \bigwedge^{n-1} \mathscr{N}_{C/\mathbf{P}^n}$ [20]. Considering degrees of both sides we get $2g - 2 = \text{degree}(\mathscr{N}_{C/\mathbf{P}^n}) - (n+1)\cdot\deg(C)$. We rewrite this expression to obtain $\text{degree}(\mathscr{N}_{C/\mathbf{P}^n}) = 2g - 2 + (n+1)\cdot\deg(C)$. Plug $\text{degree}(\mathscr{N}_{C/\mathbf{P}^n})$ into the above expression for G to yield the desired result.

Again assume that I is a prime ideal defining a smooth curve C in $\mathbf{P}^n$. As mentioned above, we are interested in schemes defined by ideals of the form $(I^2, F_1, F_2, \ldots, F_k)$. The F_i are chosen as in Lemma 1.2. One would expect that for "small" k such an expression would lead to a nonreduced scheme supported on C. This becomes more clear when one considers that $I^2 \subseteq (I^2, F_1, F_2, \ldots, F_k) \subseteq I$. To help understand this scheme, we compute the Hilbert polynomial of $(I^2, F_1, F_2, \ldots, F_k)$ in terms of the Hilbert polynomial of I and the degrees of the F_i.

Theorem 1.5. *Let I be the homogeneous ideal of a smooth irreducible curve C in $\mathbf{P}^n$. Let $\deg(C)$ denote the degree of C. Let g denote the genus of C. Assume $F_1, \ldots, F_k$ form a regular sequence and the scheme defined by $T = (F_1, \ldots, F_k)$ is smooth at the general point of C. Let d_i denote the degree of F_i. The Hilbert polynomial of $(I^2, F_1, \ldots, F_k)$ is $H(t) = ((n-k)\cdot\deg(C))t - [(n+1-\sum_1^k d_i)\cdot(\deg(C)) + (n+2-k)\cdot(g-1) + 1] + 1$.*

Proof. We have three exact sequences

1) $0 \to I^2 \cap T \to I^2 \oplus T \to I^2 + T \to 0$
2) $0 \to \bigwedge^k(\bigoplus S(-d_i)) \to \cdots \to \bigwedge^2(\bigoplus S(-d_i)) \to \bigoplus I(-d_i) \to I^2 \cap T \to 0$
3) $0 \to \bigwedge^k(\bigoplus S(-d_i)) \to \cdots \to \bigwedge^2(\bigoplus S(-d_i)) \to \bigoplus S(-d_i) \to T \to 0$

Assume $t \gg 0$. From the three exact sequences we have

$$\begin{aligned}
\dim (S/(I^2 + T))_t &= \dim (S/I^2)_t + \dim (S/T)_t - \dim (S/(I^2 \cap T))_t \\
&= \dim (S/I^2)_t - \dim (T)_t - \dim (I^2 \cap T)_t \\
&= \dim (S/I^2)_t - \textstyle\sum_i \dim (S/I)_{t-d_i} \\
&= [n \deg (C)t - ((n+1)(\deg (C) + g - 1) + g) + 1] \\
&\quad - [\textstyle\sum_i (\deg (C)(t - d_i) - g + 1)] \\
&= (n-k) \deg (C)t + 1 \\
&\quad - [(n + 1 - \textstyle\sum d_i)(\deg (C)) + (n + 2 - k)(g - 1) + 1]
\end{aligned}$$

For large t, $\dim (S/(I^2 + T))_t = H(t)$ so we conclude that the Hilbert polynomial of (I^2, T) is as predicted.

Remark 1.6. Let C be a smooth irreducible arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$ with homogeneous ideal I and ideal sheaf $\mathscr{I}$. We have $I^2 = I^{(2)}$ (cf. [31]) and $H^1(\mathbf{P}^3, \mathscr{I}(k)) = 0$ for all k. Pick an element $F \in I$ with $F \notin I^{(2)}$. Let d be the degree of F. By Corollary 1.3 we have an isomorphism $I^2 \cap (F) \simeq I(-d)$. Consider the first exact sequence in the proof of Theorem 1.5. Using the isomorphism $I^2 \cap (F) \simeq I(-d)$, we write this sequence as

$$0 \to I(-d) \to I^2 \oplus (F) \to I^2 + (F) \to 0\,.$$

If we sheafify and consider the long exact sequence in cohomology then we can conclude that $I^2 + (F)$ is saturated.

Let $J = (I^2, F_1, \ldots, F_k)$. From the Hilbert polynomial we see that J defines a multiplicity $n - k$ structure on C. The scheme defined by J may have embedded points or J may fail to be saturated. By saturating J and removing embedded points we get a new ideal J'. J' is the homogeneous ideal of a locally Cohen-Macaulay multiplicity $n - k$ structure on C. Since we have $I^2 \subseteq J' \subseteq I$ the scheme defined by J' is an $(n - k)$-rope on C. A natural question is whether every $(n - k)$-rope on a given curve C in $\mathbf{P}^n$ arises by the method above. That is, given a smooth curve C in $\mathbf{P}^n$ with homogeneous ideal I, can the homogeneous ideal of every $(n - k)$-rope supported on C be obtained by starting with an ideal $J = (I^2, F_1, \ldots, F_k)$ and then saturating and removing embedded points? This is answered by the following corollary.

Corollary 1.7. *If C is a smooth irreducible curve in $\mathbf{P}^n$ and I is the homogeneous ideal of C then every $(n - k)$-rope supported on C is defined by an ideal of the form $(I^2, F_1, \ldots, F_k)$ up to embedded points.*

Proof. Let J be the homogeneous ideal of an $(n-k)$-rope supported on C. We have $I^2 \subseteq J \subseteq I$. If $k > 0$ then $J \neq I^{(2)}$ so there is an element $F_1 \in J$ with $F_1 \notin I^{(2)}$. By Lemma 1.1, the scheme defined by F_1 is smooth at the general point of C. By Theorem 1.5 the scheme defined by (I^2, F_1) has degree $(n-1)\cdot\deg(C)$. If $k > 1$ then $J \neq (I^2, F_1)^{(1)}$ (where the "$^{(1)}$" denotes the I-primary part of the corresponding ideal) so there is an element $F_2 \in J$ with $F_2 \notin (I^2, F_1)^{(1)}$. By Lemma 1.1, the scheme defined by (F_1, F_2) is smooth at the general point of C. F_1, F_2 form a regular sequence so by Theorem 1.5 the scheme defined by (I^2, F_1, F_2) has degree $(n-2)\cdot\deg(C)$. Continuing in this manner we can pick $F_1, F_2, \ldots, F_k$ satisfying the conditions of part c) of Lemma 1.1. Note that at each step, we have to choose the F_i in sufficiently high degrees in order to guarantee that $(F_1, \ldots, F_k)$ form a regular sequence. Using Theorem 1.5, the scheme defined by $(I^2, F_1, \ldots, F_k)$ has degree $(n-k)\cdot\deg(C)$. Since $(I^2, F_1, \ldots, F_k) \subseteq J$ and since both ideals define schemes with the same degree we know the schemes differ at most by lower dimensional components. This is equivalent to saying that the $(n-k)$-rope, J, is defined by the ideal $(I^2, F_1, \ldots, F_k)$ up to embedded points.

We now want to give two examples of ribbons in $\mathbf{P}^3$ (i.e. 2-ropes in $\mathbf{P}^3$). If the scheme defined by (I^2, F) is already locally Cohen-Macaulay then (I^2, F) defines a ribbon (without having to remove embedded points). A somewhat trivial example is given by considering the scheme defined by $J = (x^2, y)$ in $k[w, x, y, z]$. It is clear that J defines a locally Cohen-Macaulay scheme of degree two supported on the line defined by (x, y) and is thus a ribbon. We can write $J = (x^2, xy, y^2, y) = ((x, y)^2, y)$. It is a natural question to ask if the homogeneous ideal of every ribbon can be obtained by saturating an ideal of the form (I^2, F). We answer this question by recalling a well-known example of a set-theoretic complete intersection [30], [22] which also turns out to be self-linked [33].

Example 1.8. Let C denote the twisted cubic curve embedded in $\mathbf{P}^3$ with homogeneous ideal $I = (x^2 - wy, xy - wz, y^2 - xz)$. C is a curve of degree 3 and genus 0. Consider the saturated ideal $J = (x^2 - wy, y^3 - 2xyz + wz^2)$. One can check that the radical of J is I and that J defines a scheme with degree 6 and genus 4. Since J is a complete intersection, we have automatically that the scheme defined by J is locally Cohen-Macaulay. We conclude that J defines a ribbon on C. Does there exist an $F \in J$ such that J is the saturation of (I^2, F)? Let d denote the degree of F. By Theorem 1.5, we can solve for d in terms of the genus of C, the degree of C, and the genus of the scheme defined by J. We find that d must equal 5/3. In conclusion, J is not the saturation of an ideal of the form (I^2, F). *See also Examples* 2.3 *and* 2.10.

Judging from the example of ribbons, it appears that $(n-k)$-ropes in $\mathbf{P}^n$ can be put into two families, those that are defined by ideals of the form $(I^2, F_1, \ldots, F_k)$ and those that are not defined by such ideals. One is led to the question, is there a special property shared by the $(n-k)$-ropes in one family but not shared by the $(n-k)$-ropes in the other family? The answer is yes. We give one such property via the following proposition.

Proposition 1.9. *Let I be the ideal of a smooth and irreducible curve C in $\mathbf{P}^n$. Let $F_1, \ldots, F_k$ be a regular sequence in I, such that the complete intersection X they define is smooth at a general point of C. Then the non-irrelevant embedded primes of the ideal $(I^2, F_1, \ldots, F_k)$ are precisely the singularities of X along C.*

Proof. Let P be a point on C, say in the affine piece $x_0 \neq 0$. Let $R = \mathcal{O}_{\mathbf{P}^n,P}$ be the local ring of $\mathbf{P}^n$ at P (i.e. the homogeneous localization $S_{(P)}$) and let f_i denote the image of $F_i/x_0^{\deg(F_i)}$ in R. Note that X is smooth at P if and only if the local ring $R/(f_1, \ldots, f_k)$ is a regular local ring. By assumption, the local ring $\mathcal{O}_{C,P}$ of C is a regular local ring. In other words, we have

$$\mathcal{O}_{C,P} \cong R/(u_1, \ldots, u_{n-1}),$$

where $u_1, \ldots, u_{n-1}$ is part of a regular system of parameters $u_1, \ldots, u_n$ of the regular local ring R. Let Y denote the scheme defined by the ideal $(I^2, F_1, \ldots, F_k)$. We will denote by $\wp$ the ideal $(u_1, \ldots, u_{n-1})$. Note that the local ring $\mathcal{O}_{Y,P} \cong R/(\wp^2, f_1, \ldots, f_k)$.

Note that P is an embedded point of (I^2, F) if and only if the local ring $\mathcal{O}_{Y,P}$ has depth 0 (to see this, localize the primary decomposition of $(I^2, F_1, \ldots, F_k)$ at P). Note also that the assumption that the complete intersection X is smooth at the general point of C implies that the elements $f_1, \ldots, f_k$ are part of a regular system of parameters of the local ring $R_\wp$ (use the Jacobian criterion for smoothness, and argue as in Lemma 1.1). In particular the classes of the f_i in $\wp/\wp^2$ are linearly independent.

Assume first that the ring $R' = R/(f_1, \ldots, f_k)$ is regular. Let $\overline{\wp}$ denote the image of $\wp$ in R'. Since the quotient $R'/\overline{\wp}$ is regular (it is simply $R/\wp$), we deduce that $\overline{\wp}$ is generated by a subset of a regular parameter system of the regular ring R'. In particular $\overline{\wp}$ is generated by a regular sequence. This implies that the ring $R'/\overline{\wp}^2$ is a Cohen-Macaulay ring (cf. for instance [25], exercise 17.4), and in particular it is unmixed. The first implication is now clear, as $R'/\overline{\wp}^2 \cong R/(\wp^2, f_1, \ldots, f_k)$.

For the second implication, assuming that the ring R' is not a regular local ring, we want to show that u_n is a zero divisor in the ring $R/(\wp^2, f_1, \ldots, f_k)$. This in turn would imply that u_n lies in an associated prime of this latter ring. But any associated prime containing u_n has to be the maximal ideal $\mathfrak{m} = (u_1, \ldots, u_n)$, and the second implication will follow.

If R' is not a regular ring, we have that the classes of the f_i are not linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. In other words, we have an expression

$$\sum f_i g_i \in \mathfrak{m}^2,$$

where not all the g_i are in $\mathfrak{m}$. As the classes of the f_i are linearly independent in $\wp/\wp^2$, we can rewrite the above expression as

$$\sum f_i g_i = u_n q + q' , \tag{$*$}$$

where $q \neq 0$ and q' is in $\wp^2$. To conclude the proof, we have to show that $q \notin (\wp^2, f_1, \dots, f_k)$ (as this will imply that u_n is a zero divisor in the ring $R/(\wp^2, f_1, \dots, f_k)$).

But this is clear, since an expression $q = \sum f_i h_i + h'$, with $h' \in \wp^2$, combined with the expression $(*)$, would give

$$\sum f_i(g_i - u_n h_i) \in \wp^2 ,$$

which is forbidden, as the classes of the f_i are linearly independent in $\wp/\wp^2$. This concludes the proof of Proposition 1.9.

Corollary 1.10. *Let J be the homogeneous ideal of an $(n-k)$-rope supported on a smooth irreducible curve C in $\mathbf{P}^n$. Let I be the homogeneous ideal of C. J is the saturation of an ideal of the form $(I^2, F_1, \dots, F_k)$ where $F_1, \dots, F_k$ form a regular sequence if and only if there exists a regular sequence in J of length k such that the scheme defined by this regular sequence is smooth along C.*

Proof. Assume there exists such a regular sequence. Let V denote this sequence. By Proposition 1.9 we know that (I^2, V) is free of non-irrelevant embedded points. The definition of a rope tells us immediately that J has no embedded points and that J has degree $(n-k)\cdot\deg(C)$. By Theorem 1.5 we know that the degree of the scheme defined by (I^2, V) is $(n-k)\cdot\deg(C)$. By the choice of V we have an inclusion $(I^2, V) \subseteq J$. Since the degree of the scheme defined by J and the degree of the scheme defined by (I^2, V) are the same and since neither of them has any embedded points, they must agree up to an irrelevant component. By assumption, J is saturated so J is the saturation of (I^2, V).

Conversely, assume J is the saturation of $(I^2, F_1, \dots, F_k)$ where $F_1, \dots, F_k$ form a regular sequence. We first claim that the scheme defined by $(F_1, \dots, F_k)$ is smooth at the *general* point of C.

Let i be an integer with $0 < i \leq k$ (when $i = 1$, set $(F_1, \dots, F_{i-1}) = (0)$ in what follows) and suppose that $(F_1, \dots, F_{i-1})$ is smooth at the general point of C (*this is trivially the case, when* $i = 1$), while $(F_1, \dots, F_i)$ is singular all along C. Then we have, by Lemma 1.1, that $F_i \in (I^2, F_1, \dots, F_{i-1})^{(1)}$ (the ideal of the corresponding $n - i + 1$ rope). Using a few times the identity $(I^2, F_1, \dots, F_j)^{(1)} = ((I^2, F_1, \dots, F_{j-1})^{(1)}, F_j)^{(1)}$, we would then have that J is equal to the ideal $(I^2, F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_k)^{(1)}$ (with the obvious modification in case $i = 1$), the ideal of at least an $(n-k+1)$ rope; contradiction.

The claim is now obvious from the above observation, and the converse follows from Proposition 1.9.

We would like to present the following example, suggested by the referee, which shows the necessity of the assumption that $F_1, \dots, F_k$ form a regular sequence in Corollary 1.10.

Example 1.11. Let $I = (G_1, G_2, G_3, G_4)$ be a homogeneous ideal defining a smooth irreducible complete intersection curve C in $\mathbf{P}^5$. Assume further that $\deg(G_1) = \deg(G_2) = \deg(G_3) = \deg(G_4) \geq 2$. Let H_1, H_2 be linear forms which are S/I-regular. The polynomials $F_1 = H_1G_1 - H_2G_2$, $F_2 = H_1G_2 - H_2G_3$, $F_3 = H_1G_3 - H_2G_4$ do not form a regular sequence but the scheme they define is smooth along C. The saturation J of (I^2, F_1, F_2, F_3) defines a ribbon, but it can be shown that J cannot contain any length three regular sequence smooth along C.

We can actually improve on Proposition 1.9 by giving a decomposition of the saturated ideal $(I^2, F)^{sat}$.

Proposition 1.12. *Let I be the ideal of a smooth curve in $\mathbf{P}^3$. Let F be a form in S not in $I^{(2)}$. Let J denote the I-primary component in the ideal (I^2, F) (the "ribbon part"). Let $J(F)$ denote the ideal $(\partial F/\partial x_0, \ldots, \partial F/\partial x_3)$. There is a decomposition*

$$(I^2, F)^{sat} = J \cap ((I, J(F))^{(2)}).$$

Proof. To prove the proposition it is enough to show that for each point P of C, we have an equality

$$(I^2, F)_{(P)} = J_{(P)} \cap (I, J(F))^2_{(P)}$$

of the homogeneous localizations.

If F is smooth at P, then the claim is obvious, thanks to Proposition 1.9, so we can assume that P is a singular point for F. We denote by R the local ring of $\mathbf{P}^3$ at P. As in the proof of Proposition 1.9, we consider a system of parameters u, v, w of the maximal ideal of R such that u, v are generators of the ideal $I_{(P)}$, and we denote by f, the image in R of $F/x_0^{\deg(F)}$ (we assume again that P lies in the affine piece $x_0 \neq 0$).

As f is in I, we can write $f = ug_1 + vg_2$. We define the integer s as follows:

$$s = \min\{\mathrm{ord}_w(\bar{g}_1), \mathrm{ord}_w(\bar{g}_2)\},$$

where $\bar{g}_i$ denotes the class of g_i in the ring $R/(u, v)$ (i.e. the local ring of C at P) and ord_w denotes the valuation induced by the class of w in this ring. As P is a singular point for F, we have that $s > 0$, and we can thus write $g_i \equiv w^s g_i'$ modulo (u, v). We set $f' = ug_1' + vg_2'$. Note that by subtracting from f an element of $(u, v)^2$, we can assume that one of the g_i', say g_1', is a unit in R. It follows from the proof of Proposition 1.9 that ideal (u^2, uv, v^2, f') is (u, v)-primary, as by construction, $f' \notin (u, v, w)^2$.

Now, let D_u, D_v, D_w be an R-basis of the module of derivations $\mathrm{Der}_k(R, R)$ corresponding to the system of parameters u, v, w (i.e. such that $D_u(u) = 1$, $D_u(v) = D_u(w) = 0$ etc.). We have that $(I, J(F))_{(P)} = (u, v, D_u(f), D_v(f), D_w(f))$. As $D_u(f) \in g_1 + (u, v)$, $D_v(f) \in g_2 + (u, v)$, and $D_w(f) \in (u, v)$, we conclude that

$$(I, J(F))_{(P)} = (u, v, g_1, g_2) = (u, v, w^s)$$

(since g_1' is a unit).

We will now show the following equality which will conclude the proof (as the ideal (u^2, uv, v^2, f') is (u, v)-primary):

$$\text{CLAIM:} \quad (u^2, uv, v^2, f) = (u^2, uv, v^2, f') \cap (u, v, w^s)^2 .$$

The claim follows easily from the fact that u, v, w^s is a regular sequence, as follows.

The inclusion "$\subset$" is obvious. For the other inclusion, consider g of the form $hf' + q$, with q in $(u, v)^2$, and assume that g is also in $(u, v, w^s)^2$, hence of the form $w^s h' + q'$, with q' in $(u, v)^2$. We obtain an equality

$$u(g_1'h + q_1) + v(g_2'h + q_3) = w^s h'$$

where the q_i are in (u, v). As u, v, w^s form a regular sequence, we have $g_1'h + q_1 = vr_1 + w^s r_2$ *for some elements* r_1 and r_2. As g_1' is a unit in R, we can assume that $h = ur_0 + vr_1 + w^s r_2$, and finally, we obtain

$$\begin{aligned} g &= r_0 uf' + r_1 vf' + r_2 w^s f' + q \\ &= (r_0 uf' + r_1 vf' + q + r_2 w^s f' - r_2 f) + r_2 f \in (u^2, uv, v^2, f) . \end{aligned}$$

This concludes the proof.

It is interesting to note that every ribbon on a line in $\mathbf{P}^3$ is defined by an ideal of the form (I^2, F) [19], [27] (for a converse statement see [12]). In Example 1.8 we saw that in general, not every ribbon could be defined by an ideal of this form. Perhaps every ribbon on a complete intersection can be defined by an ideal of the form (I^2, F). The following example shows that this is not the case.

Example 1.13. Let $J = (w^2, wx, x^2, wy - xz)$. J defines a ribbon on the line $L = (w, x)$. Pick a general element in J of degree three, call it T. Define a new ideal $CI = (w^2, T)$. CI will be a complete intersection contained in J. Using the complete intersection CI, we can link J to a new scheme J' of degree 4 and genus 0. Using the complete intersection (w, T) we can link L to a conic. Let I denote the homogeneous ideal of this conic. The operation of linkage preserves the property of being locally Cohen-Macaulay [30]. The radical of J' is I so by the above remarks J' defines a ribbon on the conic defined by I. Using Theorem 1.5 we can determine that if J' is the saturation of an ideal of the form (I^2, F) then F must have degree 5/2. We conclude that the scheme defined by J' cannot be defined by an ideal of the form (I^2, F).

2. The deficiency module of a ribbon on a smooth curve in $\mathbf{P}^n$

Recall that, for any (possibly nonreduced) curve $Y \subset \mathbf{P}^n$, the deficiency module $M(Y)$ is defined by

$$M(Y) = \bigoplus_{k \in \mathbf{Z}} H^1(\mathbf{P}^n, \mathscr{I}_Y(k)) .$$

If Y is locally Cohen-Macaulay and equidimensional, this is a graded S-module of finite length, and it measures the failure of Y to be arithmetically Cohen-Macaulay. (This, and other properties, give it the name "deficiency module." It is very important in Liaison Theory, especially for curves in $\mathbf{P}^3$, and in this context it is sometimes also called the *Hartshorne-Rao module* of Y. See [29] for more details about these modules and their submodules, and about Liaison Theory, which we will use below.)

In this section we are interested in studying the deficiency module of a ribbon Y supported on a smooth curve C in $\mathbf{P}^n$. We will always denote by I the saturated homogeneous ideal of C. Interestingly enough, our techniques are best suited to give the deficiency module of a ribbon (i.e. multiplicity two structure), even though we have set up the theory for higher multiplicity structures. Our idea is to show how to link the ribbon to a union of two curves and to express the deficiency module, accordingly, as the quotient of two ideals.

We assume throughout this section that Y is supported on a smooth curve C in $\mathbf{P}^n$. We will be able to give the "cleanest" answer in the case of a ribbon in $\mathbf{P}^3$ whose homogeneous ideal is of the form (I^2, F) supported on a complete intersection curve: in this case we get that the deficiency module is a shift of the quotient ring of a certain complete intersection of height 4, and that Y is directly linked to the disjoint union of two complete intersections (see Theorem 2.8). (Recall from § 1 that not all ribbons have a saturated ideal of this form, even on a complete intersection curve, except when C is a line.) Applying our results gets progressively more difficult as n grows.

We begin with a simple observation, in the case where C is an arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$, other than a complete intersection. (See Theorem 2.8 for the case of a complete intersection in $\mathbf{P}^3$.)

Lemma 2.1. *Let C be any smooth arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$, other than a complete intersection, with saturated homogeneous ideal I. Let Y be a scheme defined by the saturation of the ideal (I^2, F), where $F \in I$ is not in $I^{(2)}$. Then Y is not arithmetically Cohen-Macaulay.*

Proof. Of course if F has a singular point somewhere on C we already know this from Proposition 1.9 since then Y has an embedded point. Let $d = \deg(F)$. We have the exact sequence

$$0 \to I(-d) \to I^2 \oplus (F) \to (I^2, F) \to 0\,.$$

We know (cf. for instance [31]) that the scheme Z defined by the ideal I^2 is not arithmetically Cohen-Macaulay (but it is saturated). Hence we have an injection $0 \neq M(Z) \hookrightarrow M(Y)$ so we are finished.

Remark 2.2. Of course it is not true that every ribbon is non-arithmetically Cohen-Macaulay, not even every ribbon supported on a non-arithmetically Cohen-Macaulay curve. A self-linked non-arithmetically Cohen-Macaulay curve gives rise to a counterexample (and such curves exist—see [33]). However, if C is any smooth curve in $\mathbf{P}^n$ other than a complete intersection (and we will discuss the complete intersection case below), *we conjecture that if Y is any ribbon whose homogeneous ideal is the saturation of $(I_C^2, F_1, \ldots, F_{n-2})$, where $(F_1, \ldots, F_{n-2})$ is a regular sequence defining a complete intersection variety which is smooth around C, then Y is not arithmetically Cohen-Macaulay.*

This is at least true if some of the degrees of the F_i are sufficiently large. We have an exact sequence

$$0 \to I^2 \cap (F_1, \ldots, F_{n-2}) \to I^2 \oplus (F_1, \ldots, F_{n-2}) \to I^2 + (F_1, \ldots, F_{n-2}) \to 0\,.$$

Let $d_i = \deg(F_i)$ for each i. It follows from the discussion in § 1 that, if we denote by $\mathscr{J}$ the sheafification of $I^2 \cap (F_1, \ldots, F_{n-2})$, then $H^1(\mathscr{J}(t)) \cong M(C)_{t-d_1} \oplus \cdots \oplus M(C)_{t-d_{n-2}}$ and $H^2(\mathscr{J}(t)) \cong H^2(\mathscr{I}_C(t-d_1)) \oplus \cdots \oplus H^2(\mathscr{I}_C(t-d_{n-2}))$. For large d_i it is impossible that this latter vector space can inject into $H^2(\mathscr{I}_C^2(t))$ for every t, since the latter vector space can be made non-zero in arbitrarily large degree by taking at least one of the d_i large.

It is known that not all smooth curves are self-linked (i.e. admit a ribbon which is a complete intersection), and it is an open question whether every smooth curve C admits a non-reduced structure of higher multiplicity which is a complete intersection (i.e. C is a set-theoretic complete intersection). It would also be interesting to know which smooth curves admit ribbons (or, for that matter, any non-reduced structures) which are arithmetically Cohen-Macaulay. This is the sort of question which started our investigation.

Example 2.3. This example is evidence for the conjecture mentioned above, at least in $\mathbf{P}^3$. Let C be a twisted cubic curve in $\mathbf{P}^3$. (See also Example 2.10 below.) The ideal of C is generated by three quadrics, so C is an "almost complete intersection" but not a complete intersection. Consider the ideal (I^2, F) where $F \in I$ has degree 2. If F is smooth then one can check that the ribbon Y thus obtained is of type (4,2) on the smooth quadric F, and hence is not arithmetically Cohen-Macaulay. (Its deficiency module is one-dimensional, occurring in degree 2). The degree and arithmetic genus of Y are 6 and 3, respectively.

On the other hand, if F is a quadric cone then the ideal (I^2, F) defines a curve of multiplicity two on C with an embedded point, and when the embedded point is removed we obtain a ribbon of degree 6 and arithmetic genus 4, which one can show is hence a complete intersection. This helps to explain how Example 2.2 of [30] is obtained. (Note that C necessarily passes through the singular point P of F since otherwise projection from P to $\mathbf{P}^2$ projects the cubic C to a conic.)

Let C be a smooth curve in $\mathbf{P}^n$ with saturated homogeneous ideal $I = (G_1, \ldots, G_k)$ $(k \geq n-1)$, and assume that $\deg(G_1) \leq \deg(G_2) \leq \cdots \leq \deg(G_k)$. Note that since C is integral, all of the G_i are irreducible.

Now let Y be a ribbon supported on C, and let $F_1, \ldots, F_{n-2} \in I_Y$ be a regular sequence satisfying

$$F_1 \notin I^{(2)},\ F_2 \notin (I^2, F_1)^{(1)},\ \dots,\ F_{n-2} \notin (I^{(2)}, F_1, \dots, F_{n-3})^{(1)}$$

as in Lemma 1.1. Recall from § 1 that $(I^2, F_1, \dots, F_{n-2})$ defines Y as a scheme, after removing any embedded points of Y which may arise (at the singularities of F on C). Our strategy will be to find $M(Y)$ by linking Y using the complete intersection $(G_iG_j, F_1, \dots, F_{n-2})$ (explained below) and finding the deficiency module of the residual.

Now, suppose C is a complete intersection, $I = (G_1, \dots, G_{n-1})$, and assume that $F_1, \dots, F_{n-2}$ are part of a minimal generating set for I, say for instance $F_i = G_i$ for all $1 \le i \le n-2$. Then $(I^2, F_1, \dots, F_{n-2}) = (G_{n-1}^2, F_1, \dots, F_{n-2})$ is again a complete intersection, and hence Y is arithmetically Cohen-Macaulay. (Note that none of the generators can have singular points anywhere on C since C is smooth.) (See also Theorem 2.8.)

So we may assume that either C is not a complete intersection or else at least one of the F_i is not part of a minimal generating set for I. Either way, we can find two polynomials, G_i and G_j in I_C, satisfying the following conditions:

(a) $(F_1, \dots, F_{n-2}, G_i)$ and $(F_1, \dots, F_{n-2}, G_j)$ are regular sequences;

(b) these regular sequences link C to residual curves C_i and C_j respectively, each of which has no component in common with C; and

(c) C_i and C_j have no common component.

If we want to make the residual to Y be as "nice" as possible, it is best to choose G_i and G_j as small as possible.

By definition of liaison, the saturated ideals of C_i and C_j are given by

$$I_{C_i} = [(G_i, F_1, \dots, F_{n-2}) : I_C]$$
$$I_{C_j} = [(G_j, F_1, \dots, F_{n-2}) : I_C]\,.$$

Lemma 2.4. *Y is directly linked (geometrically) to $C_i \cup C_j$ by the complete intersection $(G_iG_j, F_1, \dots, F_{n-2})$.*

Proof. Let X be the complete intersection scheme defined by $(G_iG_j, F_1, \dots, F_{n-2})$. Set-theoretically, $X = C \cup C_1 \cup C_2$. We have observed that there is no component in common to any two of C, C_i and C_j, and all components have height $n-1$. Now consider the primary decomposition of $(G_iG_j, F_1, \dots, F_{n-2})$. We can group some primary components together if necessary and get that

$$(G_iG_j, F_1, \dots, F_{n-2}) = I_{C_i} \cap I_{C_j} \cap I_{\overline{Y}}$$

where $\overline{Y}$ is a locally Cohen Macaulay, equidimensional curve supported on C, and one easily computes that the degree of $\overline{Y}$ is equal to twice the degree of C. Hence $\overline{Y}$ is a ribbon. But then both Y and $\overline{Y}$ are components of X, which is pure-dimensional, and Y and $\overline{Y}$ have the same degree and support. Hence $Y = \overline{Y}$.

As a result we have our first description of $M(Y)$, thanks to the well-known invariance of the deficiency module under liaison (up to shifts and duals), first proved by Hartshorne for curves in $\mathbf{P}^3$ (cf. [32]) and proved more generally by Schenzel and subsequently by others (cf. [35], [28], [7]). For a graded S-module M of finite length, we denote by $M^\vee$ its K-dual. Under the assumptions of Lemma 2.4, we have the following corollary.

Corollary 2.5.

$$M(Y) \cong M(C_i \cup C_j)^\vee(n+1-d-d_i-d_j)$$

(*where* $d_i = \deg(G_i)$, $d_j = \deg(G_j)$ *and* $d = \deg(F_1) + \deg(F_2) + \cdots + \deg(F_{n-2})$).

Our next goal is to understand this module better in the case where C is an arithmetically Cohen-Macaulay curve. We will prove:

Theorem 2.6. *Let $C \subset \mathbf{P}^n$ be a smooth arithmetically Cohen-Macaulay curve and let $I = I_C$ be the saturated ideal of C, with $\deg(G_1) \leq \deg(G_2) \leq \cdots \leq \deg(G_k)$. Let G_i and G_j be chosen as above, with $d_i = \deg(G_i)$, $d_j = \deg(G_j)$. Let Y be a ribbon supported on C and let $(F_1, \ldots, F_{n-2})$ be a regular sequence in I_Y defining a complete intersection which is smooth at the general point of C. Let $d = \sum \deg(F_i)$. Let C_i and C_j be the residuals to C under the complete intersections $(G_i, F_1, \ldots, F_{n-2})$ and $(G_j, F_1, \ldots, F_{n-2})$ respectively. C_i and C_j have no common component. Let X be the zeroscheme defined by the scheme-theoretic intersection of C_i and C_j; the saturated ideal I_X of X is the saturation of $I_{C_i} + I_{C_j}$. Then*

(i) $M(Y)^\vee(n+1-d-d_i-d_j) \cong M(C_i \cup C_j) \cong \dfrac{I_X}{I_{C_i} + I_{C_j}}$

(ii) $\deg(X) = p_a(C_i \cup C_j) - p_a(C_i) - p_a(C_j) + 1$ (*where as usual p_a is the arithmetic genus*).

Proof. Notice that since C_i and C_j are each directly linked to C, and since the property of being arithmetically Cohen-Macaulay is preserved under liaison, it follows that $M(C_i) = M(C_j) = 0$. Consider the exact sequence

$$0 \to I_{C_i} \cap I_{C_j} \to I_{C_i} \oplus I_{C_j} \to I_{C_i} + I_{C_j} \to 0\,.$$

Sheafifying and taking cohomology we get

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & I_{C_i \cup C_j} \arrow[r] & I_{C_i} \oplus I_{C_j} \arrow[rr] \arrow[dr] & & I_X \arrow[r] & M(C_i \cup C_j) \arrow[r] & 0 \\
& & & I_{C_i} + I_{C_j} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \qquad (1)$$

so (i) follows immediately from this and Corollary 2.5.

Recall that $M(C_i \cup C_j)$ has finite length. Hence for $t \gg 0$ we can use the exact sequence (1) to get that

$$-H(C_i \cup C_j, t) + H(C_i, t) + H(C_j, t) = H(X, t)$$

and this last term is exactly $\deg(X)$. But for $t \gg 0$, the Hilbert function is equal to the Hilbert polynomial. By a degree consideration, the coefficients of t will cancel out, so we have only to consider the constant terms. This says that

$$p_a(C_i \cup C_j) - p_a(C_i) - p_a(C_j) + 1 = \deg(X),$$

as desired.

Remark 2.7. Recall that if curves C and C' are directly linked in $\mathbf{P}^n$ by a complete intersection of hypersurfaces of degrees $a_1, a_2, \ldots, a_{n-1}$ with $a = \sum a_i$, then their degrees and arithmetic genera are related by

$$p_a(C') - p_a(C) = \frac{1}{2}(a - n - 1)(\deg(C') - \deg(C)).$$

In our case, suppose that the complete intersection defined by $(F_1, \ldots, F_{n-2})$ is smooth along all of C. If we assume known the degree and arithmetic genus of the original curve C, we have a formula for the arithmetic genus of Y (see § 1). Hence using the above formula we can compute the arithmetic genus of $C_i \cup C_j$ and also that of C_i and C_j, so we can take $\deg(X)$ as known.

There is one situation in which both parts of Theorem 2.6 take a very nice form. That is, we now assume that C is a complete intersection in $\mathbf{P}^3$ defined by the ideal $I = (G_1, G_2)$ and that F is smooth along C (hence the saturated ideal I_Y is of the form $I_Y = (I^2, F)$; see Remark 1.6).

Theorem 2.8. *Let C be the complete intersection in $\mathbf{P}^3$ of G_1 and G_2,* with $d_1 = \deg(G_1)$, $d_2 = \deg(G_2)$. *Let $F = AG_1 + BG_2$ be a form of degree d which is smooth along C and let Y be the ribbon with saturated ideal (I^2, F). Then*

(i) *If $d \le d_2$ then Y is arithmetically Cohen-Macaulay. Hence from now on assume that $d > d_2$.*

(ii) $I_{C_1} = (B, G_1)$, $I_{C_2} = (A, G_2)$; *that is, both C_1 and C_2 are complete intersections.*

(iii) *C_1 and C_2 are disjoint.*

(iv) $M(Y) \cong \left(\dfrac{S}{(A, B, G_1, G_2)}\right)(d - d_1 - d_2)$.

(v) $\deg(Y) = 2d_1d_2$, $p_a(Y) = (4 - d)(d_1d_2) + 2d_1d_2(d_1 + d_2 - 4) + 1$.

Proof. First, if F is a scalar multiple of G_1 then as above $(I^2, F) = (G_1, G_2^2)$ and Y is a complete intersection. Hence it is arithmetically Cohen-Macaulay. If $F = AG_1(\deg(A) > 0)$ then F is not smooth everywhere on C. So now assume that $d = d_2$. Then B is a scalar and C_1 is the empty curve. Hence Y is linked by (G_1G_2, F) to the arithmetically Cohen-Macaulay curve C_2, so Y is arithmetically Cohen-Macaulay, as claimed.

So now we assume that $d > d_2$. As noted above, since F is smooth along C, it follows that $I_Y = (I_C^2, F)$. Part (ii) is an easy exercise. For (iii), suppose that C_1 and C_2 meet at a point P. Such a P lies on both G_1 and G_2, hence on C. We now show that this is impossible.

Because C is smooth and C is the complete intersection of G_1 and G_2, it follows that G_1 and G_2 are not tangent at any point. On the other hand, the points where C and C_i meet ($i = 1, 2$) are points where F and G_i are tangent. Hence C_1 and C_2 cannot meet at any point of C. Therefore they are disjoint as claimed.

Part (iv) follows from Theorem 2.6(i) together with the fact that the saturation of $I_{C_1} + I_{C_2}$ is the whole ring S. Then use (ii) of the current theorem and the fact that $S/(A, B, G_1, G_2)$ is self-dual after a shift of $\deg(A) + \deg(B) + d_1 + d_2 - 4$. Part (v) is an easy computation using Theorem 1.5.

Remark 2.9. The simplest case where Theorem 2.8 applies is when C is a line, and Y is a so-called "double line." This situation was studied, for example, in [19] and in [27]. The latter paper, in particular, described the liaison class of a double line. The first step, finding the deficiency module, was done in a much more complicated way than what we have done here.

However, one could take a similar approach from that starting point and ask for a description of the set of ribbons in the liaison class of a given ribbon Y supported on a complete intersection C. The first step is to note that the complete intersection (G_1^2, G_2^2) links Y to another ribbon Y' also supported on C, and having the same genus (see Remark 2.7). This is true even if F has singular points on C; however, we continue to restrict ourselves to the case where F is smooth along C so we can apply Theorem 2.8.

But are all ribbons in this liaison class supported on C? In [27], it was shown that as long as $\deg(A) > 1$, any double line in the same liaison class as Y is supported on the same line as that of Y. Indeed, with our present knowledge that is not surprising: in that case we have $\deg(G_1) = \deg(G_2) = 1$ so we can recover the complete intersection (G_1, G_2) from the deficiency module by identifying the two-dimensional component of degree one in the complete intersection (A, B, G_1, G_2) defining the module.

However, now we are expanding our search for any ribbon in the liaison class. Our observation is that this problem does not have as simple a solution now, even if we were to restrict ourselves to ribbons supported on complete intersections. Indeed, one has only to observe that the union $C_1 \cup C_2$ is also linked, via the complete intersection (AB, F), to a ribbon supported on the complete intersection (A, B).

And in fact, this is not surprising from a study of the module. Indeed, we have the relation $\deg(A) + \deg(G_1) = \deg(B) + \deg(G_2)$, but there is nothing there that distinguishes the complete intersection (G_1, G_2). In fact, note that the union of complete intersection curves given by $(A, G_1) \cap (B, G_2)$ (for example) has the same module, so by Rao's theorem ([32]) it is in the same liaison class.

Although Theorem 2.8 is the "cleanest" result in this section, something similar can be done for any smooth arithmetically Cohen-Macaulay curve when F is smooth. (We also want to assume that C and C' meet transversally; this can be done by taking G_1 and G_2 of larger degree if necessary.) The point is that in order to hope to get a handle on the deficiency module it is necessary to first describe X exactly. We illustrate the idea with a simple example.

Example 2.10. Let C be a twisted cubic curve in $\mathbf{P}^3$ and let F be a sufficiently general surface of degree d which is smooth in a neighborhood of C. Consider the ribbon Y with saturated ideal (I^2, F). Let G_1 and G_2 be general elements of I of degree 2. Then (G_1, G_2) links C to a line C' meeting C transversally in two distinct points P_1 and P_2. Also, (G_1, F) and (G_2, F) link C to arithmetically Cohen-Macaulay curves C_1 and C_2 respectively, both of degree $2d-3$ and arithmetic genus $(d-2)(d-3)$. Also, (G_1G_2, F) links Y to $C_1 \cup C_2$ as above.

Hence Theorem 2.6 applies (with $d_1 = d_2 = 2$) and we know that the deficiency module of Y is K-dual to

$$\frac{I_X}{I_{C_1} + I_{C_2}}$$

(after shifting). But what is X? Observe that G_1 and G_2 are each smooth along C, and they are transverse along C except at the two points where C meets the line C', where they are tangent. Hence by generality, F is not tangent to both G_1 and G_2 anywhere on C. It follows as above that C_1 and C_2 meet only away from C. So $C_1 \cap C_2$ is precisely the intersection of the three surfaces G_1, G_2 and F away from C, i.e. it is the intersection of F with the line C', away from the points P_1 and P_2. Hence X is a set of $d-2$ points on a line (and thus a complete intersection with two of its generators being linear).

For instance, if $d = 2$ then $I_X = S$, $C_1 \cup C_2$ is a set of skew lines, and $M(Y) \cong K(-d)$ (where K is the base field). If $d = 3$ then Y is linked to the union $C_1 \cup C_2$ of two twisted cubics meeting in one point. The deficiency module is 3-dimensional in each of degrees 1 and 2. (Compare with [26], where the liaison class of a smooth rational sextic curve, whose deficiency module has the same dimension in the same degrees as $C_1 \cup C_2$, is studied.)

3. Bounds on the regularity

Recall that, given a coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n$, the Castelnuovo-Mumford regularity $\operatorname{reg}(\mathscr{F})$ is the smallest integer r with $H^i(\mathbf{P}^n, \mathscr{F}(k-i)) = 0$ for all $k \geq r$ and $i > 0$. On the other hand, given an ideal I in S (or more generally any S-module), the regularity $\operatorname{reg}(I)$ is the smallest integer r with $\operatorname{Tor}_i(I, K)_{i+j} = 0$ for all $j > r$. Given an ideal I in S, with corresponding ideal sheaf $\mathscr{I}$ on $\mathbf{P}^n$, there is an equality (compare [8])

$$\operatorname{reg}(I) = \max\{\operatorname{reg}(\mathscr{I}), \mathbf{n}(I)\}$$

where $\mathbf{n}(I)$, the saturation degree, is the smallest integer where I and its saturation agree in all degrees $\geq r$. In this section, we will give a bound on the regularity of the ideal sheaf of a rope in $\mathbf{P}^n$. We start with an algebraic result, whose proof is similar to that of Theorem 2.6 of [15].

Lemma 3.1. *Let $J \subset S$ be an ideal with* $\dim(S/J) = 1$, *where* dim *stands for the Krull dimension, and let $J^{[n]}$ denote the saturation of J^n. Given an ideal K with $J^n \subset K \subset J^{[n]}$, there is an inequality*

$$\operatorname{reg}(K) \leq n \cdot \operatorname{reg}(J).$$

Proof. Throughout the proof, we will denote by L a linear form in S which is a non-zero divisor in $S/J^{[n]}$. (This implies that the multiplication by L is an injection $(S/K)_j \to (S/K)_{j+1}$ for all j big enough, as the saturation of K is $J^{[n]}$). It follows from [2], Theorem 1.10, that $\operatorname{reg}(K)$ is the smallest integer j such that the multiplication by L induces an surjection $(S/K)_{j-1} \to (S/K)_j$ and an isomorphism $(S/K)_j \to (S/K)_{j+1}$.

CLAIM A: For $j \geq n \cdot \operatorname{reg}(J)$, the multiplication by L induces a surjection $(S/J^n)_{j-1} \to (S/J^n)_j$.

We will prove CLAIM A by induction on n, the case $n = 1$ being trivial (the proof goes as the proof of Lemma 2.3 in [15]). Let F be an element of S_j; as $j \geq \operatorname{reg}(J)$, we can write $F = LF' + G$, with F' in S_{j-1} and G in J_j. We can write $G = \sum F_i G_i$ where the F_i are minimal generators of J. In particular, we have that $\deg(G_i) \geq (n-1) \operatorname{reg}(J)$, and hence, by the induction hypothesis, each G_i can be put in the form $LG_i' + H_i$, where H_i is in J^{n-1}. Putting all this together yields an expression of the form $F = LF'' + G'$, with G' in J^n, which is precisely what is needed to establish CLAIM A.

CLAIM B: For $j \geq n \cdot \operatorname{reg}(J)$, the multiplication by L induces an injection $(S/J^n)_j \to (S/J^n)_{j+1}$.

The proof of CLAIM B is similar to the proof of Theorem 2.6 in [15], but we will sketch it here for the convenience of the reader. The proof goes by induction on n, the case $n = 1$ being again trivial. Let F be in S_j such that LF is in J^n; we want to show that F itself must be in J^n. By the induction hypothesis, F is in J^{n-1}, and so we can write $F = \sum F_i H_i$, where the F_i are minimal generators for J^{n-1}. On the other hand, as LF is in J^n, we can write $LF = \sum F_i G_i$ where the G_i are in J. Thus, we obtain the following relation amongst the F_i,

$$\sum (LH_i - G_i) F_i = 0.$$

In other words, if we consider the free S-module $\mathbf{F}$ with basis elements e_i corresponding to the minimal generators F_i of J^{n-1}, we have that the element $\sum (LH_i - G_i) e_i$ is a syzygy; we can thus write this syzygy in the form $\sum_{k,i} P_k a_{ki} e_i$ where the elements $\sum_i a_{ki} e_i$ are minimal generators for the module of syzygies.

By the induction hypothesis and CLAIM A, $\operatorname{reg}(J^{n-1}) \leq (n-1)\operatorname{reg}(J)$. This implies, after a short computation, that we have $\deg(P_j) \geq \operatorname{reg}(J)$. We can thus write $P_j = LQ_j + G_j'$ with G_j' in J.

Fixing i we get

$$L\left(H_i - \sum_j a_{ji} Q_j\right) = G_i + \sum_j a_{ji} G_j' \in J.$$

As the degree of $H_i - \sum a_{ji}Q_j$ is $\geq \operatorname{reg}(J)$, we deduce that $H_i - \sum_j a_{ji}Q_j$ is also in J, and we have that

$$F = \sum H_i F_i = \sum_i \left(H_i - \sum_j a_{ji}Q_j \right) F_i$$

is in J^n which proves CLAIM B.

To conclude the proof of Lemma 3.1, given $j \geq n \cdot \operatorname{reg}(J)$, we consider the following diagram, where all the vertical maps denote the multiplication by L, and where the horizontal maps are the obvious surjections:

$$\begin{tikzcd}
(S/J^n)_{j-1} \arrow[r] \arrow[d, "\alpha"] & (S/K)_{j-1} \arrow[r] \arrow[d, "\alpha'"] & (S/J^{[n]})_{j-1} \arrow[d] \\
(S/J^n)_j \arrow[r] \arrow[d, "\beta"] & (S/K)_j \arrow[r] \arrow[d, "\beta'"] & (S/J^{[n]})_j \arrow[d] \\
(S/J^n)_{j+1} \arrow[r] & (S/K)_{j+1} \arrow[r] & (S/J^{[n]})_{j+1} .
\end{tikzcd}$$

By CLAIMS A and B, we know that α is onto and that β is an isomorphism. An obvious diagram chasing (using the fact that for large j, all the horizontal maps are isomorphisms) shows that α' is injective and that β' is an isomorphism. This proves Lemma 3.1.

An interesting consequence of Lemma 3.1 is the following proposition.

Proposition 3.2. *Let $\mathcal{I}$ be the ideal sheaf of a 1-dimensional subscheme of* $\mathbf{P}^n$. *Then there is an inequality*

$$\operatorname{reg}(\mathcal{I}^n) \leq n \cdot \operatorname{reg}(\mathcal{I}) .$$

Proof. Let I be the saturated ideal in S corresponding to the ideal sheaf $\mathcal{I}$, and let $I^{[n]}$ denote the saturation of the ideal I^n. (Note that if I is the ideal of a local complete intersection curve, then $I^{[n]}$ is the n-th symbolic power). We have equalities

$$\operatorname{reg}(\mathcal{I}) = \operatorname{reg}(I) \quad \text{and} \quad \operatorname{reg}(\mathcal{I}^n) = \operatorname{reg}(I^{[n]}) .$$

Let L be a linear form in S, non zero divisor in $S/I^{[n]}$. We have then equalities

$$\operatorname{reg}_S(I) = \operatorname{reg}_{S/L}((I, L)/L) \quad \text{and} \quad \operatorname{reg}_S(I^{[n]}) = \operatorname{reg}_{S/L}((I^{[n]}, L)/L) .$$

Denote by J the ideal $(I, L)/L$. We have then the inclusions

$$J^n \subset (I^{[n]}, L)/L \subset J^{[n]} .$$

The theorem follows now from Lemma 3.1.

Remark 3.3. The bounds in Proposition 3.2 and Lemma 3.1 are sometimes sharp: consider any ideal I with a linear resolution; we have then an obvious inequality $\operatorname{reg}(I^n) \geq n \cdot \operatorname{reg}(I)$, which shows that Lemma 3.1 can be sharp; the case of a curve with a linear resolution, and whose powers are saturated (such as the twisted cubic in $\mathbf{P}^3$) shows that Proposition 3.2 can also be sharp.

We can also have sharpness in the case of a non linear resolution, as shown in the following example. Consider 23 general points in $\mathbf{P}^2$, with ideal I in $S = k[x, y, z]$. As the points are general, we have that $\operatorname{reg}(I) = 7$. As the minimal resolution conjecture is true for points in $\mathbf{P}^2$ [16], [23], we know that I is generated by forms of degree 6 (as $3 \cdot \dim_k(I_6) \geq \dim_k(I_7)$). A necessary condition for I^2 to be 13-regular is to have an equality $\dim_k(S/I^2)_{13} = 69$, since this is the multiplicity of the corresponding scheme. On the other hand, by [21] Proposition 3.6.1, we know that $\dim_k(S/I^2)_{13} = 70$. Hence, Lemma 3.1 is sharp in that case. As 23 general points in $\mathbf{P}^2$ are the hyperplane section of a smooth arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$ [6], and as the square of the ideal of an arithmetically Cohen-Macaulay curve in $\mathbf{P}^3$ is saturated by [31], we can also obtain an example where Proposition 3.2 is sharp in the case of a non linear resolution.

On the other hand, Theorem 1 of [36] and Proposition 1 of [4] show that our bounds are not sharp in the case where all the generators are in degree much lower than the regularity (in many complete intersections, for instance).

Compare also [34], where related questions are studied.

Consider now, after these algebraic preliminaries, a smooth and irreducible curve C in $\mathbf{P}^n$ with saturated ideal I, and corresponding ideal sheaf $\mathscr{I}$. We want to use Proposition 3.2 to give a bound on the regularity of a rope Y, in $\mathbf{P}^n$, supported on C. Recall that such a Y can always be obtained by removing possible embedded points to a scheme defined by an ideal of the form $(I^2, F_1, \ldots, F_k)$ where the F_i are a regular sequence in I, such that the singular locus of the complete intersection they define, does not contain the curve C. Our approach will be to first find a bound on the regularity of such schemes, and then to show that the regularity cannot increase by removing embedded points.

Recall that for a curve C, we define the index of speciality to be the integer $e(C) = \max\{t \in \mathbf{Z} \mid h^2(\mathscr{I}_C(t)) \neq 0\}$. Note that $e(C) \leq \operatorname{reg}(\mathscr{I}_C) - 3$ so we could also write the following results in terms of $\operatorname{reg}(\mathscr{I}_C)$ instead of $e(C)$. This would, however, give weaker statements.

Proposition 3.4. *Let I be the ideal of a smooth and irreducible curve C in $\mathbf{P}^n$ and let $\mathscr{I}$ be the corresponding sheaf of ideals. Let $F_1, \ldots, F_k$ be a regular sequence of forms in I of degree $d_1, \ldots, d_k$ respectively. Assume that C is not contained in the singular locus of the complete intersection X defined by the F_i. Denote by Y the scheme defined by the ideal $(I^2, F_1, \ldots, F_k)$. There is then an inequality*

$$\operatorname{reg} \mathscr{I}_Y \leq \max\{2 \operatorname{reg}(\mathscr{I}), e(C) + d_i + 2, d_1 + \cdots + d_k - (k-1)\}.$$

In the discussion that follows, we wish to emphasize the case of ribbons supported on curves in $\mathbf{P}^3$, as the results take a simpler form in this context. Hence before proving Proposition 3.4, we will state as a corollary what happens in $\mathbf{P}^3$.

Corollary 3.5. *Let I be the ideal of a smooth and irreducible curve in $\mathbf{P}^3$ and let $\mathscr{I}$ be the corresponding sheaf of ideals. Let F be a form of degree d which does not contain C in its singular locus. There is an inequality*

$$\operatorname{reg}(\widetilde{(I^2, F)}) \leq \max\{2 \operatorname{reg}(\mathscr{I}), e(C) + d + 2\}.$$

Proof of Proposition 3.4. We start with the following elementary observation, which we will use tacitly throughout the argument: given a short exact sequence of sheaves on $\mathbf{P}^n$,

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0,$$

there is an inequality

$$\operatorname{reg}(\mathscr{F}'') \leq \max\{\operatorname{reg}(\mathscr{F}), s(\mathscr{F}') - 1\},$$

where $s(\mathscr{F}')$ denotes the smallest integer r with $H^i(\mathbf{P}^n, \mathscr{F}'(k - i)) = 0$ for all $k \geq r$ and $i \geq 2$. This follows at once by looking at the long exact sequence induced in cohomology.

Consider the exact sequence

$$0 \to I^2 \cap (F_1, \ldots, F_k) \to I^2 \oplus (F_1, \ldots, F_k) \to (I^2, F_1, \ldots, F_k) \to 0.$$

Sheafifying this sequence, we obtain a first inequality

$$\operatorname{reg}(\mathscr{I}_Y) \leq \max\{\operatorname{reg}(\mathscr{I}^2), \operatorname{reg}(\mathscr{I}_X), s(\mathscr{J}) - 1\},$$

where $\mathscr{J}$ denotes the sheafification of the ideal $I^2 \cap (F_1, \ldots, F_k)$.

As the F_i form a regular sequence, we have

$$\operatorname{reg}(\mathscr{I}_X) = d_1 + \cdots + d_k - (k - 1).$$

Next, we can use the sheafification of the exact sequence of Corollary 1.3

$$0 \to \mathscr{O}(-d_1 - \cdots - d_k) \to \cdots \to \bigoplus \mathscr{I}(-d_i) \to \mathscr{J} \to 0,$$

and obtain the following inequality:

$$s(\mathscr{J}) \leq \max\{s(\mathscr{I}) + d_i, d_1 + \cdots + d_k - (k - 1)\}.$$

We have that $s(\mathscr{I}) - 3$ is just the index of speciality $e(C)$ and hence the two inequalities above give us a new inequality

$$\operatorname{reg}(\mathscr{I}_Y) \leq \max\{\operatorname{reg}(\mathscr{I}^2), e(C) + d_i + 2, d_1 + \cdots + d_k - (k - 1)\}.$$

Proposition 3.4 follows now from the inequality $\operatorname{reg}(\mathscr{I}^2) \leq 2 \cdot \operatorname{reg}(\mathscr{I})$ of Proposition 3.2.

Example 3.6. Let I be the homogeneous ideal of a twisted cubic curve, C, in $\mathbf{P}^3$. Let $\mathscr{I}_C$ be the associated ideal sheaf. It is easy to check that $e(C) = -1$. Pick a form $F \in I$ such that the associated hypersurface is smooth along C. Let D be the scheme defined by (I^2, F) and let $\mathscr{I}_D$ be the associated ideal sheaf. If F has degree 2 then $\mathscr{I}_D$ has regularity 4. If F has degree $d > 2$ then $\mathscr{I}_D$ has regularity $d + 1$. This shows that Proposition 3.4 and Corollary 3.5 are in some instances sharp, and that the sharpness can be achieved by either of the two possibilities for the max.

We can now state and prove our bound on the regularity of a rope in $\mathbf{P}^n$.

Theorem 3.7. *Let C be a smooth and irreducible curve in $\mathbf{P}^n$ with ideal I and ideal sheaf $\mathscr{I}$ and let Y be an $(n-k)$-rope on C. Let $F_1, \dots, F_k$ be a regular sequence of forms of degree d_i in I_Y. Assume that C is not contained in the singular locus of the complete intersection defined by the F_i. There is then an inequality*

$$\operatorname{reg}(\mathscr{I}_Y) \leq \max\{2\operatorname{reg}(\mathscr{I}), e(C) + d_i + 2, d_1 + \cdots + d_k - (k-1)\}.$$

We also state the corresponding result for ribbons in $\mathbf{P}^3$.

Corollary 3.8. *Let C be a smooth and irreducible curve in $\mathbf{P}^3$ with ideal I and ideal sheaf $\mathscr{I}$ and let Y be a ribbon on C. Let d be the smallest degree such the ideal of Y contains a form F of degree d which does not contain C in its singular locus. Then*

$$\operatorname{reg}(\mathscr{I}_Y) \leq \max\{2\operatorname{reg}(\mathscr{I}), e(C) + d + 2\}.$$

Proof of Theorem 3.7. Let $F_1, \dots, F_k$ be as in the statement of Theorem 3.7. By Corollary 1.7, we have that Y can be obtained by removing possible embedded points from the scheme defined by the ideal $(I^2, F_1, \dots, F_k)$. Theorem 3.7 follows now from Proposition 3.4 together with Lemma 3.9 below.

Lemma 3.9. *Let Y' be a 1-dimensional subscheme of $\mathbf{P}^n$ (possibly with embedded or isolated points). Let Y be the union of the top-dimensional components of Y'. Then*

$$\operatorname{reg}(Y) \leq \operatorname{reg}(Y').$$

Proof. Consider a primary decomposition of $I_{Y'}$, and collect terms if necessary so that $I_{Y'} = I_Y \cap J$ where J is a saturated ideal whose associated primes all have height n. Let Z be the zeroscheme defined by J, so $J = I_Z$ (J is saturated).

Let Z' be the scheme defined by the ideal $I_Y + I_Z$. Note that Z' is a 0-dimensional subscheme of $\mathbf{P}^n$ with $Z' \subset Z$. The exact sequence of S-modules

$$0 \to I_Y \cap I_Z \to I_Y \oplus I_Z \to I_Y + I_Z \to 0$$

induces an exact sequence of sheaves

$$0 \to \mathscr{I}_{Y'} \to \mathscr{I}_Y \oplus \mathscr{I}_Z \to \mathscr{I}_{Z'} \to 0. \qquad (*)$$

We will show that

a) $H^1(\mathbf{P}^n, \mathcal{I}_{Y'}(t)) = 0$ implies $H^1(\mathbf{P}^n, \mathcal{I}_Y(t)) = 0$ for every t and

b) $H^2(\mathbf{P}^n, \mathcal{I}_{Y'}(t)) = 0$ implies $H^2(\mathbf{P}^n, \mathcal{I}_Y(t)) = 0$ for every t,

and this will prove Lemma 3.9.

Using the long exact cohomology sequence induced from the exact sequence $(*)$ we get

$$H^2(\mathbf{P}^n, \mathcal{I}_{Y'}(t)) \to H^2(\mathbf{P}^n, \mathcal{I}_Y(t)) \oplus H^2(\mathbf{P}^n, \mathcal{I}_Z(t)) \to H^2(\mathbf{P}^n, \mathcal{I}_{Z'}(t)).$$

Since both Z and Z' are 0 dimensional, we get a surjection

$$H^2(\mathbf{P}^n, \mathcal{I}_{Y'}(t)) \to H^2(\mathbf{P}^n, \mathcal{I}_Y(t)) \to 0,$$

which establishes Claim b).

Consider now the exact sequence

$$H^1(\mathbf{P}^n, \mathcal{I}_{Y'}(t)) \to H^1(\mathbf{P}^n, \mathcal{I}_Y(t)) \oplus H^1(\mathbf{P}^n, \mathcal{I}_Z(t)) \to H^1(\mathbf{P}^n, \mathcal{I}_{Z'}(t)). \tag{$**$}$$

Since Z' is a subscheme of Z, we have an exact sequence of sheaves

$$0 \to \mathcal{I}_Z \to \mathcal{I}_{Z'} \to \mathcal{I}_{Z'/Z} \to 0,$$

where $\mathcal{I}_{Z'/Z}$ is supported on Z. This in turn induces the exact sequence

$$H^1(\mathbf{P}^n, \mathcal{I}_Z(t)) \to H^1(\mathbf{P}^n, \mathcal{I}_{Z'}(t)) \to 0.$$

This exact sequence, together with the exact sequence $(**)$, establishes Claim a): if $H^1(\mathbf{P}^n, \mathcal{I}_{Y'}(t)) = 0$, we get an injection

$$0 \to H^1(\mathbf{P}^n, \mathcal{I}_Y(t)) \oplus H^1(\mathbf{P}^n, \mathcal{I}_Z(t)) \to H^1(\mathbf{P}^n, \mathcal{I}_{Z'}(t)),$$

which can only occur if $H^1(\mathbf{P}^n, \mathcal{I}_Y(t)) = 0$.

Remark 3.10. Lemma 3.9 and its proof remain valid if we replace Y' by any positive dimensional subscheme of $\mathbf{P}^n$ with embedded components of dimensions at most 0.

Remark 3.11. Let C be a smooth curve in $\mathbf{P}^n$ and Y an $(n-1)$-rope supported on C. If there exists a hypersurface containing Y which is smooth along C then we can bound the regularity of $\mathcal{I}_Y$ in terms of the regularity of $\mathcal{I}_C$, the degree and genus of C, and the genus of Y. Let d denote the degree of the above mentioned hypersurface, G_C the genus of C, D_C the degree of C, and G_Y the genus of Y. By Theorem 1.5 and Corollary 1.10, $G_Y = (n+1-d)\cdot D_C + (n+1)\cdot(G_C - 1) + 1$ so $d = n + 1 + ((n+1)\cdot G_C - G_Y - n)/D_C$. Use Theorem 3.7 to obtain $\operatorname{reg}(\mathcal{I}_Y) \leq \max\{2\operatorname{reg}(\mathcal{I}_C), e(C) + d + 2\} \leq \max\{2\operatorname{reg}(\mathcal{I}_C), e(C) + n + 3 + ((n+1)G_C - G_Y - n)/D_C\}$. If we note that $e(C) \leq \operatorname{reg}(\mathcal{I}_C) - 3$ and $\operatorname{reg}(\mathcal{I}_C) \leq D_C - n + 2$ [18] then we obtain

$$\begin{aligned}
\operatorname{reg} \mathcal{I}_Y &\leq \max\left\{2 \operatorname{reg}(\mathcal{I}_C), e(C) + n + 3 + \frac{(n+1)G_C - G_Y - n}{D_C}\right\} \\
&\leq \max\left\{2 \operatorname{reg}(\mathcal{I}_C), \operatorname{reg}(\mathcal{I}_C) + n + \frac{(n+1)G_C - G_Y - n}{D_C}\right\} \\
&\leq \max\left\{2D_C - 2n + 4, D_C + 1 + \frac{(n+1)G_C - G_Y - n}{D_C}\right\}
\end{aligned}$$

(The last inequality comes from the main theorem of [18].)

Furthermore, let C be a smooth curve in $\mathbf{P}^n$ and Y an $(n-k)$-rope supported on C. Assume that there exists a regular sequence $F_1, \ldots, F_k$ defining a complete intersection scheme which contains Y and is smooth along C. Order $F_1, \ldots, F_k$ such that $\deg(F_1) \geq \cdots \geq \deg(F_k)$. If $\deg(F_1) + \deg(F_2) \geq 2 \operatorname{reg}(\mathcal{I}_C) + 1$ and $\deg(F_2) \geq \operatorname{reg}(\mathcal{I}_C)$ then from Theorem 1.5, Corollary 1.10, and Theorem 3.7 we conclude that

$$\operatorname{reg}(\mathcal{I}_Y) \leq (n + 2 - k)\left(1 + \frac{G_C - 1}{D_C}\right) - \frac{G_Y - 1}{D_C}.$$

Remark 3.12. As we mentioned before, the bounds given in Theorem 3.7 are sometimes sharp. On the other hand, we would now like to give an example of an obvious situation where the bounds have no chance of being sharp. This will happen because Proposition 3.2 is not sharp, in the case of a big gap between the largest degree of a minimal generator of I and the regularity of $\mathcal{I}$ (compare Remark 3.3).

Let $C \subset \mathbf{P}^3$ be a smooth complete intersection of surfaces of degree a and b, with $a \leq b$. Let $I = I_C$ be the homogeneous ideal of C. Let Y be a ribbon defined (up to embedded points) by the ideal (I^2, F). Note that $\operatorname{reg}(I) = a + b - 1$. A minimal free resolution for I^2 is

$$0 \to S(-2a - b) \oplus S(-a - 2b) \to S(-2a) \oplus S(-a - b) \oplus S(-2b) \to I^2 \to 0$$

(cf. [31]), hence $\operatorname{reg}(\mathcal{I}^2) = 2b + a - 1$. So already there is a gap and for small d Theorem 3.7 has no chance to be sharp. In fact, under the assumption that $d = b + 1$, one can check that $\operatorname{reg}(\mathcal{I}_Y) = \operatorname{reg}(\mathcal{I}^2) = \operatorname{reg}(\mathcal{I}) + b$.

We would also like to point out that it would be nice to derive a bound on the degree d_i in the statement of Theorem 3.7, say in terms of the genus and degree of the curve and the rope.

Example 3.13. We give an easy example of Lemma 3.9. Let Y' be the scheme in $\mathbf{P}^3$ defined by $I = (w, x^2, xy)$. Y' is a line with an embedded point. If we remove the embedded point then we get the line Y defined by the ideal $I_Y = (w, x)$. It is easy to check that the regularity of the ideal sheaf of Y' is 2 and the regularity of the ideal sheaf of Y is 1.

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