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On a confluence problem for the equations of a viscous heat-conducting gas

By

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1. Introduction

The one-dimensional motion of a viscous polytropic ideal (perfect) gas is described by the following system of equations [1, 14]:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}-\frac{\partial(R\rho\theta)}{\partial y}, \tag{1.1}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial y}+\rho\frac{\partial u}{\partial y}=0, \tag{1.2}$$

$$c_v\rho\left(\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial y}\right)=\kappa\frac{\partial^2 \theta}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2-R\rho\theta\frac{\partial u}{\partial y}. \tag{1.3}$$

Here u, ρ and θ are the velocity, density and absolute temperature, respectively—the required characteristics of the medium; y is the Eulerian coordinate; t is the time; μ, c_v, κ are the viscosity, specific heat capacity and thermal conductivity—positive constants. R is the universal gas constant.

The distinctive feature of the viscous gas equations is an indeterminate type of the whole system. The continuity equation (1.2) can be treated as a first order partial differential equation with respect to ρ. Its characteristics are integral curves of the ordinary differential equation

$$\frac{dy}{dt}=u(y,t).$$

If we formulate an initial-boundary value problem for the system then, according to the boundary conditions, either the side boundaries of a domain of unknowns are characteristics of the continuity equation or they simulate permeable walls, that is the characteristics intersect the boundaries.

In accordance with the above, boundary problems for the one-dimensional viscous gas equations can be classified into two groups: characteristic and non-characteristic problems. The side boundaries are characteristics, for instance, when the boundary conditions model rigid volume walls or free surfaces. The

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problems of the second group are more delicate because of a specific form of the integral laws of conservation.

For the last twenty years free boundary problems for the one-dimensional viscous gas equations have been intensively studied. The main attention has been paid to the existence, uniqueness and asymptotic properties of solutions (see A. V. Kazhikhov [5], [7], M. Okada [12], [13], T. Nagasawa [9]). Afterwards, A. S. Tersenov [19], T. Nishida [10], T. Nishida and M. Okada [11] studied free boundary problems where the density is continuous across the interface of a gas and a vacuum.

In this paper we consider a noncharacteristic initial-boundary value problem for the system (1.1)–(1.3) in a domain, the right-hand boundary of which is fixed and simulates a permeable wall, and the left-hand boundary is a characteristic curve and simulates a free surface of a viscous compressible fluid. The same problem has been studied in [3] for the viscous gas equations of a barotropic motion. However, at first the solvability of a problem where a free boundary and a permeable one present simultaneously was proved by A. V. Kazhikhov [6] for the generalized Burgers' equations and by S. Ya. Belov [2] for the heat-conductive case. But they studied the problem modeling a filling of a vacuum with a viscous compressible fluid.

The main feature of our problem is that the characteristics of the continuity equation go out of the domain on the right-hand boundary simulating a permeable wall. Therefore the left-hand boundary being a characteristic has a chance to intersect the right-hand one and we can get some kind of degeneration of the problem.

For the model of a barotropic motion the following results have been proved in [3]:

1. It is possible that the side boundaries of the domain intersect in a finite time, that is, the domain is getting degenerated.
2. Some estimate for the time of the degeneration can be obtained.
3. There exists a solution of the problem in a degenerate domain.

In this paper we prove the same statements for the equations of a heat-conducting gas. However, the confluence problem for a perfect gas is more delicate than in the barotropic case. The main obstacle is that the domain of the problem is unknown in the Eulerian variables as well as in the Lagrangian ones. More precisely, the free boundary of the contact with the vacuum is unknown in the Eulerian variables and it is impossible to define the Lagrangian image of the permeable boundary.

The plan of the work is the same as in [3]. We state the problem in section 2. In section 3, we deduce some a priori estimates for the regular solution, which lead to results on the degeneration of a domain of unknowns. In section 4 we formulate the local existence theorem and prove the existence of a final solution (see Definition 2). The auxiliary problem, which plays an important role in the proof of the local solvability, is studied in section 5. The proof of the local existence theorem is presented in section 6.

Throughout the paper, we use well-known notations. Thus, by $C^{k,l}(\bar{Q}_T)$, $0 \le k \le \infty$, $0 \le l \le \infty$, we denote the set of all continuous functions in closed domain $\bar{Q}_T \subset R \times R^+ = \{(x, t) \in R^2, t \ge 0\}$ having derivatives with respect to x up to order k inclusively and with respect to t up to order l inclusively, which are continuous in $\bar{Q}$. By $C^{k+\alpha}(0, X)$, $0 \le k \le \infty$, we denote the Banach space of functions on $(0, X)$ having derivatives up to order k inclusively, which are uniformly Hölder continuous with exponent $\alpha \in (0, 1)$. We use the notation $C^{k+\alpha, l+\beta}(Q_T)$, $0 \le k \le \infty$, $0 \le l \le \infty$, for the Banach space of functions on Q_T having derivatives with respect to x up to order k inclusively and with respect to t up to order l inclusively, which are uniformly Hölder continuous for x with exponent α and for t with exponent β, where α, $\beta \in (0, 1)$. The symbols H_x^α and H_t^β denote the constants of uniformly Hölder continuity with respect to x and to t respectively, i.e. if $Q_T = (0, X) \times (0, T)$

$$H_x^\alpha(f(x, t)) = \sup_{\substack{x_1, x_2 \in (0, X) \\ 0 \le t \le T}} \frac{|f(x_1, t) - f(x_2, t)|}{|x_1 - x_2|^\alpha},$$

$$H_t^\beta(f(x, t)) = \sup_{\substack{t_1, t_2 \in (0, T) \\ 0 \le x \le X}} \frac{|f(x, t_1) - f(x, t_2)|}{|t_1 - t_2|^\beta}.$$

If $k = l$, $\alpha = \beta$, we use notation $C^{k+\alpha}(Q_T)$ for $C^{k+\alpha, k+\alpha}(Q_T)$.

We also use a Sobolev space $W_p^{2,1}(Q_T)$ and a Sobolev–Slobodetskiy space $W_p^l(0, T)$, $0 < l < 1$ with the norm (see [15])

$$\|f\|_{W_p^l(0,T)} = \|f\|_{L_p(0,T)} + \left(\int_0^T dt \int_0^{T-t} \frac{|f(t + \tau) - f(t)|^p}{\tau^{1+lp}} d\tau \right)^{\frac{1}{p}}.$$

The standard norm of any Banach space B will be denoted by $\|\cdot\|_B$.

2. Formulation of the problem

We consider the motion of a viscous gas in a certain region, the right-hand boundary of which is fixed and permeable. The gas is constantly pumped out through the permeable wall and is in contact with a vacuum on the left-hand side. This process can be described by a solution of system (1.1)–(1.3), which is defined in the region $Q = \{(y, t): t > 0, z(t) < y < Y\}$ and satisfies the following boundary conditions:

$$u = \tilde{u}_0(y), \quad \rho = \tilde{\rho}_0(y), \quad \theta = \tilde{\theta}_0(y) \quad \text{for } t = 0, \quad 0 \le y \le Y; \tag{2.1}$$

$$u = u_1(t) > 0, \quad \frac{\partial \theta}{\partial y} = 0 \quad \text{for } y = Y, \quad t \ge 0; \tag{2.2}$$

$$\mu \frac{\partial u}{\partial y} - p = \frac{\partial \theta}{\partial y} = 0 \quad \text{for } y = z(t), \quad t \ge 0; \tag{2.3}$$

The equation of the free boundary $y = z(t)$ is defined by the solution of the Cauchy problem:

$$\frac{dz}{dt} = u(z, t), \qquad z(0) = 0, \qquad t \geq 0. \tag{2.4}$$

The behavior of the free boundary is the main topic of our investigation.

As mentioned above, the left-hand boundary of the domain Q is a characteristic of the equation (1.2). But the characteristics go out of the domain on the right-hand boundary because they have appropriate inclination for $y = Y$. We may state a conjecture on an intersection of the side boundaries in a finite time. However, the following example shows that the positiveness of a boundary function u_1 does not ensure any confluence of the boundaries. The functions

$$u(y, t) = \frac{y - \frac{1}{2}}{1 + t}, \qquad \rho(y, t) = (1 + t)^{-1}, \qquad \theta(y, t) = \frac{\mu}{R},$$

are a solution of the problem (1.1)–(1.3), (2.1)–(2.4) with data

$$u_1 = \frac{1}{2}(1 + t)^{-1} > 0 \qquad \text{for any } t > 0;$$

$$\tilde{u}_0 = y - \frac{1}{2}, \qquad \tilde{\rho}_0 = 1, \qquad \tilde{\theta}_0 = \frac{\mu}{R},$$

over the infinite time interval. Here $Y = 1$.

Obviously,

$$z(t) = -\frac{t}{2} \qquad \text{and} \qquad \lim_{t \to \infty} z(t) = -\infty.$$

Taking into account some possibility of the degeneration of the problem, it is worth using the following definitions.

Definition 1. Let T be a real number and $T > 0$. We will say that the problem *has a solution on the interval* $[0, T]$ if the following conditions are valid:

1. there exists a function $z(t) \in C^2[0, T]$ such that $z(0) = 0$, $z(t) < Y$ for $t < T$ and $z(T) \leq Y$;
2. there exist functions $u(y, t)$, $\theta(y, t)$, $\rho(y, t)$ in domain $Q_T^Y = \{(y, t): 0 < t < T, z(t) < y < Y\}$, which have the properties:

$$(u, \theta) \in C^{2,1}(Q_T^Y) \cap C(\bar{Q}_T^Y), \qquad \rho \in C^{1,1}(Q_T^Y) \cap C(\bar{Q}_T^Y),$$

$$\frac{\partial u}{\partial x}, \frac{\partial \theta}{\partial x} \in C(\bar{Q}_T^Y) \qquad \rho > 0, \qquad \theta > 0.$$

and satisfy equations (1.1)–(1.3) and boundary conditions (2.1)–(2.4).

Definition 2. If the problem (1.1)–(1.3), (2.1)–(2.4) has a solution on an interval $[0, T]$ and $z(T) = Y$ then the solution is called *a final solution.*

3. The degeneration of the domain

In this section we consider the free boundary problem (1.1)–(1.3), (2.1)–(2.4). Suppose the problem has a solution on an interval $[0, T]$. It is not possible to follow the free boundary in the Eulerian variables because we are not able to deduce strong enough estimates for $u(z(t), t)$. For our purpose it is worth using the Lagrangian mass variables:

$$\text{(L)}\qquad \begin{aligned} x &= x(y, t) = \int_{z(t)}^{y} \rho(s, t)\, ds, \\ t' &= t'(y, t) = t \qquad \text{(the prime will be omitted).} \end{aligned}$$

Let the notations of the desired functions be preserved. Then the functions $u(x, t)$, $\rho(x, t)$ and $\theta(x, t)$ are a solution of the following system of differential equations [1]

$$\frac{\partial u}{\partial t} = \mu \frac{\partial}{\partial x}\left(\rho \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}(R\rho\theta), \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial x} = 0, \tag{3.2}$$

$$c_v \frac{\partial \theta}{\partial t} = \kappa \frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right) + \mu\rho\left(\frac{\partial u}{\partial x}\right)^2 - R\rho\theta \frac{\partial u}{\partial x} \tag{3.3}$$

in the domain $Q_T = \{(x, t): 0 < t < T, x \in \Omega_t = (0, a(t))\}$, and satisfy the initial and boundary conditions

$$(u, \theta, \rho) = (u_0(x), \theta_0(x), \rho_0(x)) \qquad \text{for } t = 0, \qquad 0 \le x \le X = \int_0^Y \tilde{\rho}_0(s)\, ds, \tag{3.4}$$

$$\text{(3.5)}\qquad \begin{aligned} &u = u_1(t) > 0, \qquad \frac{\partial \theta}{\partial x} = 0, \qquad \text{for } x = a(t), \qquad 0 \le t \le T, \\ &\mu \frac{\partial u}{\partial x} - R\theta = \frac{\partial \theta}{\partial x} = 0 \qquad \text{for } x = 0, \qquad 0 \le t \le T. \end{aligned}$$

Here

$$u_0(x) = \tilde{u}_0(y), \qquad \theta_0(x) = \tilde{\theta}_0(y), \qquad \rho_0(x) = \tilde{\rho}_0(y), \qquad \text{if } x = \int_0^y \tilde{\rho}_0(s)\, ds. \tag{3.6}$$

The equation $x = a(t)$ defines the image of the "permeable" right-hand boundary and

$$a(t) = X - \int_0^t u_1(\tau)\rho(a(\tau), \tau) d\tau.$$

The first integral estimate can be proved in an usual way. We sum equation (3.1) multiplied by $w = u - u_1$ and equation (3.3), then we integrate the result over $Q_t = \{(x, \tau): 0 < \tau < t, 0 < x < a(\tau)\}$. After simple reductions, using the Cauchy inequality and Gronwall's inequality, we can obtain the first energy estimate:

$$\max_{0 \le \tau \le t} \{\|w(\tau)\|^2_{L_2(\Omega_\tau)} + 2c_v \|\theta(\tau)\|_{L_1(\Omega_\tau)}\} - 2c_v \int_0^t \rho(a(\tau), \tau)\theta(a(\tau), \tau)u_1(\tau)d\tau \le N_1, \tag{3.7}$$

where

$$N_1 = \left(\|u_0 - u_1(0)\|^2_{L_2(0,X)} + 2c_v \|\theta_0\|_{L_1(0,X)} + X \int_0^\infty |u_1'|\, d\tau \right) \exp \int_0^\infty |u_1'|\, d\tau,$$

The following Kazhikhov's representation for the function $\rho(x, t)$ is also well-known [1]:

$$\rho(x, t) = \frac{\rho_0(x) \exp\left\{\frac{1}{\mu} \int_0^x [u_0(s) - u(s, t)]ds\right\}}{1 + \rho_0(x)\frac{R}{\mu} \int_0^t \theta(x, \tau) \exp\left\{\frac{1}{\mu} \int_0^x [u_0(s) - u(s, \tau)]ds\right\} d\tau}. \tag{3.8}$$

Let us introduce the notations

$$M_\rho(t) = \max_{x \in \overline{\Omega}_t} \rho(x, t), \qquad M_\theta(t) = \max_{x \in \overline{\Omega}_t} \theta(x, t),$$

$$m_\rho(t) = \min_{x \in \overline{\Omega}_t} \rho(x, t), \qquad m_\theta(t) = \min_{x \in \overline{\Omega}_t} \theta(x, t)$$

and suppose that

$$0 < m_0 \le (\rho_0(x), \theta_0(x)) \le M_0 < \infty \qquad \text{for } 0 \le x \le X.$$

Using an obvious relation for the argument of the exponential function in (3.8)

$$\left| \int_0^x [u(s, t) - u_0(s)]ds \right|$$

$$= \left| \int_0^x [u(s, t) - u_1(t)]ds + \int_0^x [u_1(t) - u_1(0)]ds + \int_0^x [u_1(0) - u_0(s)]ds \right|$$

$$\le X^{\frac{1}{2}} \|w(t)\|_{L_2(\Omega_t)} + X \int_0^t |u_1'|\, d\tau + X^{\frac{1}{2}} \|u_0 - u_1(0)\|_{L_2(0,X)},$$

we estimate the right-hand side of (3.8) from above and from below and obtain the following assertion.

Lemma 1. *For any $t < T$ the following relations hold:*

$$M_\rho(t) \le M_0 \exp\{N_2\}, \tag{3.9}$$

$$m_\rho(t) \ge m_0 \exp\{-N_2\}\left[1 + \frac{M_0 R}{\mu} \exp\{N_2\} \int_0^t M_\theta(\tau)d\tau\right]^{-1}, \tag{3.10}$$

where

$$N_2 = \frac{1}{\mu}(N_1 + 2X^{\frac{1}{2}}N_1^{\frac{1}{2}}).$$

After that we can deduce a stronger integral estimate. Having multiplied the equation (3.1) by $w = u - u_1$ and the equation (7.3) by $(1 - \theta^{-1})$ then integrating their sum over $Q_t = \{(x, \tau): 0 < \tau < t, 0 < x < a(\tau)\}$, after simple reductions, we come to the inequality

$$\text{(3.11)} \qquad \frac{1}{2}\|w\|^2_{L_2(\Omega_t)} + \|\theta - \ln\theta\|_{L_1(\Omega_t)} + \iint_{Q_t}\left[\frac{\mu\rho u_x^2}{\theta} + \frac{\kappa\rho\theta_x^2}{\theta^2}\right]dxdt$$
$$\leq -\iint_{Q_t} u_1' w\,dxd\tau + \frac{1}{2}\|u_0 - u_1(0)\|^2_{L_2(0,X)}$$
$$+ \|\theta_0 - \ln\theta_0\|_{L_2(0,X)} + \iint_{Q_t} R\rho u_x\,dxd\tau.$$

The last term of the right-hand part of (3.11) is estimated with the help of the Cauchy inequality so as

$$\text{(3.12)} \qquad \iint_{Q_t} R\rho u_x\,dxd\tau \leq \frac{1}{2}\iint_{Q_t}\frac{\mu\rho u_x^2}{\theta}dxd\tau + \frac{t}{4\mu c_v}R^2 M_0 N_1 \exp\{N_2\}.$$

Hence,

$$\text{(3.13)} \qquad \frac{1}{2}\iint_{Q_t}\frac{\mu\rho u_x^2}{\theta}dxd\tau + \iint_{Q_t}\frac{\kappa\rho\theta_x^2}{\theta^2}dxdt \leq N_3 + \frac{t}{4\mu c_v}R^2 M_0 N_1 \exp\{N_2\},$$

where

$$N_3 = X^{\frac{1}{2}}N_1^{\frac{1}{2}}\int_0^t |u_1'|\,d\tau + \frac{1}{2}N_1 + \|\theta_0 - \ln\theta_0\|_{L_2(0,X)}.$$

Finally we can prove that ρ is bounded away zero. Let us consider the inequality

$$\text{(3.14)} \qquad M_\theta(t) \leq \theta(s(t), t) + \int_{\Omega_t}|\theta_x|\,dx$$
$$\leq \frac{1}{2}M_\theta(t) + m_\rho^{-1}(t)\left\{\theta(s(t), t)\rho(s(t), t) + \frac{1}{4c_v}N_1\int_{\Omega_t}\frac{\rho\theta_x^2}{\theta^2}dx\right\}.$$

Hence, the function $M_\theta(t)$ may be bounded from above by means of the function $m_\rho(t)$. Then inequality (3.10) may be rewritten in the following way

$$\text{(3.15)} \qquad m_\rho^{-1}(t) \leq A + \int_0^t B(\tau)m_\rho^{-1}(\tau)\,d\tau,$$

where

$$A = m_0^{-1} \exp\{N_2\},$$

$$B(t) = \frac{M_0 R}{\mu m_0} \exp\{2N_2\}\left(2\theta(a(t), t)\rho(a(t), t) + \frac{N_1}{2c_v}\int_{\Omega_t} \frac{\rho\theta_x^2}{\theta^2} dx\right).$$

Using Gronwall's inequality, we obtain the important relation:

$$m_\rho^{-1}(t) \leq A \exp\left\{\int_0^t B(\tau) d\tau\right\}. \tag{3.16}$$

Consequently,

$$(3.17)\quad m_\rho(t) \geq N_4(t) = m_0 \exp\{-N_2\}$$
$$\times \exp\left\{-\frac{M_0 R}{\mu m_0} \exp\{2N_2\} N_1 \left(\frac{1}{c_v \min\limits_{0 \leq \tau \leq t} |u_1|} + \frac{N_3}{2\kappa c_v} + \frac{t}{8\mu c_v^2 \kappa} R^2 M_0 N_1 \exp\{N_2\}\right)\right\}.$$

Now we are able to estimate the function $a(t)$. Obviously,

$$a(t) = X - \int_0^t \rho(a(\tau), \tau) u_1(\tau) d\tau \geq I_1(t) = X - M_0 \exp\{N_2\} \int_0^t |u_1(\tau)| d\tau,$$

$$a(t) \leq I_2(t) = X - \int_0^t N_4(\tau) u_1(\tau) d\tau.$$

Suppose

$$I_2(\infty) < 0,$$

then

$$I_1(\infty) < 0.$$

Let us take $T^* > T_* > 0$ such that

$$I_1(T_*) = 0,$$
$$I_2(T^*) = 0.$$

The form of the Lagrangian transformation (L) and the estimates (3.9), (3.17) allow us to write the following relations:

$$M^{-1} \exp\{-N_2\} a(t) \leq Y - z(t) \leq N_4^{-1}(T) a(t). \tag{3.18}$$

Therefore, if $T < T_*$ then $a(T) = \lim\limits_{t \to T-0} a(t) > 0$ and, hence, $z(T) = \lim\limits_{t \to T-0} z(t) < Y$. If $z(T) = \lim\limits_{t \to T-0} z(t) < Y$ then $a(T) = \lim\limits_{t \to T-0} a(t) > 0$ and, hence, $T < T^*$.

We have thus proved:

Theorem 1. *Suppose the functions* $\tilde{u}_0$, $\tilde{\rho}_0$, $\tilde{\theta}_0$, $u_1 > 0$ *are sufficiently smooth and:*

$$0 < m \le (\tilde{\rho}_0, \tilde{\theta}_0) \le M < \infty,$$

(3.19) $$\int_0^\infty |u_1'|\,d\tau < \infty.$$

Let the data satisfy one of the additional conditions

i) $$m \exp\{-N_2\} \times \int_0^\infty \exp\left\{-\frac{MR}{\mu m}\exp\{2N_2\}N_1\left(\frac{1}{c_v \min\limits_{0\le s\le t}|u_1|} + \frac{N_3}{2\kappa c_v} + \frac{t}{16c_v^2}MN_1\exp\{N_2\}\right)\right\}u_1(t)dt > X$$

ii) $$M\exp\{N_2\}\int_0^\infty |u_1(t)|\,dt < X.$$

Then in the case i) *there exist such constants* $T_* > 0$ *and* $T^* > 0$ *that*

1. *if* $0 < T < T_*$ *then the problem* (1.1)–(1.3), (2.1)–(2.4) *has no final solution on the interval* $[0, T]$;
2. *if* $T > T^*$ *then the problem has no solution on the interval* $[0, T]$.

In the case ii), *the problem has no final solution at all.*

Remark 1. The condition ii) can be made more delicate if we use a more exact estimate for ρ from above (see [1], pp. 49–50).

4. The existence of a final solution

Our goal in this section is to prove the existence of a final solution to the problem (1.1)–(1.3), (2.1)–(2.4). As mentioned above, a striking obstacle is that the domain of definition is unknown in the Eulerian variables as well as in the Lagrangian ones. However, if we prove both the existence of a solution locally in time and some a priori estimates, we will be able to obtain the final solution using a standard continuation method.

The local existence theorem, which will be proved in section 6, has the following formulation.

Theorem 2. *Let*

$$\tilde{\rho}_0 \in C^{1+\alpha}(0, Y), \qquad \tilde{u}_0, \tilde{\theta}_0 \in C^{2+\alpha}(0, Y),$$

$$u_1 \in C^{1+\frac{\alpha}{2}}(0, T), \qquad u_1 > 0 \qquad T > 0,\ 0 < \alpha < 1,$$

$$0 < m_0 \le (\tilde{\rho}_0(y), \tilde{\theta}_0(y)) \le M_0 < \infty,$$

and the following compatibility conditions be valid

$$u_1(0) = \tilde{u}_0(Y), \qquad \mu\tilde{u}_0'(0) = R\tilde{\rho}_0(0)\tilde{\theta}_0(0), \qquad \tilde{\theta}_0'(0) = \tilde{\theta}_0'(Y) = 0$$

$$u_1'(0) = -\tilde{u}_0(Y)\tilde{u}_0'(Y) + \mu\tilde{\rho}_0^{-1}(Y)\tilde{u}_0''(Y) - R\tilde{\theta}_0'(Y) - R\tilde{\rho}_0'(Y)\tilde{\rho}_0'(Y)\tilde{\theta}_0(Y).$$

Then the problem (1.1)–(1.3), (2.1)–(2.4) *has a solution on some interval* $[0, T_1]$, *where* $T_1 \in (0, T)$, *such that:*

$$u,\ \theta \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q^Y_{T_1}), \qquad \rho \in C^{1+\alpha}(Q^Y_{T_1}), \qquad z(T_1) < Y.$$

We are able to deduce the same a priori estimates for a solution of the problem (1.1)–(1.3), (2.1)–(2.4) as for a solution of the auxiliary problem (section 5). The estimates are dependent on the data of the problem and $\min\limits_{0 \le t \le T} (Y - z(t))$. They guarantee that any extension may be impossible only if $z(t)$ becomes equal to Y.

Let the conditions of Theorem 2 for any $T > 0$ and conditions (3.19), i) be satisfied. In this case we have proved the degeneration of the domain of unknowns. The existence of a solution in the degenerate domain is proved in the following way.

We choose the sequence $\{z_n\}$ which tends to Y. Let $z_n = Y\left(1 - \dfrac{1}{2n}\right)$, $n > 1$. On $n^{-\text{th}}$ step the solution is produced over the time interval $[0, T_n]$, where $z(T_n) = z_n$, but $z(t) < z_n$ for $t < T_n$. Such a moment exists without fail. Otherwise the a priori estimates ensure the extension of the solution over any time interval, but it is impossible (Theorem 1).

We have a non-decreasing sequence $\{T_n\}$, which is bounded from above: $T_n \le T^*$, $n > 1$. Consequently, there exists value $T_f = \lim\limits_{n \to \infty} T_n$ and the function $z(t)$ is defined over the open time interval $[0, T_f)$.

Obviously,

$$\lim_{n \to \infty} z(T_n) = Y. \tag{4.1}$$

We state that $z(t)$ can be determined over the closed time interval $[0, T_f]$ and $z(T_f) = Y$. Really, we see that

$$N_4(T) \ge N_4(T_f) > 0 \qquad \text{for } T \le T_f. \tag{4.2}$$

Hence, because of (3.18), (4.1), (4.2) there exists

$$\lim_{n \to \infty} a(T_n) = 0.$$

However, the function $a(t)$ is monotone. Therefore there exists

$$\lim_{t \to T_f - 0} a(t) = 0.$$

Then the relations (3.18) and (4.2) ensure that there exists

$$\lim_{t \to T_f - 0} z(t) = Y.$$

and we take $z(T_f) = \lim\limits_{t \to T_f - 0} z(t) = Y$.

We have thus proved

Theorem 3. *Suppose that the conditions of Theorem 2 are valid for any* $T > 0$ *and the data satisfy* (3.19) *and* i). *Then there exists a final solution to the problem* (1.1)–(1.3), (2.1)–(2.4), *which has properties*:

$$u,\ \theta \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q^Y_{T_f-\varepsilon}), \qquad \rho \in C^{1+\alpha}(Q^Y_{T_f-\varepsilon}), \qquad \textit{for any } \varepsilon,\ 0 < \varepsilon < T_f.$$

Here $[0, T_f]$ *is the time interval of the final solution.*

Remark 2. If the condition ii) is valid then Theorem 2 and the a priori estimates ensure that a solution exists over any time interval $[0, T]$.

5. The auxiliary problem

Let

$$a(t) \in C^{1+\frac{\alpha}{2}}(0, T), \qquad \text{for some } T > 0,$$

and

(5.1)
$$a(0) = X, \qquad a'(0) = -u_0(X)\rho_0(X),$$
$$a(t) \geq x_0 > 0, \qquad 0 < m_1 \leq -a'(t) \leq M_1 < \infty$$

for any $0 \leq t \leq T$.

We denote by P_a the following problem in a domain with a known right-hand boundary:

(5.2)
$$\frac{\partial u}{\partial t} = \mu \frac{\partial}{\partial x}\left(\rho \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}(R\rho\theta),$$

(5.3)
$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial x} = 0,$$

(5.4)
$$c_v \frac{\partial \theta}{\partial t} = \kappa \frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right) + \mu\rho\left(\frac{\partial u}{\partial x}\right)^2 - R\rho\theta\frac{\partial u}{\partial x}$$

in $Q^a_T = \{(x, t): 0 < t < T, x \in \Omega_t = (0, a(t))\}$,

(5.5)
$$(u, \theta, \rho) = (u_0(x), \theta_0(x), \rho_0(x)) \qquad \text{for } t = 0,\ 0 \leq x \leq X,$$

(5.6)
$$u = u_1(t) > 0, \qquad \frac{\partial \theta}{\partial x} = 0, \qquad \text{for } x = a(t),\ 0 \leq t \leq T,$$
$$\mu\frac{\partial u}{\partial x} - R\theta = \frac{\partial \theta}{\partial x} = 0 \qquad \text{for } x = 0,\ 0 \leq t \leq T.$$

We suppose that the functions u_1, $\tilde{u}_0$, $\tilde{\theta}_0$, $\tilde{\rho}_0$ (see (3.6)) satisfy the conditions of Theorem 2.

The local solvability of the problem (5.2)–(5.6) is proved in a way presented in [17], [18]. We have to deduce global a priori estimates to extend a local solution over the whole time interval $[0, T]$. The estimates also help to prove the local solvability of the main problem and to extend a local solution over some time interval $[0, T_f]$.

First of all, we should note that estimates similar to (3.7), (3.9), (3.13), (3.17) are also valid for a solution of the auxiliary problem (5.2)–(5.6).

More precisely, we have:

Lemma 2.

$$\max_{0\le t\le T}\{\|u(t)\|_{L_2(\Omega_{a(t)})}+\|\theta(t)\|_{L_1(\Omega_{a(t)})}\}\le c_1, \tag{5.7}$$

$$c_2\le\rho(x,t)\le c_3, \qquad (x,t)\in Q_T^a, \tag{5.8}$$

where positive constants c_i, $i=1, 2, 3$, are only dependent on the given functions u_1, u_0, ρ_0, θ_0 and T.

We now prove that $\theta(x, t)$ is bounded away from zero.

Lemma 3. *There exists a constant $m>0$ such that*

$$m_\theta(t)\ge m \tag{5.9}$$

for any $t\in[0, T]$.

Proof. Using an idea of A. V. Kazhikhov we multiply (5.4) by $-\theta^{-2}$ to obtain the following equation for $\omega=\theta^{-1}$:

$$\frac{\partial\omega}{\partial t}=\kappa\frac{\partial}{\partial x}\left(\rho\frac{\partial\omega}{\partial x}\right)+f+\frac{R^2}{4\mu}\rho, \tag{5.10}$$

where

$$f(x,t)=-\frac{\mu\rho}{\theta^2}\left(\frac{\partial u}{\partial x}-\frac{R}{2\mu}\theta\right)^2-\frac{2\kappa\rho}{\theta^3}\left(\frac{\partial\theta}{\partial x}\right)^2.$$

Now we multiply (5.10) by $2p\omega^{2p-1}$ and integrate with respect to x over $(0, a(t))$. Since $f(x,t)\le 0$, using the Hölder inequality and (5.8) we get:

$$\frac{d}{dt}\|\omega(t)\|_{L_{2p}(0,X)}\le c_5.$$

Then,

$$\|\omega(t)\|_{L_{2p}(0,X)}\le\|\omega(0)\|_{L_{2p}(0,X)}+c_5t.$$

Passing to the limit as $p\to\infty$, we see

$$\|\omega(t)\|_{L_\infty(0,X)}\le\|\omega(0)\|_{L_\infty(0,X)}+c_6t.$$

and

$$\theta(x,t)\ge(m_0^{-1}+c_6T)^{-1}=m>0, \qquad 0\le x\le 1, \qquad 0\le t\le T.$$

It follows from (3.13), (5.8), (5.9) that

$$\iint_{Q_T^a}\left(\frac{\partial u}{\partial x}\right)^2 dxdt + \iint_{Q_T^a}\left(\frac{\partial \theta}{\partial x}\right)^2 dxdt \le c_7, \tag{5.11}$$

It would be observed that estimates (5.7)–(5.9), (5.11) are independent of a function $a(t)$. It is enough to have a smooth function with the property $a'(t) \le 0$.

Using a simple inequality

$$\theta(x, t) - \frac{1}{a(t)}\int_{\Omega_t} \theta(s, t)ds \le a(t)^{\frac{1}{2}}\left(\int_{\Omega_t}\left(\frac{\partial \theta}{\partial x}\right)^2 dx\right)^{\frac{1}{2}}$$

we deduce

$$\int_0^T M_\theta(t)^2 dt \le c_8. \tag{5.12}$$

We are in position to prove the final integral estimates.

Lemma 4.

$$\max_{0\le t\le T}\int_0^{a(t)}\left(\frac{\partial u}{\partial x}\right)^2 dx + \iint_{Q_T^a}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]dxdt \le c_9, \tag{5.13}$$

$$\max_{0\le t\le T}\int_0^{a(t)}\left(\frac{\partial \theta}{\partial x}\right)^2 dx + \iint_{Q_T^a}\left[\left(\frac{\partial^2 \theta}{\partial x^2}\right)^2 + \left(\frac{\partial \theta}{\partial t}\right)^2\right]dxdt \le c_{10}, \tag{5.14}$$

$$\max_{0\le t\le T}\int_0^{a(t)}\left[\left(\frac{\partial \rho}{\partial x}\right)^2 + \left(\frac{\partial \rho}{\partial t}\right)^2\right]dx + \iint_{Q_T^a}\left(\frac{\partial^2 \rho}{\partial x\partial t}\right)^2 dxdt \le c_{11}. \tag{5.15}$$

Proof. Using the translation $u = w + u_1$ into equations (5.2), (5.3), (5.4) and differentiating the second equation, written in the form

$$\frac{\partial \ln \rho}{\partial t} = -\rho\frac{\partial w}{\partial x},$$

with respect to x, we obtain the system

$$\frac{\partial w}{\partial t} = \mu\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right) - \gamma\rho^\gamma\frac{\partial \ln \rho}{\partial x} - u_1', \tag{5.16}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \ln \rho}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right), \tag{5.17}$$

$$c_v\frac{\partial \theta}{\partial t} = \kappa\frac{\partial}{\partial x}\left(\rho\frac{\partial \theta}{\partial x}\right) + \mu\rho\left(\frac{\partial w}{\partial x}\right)^2 - R\rho\theta\frac{\partial w}{\partial x}. \tag{5.18}$$

We multiply equation (5.16) by $\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)$ and equation (5.17) by $\frac{\partial \ln \rho}{\partial x}$ and then we integrate their sum over $Q_t^a = \{(x, \tau): 0 < \tau < t,\ 0 < x < a(\tau)\}$. After simple reductions we see

$$(5.19)\quad \frac{1}{2}\int_0^{a(\tau)}\left[\rho\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial \ln\rho}{\partial x}\right)^2\right]dx\Big|_0^t+\mu\int_0^t\int_0^{a(\tau)}\left[\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)\right]^2dxd\tau$$
$$=\int_0^t\left[\frac{\partial w}{\partial t}\rho\frac{\partial w}{\partial x}+\frac{1}{2}a'(\tau)\rho\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}a'(\tau)\left(\frac{\partial \ln\rho}{\partial x}\right)^2\right]\Big|_{x=a(\tau)}d\tau$$
$$+\int_0^t\int_0^{a(\tau)}u_1'\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)dxd\tau+\int_0^t\int_0^{a(\tau)}R\left(\rho\frac{\partial\theta}{\partial x}+\theta\frac{\partial\rho}{\partial x}\right)\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)dxd\tau$$
$$-\frac{1}{2}\int_0^t\int_0^{a(\tau)}\rho^2\left(\frac{\partial w}{\partial x}\right)^3dxd\tau.$$

Let us note that the integral

$$\int_0^t\left[a'(\tau)\left(\frac{\partial \ln\rho}{\partial x}\right)^2\right]\Big|_{x=a(\tau)}d\tau$$

is negative and

$$\frac{\partial w}{\partial t}\Big|_{x=a(t)}=-a'\frac{\partial w}{\partial x}\Big|_{x=a_1(t)}.$$

Therefore, using the Cauchy inequality with ε, we can estimate the right-hand side of (5.19) so as to obtain:

$$(5.20)\quad \int_0^{a(t)}\left[\rho\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial \ln\rho}{\partial x}\right)^2\right]dx+\mu\int_0^t\int_0^{a(\tau)}\left[\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)\right]^2dxd\tau$$
$$\le c_{12}\left[1+\int_0^t M_\theta(\tau)^2\int_0^{a(\tau)}\left[\frac{\partial\ln\rho}{\partial x}\right]^2dxd\tau+\int_0^t\max_{0\le x\le a(\tau)}\left|\rho\frac{\partial w}{\partial x}\right|^2d\tau\right.$$
$$\left.+\int_0^t\max_{0\le x\le a(\tau)}\left|\rho\frac{\partial w}{\partial x}\right|\int_0^{a(\tau)}\rho\left(\frac{\partial w}{\partial x}\right)^2dxd\tau\right].$$

By an embedding inequality we see

$$(5.21)\quad \max_{0\le x\le a(t)}\left|\rho\frac{\partial w}{\partial x}\right|\le N\left(\left\|\rho\frac{\partial w}{\partial x}\right\|_{L_2(0,a)}^{\frac{1}{2}}\left\|\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)\right\|_{L_2(0,a)}^{\frac{1}{2}}+\left\|\rho\frac{\partial w}{\partial x}\right\|_{L_2(0,a)}\right).$$

Combining (5.20), (5.21) and using the Cauchy inequality with ε and (5.8), (5.11), (5.12), we have

$$z(t)\le C+\int_0^t A(\tau)z(\tau)d\tau,$$

where

$$z(t)=\int_0^{a(t)}\left[\rho\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial \ln\rho}{\partial x}\right)^2\right]dx+\mu\int_0^t\int_0^{a(\tau)}\left[\frac{\partial}{\partial x}\left(\rho\frac{\partial w}{\partial x}\right)\right]^2dxd\tau$$

and

$$0<C=const,\qquad A(t)>0,\qquad A(t)\in L_1(0,T).$$

Thanks to Gronwall's inequality, the representation for $w(x, t)$ and equations (5.2), (5.3), we obtain the desired relations for $u(x, t)$ and $\rho(x, t)$.

We now multiply equation (5.18) by $\frac{\partial}{\partial x}\left(\rho \frac{\partial \theta}{\partial x}\right)$ and integrate over Q_t^a. Using the same arguments as for (5.19)–(5.20) we deduce integral estimate (5.14).

Finally, we have to estimate the Hölder constants for a solution because the local solvability of the auxiliary problem is established in space of Hölder continuous functions.

Lemma 5. *For any solution of the auxiliary problem*

$$\|\rho\|_{C^{\frac{1}{2},\frac{3}{4}}(Q_T^a)} \le c_{13}. \tag{5.22}$$

Proof. The inequality is obtained by means of straightforward calculations using (5.15) and the imbedding inequality (5.21) rewritten for $\frac{\partial \rho}{\partial t}$.

Remark 3. It would be observed that m, N and c_i, $i = 7, \ldots, 11, 13$, are solely dependent on the problem data and the values x_0, M_1 in (5.1).

For our further considerations it is important to have the exact representation for the partial derivative $\frac{\partial \rho}{\partial x}$. Using relation (3.8), we see

(5.23)

$$\begin{aligned}
\frac{\partial \rho}{\partial x}(x, t) &= \rho_0'(x)\rho_0(x)^{-1}\rho(x, t) + \frac{1}{\mu}[u_0(x) - u(x, t)]\rho(x, t) \\
&\quad - \frac{R}{\mu}\rho(x, t)\left(1 + \frac{R}{\mu}\int_0^t \rho_0(x)\theta(x, \tau)\exp\left\{\frac{R}{\mu}\int_0^x [u_0(s) - u(s, \tau)]ds\right\}d\tau\right)^{-1} \\
&\quad \times \int_0^t [\rho_0'(x)\theta(x, \tau) + \rho_0(x)\frac{\partial \theta}{\partial x}(x, \tau) \\
&\quad + \frac{R}{\mu}\rho_0(x)\theta(x, \tau)(u_0(x) - u(x, \tau))]\exp\left\{\frac{R}{\mu}\int_0^x [u_0(s) - u(s, \tau)]ds\right\}d\tau,
\end{aligned}$$

Lemma 6. *If the data of the auxiliary problem satisfy the conditions of Theorem 2 then for any solution the following estimates are valid*

$$\|u(x, t)\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T^a)} + \|u(x, t)\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T^a)} + \|\rho(x, t)\|_{C^{1+\alpha}(Q_T^a)} \le c_{14}.$$

where c_{14} is dependent on the data of the problem and values T, x_0, m_1, M_1, $\|a\|_{C^{1+\frac{\alpha}{2}}(0,T)}$.

Proof. We use the Schauder estimates for a solution of a linear parabolic equation. We can rewrite equation (5.2) and the boundary conditions for $u(x, t)$ in the following form

$$
(5.24)\qquad \begin{gathered}
\frac{\partial u}{\partial t} = A_1(x,t)\frac{\partial^2 u}{\partial x^2} + B_1(x,t)\frac{\partial u}{\partial x} + F_1(x,t),\\
u(x,0) = u_0(x), \qquad u(a(t),t) = u_1(t), \qquad \frac{\partial u}{\partial x}(0,t) = \Psi(t),
\end{gathered}
$$

where

$$
A_1(x,t) = \mu\rho(x,t), \quad B_1(x,t) = \mu\frac{\partial\rho}{\partial x}(x,t), \quad F_1(x,t) = -\frac{\partial(R\rho\theta)}{\partial x}, \quad \Psi(t) = \frac{R}{\mu}\rho\theta(0,t).
$$

Obviously, $A_1(x,t)$ is a continuous and bounded function, and

$$
\|B_1(x,t)\|_{L_4(Q_T^a)} + \|F_1(x,t)\|_{L_4(Q_T^a)} \le c_{15}.
$$

Using the Schauder estimates [15], we have

$$
(5.25)\qquad \|u(x,t)\|_{W_4^{2,1}(Q_T^a)} \le c_{16}(1 + \|\theta|_{x=0}\|_{W_4^{\frac{3}{8}}(0,T)}).
$$

We can also rewrite equation (5.4) in a similar form

$$
(5.26)\qquad \begin{gathered}
\frac{\partial\theta}{\partial t} = A_2(x,t)\frac{\partial^2\theta}{\partial x^2} + B_2(x,t)\frac{\partial\theta}{\partial x} + F_2(x,t),\\
\theta(x,0) = \theta_0(x), \qquad \frac{\partial\theta}{\partial x}(0,t) = \frac{\partial\theta}{\partial x}(a(t),t) = 0,
\end{gathered}
$$

where

$$
A_2(x,t) = \frac{\kappa}{c_v}\rho(x,t), \qquad B_2(x,t) = \frac{\kappa}{c_v}\frac{\partial\rho}{\partial x}(x,t), \qquad F_2(x,t) = \frac{\mu}{c_v}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{R}{c_v}\rho\theta\frac{\partial u}{\partial x}.
$$

The Schauder estimates give

$$
(5.27)\qquad \|\theta\|_{W_4^{2,1}(Q_T^a)} \le c_{17}(1 + \|F_2\|_{L_4(Q_T^a)}) \le c_{17}\left(1 + \left\|\frac{\partial u}{\partial x}\right\|_{C^0(Q_T^a)}^{\frac{1}{2}}\right).
$$

The second inequality (5.27) is obtained with the help of (5.13) and inequality (5.21) rewritten for $\frac{\partial u}{\partial x}$.

By embedding theorem ([8], p. 80) $W_4^{2,1}(Q_T^a) \subset C^{1+\frac{1}{4},\frac{1}{8}}(Q_T^a)$ and

$$
(5.28)\qquad \|u(x,t)\|_{C^{1+\frac{1}{4},\frac{1}{8}}(Q_T^a)} \le c_{18}\|u(x,t)\|_{W_4^{2,1}(Q_T^a)},
$$

$$
(5.29)\qquad \|\theta(x,t)\|_{C^{1+\frac{1}{4},\frac{1}{8}}(Q_T^a)} \le c_{18}\|\theta(x,t)\|_{W_4^{2,1}(Q_T^a)}.
$$

In our case $W_4^{2,1}(Q_T^a) \subset W_4^{\frac{3}{8}}(\{x=0\}\times(0,t))$. Hence, it follows from (5.25), (5.28), (5.27) that

$$
(5.30)\qquad \|u(x,t)\|_{C^{1+\frac{1}{4},\frac{1}{8}}(Q_T^a)} \le c_{19}\left(1 + \left\|\frac{\partial u}{\partial x}\right\|_{C^0(Q_T^a)}^{\frac{1}{2}}\right).
$$

Using the Cauchy inequality and the obvious imbedding $C^{\frac{1}{4},\frac{1}{8}}(Q_T^a) \subset C^0(Q_T^a)$ we deduce from (5.30)

$$\|u(x, t)\|_{C^{1+\frac{1}{4},\frac{1}{8}}(Q_T^a)} \leq c_{20}.$$

Therefore, (5.27), (5.29) imply

$$\|\theta(x, t)\|_{C^{1+\frac{1}{4},\frac{1}{8}}(Q_T^a)} \leq c_{21}.$$

Then representation (5.23) helps to obtain the following relation

$$\left\| \frac{\partial \rho}{\partial x} \right\|_{C^{\beta,\frac{\beta}{2}}(Q_T^a)} \leq c_{22},$$

where $\beta = \min\{\frac{1}{4}, \alpha\}$.

Hence, equations (5.24), (5.26) have the coefficients with the properties:

$$\|A_i(x, t)\|_{C^{\beta,\frac{\beta}{2}}(Q_T^a)} + \|B_i(x, t)\|_{C^{\beta,\frac{\beta}{2}}(Q_T^a)} + \|F_i(x, t)\|_{C^{\beta,\frac{\beta}{2}}(Q_T^a)} \leq c_{23}, \qquad i = 1, 2$$

The Schauder estimates for the Hölder continuity constants of a solution of a linear parabolic equation [16] give

$$\|u(x, t)\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_T^a)} + \|\theta(x, t)\|_{C^{2+\beta,1+\frac{\beta}{2}}(Q_T^a)} \leq c_{24}, \tag{5.31}$$

If $\beta = \alpha$, then we have the desired estimates. If $\beta < \alpha$, then we use representation (5.23) again.

We see, from (5.23), (5.31) and (5.3), that

$$\|\rho(x, t)\|_{C^{1+\alpha,1+\frac{\beta}{2}}(Q_T^a)} \leq c_{23}. \tag{5.32}$$

Therefore, we can repeat the relation (5.31) with β replaced by α. Moreover, from (5.23) and (5.3), we deduce the desired estimate for ρ:

$$\|\rho(x, t)\|_{C^{1+\alpha}(Q_T^a)} \leq c_{24}.$$

The lemma has been proved.

6. Local existence

In this section we prove Theorem 2. The local solution to (1.1)–(1.3), (2.1)–(2.4) will be constructed using the Lagrangian formulation of the problem. The existence of a solution over a time interval $(0, T_1)$ will be established by means of the Tikhonov-Schauder fixed point theorem, which we recall for easy reference:

Theorem 4 [4, p. 221]. *Let $\mathscr{M}$ be a compact convex set of a Banach space $\mathscr{B}$ and let $\mathscr{L}$ be a continuous mapping of $\mathscr{M}$ into itself. Then $\mathscr{L}$ has a fixed point, that is, $\mathscr{L}f = f$ for some $f \in \mathscr{M}$.*

For our purpose, $\mathscr{B} = C^1[0, T_1]$, where T_1 will be chosen in the sequel. The set $\mathscr{M}$ is determined in the following way:

$$\mathscr{M} = \{a(t) \in C^{1+\frac{\alpha}{2}}[0, T_1]\colon a(0) = X, a'(0) = -u_0(X)\rho_0(X),$$
$$a(t) \geq x_0, 0 < m_1 \leq -a'(t) \leq M_1 < \infty, \|a\|_{C^{1+\frac{\alpha}{2}}[0,T_1]} \leq M_2\}$$

It is clear that $\mathscr{M}$ is convex and closed in $\mathscr{B}$. Moreover, according to Ascoli's theorem, $\mathscr{M}$ is precompact in $\mathscr{B}$.

Consider now the map $\mathscr{L}$ defined on $\mathscr{M}$ in the following way. Let $a_0 \in \mathscr{M}$. We can formulate the auxiliary Problem P_{a_0}. If $u(x, t)$, $\rho(x, t)$ is its classical solution, then

$$a_1(t) = \mathscr{L}a_0(t) = X - \int_0^t u_1(\tau)\rho(a_0(\tau), \tau)d\tau.$$

Obviously,

$$a_1(0) = X,$$
$$a_1'(t) = -u_1(t)\rho(a_0(t), t), \qquad a'(0) = -u_0(X)\rho_0(X),$$
$$\begin{aligned} a_1'(t_2) - a_1'(t_1) = &-u_1'(t^*)\rho(a_0(t_2), t_2)|t_2 - t_1| \\ &- u_1(t_1)H_x^{\frac{1}{2}}(\rho(x, t))a_0'(t^{**})^{\frac{1}{2}}|t_2 - t_1|^{\frac{1}{2}} \\ &- u_1(t_1)H_t^{\frac{1}{2}}(\rho(x, t))|t_2 - t_1|^{\frac{1}{2}}, \end{aligned}$$

where $0 \leq t_1 < t_2 \leq T_1$ and t^*, $t^{**} \in [t_1, t_2]$.

Let us take $x_0 = \frac{1}{2}X$. According to the a priori estimates for the auxiliary problem, the bounds for $\rho(x, t)$ are only determined by the data of the problem and T_1. Using the bounds for $\rho(x, t)$ if $T_1 = 1$, we can choose values for m_1 and M_1. Therefore, we can establish bounds for $H_x^{\frac{1}{2}}(\rho(x, t))$ and $H_t^{\frac{1}{2}}(\rho(x, t))$ because of (5.22). Hence, if we take M_2 large enough and T_1 small enough, we get that $a_1(t)$ satisfies the relations required in the definition of the set $\mathscr{M}$. Thus, the operator $\mathscr{L}$ is a mapping of $\mathscr{M}$ into itself.

Finally, we only need to verify that $\mathscr{L}$ is continuous on $\mathscr{M}$ in the norm of the space $\mathscr{B}$.

Let elements of sequence $\{a_n\}$, $n \geq 2$, belong to $\mathscr{M}$ and

$$a_n \to a \qquad \text{in } C^1[0, T_1].$$

Because $\mathscr{M}$ is closed in $C^1[0, T_1]$, we have $a \in \mathscr{M}$. We consider the auxiliary problems. The functions u_n, ρ_n and u, ρ are solutions of Problem P_{a_n} and Problem P_a respectively.

Using simple transformations

$$x_1 = \frac{x}{a_n}, \qquad t_1 = t, \qquad (n \geq 2), \tag{6.1}$$

we obtain problems formulated in the fixed domain:

$$\frac{\partial u_n}{\partial t} - \frac{x_1 a_n'}{a_n}\frac{\partial u_n}{\partial x_1} = \frac{\mu}{a_n^2}\frac{\partial}{\partial x_1}\left(\rho_n\frac{\partial u_n}{\partial x_1}\right) - \frac{1}{a_n}\frac{\partial}{\partial x_1}(Rp_n\theta_n),$$

$$\frac{\partial \rho_n}{\partial t} - \frac{x_1 a_n'}{a_n}\frac{\partial \rho_n}{\partial x_1} + \frac{1}{a_n}\rho_n^2\frac{\partial u_n}{\partial x_1} = 0,$$

$$c_v\frac{\partial \theta_n}{\partial t} - \frac{x_1 a_n'}{a_n}\frac{\partial \theta_n}{\partial x_1} = \frac{\kappa}{a_n^2}\frac{\partial}{\partial x_1}\left(\rho_n\frac{\partial \theta_n}{\partial x_1}\right) + \frac{\mu}{a_n^2}\left(\frac{\partial u_n}{\partial x_1}\right)^2 + \frac{1}{a_n}R\rho_n\theta_n\frac{\partial u_n}{\partial x_1},$$

in $Q_1 = \{(x_1, t): 0 < x_1 < 1, 0 < t < T_1\}$

$$u_n = u_0(x_1X), \quad \rho_n = \rho_0(x_1X), \quad \theta_n = \theta_0(x_1X) \quad \text{for } t = 0,\ 0 \le x_1 \le 1;$$

$$u_n = u_1(t) > 0, \quad \frac{\partial \theta_n}{\partial x_1} = 0 \quad \text{for } x_1 = 1,\ 0 \le t \le T_1.$$

$$\frac{\mu\rho_n}{a_n}\frac{\partial u_n}{\partial x_1} - R\rho_n\theta_n = \frac{\partial \theta_n}{\partial x_1} = 0 \quad \text{for } x_1 = 0,\ 0 \le t \le T_1.$$

Problem P_a is transformed in the same way.

It is clear that

$$\mathscr{L}a_n(t) = X - \int_0^t u_1(\tau)\rho_n(0, \tau)d\tau,$$

(6.2)

$$\mathscr{L}a(t) = X - \int_0^t u_1(\tau)\rho(0, \tau)d\tau,$$

where u_n, ρ_n, θ_n and u, ρ, θ are solutions of the transformed problems.

Let

$$w_n = u_n - u, \qquad v_n = \frac{1}{\rho_n} - \frac{1}{\rho}, \qquad \Phi_n = \theta_n - \theta.$$

Then the functions w_n, v_n, Φ_n satisfy the following equations:

$$\frac{\partial w_n}{\partial t} - \frac{x_1 a_n'}{a_n}\frac{\partial w_n}{\partial x_1} = \frac{\mu}{a_n}\frac{\partial}{\partial x_1}\left(\frac{\rho_n}{a_n}\frac{\partial u_n}{\partial x_1} - R\rho_n\theta_n - \frac{\rho}{a}\frac{\partial u}{\partial x_1} + R\rho\theta\right)$$
$$+ \left(\frac{\mu}{a_n} - \frac{\mu}{a}\right)\frac{\partial}{\partial x_1}\left(\frac{\rho}{a}\frac{\partial u}{\partial x_1} - R\rho\theta\right) + \left(\frac{x_1 a_n'}{a_n} - \frac{x_1 a'}{a}\right)\frac{\partial u}{\partial x_1},$$

$$\frac{\partial v_n}{\partial t} - \frac{x_1 a_n'}{a_n}\frac{\partial v_n}{\partial x_1} - \frac{1}{a_n}\frac{\partial w_n}{\partial x_1} = \left(\frac{x_1 a_n'}{a_n} - \frac{x_1 a'}{a}\right)\frac{\partial}{\partial x_1}\left(\frac{1}{\rho}\right) + \left(\frac{1}{a_n} - \frac{1}{a}\right)\frac{\partial u}{\partial x_1},$$

$$\frac{\partial \Phi_n}{\partial t} - \frac{x_1 a_n'}{a_n}\frac{\partial \Phi_n}{\partial x_1} = \frac{\kappa}{a_n^2}\frac{\partial}{\partial x_1}\left(\rho_n\frac{\partial \theta_n}{\partial x_1} - \rho\frac{\partial \theta}{\partial x_1}\right) + \kappa\left(\frac{1}{a_n^2} - \frac{1}{a^2}\right)\frac{\partial}{\partial x_1}\left(\rho\frac{\partial \theta}{\partial x_1}\right)$$
$$+ \frac{\mu}{a_n^2}\frac{\partial w_n}{\partial x_1}\left(\frac{\partial u_n}{\partial x_1} + \frac{\partial u}{\partial x_1}\right) + \mu\left(\frac{1}{a_n^2} - \frac{1}{a^2}\right)\left(\frac{\partial u}{\partial x_1}\right)^2$$
$$- \frac{R}{a_n}\left(\rho_n\theta_n\frac{\partial u_n}{\partial x_1} - \rho\theta\frac{\partial u}{\partial x_1}\right) - \left(\frac{1}{a_n} - \frac{1}{a}\right)R\rho\theta\frac{\partial u}{\partial x_1}$$
$$+ \left(\frac{x_1 a_n'}{a_n} - \frac{x_1 a'}{a}\right)\frac{\partial \theta}{\partial x_1}.$$

We multiply the equations by w_n, v_n and Φ_n respectively and then integrate their sum over $(0, 1) \times (0, t)$. After using the following relations:

$$-\int_0^t \int_0^1 \frac{x_1 a_n'}{a_n} \frac{\partial f}{\partial x_1} f\, dx_1 d\tau = -\int_0^t \frac{a_n'}{2a_n} f^2|_{x_1=1} d\tau + \int_0^t \frac{a_n'}{2a_n} \int_0^1 f^2 dx_1 d\tau$$

$$\geq -N_1 \int_0^t \int_0^1 f^2 dx_1 d\tau \qquad \text{for any } f \in C^1(Q_1) \cap C(\bar{Q}_1),$$

$$\int_0^t \int_0^1 \frac{\mu}{a_n} \frac{\partial}{\partial x_1}\left(\frac{\rho_n}{a_n}\frac{\partial u_n}{\partial x_1} - R\rho_n\theta_n - \frac{\rho}{a}\frac{\partial u}{\partial x_1} + R\rho\theta\right) w_n dx_1 d\tau$$

$$= -\int_0^t \int_0^1 \frac{\mu}{a_n}\left[R(\rho - \rho_n)\theta \frac{\partial w_n}{\partial x_1} - R\rho_n \Phi_n \frac{\partial w_n}{\partial x_1} + \left(\frac{\rho_n}{a_n} - \frac{\rho_n}{a}\right)\frac{\partial u_n}{\partial x_1}\frac{\partial w_n}{\partial x_1}\right] dx_1 d\tau$$

$$+ \int_0^t \int_0^1 \frac{\mu}{a_n}\left[\frac{\rho_n - \rho}{a}\frac{\partial u_n}{\partial x_1}\frac{\partial w_n}{\partial x_1} - \frac{\rho}{a}\left(\frac{\partial w_n}{\partial x_1}\right)^2\right] dx_1 d\tau$$

$$\leq N_{\varepsilon_1} \int_0^t \left(1 + \max_{0 \leq x_1 \leq 1}\left|\frac{\partial u_n}{\partial x_1}\right|^2\right) \int_0^1 [(\rho_n - \rho)^2 + \Phi_n^2] dx_1 d\tau$$

$$+ N_2 \max_{0 \leq t \leq T_1} |a_n(t) - a(t)| + \left(\varepsilon_1 - \frac{\mu c_2}{X^2}\right) \int_0^t \int_0^1 \left(\frac{\partial w_n}{\partial x_1}\right)^2 dx_1 d\tau,$$

$$-\int_0^t \int_0^1 \frac{1}{a_n}\frac{\partial w_n}{\partial x}\left(\frac{1}{\rho} - \frac{1}{\rho_n}\right) dx_1 d\tau \leq N_{\varepsilon_2} \int_0^t \int_0^1 (\rho_n - \rho)^2 dx_1 d\tau + \varepsilon_2 \int_0^t \int_0^1 \left(\frac{\partial w_n}{\partial x_1}\right)^2 dx_1 d\tau,$$

$$\int_0^t \int_0^1 \frac{\mu}{a_n^2}\frac{\partial w_n}{\partial x_1}\left(\frac{\partial u_n}{\partial x_1} + \frac{\partial u}{\partial x_1}\right)\Phi_n dx_1 d\tau \leq N_{\varepsilon_3} \int_0^t \left(\max_{0 \leq x_1 \leq 1}\left|\frac{\partial u_n}{\partial x_1}\right|^2 + \max_{0 \leq x_1 \leq 1}\left|\frac{\partial u}{\partial x_1}\right|^2\right)$$

$$\times \int_0^1 \Phi_n^2 dx_1 d\tau + \varepsilon_3 \int_0^t \int_0^1 \left(\frac{\partial w_n}{\partial x_1}\right)^2 dx_1 d\tau,$$

$$\int_0^t \int_0^1 \frac{R}{a_n}\left(\rho_n\theta_n\frac{\partial u_n}{\partial x_1} - \rho\theta\frac{\partial u}{\partial x_1}\right)\Phi_n dx_1 d\tau$$

$$\leq N_{\varepsilon_4} \int_0^t \left(1 + \max_{0 \leq x_1 \leq 1}\left|\frac{\partial u_n}{\partial x_1}\right|^2\right) \int_0^1 [(\rho_n - \rho)^2 + \Phi_n^2] dx_1 d\tau + \varepsilon_4 \int_0^t \int_0^1 \left(\frac{\partial w_n}{\partial x_1}\right)^2 dx_1 d\tau,$$

$$\int_0^t \int_0^1 \frac{\kappa}{a_n^2}\frac{\partial}{\partial x_1}\left(\rho_n\frac{\partial \theta_n}{\partial x_1} - \rho\frac{\partial \theta}{\partial x_1}\right)\Phi_n dx_1 d\tau$$

$$\leq N_{\varepsilon_5} \int_0^t \max_{0 \leq x_1 \leq 1}\left|\frac{\partial \theta_n}{\partial x_1}\right|^2 \int_0^1 (\rho_n - \rho)^2 dx_1 d\tau + \left(\varepsilon_5 - \frac{\kappa c_2}{X^2}\int_0^t \int_0^1 \left(\frac{\partial \Phi_n}{\partial x_1}\right)^2 dx_1 d\tau,\right.$$

and taking $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 < \frac{\mu c_2}{X^2}$, $\varepsilon_5 < \frac{\kappa c_2}{X^2}$ we obtain

$$Z(t) \leq N_3 \|a_n - a\|_{C^1[0, T_1]} + \int_0^t A(\tau)Z(\tau)d\tau, \tag{6.3}$$

where

$$Z(t) = \int_0^1 [w_n^2 + (\rho_n - \rho)^2 + \Phi_n^2] dx_1,$$

and

$$A(t) > 0, \qquad A(t) \in L_1(0, T_1).$$

Then Gronwall's inequality ensures that

$$\int_0^1 [\rho_n(x_1, t) - \rho(x_1, t)]^2 dx_1 \leq N_4 \|a_n - a\|_{C^1[0,T_1]},$$

where N_4 is only dependent on the problem data and constants determining the set $\mathscr{M}$.

Thus, ρ_n converges to ρ in $L_2(Q_1)$.

Taking into account the estimate (5.22) for a solution of the auxiliary problem, we have

$$\|\rho_n\|_{C^{\frac{1}{2}}(Q_1)} \leq N_5,$$

where N_5 is independent of n.

Therefore, from Ascoli's theorem, the sequence $\{\rho_n\}$ is precompact in $C(\bar{Q}_1)$ and we have stronger convergence:

$$\rho_n \to \rho \qquad \text{in } C(\bar{Q}_1).$$

Hence,

$$\rho_n(1, t) \to \rho(1, t) \qquad \text{in } C[0, T_1],$$

and, according to (6.2),

$$\mathscr{L}a_n \to \mathscr{L}a \qquad \text{in } C^1[0, T_1].$$

That is, the operator $\mathscr{L}$ is continuous on $\mathscr{M}$ in the norm of $\mathscr{B}$.

We have established the existence of a fixed point for operator $\mathscr{L}$. Any fixed point determines a solution to the problem (3.1)–(3.5) ((1.1)–(1.3), (2.1)–(2.4)).

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