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The embeddings of discrete series into principal series for an exceptional real simple Lie group of type G_2

Dedicated to Professor Takeshi Hirai on his 60th birthday

By

Tetsumi YOSHINAGA and Hiroshi YAMASHITA*

Introduction

Discrete series representations of a semisimple Lie group have been studied for a long time. In [1], Harish-Chandra gave their parametrization by means of the theory of characters. For realization of discrete series, R. Hotta and R. Parthasarathy [4], W. Schmid [7] and others gave geometric construction of those representations. Let G be a connected semisimple Lie group with finite center, K its maximal compact subgroup, π_Λ the discrete series representation of G with Harish-Chandra parameter Λ and $(\tau_\lambda, V_\lambda)$ the lowest K-type of π_Λ. A differential operator $\mathscr{D}$ on the space $C^\infty_{\tau_\lambda}(G)$ of V_λ-valued smooth functions f on G satisfying $f(kg) = \tau_\lambda(k)f(g)$ for all $k \in K$ and for all $g \in G$, denoted by $\mathscr{D}_\lambda$ in this paper, is introduced in [7] by Schmid. He showed, also in [7], that the discrete series π_Λ is realized as the space of L_2-kernel of $\mathscr{D}$. This result was shown in a simpler way in the paper [4] of Hotta and Parthasarathy. By means of $\mathscr{D}$ and "Szegö kernel", A. W. Knapp and N. R. Wallach [5] found that each discrete series is expressed as a quotient of some principal series representations, determined relative to the discrete series. Considering duality, one can obtain certain principal series into which a given discrete series can be embedded.

Modifying and extending the idea in [5], the second-named author of the present paper gave, in [9], a method to determine the embeddings of discrete series into various induced representations, as $(\mathfrak{g}, K)$-modules. Here $\mathfrak{g}$ denotes the complexified Lie algebra of G.

For a closed subgroup P of G and its representation η on a Fréchet space F, define $C^\infty_{\tau_\lambda}(G;\eta)$ to be the space of $V_\lambda \otimes F$-valued smooth functions f on G with the condition $f(kgp) = \delta_P(p)^{1/2}(\tau_\lambda(k) \otimes \eta(p)^{-1})f(g)$ for $k \in K$, $g \in G$ and $p \in P$, where δ_P is a modular function of P. Since $C^\infty_{\tau_\lambda}(G;\eta)$ is canonically embedded in $C^\infty_{\tau_\lambda}(G) \otimes F$, a differential operator $\mathscr{D}_{\lambda,\eta}$ on $C^\infty_{\tau_\lambda}(G;\eta)$ is defined as $\mathscr{D}_{\lambda,\eta} = \mathscr{D}_\lambda \otimes \mathrm{id}_F$, where id_F is the identity map on F. Under these notations, one of the important results given there is the isomorphism

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$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathrm{Ind}_P^G(\eta)) \simeq \mathrm{Ker}\,\mathscr{D}_{\lambda,\eta},$$

as linear spaces, provided η is weakly cyclic and the Blattner parameter λ of π_Λ is "far from the walls".

In the case where P is a parabolic subgroup of G, the above result gives that

$$(*)\qquad \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \mathrm{Ind}_P^G(\xi\otimes 1_N)) \simeq \mathrm{Hom}_{(\mathfrak{l},K_L)}(\tilde{\xi}^*, \mathrm{Ker}\,\mathscr{D}_{\lambda,1_N})$$

as linear spaces, provided λ is "far from the walls". Here ξ is an irreducible, admissible representation of the Levi part L of P, $\mathfrak{l}=\mathrm{Lie}(L)\otimes\mathbf{C}$, $\tilde{\xi}=\xi\otimes e^\rho$ with $\rho(H)=\frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}(H)|_{\mathfrak{n}}$, $\mathfrak{n}=\mathrm{Lie}(N)\otimes\mathbf{C}$, N is the nilpotent radical of P, $K_L=K\cap L$, and 1_N is the trivial character of N. In order to obtain the embeddings for discrete series whose Blattner parameter is not far from the walls, the "translation functor" introduced by G. J. Zukerman in [10] can be used.

For the problem of determining all the embeddings of discrete series into principal series, W. B. Silva investigated in her paper [8], the case of real rank one groups. For the groups with higher real rank, the second-named author studied the case of $SU(2,2)$ in [9]. Our main aim in this paper is to determine the embeddings of discrete series into principal series induced from a minimal parabolic subgroup for the normal real form of a comlex Lie group of type G_2. Our method is elementary one, which is the same as that in [9, Part A].

We are now going to explain the contents of each section.

In §1, we summarize general theory: parametrization of discrete series, differential operators $\mathscr{D}_\lambda$ and $\mathscr{D}_{\lambda,\eta}$, the results in [9] and Zuckerman's translation functor.

We specialize G and K as the normal real form of a complex Lie group of type G_2 and a maximal compact subgroup of G respectively. Then the structure of G and that of $\mathfrak{g}_0=\mathrm{Lie}(G)$ are explicitly described in §2. The parametrization and structures of irreducible K-modules are also given in §2. According to the standard facts in §1, one sees that it may be assumed that the Harish-Chandra parameter Λ of a given discrete series of G is dominant with respect to one of the following three different Δ_J^+ $(J=I, II, III)$ of positive systems of the root system Δ.

$$\Delta_I^+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2,2\alpha_1+\alpha_2,3\alpha_1+\alpha_2,3\alpha_1+2\alpha_2\},$$

$$\Delta_{II}^+=\{\alpha_1+\alpha_2,-\alpha_2,\alpha_1,2\alpha_1+\alpha_2,3\alpha_1+2\alpha_2,3\alpha_1+\alpha_2\},$$

$$\Delta_{III}^+=\{-\alpha_1-\alpha_2,3\alpha_1+2\alpha_2,2\alpha_1+\alpha_2,\alpha_1-\alpha_2,3\alpha_1+\alpha_2\}.$$

Here α_1 is short compact root and α_2 is long noncompact root, which are both simple in Δ_I^+.

For a function f of $C_{\tau_\lambda}^\infty(G)$, f is expressed uniquely in the form $f(g)=\sum_{p,q}c_{pq}(g)e_{pq}^{(rs)}$ with a certain basis $\{e_{pq}^{(rs)}\}$ of V_λ and smooth functions c_{pq} on G. In §3, we write $\mathscr{D}_\lambda f$ explicitly in terms of c_{pq}.

Subsequently we solve the differential equation $\mathscr{D}_{\lambda,1_N}f=0$ in §4. There the explicit form of $\mathrm{Ker}\,\mathscr{D}_{\lambda,1_N}$ and its $(\mathfrak{l},K_L)$-module structure are given.

In §5 we rewrite the results in §4 by means of $(*)$ and obtain the main result:

Theorem (see Theorem 5.1). *For $J = I, II, III$, let Ξ_J be the totality of linear forms $\Lambda \in \mathfrak{t}^*$ which are Δ_J^+-dominant, regular and K-integral. If $\Lambda \in \Xi_I$ (resp. Ξ_{II}, Ξ_{III}), there exist three (resp. thirteen, four) distinct principal series representations $W = \mathrm{Ind}_P^G (\xi \otimes 1_N)$ satisfying*

$$\dim \mathrm{Hom}_{(\mathfrak{g}, K)} (\pi_\Lambda, W) = 1,$$

and for other principal series W', $\dim \mathrm{Hom}_{(\mathfrak{g}, K)} (\pi_\Lambda, W') = 0$.

In this theorem, we can determine W explicitly according to Λ.

Finally in §6, we compare our results with those in [5]. The method in [5] gives two embeddings out of three, thirteen or four embeddings described in the above theorem.

1. General theory

In this section, we summarize the results on which the later calculations rely. The arguments in §§3–5 are based on the results in §1.4.

1.1. Notations. We explain some notations. Let G be a connected semisimple Lie group with finite center and $\mathfrak{g}_0$ its Lie algebra. Denote by $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0$, and θ stands for the corresponding Cartan involution, where $\mathfrak{k}_0 = \{X \in \mathfrak{g}_0 | \theta X = X\}$, $\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 | \theta X = -X\}$. We denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$ etc. by $\mathfrak{g}$, $\mathfrak{k}$ etc., omitting the subscript $_0$. For a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{p}_0$, let Ψ be the restricted root system for $(\mathfrak{g}_0, \mathfrak{a}_0)$ and Ψ^+ the set of all positive roots in Ψ. Then we have an Iwasawa decomposition of $\mathfrak{g}_0$ as $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$. Here, $\mathfrak{n}_0 = \sum_{\lambda \in \Psi^+} (\mathfrak{g}_0)_\lambda$, $(\mathfrak{g}_0)_\lambda = \{X \in \mathfrak{g}_0 | [H, X] = \lambda(H) X (\forall H \in \mathfrak{a}_0)\}$. Let $G = KAN$ be the Iwasawa decomposition of G corresponding to the decomposition of $\mathfrak{g}_0$.

From now on, we assume that $\mathrm{rank}\, G = \mathrm{rank}\, K$. It is known that this condition is necessary and sufficient for G to have discrete series representations (cf. [1, Theorem 13]). By virtue of this assumption, there is a compact Cartan subalgebra $\mathfrak{t}_0 (\subset \mathfrak{k}_0)$ of $\mathfrak{g}_0$. We denote the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$ by Δ, and let $\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$, where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H) X (\forall H \in \mathfrak{t})\}$. Define Δ_c (resp. Δ_n) as the set of compact (resp. noncompact) roots, and we denote the Weyl groups of Δ and Δ_c by W and W_c respectively. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$, $U(\mathfrak{t})^W$ the set of the elements in $U(\mathfrak{t})$ invariant under the action of the Weyl group W. The Harish-Chandra isomorphism of $Z(\mathfrak{g})$ onto $U(\mathfrak{t})^W$ is denoted by γ. As usual, $\mathbf{R}$ and $\mathbf{C}$ stand for the field of real numbers and the field of complex numbers respectively.

1.2. Parameterization of discrete series representations. Let $\mathfrak{t}_0$ be a compact Cartan subalgebra of $\mathfrak{g}_0$ as above, and T the maximal torus of K corresponding to $\mathfrak{t}_0$.

Definition 1.1. For a linear form λ on $\mathfrak{t}$, we say that λ is *K-integral* if the following assignment gives a unitary character of T:

$$T \ni \exp H \mapsto e^{\lambda(H)} \in \mathbf{C}^{\times} \qquad (H \in \mathfrak{t}_0).$$

Let Ξ be the totality of linear forms Λ on $\mathfrak{t}$ satisfying the following two conditions:

(i) $\Lambda + \rho$ is K-integral,

(ii) Λ is regular.

Here, $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ for some fixed positive system Δ^+ of Δ. Note that condition (i) is independent of the choice of a positive system Δ^+. By Harish-Chandra, there exists a tempered invariant distribution Θ_Λ on G, which is in fact a locally integrable function on G, satisfying the following conditions:

(i) Choose a positive system Δ^+ so that Λ is Δ^+-dominant, then

$$\Theta_\Lambda(\exp H)\cdot\left(\sum_{w \in W} (\det w) e^{w \cdot \rho(H)}\right) = (-1)^{(1/2)\dim \mathfrak{p}_0} \sum_{w \in W_c} (\det w) e^{w \cdot \Lambda(H)} (H \in \mathfrak{t}_0).$$

(ii) Let l be the rank of $\mathfrak{g}$, D_l the coefficient of x^l in $\det(x - 1 - \mathrm{Ad}(g))$, $G' = \{g \in G \mid D_l(g) \neq 0\}$, then

$$\sup_{g \in G'} |\Theta_\Lambda(g)| \cdot |D_l(g)|^{1/2} < \infty.$$

(iii) Put $\chi_\Lambda(Z) = \Lambda(\gamma(Z))$, then

$$Z \cdot \Theta_\Lambda = \chi_\Lambda(Z) \cdot \Theta_\Lambda \qquad (Z \in Z(\mathfrak{g})).$$

Distribution Θ_Λ is uniquely determined under the above conditions.

The following facts give the parametrization of discrete series. For the case where G is acceptable (i.e. ρ is K-integral), see [1, Theorem 16].

(i) For any $\Lambda \in \Xi$, there exists a unique, up to isomorphisms, discrete series representation π_Λ of G with character Θ_Λ.

(ii) For any discrete series representation π of G, there exists an element Λ of Ξ such that π is unitarily equivalent to π_Λ.

(iii) For two elements Λ_1 and Λ_2 of Ξ, π_{Λ_1} is unitarily equivalent to π_{Λ_2} if and only if $W_c \cdot \Lambda_1 = W_c \cdot \Lambda_2$.

The linear form Λ in (i) is called the *Harish-Chandra parameter* of π_Λ.

Now, define an equivalence relation $\sim$ on Ξ as follows:

$$\Lambda_1 \sim \Lambda_2 \qquad \text{if and only if } W_c \cdot \Lambda_1 = W_c \cdot \Lambda_2.$$

Then discrete series representations of G are parametrized by $\Xi/\sim$.

1.3. Gradient type differential operators. Let τ be a finite-dimensional unitary representation of K on a Hilbert space H. Take a closed subgroup P of G, and a continuous representation (η, F) of P on a Fréchet space F. We define three function spaces $C_\tau^\infty(G)$, $C^\infty(G; \eta)$ and $C_\tau^\infty(G; \eta)$ as follows:

$$C_\tau^\infty(G) = \{f\colon G \overset{C^\infty}{\to} H \mid f(kg) = \tau(k)f(g)(\forall(k, g) \in K \times G)\},$$

$$C^\infty(G; \eta) = \{f\colon G \overset{C^\infty}{\to} F \mid f(gp) = \delta_P(p)^{1/2}\eta(p)^{-1}f(g)(\forall(g, p) \in G \times P)\},$$

$$C_\tau^\infty(G; \eta) = \{f\colon G \overset{C^\infty}{\to} H \otimes F \mid$$

$$f(kgp) = \delta_P^{1/2}(\tau(k) \otimes \eta(p)^{-1})f(g)(\forall(k, g, p) \in k \times G \times P)\}.$$

Here, δ_P stands for the modular function of P with respect to the left Haar measure of P.

Equip these spaces with the topology of uniform convergence of functions and their partial derivatives on any compact subset of G. Now G acts on the space $C^\infty(G; \eta)$ as follows:

$$(g \cdot f)(x) = f(g^{-1}x) \qquad (f \in C^\infty(G; \eta), g, x \in G),$$

and we have a smooth representation of G on $C^\infty(G; \eta)$. We call it the representation of G *induced from* (η, F) *in* C^∞*-context* and denote it by $(C^\infty\text{-})\,\mathrm{Ind}_P^G(\eta)$. By differentiating the action of G, we can introduce a $(\mathfrak{g}, K)$-module structure on $C^\infty(G; \eta)$. This $(\mathfrak{g}, K)$-module is also denoted by $(C^\infty\text{-})\,\mathrm{Ind}_P^G(\eta)$.

Next, for a fixed positive system Δ^+ of Δ, put $\Delta_c^+ = \Delta_c \cap \Delta^+$, and $\Delta_n^+ = \Delta_n \cap \Delta^+$. For a Δ_c^+-dominant, K-integral linear form λ on $\mathfrak{t}$, let $(\tau_\lambda, V_\lambda)$ be the finite-dimensional irreducible representation of K with highest weight λ. Consider the adjoint representation $(\mathrm{Ad}\,|_{\mathfrak{p}}, \mathfrak{p})$ of K on $\mathfrak{p}$. Then the tensor product representation $\tau_\lambda \otimes \mathrm{Ad}\,|_{\mathfrak{p}}$ is decomposed into irreducible as

$$\tau_\lambda \otimes \mathrm{Ad}\,|_{\mathfrak{p}} \simeq \bigoplus_{\beta \in \Delta_n} m(\beta)\tau_{\lambda+\beta}. \tag{1.1}$$

Here, $m(\beta)$ is the multiplicity of $\tau_{\lambda+\beta}$ in $\tau_\lambda \otimes \mathrm{Ad}\,|_{\mathfrak{p}}$, and is 0 or 1 for any β in Δ_n. By using decomposition (1.1), we define a subrepresentation $(\tau_\lambda^-, V_\lambda^-)$ (resp. $\tau_\lambda^+, V_\lambda^+)$) of $\tau_\lambda \otimes \mathrm{Ad}\,|_{\mathfrak{p}}$ by $\tau_\lambda^- = \otimes_{\beta \in \Delta_n^+} m(-\beta)\tau_{\lambda-\beta}$ (resp. $\tau_\lambda^+ = \oplus_{\beta \in \Delta_n^+} m(\beta)\tau_{\lambda+\beta}$). Then, $V_\lambda \otimes \mathfrak{p}$ is decomposed as

$$V_\lambda \otimes \mathfrak{p} = V_\lambda^+ \oplus V_\lambda^-. \tag{1.2}$$

Let P_λ be the projection onto V_λ^- along decomposition (1.2).

We are now ready to define certain differential operators playing an important role in the determination of embeddings of discrete series representations into principal series. Let L_X $(X \in \mathfrak{g})$ be the differentiation with respect to the right invariant vector field defined by X. The Killing form of $\mathfrak{g}$ is denoted by $B(\cdot, \cdot)$, and $(\cdot, \cdot)$ stands for the inner product on $\mathfrak{g}$ defined by $(X, Y) = -B(X, \overline{Y})$. Here, $\overline{Y}$ denotes the complex conjugate of Y relative to the compact real form $\mathfrak{k}_0 + \sqrt{-1}\mathfrak{p}_0$ of $\mathfrak{g}$. Then we can define first order differential operators $\nabla\colon C_{\tau_\lambda}^\infty(G) \to C_{\tau_\lambda \otimes \mathrm{Ad}|_{\mathfrak{p}}}^\infty(G)$ and $\mathscr{D}_\lambda\colon C_{\tau_\lambda}^\infty(G) \to C_{\tau_\lambda^-}^\infty(G)$ as follows:

$$\nabla f(g) = \sum_{j=1}^{2n} L_{X_j} f(g) \otimes \overline{X}_j,$$

$$\mathscr{D}_\lambda f(g) = P_\lambda(\nabla f(g)).$$

Here, $\{X_j | j = 1, 2, \ldots, 2n\}$ is an orthonormal basis of $\mathfrak{p}$ relative to the inner product $(\cdot, \cdot)$. Note that ∇ and $\mathscr{D}_\lambda$ are independent of a choice of $\{X_j\}$. Since the space $C^\infty_{\tau_\lambda}(G; \eta)$ is canonically embedded into $C^\infty_{\tau_\lambda}(G) \otimes F$, we can define an operator $\mathscr{D}_{\lambda,\eta}$: $C^\infty_{\tau_\lambda}(G; \eta) \to C^\infty_{\tau_\lambda^-}(G; \eta)$ by $\mathscr{D}_{\lambda,\eta} = \mathscr{D}_\lambda \otimes \mathrm{id}_F$.

1.4. Description of embeddings. For $\Lambda \in \Xi$, we take a positive system Δ^+ of Δ so that Λ is Δ^+-dominant. For such Δ^+, put $\rho_c = \frac{1}{2}\sum_{\alpha \in \Delta_c^+} \alpha$, $\rho_n = \frac{1}{2}\sum_{\alpha \in \Delta_n^+} \alpha$, and $\lambda = \Lambda - \rho_c + \rho_n$. Note that λ is also K-integral and Δ_c^+-dominant.

Definition 1.2. For a Δ_c^+-dominant, integral linear form λ on $\mathfrak{t}$, λ is said to be *far from the walls* if the following condition holds:

$$\lambda - \textstyle\sum_{\beta \in Q} \beta \text{ is } \Delta_c^+\text{-dominant for any subset } Q \text{ of } \Delta_n^+.$$

Definition 1.3. Let P be a closed subgroup of G. A continuous representation (η, F) of P on a Fréchet space F, is said to be *weakly cyclic* if there exists a continuous linear functional T on F such that for an element v of F, $T(\eta(p)v) = 0 (\forall p \in P)$ implies $v = 0$.

For instance, every irreducible unitary representation is weakly cyclic.

For an element Λ of Ξ, take ρ_c, ρ_n, λ as above. For a linear form μ on $\mathfrak{t}$, define an integer $N_\lambda(\mu)$ by

$$N_\lambda(\mu) = \sum_{w \in W_c} \det(w) Q(w \cdot (\mu + \rho_c) - (\lambda + \rho_c)).$$

Here, $Q(\nu)$ $(\nu \in \mathfrak{t}^*)$ is the number of distinct ways that ν can be written as a sum of elements of Δ_n^+. Then next theorem gives the K-multiplicity formula for discrete series.

Theorem 1.1 (cf. [2, Theorem (1.3)]). *The discrete series π_Λ is decomposed as a K-module in the following way*:

$$\pi_\Lambda|_K = \sum_\mu N_\lambda(\mu) \tau_\mu.$$

Here, μ runs through the set of linear forms ν on $\mathfrak{t}$ for which $\nu - \lambda$ can be expressed as a sum of (not necessarily distinct) positive roots.

Note that $N_\lambda(\lambda) = 1$, and that τ_λ is the lowest K-type of π_Λ. The linear form λ is called the *Blattner parameter* of π_Λ. For a discrete series π_Λ of G, the contragredient representation π_Λ^* of π_Λ is also a discrete series. Therefore we may consider the embeddings of π_Λ^* instead of those of π_Λ, and they are described as in the following theorem.

Theorem 1.2 (cf. [9, Theorem 2.4]). *Let Λ be in Ξ and (η, F) a weakly cyclic representation of a closed subgroup P of G. Then there exists a natural isomorphism*

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(\pi_\Lambda^*, \mathrm{Ind}_P^G(\eta)) \simeq \mathrm{Ker}\, \mathscr{D}_{\lambda,\eta}$$

as linear spaces, if λ is far from the walls.

When P is a parabolic subgroup of G, let $P = M_P A_P N_P$ be a Langlands decomposition of P and L_P the Levi part $M_P A_P$ of P. We denote the Lie algebra of L_P (resp. A_P, N_P) by $\mathfrak{l}_0$ (resp. $\mathfrak{a}_{P0}$, $\mathfrak{n}_{P0}$) and put $K_L = L_P \cap K$. The complexification of $\mathfrak{l}_0$ (resp. $\mathfrak{a}_{P0}$, $\mathfrak{n}_{P0}$) is denoted by $\mathfrak{l}$ (resp. $\mathfrak{a}_P$, $\mathfrak{n}_P$) as usual. The Levi part L_P acts on Ker $\mathscr{D}_{\lambda, 1_N}$ by right translation as

$$(x \cdot f)(g) = f(gx) \qquad (x \in L_P, g \in G, f \in \text{Ker } \mathscr{D}_{\lambda, 1_N}). \tag{1.3}$$

Here, 1_N stands for the trivial character of N_P. As in the case of $\text{Ind}_P^G(\eta)$, Ker $\mathscr{D}_{\lambda, 1_N}$ has an $(\mathfrak{l}, K_L)$-module structure.

Now we are going to see that Ker $\mathscr{D}_{\lambda, 1_N}$ is stable under L_P. Let g, $x \in G$, $f \in C_{\tau_\lambda}^\infty(G)$ and $X \in \mathfrak{g}_0$, then

$$\begin{aligned}(x \cdot L_X f)(g) &= (L_X f)(gx) \\ &= \frac{d}{dt} f(\exp(-tX) \cdot gx)|_{t=0} \\ &= \frac{d}{dt}(x \cdot f)(\exp(-tX) \cdot g)|_{t=0} \\ &= (L_X x \cdot f)(g).\end{aligned}$$

So, $x \cdot (L_X f) = L_X(x \cdot f)$. Here, $x \cdot f$ is defined as in (1.3). We see that for $g \in G$, $x \in L_P$, $n \in N_P$,

$$\begin{aligned}(\mathscr{D}_\lambda x \cdot f) &= P_\lambda\left(\sum_{j=1}^{2n} L_{x_j}(x \cdot f)(g) \otimes \overline{X_j}\right) \\ &= P_\lambda\left(\sum_{j=1}^{2n} (L_{x_j} f)(gx) \otimes \overline{X_j}\right) \\ &= (\mathscr{D}_\lambda f)(gx) \\ &= 0 \qquad (\because f \in \text{Ker } \mathscr{D}_\lambda).\end{aligned}$$

Since Ker $\mathscr{D}_{\lambda, 1_N} = \{f \in \text{Ker } \mathscr{D}_\lambda | f(gn) = f(g)(\forall (g, n) \in G \times N_P)\}$ and L_P normalizes N_P,

$$\begin{aligned}(x \cdot f)(gn) &= f(gnx) \\ &= f(gxx^{-1}nx) \\ &= f(gx) \qquad (\because x^{-1}nx \in N_P) \\ &= (x \cdot f)(g).\end{aligned}$$

Therefore, $x \cdot f$ is also an element of Ker $\mathscr{D}_{\lambda, 1_N}$, and L_P actually acts on Ker $\mathscr{D}_{\lambda, 1_N}$. Smoothness of this representation is deduced from the topological structure of Ker $\mathscr{D}_{\lambda, 1_N}$.

For an irreducible admissible representation σ of M_P and a linear form μ

on $\mathfrak{a}_P$, $\xi = \sigma \otimes e^{\mu}$ is an irreducible admissible representation of L_P. Put $\tilde{\xi} = \sigma \otimes e^{\mu+\rho_P}$. Here, $\rho_P(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} H|_{\mathfrak{n}_P})$ $(H \in \mathfrak{a}_P)$. Then, next theorem gives a kind of Frobenius reciprocity on the embeddings into principal series, and our argument in §4 is based on it.

Theorem 1.3 (cf. [9, Theorem 3.5]). *Notations are as before, and assume that the Blattner parameter λ is far from the walls. Then,*

$$\operatorname{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \operatorname{Ind}_P^G(\xi \otimes 1_N)) \simeq \operatorname{Hom}_{(\mathfrak{l},K_L)}(\tilde{\xi}^*, \operatorname{Ker} \mathscr{D}_{\lambda,1_N})$$

as linear spaces.

Note that, $\operatorname{Ker} \mathscr{D}_{\lambda,1_N}$ is expressed as follows:

$$\operatorname{Ker} \mathscr{D}_{\lambda,1_N} = \{f \in C_{\tau_\lambda}^\infty(G) \mid \mathscr{D}_\lambda f = 0, f(gn) = f(g) (\forall (g,n) \in G \times N)\}. \tag{1.4}$$

1.5. Zuckerman's translation functor. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Delta_{\mathfrak{h}}$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, $\Delta_{\mathfrak{h}}^+$ a positive system of $\Delta_{\mathfrak{h}}$ and $W_{\mathfrak{h}}$ the Weyl group of $\Delta_{\mathfrak{h}}$. We denote the universal enveloping algebra of $\mathfrak{g}$ by $U(\mathfrak{g})$. The center of $U(\mathfrak{g})$ is denoted by $Z(\mathfrak{g})$ and $U(\mathfrak{h})^{W_{\mathfrak{h}}}$ means the set of the elements in $U(\mathfrak{h})$ invariant under the action of Weyl group $W_{\mathfrak{h}}$. We use the symbol γ for the Harish–Chandra isomorphism of $Z(\mathfrak{g})$ onto $U(\mathfrak{h})^{W_{\mathfrak{h}}}$. For a linear form λ on $\mathfrak{h}$, the $W_{\mathfrak{h}}$-orbit through λ is denoted by $[\lambda]$ and define a character $\chi_{[\lambda]}$ of $Z(\mathfrak{g})$ by

$$\chi_{[\lambda]}(z) = \lambda(\gamma(z)) \qquad (z \in Z(\mathfrak{g})),$$

where λ is extended to an algebra homomorphism on $U(\mathfrak{h})$ with $\lambda(1) = 1$. Note that this definition is independent of the representative λ of $[\lambda]$ and $\chi_{[\lambda_1]} = \chi_{[\lambda_2]}$ if and only if $[\lambda_1] = [\lambda_2]$.

Let A be a $(\mathfrak{g}, K)$-module finitely generated over $U(\mathfrak{g})$ and assume that each K-isotypic subspace of A is finite-dimensional. The category of such $(\mathfrak{g}, K)$-modules is denoted by $\mathfrak{U}$. For a $W_{\mathfrak{h}}$-orbit $[\lambda]$ in $\mathfrak{h}^*$, define $A_{[\lambda]}$ to be the maximal submodule of A on which $z - \chi_{[\lambda]}(z)\cdot\mathrm{id}$ is locally nilpotent for all $z \in Z(\mathfrak{g})$. Then A is decomposed as

$$A \simeq \bigoplus_{[\lambda] \in \mathfrak{h}^*/W_{\mathfrak{h}}} A_{[\lambda]}. \tag{1.5}$$

Let $p_{[\lambda]}$ be the projection of A onto $A_{[\lambda]}$ with respect to decomposition (1.5) and put $\mathfrak{U}_{[\lambda]} = \{A \in \mathfrak{U} \mid A = p_{[\lambda]}(A)\}$.

We denote the linear span of $\Delta_{\mathfrak{h}}$ over $\mathbf{R}$ by $\mathfrak{h}_{\mathbf{R}}^*$ and decompose an element λ in $\mathfrak{h}^*$ as $\lambda = \operatorname{Re}\lambda + \operatorname{Im}\lambda$ along the decomposition $\mathfrak{h}^* = \mathfrak{h}_{\mathbf{R}}^* + \sqrt{-1}\mathfrak{h}_{\mathbf{R}}^*$. For a $\Delta_{\mathfrak{h}}^+$-dominant, integral linear form μ on $\mathfrak{h}$, F_μ stands for the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight μ. We assume that the $\mathfrak{g}$-action on F_μ induces a G-action. For $\lambda \in \mathfrak{h}^*$ with $\Delta_{\mathfrak{h}}^+$-dominant real part $\operatorname{Re}\lambda$ and $\Delta_{\mathfrak{h}}^+$-dominant linear form μ on $\mathfrak{h}$, define two functors $\varphi_{[\lambda+\mu]}^{[\lambda]}$ and $\psi_{[\lambda]}^{[\lambda+\mu]}$ on the category $\mathfrak{U}$ as follows:

$$\varphi_{[\lambda+\mu]}^{[\lambda]}(A) = p_{[\lambda+\mu]}(p_{[\lambda]}(A) \otimes F_\mu) \qquad \text{for } A \in \mathfrak{U},$$

$$\psi_{[\lambda]}^{[\lambda+\mu]}(A) = p_{[\lambda]}(p_{[\lambda+\mu]}(A) \otimes F_\mu^*) \qquad \text{for } A \in \mathfrak{U}.$$

One can see the following Proposition 1.1 in almost the same manner as the proof of Lemma 4.1 in [10]. For the problem of determining the embeddings of discrete series of G into principal series, this proposition and Proposition 1.2 below allow us to reduce the problem to the case where the Blattner parameter of the discrete series in far from the walls.

Proposition 1.1 (cf. [10, Lemma 4.1]). *Let λ be a linear form on $\mathfrak{h}$ with $\Delta_{\mathfrak{h}}^{+}$-dominant real part $\operatorname{Re}\lambda$ and μ a $\Delta_{\mathfrak{h}}^{+}$-dominant, integral linear form on $\mathfrak{h}$ so that the $\mathfrak{g}$-action on F_μ lifts to a G-action. Then for $(\mathfrak{g}, K)$-modules $X \in \mathfrak{U}_{[\lambda]}$ and $Y \in \mathfrak{U}_{[\lambda+\mu]}$, there exists an isomorphism*

$$\operatorname{Hom}_{(\mathfrak{g},K)}(X, \psi_{[\lambda]}^{[\lambda+\mu]}Y) \simeq \operatorname{Hom}_{(\mathfrak{g},K)}(\varphi_{[\lambda+\mu]}^{[\lambda]}X, Y),$$

as linear spaces.

Proposition 1.2 (cf. [10, Theorem 1.2, Corollary 5.5]). *Assume that $\mathfrak{h} = \mathfrak{t}$, and let Λ be in Ξ, Δ^{+} the unique positive system so that Λ is Δ^{+}-dominant and μ a Δ^{+}-dominant, K-integral linear form on $\mathfrak{t}$, then*

$$\varphi_{[\Lambda+\mu]}^{[\Lambda]}(\pi_\Lambda) \simeq \pi_{\Lambda+\mu}.$$

Here π_Λ is the discrete series with Harish-Chandra parameter Λ. See § 1.2.

Let P be a parabolic subgroup of G with Langlands decomposition $P = M_P A_P N_P$ as before. Since the Levi part $L_P = M_P A_P$ of P has the same rank as G, we can take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{l} = \operatorname{Lie}(L_P) \otimes \mathbf{C}$. Define two categories ${}^{L_P}\mathfrak{U}$ and ${}^{L_P}\mathfrak{U}_{[\lambda]}$ of Harish-Chandra module of L_P and projection ${}^{L_P}p_{[\lambda]}$ as above. Though $\mathfrak{l}$ is not necessarily semisimple, but since $\mathfrak{l}$ is reductive the definition goes through without change. For the finite-dimensional irreducible $\mathfrak{g}$-module F_μ, let v_0 be a nonzero highest weight vector of F_μ and define $\tilde{F}_\mu$ to be the $M_P A_P$-cyclic subspace of F_μ generated by v_0. Then we have an irreducible L_P-module $\tilde{F}_\mu$. By using this module, we define a functor ${}^{L_P}\psi_{[\lambda]}^{[\lambda+\mu]}$ on ${}^{L_P}\mathfrak{U}$ by

$${}^{L_P}\psi_{[\lambda]}^{[\lambda+\mu]}(A) = {}^{L_P}p_{[\lambda]}({}^{L_P}p_{[\lambda+\mu]}(A) \otimes (\tilde{F}_\mu)^*).$$

We denote the functor $\psi_{[\lambda]}^{[\lambda+\mu]}$ by ${}^{G}\psi_{[\lambda]}^{[\lambda+\mu]}$ to distinguish two functors ${}^{G}\psi_{[\lambda]}^{[\lambda+\mu]}$ and ${}^{L_P}\psi_{[\lambda]}^{[\lambda+\mu]}$.

Next proposition tells us the relation of translation functors and parabolic induction.

Proposition 1.3 (cf. [6, Theorem B.1]). *Let notations be as above, then for a $\Delta_{\mathfrak{h}}^{+}$-dominant linear form λ on $\mathfrak{h}$, a $\Delta_{\mathfrak{h}}^{+}$-dominant, integral linear form μ on $\mathfrak{h}$ and an L_P-module $X \in {}^{L_P}\mathfrak{U}_{[\lambda+\mu]}$, we have*

$${}^{G}\psi_{[\lambda]}^{[\lambda+\mu]} \operatorname{Ind}_P^G(X \otimes 1_N) \simeq \operatorname{Ind}_P^G({}^{L_P}\psi_{[\lambda]}^{[\lambda+\mu]}(X) \otimes 1_N).$$

We use these propositions later in § 5.

2. Real simple Lie group of type G_2

2.1. Structure of a Lie algebra of type G_2. We keep to the notations in the last section and specialize $\mathfrak{g}$ as a complex simple Lie algebra of type G_2, and $\mathfrak{g}_0$ as a normal real form of $\mathfrak{g}$. It is known that a noncompact real form of $\mathfrak{g}$ is unique up to isomorphisms. (cf. [3, Chapter X]).

For a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, we have $\mathfrak{t}_0 \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and $\dim \mathfrak{p}_0 = 8$. So, rank $\mathfrak{g}$ = rank $\mathfrak{k} = 2$, and we can take a compact Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{g}_0$. The root system Δ of $\mathfrak{g}$ relative to $\mathfrak{t}$ is

$$\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\}.$$

Here, α_1 is the short simple root of Δ, and α_2 is the long simple root of Δ. They satisfy the following relations:

$$|\alpha_2|^2 = 3|\alpha_1|^2 = \tfrac{1}{4}, \qquad \langle \alpha_1, \alpha_2^\vee \rangle = -1, \qquad \langle \alpha_2, \alpha_1^\vee \rangle = -3.$$

Since $\mathfrak{k}_0 \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, Δ_c is generated from two mutually orthogonal roots. So we may assume that $\Delta_c = \{\pm\alpha_1, \pm(3\alpha_1 + 2\alpha_2)\}$ without losing generality.

See Figure 1, and note that "rotation of angle $\pi/3$" is an element of W. Consider the root space decomposition $\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ as before, then there exists an element E_α of $\mathfrak{g}_\alpha$ for each root α such that

$$B(E_\alpha, E_{-\alpha}) = 2/|\alpha|^2, \qquad E_{-\alpha} = -\bar{E}_\alpha. \tag{2.1}$$

Moreover, we can take E_α's in the following way:

$$[E_{10}, E_{01}] = E_{11}, \tag{2.2}$$

$$[E_{10}, E_{11}] = 2E_{21}, \tag{2.3}$$

$$[E_{10}, E_{21}] = 3E_{31}, \tag{2.4}$$

$$[E_{32}, E_{-3,-1}] = E_{01}. \tag{2.5}$$

Here, E_{ij} stands for $E_{i\alpha_1 + j\alpha_2}$, and E_{ij}'s are uniquely determined under conditions (2.1)–(2.5) above, when E_{10} and E_{01} are given. From now on, E_{ij}'s are assumed to satisfy relations (2.1)–(2.5), and define H_{ij}, $\tilde{H}_1$, $\tilde{H}_2$ and $\mathfrak{a}_0$ by

$$\begin{aligned} H_{ij} &= [E_{ij}, E_{-i,-j}], \\ \tilde{H}_1 &= E_{01} + E_{0,-1}, \\ \tilde{H}_2 &= E_{21} + E_{-2,-1}, \\ \mathfrak{a}_0 &= \mathbf{R}\tilde{H}_1 + \mathbf{R}\tilde{H}_2. \end{aligned}$$

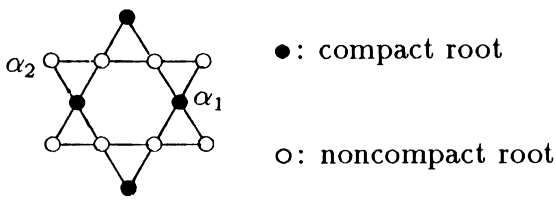

Figure 1: The root system Δ

Then we see that $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$. We equip $\mathfrak{a}_0^*$ with the lexicographic order with respect to the ordered basis $(\tilde{H}_1, \tilde{H}_2)$ of $\mathfrak{a}_0$. Relative to this order, Ψ^+ is

$$\{\lambda_1, \lambda_2, \lambda_1 + \lambda_2, 2\lambda_1 + \lambda_2, 3\lambda_1 + \lambda_2, 3\lambda_1 + 2\lambda_2\}.$$

Here, λ_1 and λ_2 are linear forms on $\mathfrak{a}$ defined through the conditions:

$$\lambda_1(\tilde{H}_1) = 0,\ \lambda_1(\tilde{H}_2) = 2,\ \lambda_2(\tilde{H}_1) = 1,\ \lambda_2(\tilde{H}_2) = -3.$$

Using this Ψ^+, we have the Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ as before.

For an element X of $\mathfrak{g}$, let $X = \mathfrak{k}(X) + \mathfrak{a}(X) + \mathfrak{n}(X)$ be the decomposition of X with respect to the complexified Iwasawa decomposition with $\mathfrak{k}(X) \in \mathfrak{k}$, $\mathfrak{a}(X) \in \mathfrak{a}$, $\mathfrak{n}(X) \in \mathfrak{n}$, and put $\mathfrak{s}(X) = \mathfrak{a}(X) + \mathfrak{n}(X)$.

If X is one of the root vectors E_α $(\alpha \in \Delta_n)$, then according to Proposition 5.2 in [5], $\mathfrak{k}(X)$, $\mathfrak{a}(X)$, $\mathfrak{n}(X)$ are given as in the following table.

X	$\mathfrak{k}(X)$	$\mathfrak{a}(X)$	$\mathfrak{n}(X)$
E_{01}	$\frac{1}{2}H_{01}$	$\frac{1}{2}\tilde{H}_1$	$-\frac{1}{2}(H_{01} - E_{01} + E_{0,-1})$
$E_{0,-1}$	$-\frac{1}{2}H_{01}$	$\frac{1}{2}\tilde{H}_1$	$\frac{1}{2}(H_{01} - E_{01} + E_{0,-1})$
E_{21}	$\frac{1}{2}H_{21}$	$\frac{1}{2}\tilde{H}_2$	$-\frac{1}{2}(H_{21} - E_{21} + E_{-2,-1})$
$E_{-2,-1}$	$-\frac{1}{2}H_{21}$	$\frac{1}{2}\tilde{H}_2$	$\frac{1}{2}(H_{21} - E_{21} + E_{-2,-1})$
E_{11}	E_{10}	0	$E_{11} - E_{10}$
$E_{-1,-1}$	$-E_{-1,0}$	0	$E_{-1,-1} + E_{-1,0}$
E_{31}	$-E_{32}$	0	$E_{31} + E_{32}$
$E_{-3,-1}$	$E_{-3,-2}$	0	$E_{-3,-1} - E_{-3,-2}$

Define an automorphism u of $\mathfrak{g}$ by

$$u = \left(\exp\frac{\pi}{4}\operatorname{ad}(E_{01} - E_{0,-1})\right)\cdot\left(\exp\frac{\pi}{4}\operatorname{ad}(E_{21} - E_{-2,-1})\right).$$

For this u, there holds $u(H_{01}) = -\tilde{H}_1$ and $u(H_{21}) = -\tilde{H}_2$. Therefore, u induces a linear bijection of $\mathfrak{t}$ onto $\mathfrak{a}$. The linear forms λ_1, λ_2, α_1 and α_2 are related through u as $\lambda_1 \circ u = -(2\alpha_1 + \alpha_2)$, $\lambda_2 \circ u = 3\alpha_1 + \alpha_2$.

2.2. Structures of group and its minimal parabolic subgroup. Let G_C be a connected, simply connected complex simple Lie group with Lie algebra $\mathfrak{g}$, and G the analytic subgroup of G_C corresponding to the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. The Iwasawa decomposition of G corresponding to that of $\mathfrak{g}_0$ is $G = KAN$. Put

$$(\mathfrak{k}_1)_0 = \sqrt{-1}\mathbf{R}H_{10} + \mathbf{R}(E_{10} - E_{-1,0}) + \sqrt{-1}\mathbf{R}(E_{10} + E_{-1,0}),$$
$$(\mathfrak{k}_2)_0 = \sqrt{-1}\mathbf{R}H_{32} + \mathbf{R}(E_{32} - E_{-3,-2}) + \sqrt{-1}\mathbf{R}(E_{32} + E_{-3,-2}),$$

then $\mathfrak{k}_0 = (\mathfrak{k}_1)_0 \oplus (\mathfrak{k}_2)_0$. For the structures of $(\mathfrak{k}_j)_0$, $j = 1, 2$, they are isomorphic to $\mathfrak{su}(2)$, and the isomorphims $\zeta_j\colon \mathfrak{su}(2) \to (\mathfrak{k}_j)_0$, $j = 1, 2$, are given as follows:

$$\begin{array}{rll} & \mathfrak{su}(2) \to (\mathfrak{k}_1)_0 & (\text{resp. } (\mathfrak{k}_2)_0) \\ \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} & \to \sqrt{-1}H_{10} & (\text{resp. } \sqrt{-1}H_{32}) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \to E_{10} - E_{-1,0} & (\text{resp. } E_{32} - E_{-3,-2}) \\ \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} & \to \sqrt{-1}(E_{10} + E_{-1,0}) & (\text{resp. } \sqrt{-1}(E_{32} + E_{-3,-2})). \end{array}$$

Then there exists a covering homomorphism σ of $SU(2) \times SU(2)$ onto K whose differential is $\zeta_1 \oplus \zeta_2$. For an element g of $SU(2) \times SU(2)$, the image of g under σ is denoted by $g^{\ddagger}$. Now, by comparing the unit lattice $\{H \in \mathfrak{t}_0 \mid \exp H = 1\}$ of K with that of $SU(2) \times SU(2)$, we can see that $K \simeq (SU(2) \times SU(2))/D$ with $D = \{1, (-1_2, -1_2)\}$. Here 1_2 is the unit matrix of degree 2. Put $M = \{m \in K \mid \mathrm{Ad}(m)|_{\mathfrak{a}_0} = \mathrm{id}_{\mathfrak{a}_0}\}$. Note that the $(d+1)$-dimensional irreducible $\mathfrak{su}(2)$-module V_d is realized on the space of homogeneous polynomials of degree d in two variables. Using this realization for K-module $\mathfrak{p}$, which is isomorphic to the exterior tensor product of V_3 and V_1, and computing the condition $\mathrm{Ad}(m)\tilde{H}_j = \tilde{H}_j$, $j = 1, 2$, for $m \in K$, we find that $M = \{1, m_1, m_2, m_1 m_2\}$. Here, m_1 and $m_2 \in K$ are given by

$$\begin{aligned} m_1 &= \left(\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}\right)^{\ddagger} \\ &= \exp(\sqrt{-1}\pi/2)(H_{10} - H_{32}), \\ m_2 &= \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)^{\ddagger} \\ &= \exp\left(-\frac{\pi}{2}\right)(E_{10} - E_{-1,0} + E_{32} - E_{-3,-2}). \end{aligned}$$

Therefore M is generated by two elements m_1 and m_2 with $m_1^2 = m_2^2 = 1$, $m_1 m_2 = m_2 m_1$, and $M \simeq \mathbf{Z}/2\mathbf{Z} \otimes \mathbf{Z}/2\mathbf{Z}$. Next, define a character $\sigma_{\varepsilon_1, \varepsilon_2}$ of M through $\sigma_{\varepsilon_1, \varepsilon_2}(m_i) = \varepsilon_i$ for $i = 1, 2$. Then, $\hat{M} = \{\sigma_{\varepsilon_1, \varepsilon_2} \mid \varepsilon_i = \pm 1 (i = 1, 2)\}$.

Put $P = MAN$ for M defined above, then we have a minimal parabolic subgroup of G and consider principal series $\mathrm{Ind}_P^G(\sigma_{\varepsilon_1, \varepsilon_2} \otimes e^{\mu} \otimes 1_N)$ for this P.

2.3. Structure of an irreducible K-module. Since $\mathfrak{k}_0 \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{k} \simeq \mathfrak{sl}(2, \mathbf{C}) \oplus \mathfrak{sl}(2, \mathbf{C})$, every finite-dimensional irreducible $\mathfrak{k}$-module is an exterior tensor product of two finite-dimensional irreducible $\mathfrak{sl}(2, \mathbf{C})$-modules. So we first explain some facts about irreducible $\mathfrak{sl}(2, \mathbf{C})$-modules.

Let X, Y and H be elements of $\mathfrak{sl}(2, \mathbf{C})$ satisfying the following relations:

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

For $(d+1)$-dimensional irreducible $\mathfrak{sl}(2, \mathbf{C})$-module (say V_d), there exists a basis $\{e_p^{(d)} \mid p = -d, -d+2, \dots, d\}$ of V_d such that

$$\begin{cases} H \cdot e_p^{(d)} = p e_p^{(d)} \\ X \cdot e_p^{(d)} = x_p^{(d)} e_{p+2}^{(d)} \qquad (p = -d, -d+2, \dots, d) \\ Y \cdot e_p^{(d)} = x_{p-2}^{(d)} e_{p-2}^{(d)} \end{cases}$$

with $x_p^{(d)} = (1/2)\sqrt{(d-p)(d+p+2)}$, where we regard $e_p^{(d)}$ as 0 if $p \notin \{-d, -d+2, \dots, d\}$. Denote by $(\cdot, \cdot)$ the inner product on V_d for which $\{e_p^{(d)} \mid p = -d, -d+2, \dots, d\}$ gives an orthonormal basis of V_d. Then $(\cdot, \cdot)$ is invariant under $\mathfrak{su}(2) \simeq \sqrt{-1}\mathbf{R}H + \mathbf{R}(X - Y) + \sqrt{-1}\mathbf{R}(X + Y)$.

Now, $\mathfrak{p} \simeq V_3 \hat{\otimes} V_1$ as $\mathfrak{k}$-modules, where $\hat{\otimes}$ means an exterior tensor product. Here,

$$\mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_1+\alpha_2} + \mathfrak{g}_{2\alpha_1+\alpha_2} + \mathfrak{g}_{3\alpha_1+\alpha_2} \simeq V_3 \qquad \text{as } \mathfrak{k}_1\text{-modules,}$$

$$\mathfrak{g}_{-3\alpha_1-\alpha_2} + \mathfrak{g}_{\alpha_2} \simeq V_1 \qquad \text{as } \mathfrak{k}_2\text{-modules.}$$

Put $\Delta_c^+ = \{\alpha_1, 3\alpha_1 + 2\alpha_2\}$, and let λ be a Δ_c^+-dominant, integral linear form on $\mathfrak{t}$ and V_λ a finite-dimensional irreducible $\mathfrak{k}$-module with highest weight λ. Then,

$$V_\lambda \simeq V_r \hat{\otimes} V_s \qquad \text{for } r = \lambda(H_{10}),\ s = \lambda(H_{32}),$$

$$V_\lambda \otimes \mathfrak{p} \simeq (V_r \otimes V_3) \hat{\otimes} (V_s \otimes V_1).$$

Now, we are going to give an irreducible decomposition of a tensor product of two finite-dimensional irreducible $\mathfrak{sl}(2, \mathbf{C})$-modules. For two nonnegative integers m and n with $m \geq n$, $V_m \otimes V_n$ is decomposed as follows:

$$V_m \otimes V_n \simeq V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n}. \tag{2.6}$$

Let $P^{(m,n;j)}$ be the projection of $V_m \otimes V_n$ onto V_{m+n-2j} in (2.6), and $u_{m+n-2j}^{(m,n;j)}$ a highest weight vector of V_{m+n-2j} in (2.6) with length 1. Here, $V_m \otimes V_n$ is equipped with an inner product for which $\{e_p^{(m)} \otimes e_q^{(n)}\}$ is an orthonormal basis, and we denote $e_p^{(m)} \otimes e_q^{(n)}$ by $e_{pq}^{(mn)}$. We define vectors $u_w^{(m,n;j)}$ $(w = m+n-2j-2, m+n-2j-4, \dots, -(m+n-2j))$ through the following recursion formula:

$$u_w^{(m,n;j)} = (x_w^{(m+n-2j)})^{-1} Y \cdot u_{w+2}^{(m,n;j)}.$$

Then, $u_w^{(m,n;j)}$ is a w-weight vector of V_{m+n-2j} in (2.6) with length 1.

The weight vectors $u_w^{(m,n;j)}$ $(j = 0, 1, \dots, n;\ w = -(m+n-2j), -(m+n-2j)+2, \dots, m+n-2j)$ are expressed uniquely in terms of the basis $\{e_{pq}^{(mn)}\}$ of $V_m \otimes V_n$ as

$$u_w^{(m,n;j)} = \sum_{p+q=w} z_{pq}^{(m,n;j)} e_{pq}^{(mn)} \qquad (z_{pq}^{(m,n;j)} \in \mathbf{C}).$$

By a straightforward calculation, we have $P^{(m,n;j)} e_{pq}^{(mn)} = \overline{z_{pq}^{(m,n;j)}} u_{p+q}^{(m,n;j)}$. So the projection $P^{(m,n;j)}$ is determined if $u_w^{(m,n;j)}$'s are explicitly described.

For the purpose of determining the projection P_λ defined in §1.3, calculate $u_w^{(s,1;j)}$ $(s \geq 1)$, $u_w^{(r,3;j)}$ $(r \geq 3)$ under the conditions $z_{s,1-2j}^{(s,1;j)} > 0$, $z_{r,3-2j}^{(r,3;j)} > 0$, then we have the following equalities:

$$u_w^{(s,1;0)} = \sqrt{\frac{s+1+w}{2(s+1)}}e_{w-1,1}^{(s1)} + \sqrt{\frac{s+1-w}{2(s+1)}}e_{w+1,-1}^{(s1)},$$

$$u_w^{(s,1;1)} = -\sqrt{\frac{s+1-w}{2(s+1)}}e_{w-1,1}^{(s1)} + \sqrt{\frac{s+1+w}{2(s+1)}}e_{w+1,-1}^{(s1)},$$

$$\begin{aligned}
\sqrt{8(r+1)(r+2)(r+3)}u_w^{(r,3;0)} &= \sqrt{(r-1+w)(r+1+w)(r+3+w)}e_{w-3,3}^{(r3)} \\
&\quad + \sqrt{3(r+3-w)(r+1+w)(r+3+w)}e_{w-1,1}^{(r3)} \\
&\quad + \sqrt{3(r+1-w)(r+3-w)(r+3+w)}e_{w+1,-1}^{(r3)} \\
&\quad + \sqrt{(r-1-w)(r+1-w)(r+3+w)}e_{w+3,-3}^{(r3)}, \\
\sqrt{8r(r+1)(r+3)}u_w^{(r,3;1)} &= -\sqrt{3(r+3-w)(r-1+w)(r+1+w)}e_{w-3,3}^{(r3)} \\
&\quad - (r+3-3w)\sqrt{r+1+w}e_{w-1,1}^{(r3)} \\
&\quad + (r+3+3w)\sqrt{r+1-w}e_{w+1,-1}^{(r3)} \\
&\quad + \sqrt{3(r-1-w)(r+1-w)(r+3+w)}e_{w+3,-3}^{(r3)}, \\
\sqrt{8(r-1)(r+1)(r+2)}u_w^{(r,3;2)} &= \sqrt{3(r+1-w)(r+3-w)(r-1+w)}e_{w-3,3}^{(r3)} \\
&\quad - (r-1+3w)\sqrt{r+1-w}e_{w-1,1}^{(r3)} \\
&\quad - (r-1-3w)\sqrt{r+1+w}e_{w+1,-1}^{(r3)} \\
&\quad + \sqrt{3(r-1-w)(r+1+w)(r+3+w)}e_{w+3,-3}^{(r3)}, \\
\sqrt{8(r-1)r(r+1)}u_w^{(r,3;3)} &= -\sqrt{(r-1-w)(r+1-w)(r+3-w)}e_{w-3,3}^{(r3)} \\
&\quad + \sqrt{3(r-1-w)(r+1-w)(r-1+w)}e_{w-1,1}^{(r3)} \\
&\quad - \sqrt{3(r-1-w)(r-1+w)(r+1+w)}e_{w+1,-1}^{(r3)} \\
&\quad + \sqrt{(r-1+w)(r+1+w)(r+3+w)}e_{w+3,-3}^{(r3)}.
\end{aligned}$$

In the above formulae, the coefficient of $e_{q*}^{(s1)}$ with $q < -s$ or $q > s$ is 0. Similarly, the coefficient of $e_{p*}^{(r3)}$ with $p < -r$ or $p > r$ becomes 0.

3. Explicit expression of the differential equation

Since $\{e_{pq}^{(rs)} \mid p = -r, \ldots, r;\ q = -s, \ldots, s\}$ is a basis of $V_r \hat{\otimes} V_s \simeq V_\lambda$, an element f of $C_{\tau_\lambda}^\infty(G)$ is expressed uniquely in the following form:

$$f(g) = \sum_{p,q} c_{pq}(g)e_{pq}^{(rs)}, \tag{3.1}$$

where the sum is taken for $p = -r, -r+2, \ldots, r$; $q = -s, -s+2, \ldots, s$ and the coefficients c_{pq} are smooth functions on G.

In this section, we give the differential operator $\mathscr{D}_\lambda$ explicitly, and rewrite the condition $\mathscr{D}_\lambda f=0$ for $f\in C^\infty_{\tau_\lambda}(G)$ in terms of the coefficient functions c_{pq} in (3.1). Then we have certain systems of differential equations for c_{pq}'s, and they are solved in next section.

3.1. Explicit description of the projection P_λ. In § 1.2, we gave the parametrization of the discrete series. Since the present group G is the real form of a simply connected complex Lie group, ρ is K-integral and the elements in Ξ are all K-integral. Fix an element Λ of Ξ and take the unique positive system Δ^+ so that Λ is Δ^+-dominant. Set $\Xi_c=\{\Lambda\in\Xi\,|\,\Delta^+\supset\Delta_c^+\}$ with $\Delta_c^+=\{\alpha_1, 3\alpha_1+2\alpha_2\}$. Then Ξ_c is a complete system of representatives for $\Xi/\sim$. The positive system Δ^+ containing Δ_c^+ is one of the following Δ_J^+'s

$$\Delta_I^+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2,2\alpha_1+\alpha_2,3\alpha_1+\alpha_2,3\alpha_1+2\alpha_2\},$$

$$\Delta_{II}^+=\{\alpha_1+\alpha_2,-\alpha_2,\alpha_1,2\alpha_1+\alpha_2,3\alpha_1+2\alpha_2,3\alpha_1+\alpha_2\},$$

$$\Delta_{III}^+=\{-\alpha_1-\alpha_2,3\alpha_1+2\alpha_2,2\alpha_1+\alpha_2,\alpha_1-\alpha_2,3\alpha_1+\alpha_2\}.$$

Set $r'=\Lambda(H_{10})$ and $s'=\Lambda(H_{32})$ and put $r=\lambda(H_{10})$, $s=\lambda(H_{32})$ as before. Through the isomorphism $V_\lambda\otimes\mathfrak{p}\simeq(V_r\otimes V_1)\,\hat{\otimes}\,(V_s\otimes V_3)$, we identify these two $\mathfrak{k}$-modules. In the rest of this subsection, we give the condition for Λ being an element of Ξ with positive system Δ_J^+ $(J=I, II, III)$, the Blattner parameter λ of π_Λ, the condition for λ being far from the walls and the projection P_λ.

Case I: $\Delta^+=\Delta_I^+$

For $\Lambda\in\mathfrak{t}^*$,

$\Lambda\in\Xi$ and $\Delta^+=\Delta_I^+$

$\Leftrightarrow r'$, s' are positive integers with $s'-r'\geq 2$ and $s'-r'$ is even.

$\Leftrightarrow r$, s are nonnegative integers with $s-r\geq 4$ and $s-r$ is even. $(*1)$

In this case, we have

$\rho_n=3\alpha_1+2\alpha_2$, $\lambda=\Lambda+\alpha_1+\alpha_2$.

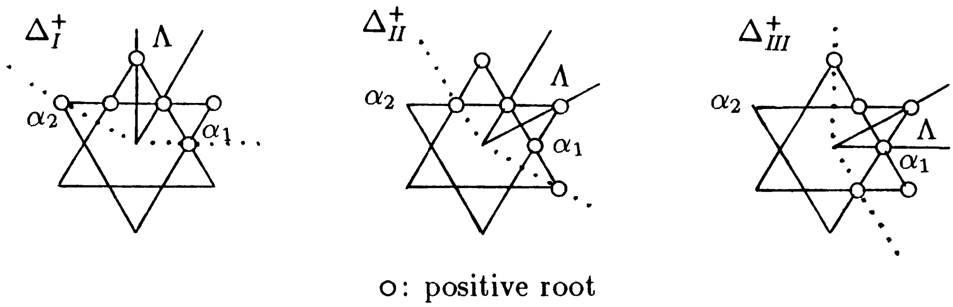

Figure 2: Three possible positive systems

λ is far from the walls $\Leftrightarrow r \geq 4,\ s \geq 4$ (with (*1))

$P_\lambda = I \otimes P^{(s,1;1)}$

Case II: $\Delta^+ = \Delta^+_{II}$

For $\Lambda \in \mathfrak{t}^*$,

$\Lambda \in \Xi$ and $\Delta^+ = \Delta^+_{II}$

$\Leftrightarrow r',\ s'$ are integers with $r' - s' \geq 2,\ r' \leq 3s' - 2$ and $r' - s'$ is even.

$\Leftrightarrow r,\ s$ are nonnegative integers with $r - s \geq 4,\ r \leq 3s$ and $r - s$ is even. (*2)

In this case, we have

$\rho_n = 3\alpha_1 + \alpha_2,\ \lambda = \Lambda + \alpha_1.$

λ is far from the walls $\Leftrightarrow r \geq 7,\ s \geq 3$ (with (*2))

$P_\lambda = P^{(r,3;1)} \otimes P^{(s,1;1)} + P^{(r,3;2)} \otimes P^{(s,1;1)} + P^{(r,3;3)} \otimes P^{(s,1;1)} + P^{(r,3;3)} \otimes P^{(s,1;0)}$

Case III: $\Delta^+ = \Delta^+_{III}$

For $\Lambda \in \mathfrak{t}^*$,

$\Lambda \in \Xi$ and $\Delta^+ = \Delta^+_{III}$

$\Leftrightarrow r',\ s'$ are positive integers with $s' \geq 1,\ r' - 3s' \geq 2$ and $r' - s'$ is even.

$\Leftrightarrow r,\ s$ are nonnegative integers with $r - 3s \geq 8$ and $r - s$ is even. (*3)

In this case, we have

$\rho_n = 2\alpha_1,\ \lambda = \Lambda - \alpha_2.$

λ is far from the walls $\Leftrightarrow r \geq 8,\ s \geq 2$ (with (*3))

$P_\lambda = P^{(r,3;2)} \otimes I + P^{(r,3;2)} \otimes I$

3.2. Differential operator ∇. In view of Theorem 1.3 and equation (1.4), we need to know the explicit form of the differential operator $\mathscr{D}_\lambda = P_\lambda \circ \nabla$. Since the projection P_λ is determined in the last subsection, here we calculate the differential operator ∇.

For $f \in C^\infty_{\tau_\lambda}(G)$ and $X \in \mathfrak{k}_0$, $L_X f$ is computed as follows:

$$\begin{aligned}(L_X f)(g) &= \frac{d}{dt} f(\exp(-tX)\cdot g)|_{t=0} \\ &= \frac{d}{dt}(\tau_\lambda(\exp(-tX))f(g))|_{t=0} \\ &= -\tau_\lambda(X)f(g).\end{aligned}$$

where we denote the differential of τ_λ by the same symbol. So the first term of the right-hand side of $L_X f = L_{\mathfrak{k}(X)} f + L_{\mathfrak{s}(X)} f$ is expressed in terms of τ_λ.

Compute $L_{\mathfrak{s}(E_{ij})}$'s by using the values of $\mathfrak{a}(E_{ij})$ and $\mathfrak{n}(E_{ij})$ in the table in §2.1, then we see that ∇f is explicitly written as

$$8\sqrt{6}\nabla f = \sum_{(i,j)\in \mathrm{I}} \Phi_{ij}(f),$$

where $\mathrm{I} = \{(0,1), (0,-1), (1,1), (-1,-1), (2,1), (-2,-1), (3,1), (-3,-1)\}$, and $\Phi_{ij}(f)$'s are defined by

$$\Phi_{01}(f) = -\sqrt{3}\sum\nolimits_{p,q} (p - q + 4\mathscr{L}_{01}) c_{pq} e^{(r3)}_{p3} \otimes e^{(s1)}_{q,-1},$$

$$\Phi_{0,-1}(f) = \sqrt{3}\sum\nolimits_{p,q} (-p + q + 4\mathscr{L}_{0,-1}) c_{pq} e^{(r3)}_{p,-3} \otimes e^{(s1)}_{q1},$$

$$\Phi_{21}(f) = -\sum\nolimits_{p,q} (-p - 3q + 4\mathscr{L}_{21}) c_{pq} e^{(r3)}_{p,-1} \otimes e^{(s1)}_{q,-1},$$

$$\Phi_{-2,-1}(f) = \sum\nolimits_{p,q} (p + 3q + 4\mathscr{L}_{-2,-1}) c_{pq} e^{(r3)}_{p1} \otimes e^{(s1)}_{q1},$$

$$\Phi_{11}(f) = 2\sum\nolimits_{p,q} (2\mathscr{L}_{11} c_{pq} - \sqrt{(r+2-p)(r+p)}\, c_{p-2,q}) e^{(r3)}_{p1} \otimes e^{(s1)}_{q,-1},$$

$$\Phi_{-1,-1}(f) = 2\sum\nolimits_{p,q} (2\mathscr{L}_{-1,-1} c_{pq} + \sqrt{(r-p)(r+2+p)}\, c_{p+2,q}) e^{(r3)}_{p,-1} \otimes e^{(s1)}_{q1},$$

$$\Phi_{31}(f) = 2\sqrt{3}\sum\nolimits_{p,q} (2\mathscr{L}_{31} c_{pq} + \sqrt{(s+2-q)(s+q)}\, c_{p,q-2}) e^{(r3)}_{p,-3} \otimes e^{(s1)}_{q,-1},$$

$$\Phi_{-3,-1}(f) = 2\sqrt{3}\sum\nolimits_{p,q} 2\mathscr{L}_{-3,-1} c_{pq} - \sqrt{(s-q)(s+2+q)}\, c_{p,q+2}) e^{(r3)}_{p3} \otimes e^{(s1)}_{q1},$$

where $\mathscr{L}_{ij} = I_{\mathfrak{s}(E_{ij})}$.

3.3. The equation $\mathscr{D}_\lambda f = 0$: Δ_I^+-case. Here we assume that $\Delta^+ = \Delta_I^+$. In this case, for $f \in C^\infty_{\tau_\lambda}(G)$, $\mathscr{D}_\lambda f$ is expressed as

$$\mathscr{D}_\lambda f = \sum\nolimits_{j,p,q} \alpha_j A^{(j)}_{pq} e^{(r3)}_{p,3-2j} \otimes u^{(s,1;1)}_{q+1},$$

where the sum is taken for $j = 0, 1, 2, 3$; $p = -r, -r+2, \ldots, r$; $q = -s, -s+2, \ldots, s-2$, with nonzero constants α_j's independent of p, q, and smooth functions $A^{(j)}_{pq}$'s on G.

This expression gives that the condition $\mathscr{D}_\lambda f = 0$ is equivalent to the equations

$$A^{(0)}_{pq} = A^{(1)}_{pq} = A^{(2)}_{pq} = A^{(3)}_{pq} = 0 \qquad \text{for } \begin{matrix} p = -r, -r+2, \ldots, r \\ q = -s, -s+2, \ldots, s-2 \end{matrix}, \tag{3.2}$$

and functions $A^{(j)}_{pq}$ are given by

$$A^{(0)}_{pq} = 4\sqrt{s-q}\,\mathscr{L}_{-3,-1} c_{pq} + \sqrt{s+2+q}\,(4\mathscr{L}_{01} - 2s + p + q - 2) c_{p,q+2},$$

$$A^{(1)}_{pq} = \sqrt{s-q}\,(4\mathscr{L}_{-2,-1} + p + 3q) c_{pq} - 4\sqrt{s+2+q}\,\mathscr{L}_{11} c_{p,q+2},$$
$$+ 2\sqrt{(r+2-p)(r+p)(s+2+q)}\, c_{p-2,q+2},$$

$$A^{(2)}_{pq} = 4\sqrt{s-q}\,\mathscr{L}_{-1,-1} c_{pq} - \sqrt{s+2+q}\,(p + 3q + 6 - 4\mathscr{L}_{21}) c_{p,q+2},$$
$$+ 2\sqrt{(r-p)(r+2+p)(s-q)}\, c_{p+2,q},$$

$$A^{(3)}_{pq} = \sqrt{s-q}\,(2s + p + q + 4 - 4\mathscr{L}_{0,-1}) c_{pq} + 4\sqrt{s+2+q}\,\mathscr{L}_{31} c_{p,q+2}.$$

In condition (3.2), we regard c_{pq} as 0 if $p \notin \{-r, -r+2, \ldots, r\}$ or $q \notin \{-s, -s+2, \ldots, s\}$.

3.4. The equation $\mathscr{D}_\lambda f = 0$: Δ_{II}^+-case. In this subsection, we calculate $\mathscr{D}_\lambda f$ with $\Delta^+ = \Delta_{II}^+$. Composite P_λ with ∇ by using the results in §3.2, then we see that $\mathscr{D}_\lambda f$ is of the form

$$\mathscr{D}_\lambda f = \textstyle\sum_{i,j,p,q} \beta_{ij} B_{pq}^{(ij)} u_p^{(r,3;i)} \otimes u_q^{(s,1;j)},$$

where the sum is taken for $(i, j) \in \{(1, 1), (2, 1), (3, 1), (3, 0)\}$; $p = -r-3, -r-1, \ldots, r+3$ and $q = -s-1, -s+1, \ldots, s+1$, with nonzero constants β_{ij}'s depending only on i, j, and smooth functions $B_{pq}^{(ij)}$'s on G.

In this case, $\mathscr{D}_\lambda f = 0$ if and only if

$$B_{pq}^{(11)} = 0 \quad \text{for } p = -r-1, \ldots, r+1;\ q = -s+1, \ldots, s-1, \tag{3.3}$$

$$B_{pq}^{(21)} = 0 \quad \text{for } p = -r+1, \ldots, r-1;\ q = -s+1, \ldots, s-1, \tag{3.4}$$

$$B_{pq}^{(31)} = 0 \quad \text{for } p = -r+3, \ldots, r-3;\ q = -s+1, \ldots, s-1, \tag{3.5}$$

$$B_{pq}^{(30)} = 0 \quad \text{for } p = -r+3, \ldots, r-3;\ q = -s-1, \ldots, s+1. \tag{3.6}$$

In conditions (3.3)–(3.6), we regard c_{pq} as 0 if $p \notin \{-r, -r+2, \ldots, r\}$ or $q \notin \{-s, -s+2, \ldots, s\}$.

The coefficient functions $B_{pq}^{(ij)}$ are defined as follows:

$$\begin{aligned}
B_{pq}^{(11)} = {} & 12\sqrt{(r+3-p)(r-1+p)(r+1+p)(s+1-q)}\mathscr{L}_{-3,-1}c_{p-3,q-1} \\
& + \sqrt{(r+3-p)(r-1+p)(r+1+p)(s+1+q)} \\
& \times (2r - 6s - 3p + 3q - 12 + 12\mathscr{L}_{01})c_{p-3,q+1} \\
& + (r+3-3p)\sqrt{(r+1+p)(s+1-q)}(p + 3q - 4 + 4\mathscr{L}_{-2,-1})c_{p-1,q-1} \\
& - 4(r+3-3p)\sqrt{(r+1+p)(s+1+q)}\mathscr{L}_{11}c_{p-1,q+1} \\
& - 4(r+3+3p)\sqrt{(r+1-p)(s+1-q)}\mathscr{L}_{-1,-1}c_{p+1,q-1} \\
& + (r+3+3p)\sqrt{(r+1-p)(s+1+q)}(p + 3q + 4 - 4\mathscr{L}_{21})c_{p+1,q+1} \\
& + \sqrt{(r-1-p)(r+1-p)(r+3+p)(s+1-q)} \\
& \times (-2r + 6s - 3p + 3q + 12 - 12\mathscr{L}_{0,-1})c_{p+3,q-1} \\
& + 12\sqrt{(r-1-p)(r+1-p)(r+3+p)(s+1+q)}\mathscr{L}_{31}c_{p+3,q+1}, \\
B_{pq}^{(21)} = {} & -12\sqrt{(r+1-p)(r+3-p)(r-1+p)(s+1-q)}\mathscr{L}_{-3,-1}c_{p-3,q-1} \\
& + \sqrt{(r+1-p)(r+3-p)(r-1+p)(s+1+q)} \\
& \times (2r + 6s + 3p - 3q + 16 - 12\mathscr{L}_{01})c_{p-3,q+1} \\
& + (r-1+3p)\sqrt{(r+1-p)(s+1-q)}(p + 3q - 4 + 4\mathscr{L}_{-2,-1})c_{p-1,q-1} \\
& - 4(r-1+3p)\sqrt{(r+1-p)(s+1+q)}\mathscr{L}_{11}c_{p-1,q+1}
\end{aligned}$$

$$+ 4(r-1-3p)\sqrt{(r+1+p)(s+1-q)}\mathscr{L}_{-1,-1}c_{p+1,q-1}$$
$$- (r-1-3p)\sqrt{(r+1+p)(s+1+q)}(p+3q+4-4\mathscr{L}_{21})c_{p+1,q+1}$$
$$+ \sqrt{(r-1-p)(r+1+p)(r+3+p)(s+1-q)}$$
$$\times (2r+6s-3p+3q+16-12\mathscr{L}_{0,-1})c_{p+3,q-1}$$
$$+ 12\sqrt{(r-1-p)(r+1+p)(r+3+p)(s+1+q)}\mathscr{L}_{31}c_{p+3,q+1},$$

$$B_{pq}^{(31)} = 4\sqrt{(r-1-p)(r+1-p)(r+3-p)(s+1-q)}\mathscr{L}_{-3,-1}c_{p-3,q-1}$$
$$- \sqrt{(r-1-p)(r+1-p)(r+3-p)(s+1+q)}$$
$$\times (2r+2s+p-q+4-4\mathscr{L}_{01})c_{p-3,q+1}$$
$$- \sqrt{(r-1-p)(r+1-p)(r-1+p)(s+1-q)}$$
$$\times (p+3q-4+4\mathscr{L}_{-2,-1})c_{p-1,q-1}$$
$$+ 4\sqrt{(r-1-p)(r+1-p)(r-1+p)(s+1+q)}\mathscr{L}_{11}c_{p-1,q+1}$$
$$+ 4\sqrt{(r-1-3p)(r-1+p)(r+1+p)(s+1-q)}\mathscr{L}_{-1,-1}c_{p+1,q-1}$$
$$- \sqrt{(r-1-p)(r-1+p)(r+1+p)(s+1+q)}$$
$$\times (p+3q+4-4\mathscr{L}_{21})c_{p+1,q+1}$$
$$+ \sqrt{(r-1+p)(r+1+p)(r+3+p)(s+1-q)}$$
$$\times (2r+2s-p+q+4-4\mathscr{L}_{0,-1})c_{p+3,q-1}$$
$$+ 4\sqrt{(r-1+p)(r+1+p)(r+3+p)(s+1+q)}\mathscr{L}_{31}c_{p+3,q+1},$$

$$B_{pq}^{(30)} = -4\sqrt{(r-1-p)(r+1-p)(r+3-p)(s+1+q)}\mathscr{L}_{-3,-1}c_{p-3,q-1}$$
$$- \sqrt{(r-1-p)(r+1-p)(r+3-p)(s+1-q)}$$
$$\times (2r-2s+p-q-4\mathscr{L}_{01})c_{p-3,q+1}$$
$$+ \sqrt{(r-1-p)(r+1-p)(r-1+p)(s+1+q)}$$
$$\times (p+3q-4+4\mathscr{L}_{-2,-1})c_{p-1,q-1}$$
$$+ 4\sqrt{(r-1-p)(r+1-p)(r-1+p)(s+1-q)}\mathscr{L}_{11}c_{p-1,q+1}$$
$$- 4\sqrt{(r-1-p)(r-1+p)(r+1+p)(s+1+q)}\mathscr{L}_{-1,-1}c_{p+1,q-1}$$
$$- \sqrt{(r-1-p)(r-1+p)(r+1+p)(s+1-q)}$$
$$\times (p+3q+4-4\mathscr{L}_{21})c_{p+1,q+1}$$
$$- \sqrt{(r-1+p)(r+1+p)(r+3+p)(s+1+q)}$$
$$\times (2r-2s-p+q-4\mathscr{L}_{0,-1})c_{p+3,q-1}$$
$$+ 4\sqrt{(r-1+p)(r+1+p)(r+3+p)(s+1-q)}\mathscr{L}_{31}c_{p+3,q+1}.$$

3.5. The equation $\mathscr{D}_\lambda f = 0$: Δ_{III}^+-case. In this subsection, we assume that $\Delta^+ = \Delta_{III}^+$, and compute $\mathscr{D}_\lambda f$ as before. Then we see that $\mathscr{D}_\lambda f$ is of the form

$$\mathscr{D}_\lambda f = \textstyle\sum_{i,j,p,q} \gamma_{ij} C_{pqj}^{(i)} u_p^{(r,3;i)} \otimes e_{qj}^{(s1)},$$

where the sum is taken for $i = 2, 3$; $j = \pm 1$; $p = -r+1, -r+3, \ldots, r-1$ and $q = -s, -s+2, \ldots, s$ with nonzero constants γ_{ij} depending only on i, j, and smooth functions $C_{pqj}^{(i)}$ on G.

In this case, for $f \in C_{\tau_\lambda}^\infty(G)$, it is necessary and sufficient for $\mathscr{D}_\lambda f = 0$ that

$$C_{pq1}^{(2)} = 0 \quad \text{for } p = -r+1, \ldots, r-1; \quad q = -s, \ldots, s, \tag{3.7}$$

$$C_{p,q,-1}^{(2)} = 0 \quad \text{for } p = -r+1, \ldots, r-1; \quad q = -s, \ldots, s, \tag{3.8}$$

$$C_{pq1}^{(3)} = 0 \quad \text{for } p = -r+3, \ldots, r-3; \quad q = -s, \ldots, s, \tag{3.9}$$

$$C_{p,q,-1}^{(3)} = 0 \quad \text{for } p = -r+3, \ldots, r-3; \quad q = -s, \ldots, s. \tag{3.10}$$

In conditions (3.7)–(3.10), we regard c_{pq} as 0 if $p \notin \{-r, -r+2, \ldots, r\}$ or $q \notin \{-s, -s+2, \ldots, s\}$. Here the functions $C_{pqj}^{(i)}$ are defined by

$$\begin{aligned}
C_{pq1}^{(2)} = {} & 12\sqrt{(r+1-p)(r+3-p)(r-1+p)}\mathscr{L}_{-3,-1} c_{p-3,q} \\
& - 6\sqrt{(r+1-p)(r+3-p)(r-1+p)(s-q)(s+2+q)}c_{p-3,q+2} \\
& - (r+3p-1)\sqrt{r+1-p}(p+3q-1+4\mathscr{L}_{-2,-1})c_{p-1,q} \\
& - 4(r-3p-1)\sqrt{r+1+p}\mathscr{L}_{-1,-1}c_{p+1,q} \\
& - \sqrt{(r-1-p)(r+1+p)(r+3+p)} \\
& \times (2r-3p-3q+7-12\mathscr{L}_{0,-1})c_{p+3,q}, \\
C_{p,q,-1}^{(2)} = {} & \sqrt{(r+1-p)(r+3-p)(r-1+p)} \\
& \times (2r+3p+3q+7-12\mathscr{L}_{01})c_{p-3,q} \\
& - 4(r+3p-1)\sqrt{r+1-p}\mathscr{L}_{11}c_{p-1,q} \\
& - (r-3p-1)\sqrt{r+1+p}(p+3q+1-4\mathscr{L}_{21})c_{p+1,q} \\
& + 6\sqrt{(r-1-p)(r+1+p)(r+3+p)(s+2-q)(s+q)}c_{p+3,q-2} \\
& + 12\sqrt{(r-1-p)(r+1+p)(r+3+p)}\mathscr{L}_{31}c_{p+3,q}, \\
C_{pq1}^{(3)} = {} & -4\sqrt{(r-1-p)(r+1-p)(r+3-p)}\mathscr{L}_{-3,-1}c_{p-3,q} \\
& + 2\sqrt{(r-1-p)(r+1-p)(r+3-p)(s-q)(s+2+q)}c_{p-3,q+2} \\
& + \sqrt{(r-1-p)(r+1-p)(r-1+p)} \\
& \times (p+3q-1+4\mathscr{L}_{-2,-1})c_{p-1,q} \\
& - 4\sqrt{(r-1-p)(r-1+p)(r+1+p)}\mathscr{L}_{-1,-1}c_{p+1,q} \\
& - \sqrt{(r-1+p)(r+1+p)(r+3+p)} \\
& \times (2r-p-q+1-4\mathscr{L}_{0,-1})c_{p+3,q},
\end{aligned}$$

$$\begin{aligned}C^{(3)}_{p,q,-1} = &-\sqrt{(r-1-p)(r+1-p)(r+3-p)}\\ &\times (2r+p+q+1-4\mathscr{L}_{01})c_{p-3,q}\\ &+4\sqrt{(r-1-p)(r+1-p)(r-1+p)}\mathscr{L}_{11}c_{p-1,q}\\ &-\sqrt{(r-1-p)(r-1+p)(r+1+p)}\\ &\times (p+3q+1-4\mathscr{L}_{21})c_{p+1,q}\\ &+2\sqrt{(r-1+p)(r+1+p)(r+3+p)(s+2-q)(s+q)}c_{p+3,q-2}\\ &+4\sqrt{(r-1+p)(r+1+p)(r+3+p)}\mathscr{L}_{31}c_{p+3,q}.\end{aligned}$$

4. Solutions for the differential equation

4.1. Preparation to solve the equation $\mathscr{D}_{\lambda,1_N}f=0$. In this section, we solve the equation $\mathscr{D}_{\lambda,1_N}f=0$ and determine the $(\mathfrak{a}, M)$-module structure of Ker $\mathscr{D}_{\lambda,1_N}$ in § 1.4. First we state the facts commonly hold for the three cases. For $f\in C^\infty_{\tau_\lambda}(G)$, we write $f=\sum_{p,q}c_{pq}e^{(rs)}_{pq}$ as before, then

$$\begin{aligned}&f(gn)=f(g) \qquad (\forall(g,n)\in G\times N)\\ &\Leftrightarrow c_{pq}(gn)=c_{pq}(g) \qquad (\forall(g,n)\in G\times N, \forall p, \forall q).\end{aligned} \tag{4.1}$$

If $f\in C^\infty_{\tau_\lambda}(G)$ satisfies (4.1), then for $X\in\mathfrak{n}_0$, $a\in A$, $n\in N$ we have,

$$\begin{aligned}(L_Xc_{pq})(an) &= \frac{d}{dt}c_{pq}(\exp(-tX)\cdot an)|_{t=0}\\ &= \frac{d}{dt}c_{pq}(a\exp(-t\operatorname{Ad}(a)^{-1}X)\cdot n)|_{t=0}\\ &= \frac{d}{dt}c_{pq}(a)|_{t=0} \qquad (\because \operatorname{Ad}(a)^{-1}X\in\mathfrak{n}_0).\end{aligned}$$

So

$$(L_Xc_{pq})(an)=0. \tag{4.2}$$

Note that, for $f\in C^\infty_{\tau_\lambda}(G)$,

$$\mathscr{D}_\lambda f=0 \Leftrightarrow \mathscr{D}_\lambda f|_{AN}=0. \tag{4.3}$$

4.2. Solutions for the equation $\mathscr{D}_{\lambda,1_N}f=0$: Δ_I^+-case. Here we assume that $\Delta^+=\Delta_I^+$. In this case, by using (3.2) and the remarks in the last subsection, we see that for $f\in C^\infty_{\tau_\lambda}(G)$,

$$\begin{aligned}&f\in \operatorname{Ker}\mathscr{D}_{\lambda,1_N}\\ &\Leftrightarrow \mathscr{D}_\lambda f=0 \quad \text{and} \quad (4.1)\\ &\Leftrightarrow (4.4)\text{–}(4.7) \quad \text{and} \quad (4.1),\end{aligned}$$

with

$$(2L_{\tilde{H}_1} - 2s + p + q - 2)c_{p,q+2} = 0, \tag{4.4}$$

$$\sqrt{s-q}(2L_{\tilde{H}_2} + p + 3q)c_{pq} + 2\sqrt{(r+2-p)(r+p)(s+2+q)}c_{p-2,q+2} = 0, \tag{4.5}$$

$$-\sqrt{s+2+q}(p + 3q + 6 - 2L_{\tilde{H}_2})c_{p,q+2} + 2\sqrt{(r-p)(r+2+p)(s-q)}c_{p+2,q} = 0, \tag{4.6}$$

$$(2s + p + q + 4 - 2L_{\tilde{H}_1})c_{pq} = 0, \tag{4.7}$$

on A for $p = -r, -r+2, \ldots, r$; $q = -s, -s+2, \ldots, s-2$.

From now on, unless otherwise stated, we consider the equations for c_{pq}'s such as (4.4)–(4.7) only on A, though c_{pq}'s are functions on G.

Now define two linear forms μ_j $(j = 1, 2)$ on $\mathfrak{a}$ through

$$\mu_1(\tilde{H}_1) = -(s+2), \qquad \mu_1(\tilde{H}_2) = r,$$

$$\mu_2(\tilde{H}_1) = -(s - r + 4)/2, \qquad \mu_2(\tilde{H}_2) = -(r + 3s)/2,$$

and put for $a \in A$,

$$f_0(a) = \sum_p \alpha_p a^{\mu_1} e^{(rs)}_{p,-p},$$

$$f_+(a) = a^{\mu_2}(e^{(rs)}_{rs} + e^{(rs)}_{-r,-s}),$$

$$f_-(a) = a^{\mu_2}(e^{(rs)}_{rs} - e^{(rs)}_{-r,-s}),$$

where the sum is taken for $p = -r, -r+2, \ldots, r$. Here $a^{\mu_j} = e^{\mu_j(\log(a))}$ and

$$\alpha_p = \sqrt{\frac{2r(2r-2)\cdots(r+p+2)}{(r-p)(r-p-2)\cdots 2} \cdot \frac{(s+r)(s+r-2)\cdots(s+p+2)}{(s-p)(s-p-1)\cdots(s-r+2)}}. \tag{4.8}$$

Extend f_*'s to G by $f_*(kan) = \tau_\lambda(k)f_*(a)$ for $k \in K$, $a \in A$, $n \in N$. Then f_*'s are elements of $C^\infty_{\tau_\lambda}(G)$, and $\operatorname{Ker} \mathcal{D}_{\lambda, 1_N}$ is described in the following lemma.

Lemma 4.1. *The functions $f_*(* = 0, +, -)$ belong to* $\operatorname{Ker} \mathcal{D}_{\lambda, 1_N}$ *and form its basis. Moreover,* $\operatorname{Ker} \mathcal{D}_{\lambda, 1_N}$ *is decomposed as an MA-module in the following way:*

$$\operatorname{Ker} \mathcal{D}_{\lambda, 1_N} \simeq \mathbf{C}f_0 \oplus \mathbf{C}f_+ \oplus \mathbf{C}f_-,$$

with

$$\mathbf{C}f_0 \simeq (\sigma_{(-1)^r, (-1)^{(r+s)/2}}) \otimes e^{\mu_1},$$

$$\mathbf{C}f_+ \simeq (\sigma_{(-1)^{(s-r)/2}, 1}) \otimes e^{\mu_2},$$

$$\mathbf{C}f_- \simeq (\sigma_{(-1)^{(s-r)/2}, -1}) \otimes e^{\mu_2}.$$

4.3. Proof of Lemma 4.1. In the following, we give a proof of Lemma 4.1. In this proof, we use the symbol "$(\cdot)_{x \to y}$" instead of "the equation $(\cdot)$ substituted x with y". In case $q \neq \pm s$, we have $(p+q)c_{pq} = 0$ because of $(4.4)_{q \to q-2}$ and (4.7). So, if $q \neq \pm s$ and $p + q \neq 0$, then $c_{pq} = 0$.

Next, by using $(4.5)_{p\to s-2}$, we have

$$\sqrt{2}(2L_{\tilde{H}_2} + p + 3s - 6)c_{p,s-2} + 2\sqrt{(r+2-p)(r+p)\cdot 2s}\,c_{p-2,s} = 0.$$

Since $p + s - 2 \geq s - r - 2 > 0$, $c_{p,s-2} = 0$ holds, therefore $c_{p-2,s}$ is also 0. That is, $c_{ps} = 0$ for $p = -r, \ -r+2, \ \ldots, \ r-2$. Similarly, we have $c_{p,-s} = 0$ for $p = -r+2, \ldots, r-2, r$. By the above remarks we see that

$$c_{pq} = 0 \qquad \text{if } (p, q) \neq (r, s), \ (-r, -s) \quad \text{and} \quad p + q \neq 0.$$

In (4.4)–(4.7), c_{rs} and $c_{-r,-s}$ appear only in the following equations.

$$(4.4)_{(p,q)\to(r,s-2)}, \qquad \text{i.e., } (2L_{\tilde{H}_1} - s + r - 4)c_{rs} = 0, \tag{4.9}$$

$$(4.6)_{(p,q)\to(r,s-2)}, \qquad \text{i.e., } (r + 3s - 2L_{\tilde{H}_2})c_{rs} = 0, \tag{4.10}$$

$$(4.5)_{(p,q)\to(-r,-s)}, \qquad \text{i.e., } (2L_{\tilde{H}_2} - r - 3s)c_{-r,-s} = 0, \tag{4.11}$$

$$(4.7)_{(p,q)\to(-r,-s)}, \qquad \text{i.e., } (s - r + 4 - 2L_{\tilde{H}_1})c_{-r,-s} = 0. \tag{4.12}$$

For real numbers x_1 and x_2, put $\tilde{c}_{rs}(x_1, x_2) = c_{rs}(\exp(x_1\tilde{H}_1 + x_2\tilde{H}_2))$. Then we can rewrite (4.9) and (4.10) as

$$\left(-2\frac{\partial}{\partial x_1} - s + r - 4\right)\tilde{c}_{rs} = 0,$$

$$\left(r + 3s + 2\frac{\partial}{\partial x_2}\right)\tilde{c}_{rs} = 0.$$

These two equations imply that

$$\tilde{c}_{rs}(x_1, x_2) = c\exp\left(-\frac{s-r+4}{2}x_1 - \frac{r+3s}{2}x_2\right),$$

with a constant c. Therefore for $a \in A$,

$$c_{rs}(a) = c\cdot a^{\mu_2} \quad (c \in \mathbf{C}). \tag{4.13}$$

Similarly $c_{-r,-s}$ is of the form

$$c_{-r,-s}(a) = c'\cdot a^{\mu_2} \quad (a \in A, c' \in \mathbf{C}). \tag{4.14}$$

For $c_{p,-p}$'s, equations (4.4)–(4.7) give

$$(4.7)_{p\to -p}, \qquad \text{i.e., } (L_{\tilde{H}_1} - s - 2)c_{p,-p} = 0, \tag{4.15}$$

$(4.5)_{p\to -p}$, i.e.,

$$\sqrt{s+p}(L_{\tilde{H}_2} - p)c_{p,-p} + \sqrt{(r+2-p)(r+p)(s+2-p)}\,c_{p-2,-(p-2)} = 0, \tag{4.16}$$

$(4.6)_{q\to -p-2}$, i.e.,

$$\sqrt{s-p}(L_{\tilde{H}_2} + p)c_{p,-p} + \sqrt{(r-p)(r+2+p)(s+2+p)}\,c_{p+2,-(p+2)} = 0, \tag{4.17}$$

By using $(4.15)_{p\to r}$ and $(4.17)_{p\to r}$, we see, as in the case of c_{rs}, that

$$c_{r,-r}(a) = c'' \cdot a^{\mu_1} \qquad (a \in A, c'' \in \mathbf{C}), \tag{4.18}$$

and we can determine $c_{p,-p}$'s inductively on p, by means of (4.16) and (4.18). The result is

$$c_{p,-p} = \alpha_p c_{r,-r}$$

with α_p defined in (4.8).

Now, define f_0, f_+, f_- as in §4.2, we can conclude, by the above arguments, that $\mathrm{Ker}\,\mathscr{D}_{\lambda,1_N}$ is contained in the linear span of f_*'s. Conversely, we can easily verify that f_*'s actually give elements of $\mathrm{Ker}\,\mathscr{D}_{\lambda,1_N}$ and that they are linearly independent.

Finally we consider the MA-module structure of $\mathrm{Ker}\,\mathscr{D}_{\lambda,1_N}$. For the representation τ_λ, it is easily seen that

$$\tau_\lambda(m_1)e_{pq}^{(rs)} = (\sqrt{-1})^{p-q}e_{pq}^{(rs)}, \qquad \tau_\lambda(m_2)e_{pq}^{(rs)} = (-1)^{(r+s-p-q)/2}e_{-p,-q}^{(rs)}.$$

So we have for $k \in K$, $a \in A$, $n \in N$,

$$\begin{aligned}(m_2 \cdot f_0)(kan) &= f_0(kanm_2)\\ &= f_0(km_2am_2^{-1}nm_2)\\ &= \tau_\lambda(k)\tau_\lambda(m_2)f_0(a) \qquad (\because m_2^{-1}nm_2 \in N),\end{aligned}$$

and

$$\begin{aligned}\tau_\lambda(m_2)f_0(a) &= \textstyle\sum_p \alpha_p a^{\mu_1}\tau_\lambda(m_2)e_{p,-p}^{(rs)}\\ &= \textstyle\sum_p \alpha_p a^{\mu_1}(-1)^{(r+s)/2}e_{-p,p}^{(rs)}\\ &= \textstyle\sum_p \alpha_{-p} a^{\mu_1}(-1)^{(r+s)/2}e_{-p,p}^{(rs)} \qquad (\text{note that } \alpha_p = \alpha_{-p})\\ &= (-1)^{(r+s)/2}\textstyle\sum_p \alpha_{-p}a^{\mu_1}e_{-p,p}^{(rs)}\\ &= (-1)^{(r+s)/2}f_0(a),\end{aligned}$$

so

$$\begin{aligned}(m_2 \cdot f_0)(kan) &= \tau_\lambda(k)((-1)^{(r+s)/2}f_0(a))\\ &= (-1)^{(r+s)/2}f_0(kan).\end{aligned}$$

Therefore $m_2 \cdot f_0 = (-1)^{(r+s)/2}f_0$. By similar computations, we see that M-action on $\mathrm{Ker}\,\mathscr{D}_{\lambda,1_N}$ is given as follows:

$$\begin{aligned}&m_1 \cdot f_0 = (-1)^r f_0, && m_1 \cdot f_+ = (-1)^{(s-r)/2}f_+, && m_1 \cdot f_- = (-1)^{(s-r)/2}f_-,\\ &m_2 \cdot f_0 = (-1)^{(r+s)/2}f_0, && m_2 \cdot f_+ = f_+, && m_2 \cdot f_- = -f_-.\end{aligned} \tag{4.19}$$

Also we have, for a, $a_0 \in A$, $k \in K$, $n \in N$,

$$\begin{aligned}(a_0\cdot f_0)(kan) &= f_0(kana_0)\\ &= f_0(kaa_0a_0^{-1}na_0)\\ &= \tau_\lambda(k)(aa_0)^{\mu_1}f_0(1) \qquad (\because\ a_0^{-1}na_0\in N)\\ &= a_0^{\mu_1}\tau_\lambda(k)a^{\mu_1}f_0(1)\\ &= a_0^{\mu_1}f_0(kan),\end{aligned}$$

so

$$a\cdot f_0 = a^{\mu_1}f_0.$$

Compute $a\cdot f_+$ and $a\cdot f_-$ similarly, we see that

$$a\cdot f_0 = a^{\mu_1}f_0, \qquad a\cdot f_+ = a^{\mu_2}f_+, \qquad a\cdot f_- = a^{\mu_2}f_- \qquad (a\in A). \tag{4.20}$$

By (4.19) and (4.20), we find that three subspaces $\mathbf{C}f_*$ ($*=0,+,-$) are MA-invariant, and their MA-module structures are those given in the lemma.

Thus we have completed the proof of the lemma.

4.4. Solutions for the equation $\mathscr{D}_{\lambda,1_N}f=0$: Δ_{II}^+-case. Here we give explicit form of $\operatorname{Ker}\mathscr{D}_{\lambda,1_N}$ in case $\Delta^+=\Delta_{II}^+$. First we prepare some symbols. We use the symbol "$u\equiv v$" for "$u\equiv v \pmod 4$", unless otherwise stated. Define $[m;n]$ for two integers m, n with $n\geq 0$ by

$$[m;n]=\begin{cases}1 & \text{if } n=0\\ \prod_{j=0}^{n-1}(m-4j) & \text{if } n\geq 1\end{cases},$$

and put

$$\beta_{pq}=\begin{cases}(-1)^{(r+s-p-q)/4}\sqrt{\begin{pmatrix} r\\ \frac12(r-p)\end{pmatrix}\cdot\begin{pmatrix} s\\ \frac12(s-q)\end{pmatrix}} & \text{if } p+q\equiv r+s\\ (-1)^{(r+s-2-p-q)/4}\sqrt{\dfrac1r\begin{pmatrix} r\\ \frac12(r-p)\end{pmatrix}\cdot\begin{pmatrix} s\\ \frac12(s-q)\end{pmatrix}} & \text{if } p+q\equiv r+s-2.\end{cases}$$

Next, set linear forms μ_j ($1\leq j\leq 5$) on $\mathfrak{a}$ as in the following table.

μ	$\mu(\tilde H_1)$	$\mu(\tilde H_2)$
μ_1	$-(s+3)$	$r-3$
μ_2	$-(s+3)$	$-(r-1)$
μ_3	$-\frac12(r+s+4)$	$-\frac12(r-3s)$
μ_4	$-\frac12(r+s+4)$	$\frac12(r-3s-4)$
μ_5	$-\frac12(r-s+4)$	$-\frac12(r+3s)$

Finally, for $a \in A$ define $f_{j,*}(a)$ as

$$f_{1,0}(a) = \sum_{p+q \equiv r+s} \beta_{pq} a^{\mu_1} e_{pq}^{(rs)},$$

$$f_{1,1}(a) = \sum_{p+q \equiv r+s-2} \beta_{pq} a^{\mu_1} e_{pq}^{(rs)},$$

$$f_{2,0}(a) = \begin{cases} \displaystyle\sum_{p+q \equiv 0} \beta_{pq} \frac{[3s - r + 2; (r + 3s - p - 3q)/4]}{[3s + 3r - 6; (r + 3s - p - 3q)/4]} a^{\mu_2} e_{pq}^{(rs)} & \text{if } r + s \equiv 0 \\ \displaystyle\sum_{p+q \equiv 2} \beta_{pq} \frac{[3s - r; (r + 3s - 2 - p - 3q)/4]}{[3s + 3r - 8; (r + 3s - 2 - p - 3q)/4]} a^{\mu_2} e_{pq}^{(rs)} & \text{if } r + s \equiv 2, \end{cases}$$

$$f_{2,\pm}(a) = \begin{cases} \displaystyle\sum_{p+q \equiv 2,\, p+q \geq 0} \beta_{pq} \frac{[3s - r; (r + 3s - 2 - p - 3q)/4]}{[3s + 3r - 8; (r + 3s - 2 - p - 3q)/4]} a^{\mu_2} (e_{pq}^{(rs)} \pm e_{-p,-q}^{(rs)}) & \text{if } r + s \equiv 0 \\ \displaystyle\sum_{p+q \equiv 0,\, p+q \geq 0} \beta_{pq} \frac{[3s - r + 2; (r + 3s - p - 3q)/4]}{[3s + 3r - 6; (r + 3s - p - 3q)/4]} a^{\mu_2} (e_{pq}^{(rs)} \pm e_{-p,-q}^{(rs)}) & \text{if } r + s \equiv 2, \end{cases}$$

$$f_{3,0}(a) = \begin{cases} \displaystyle\sum_{p+q \equiv r+s-2} \beta_{pq} \frac{[2s - 2; (r + s - 2 - p - q)/4]}{[2r - 6; (r + s - 2 - p - q)/4]} a^{\mu_3} e_{pq}^{(rs)} & \text{if } r, s \text{ even} \\ \displaystyle\sum_{p+q \equiv r+s} \beta_{pq} \frac{[2s; (r + s - p - q)/4]}{[2r - 4; (r + s - p - q)/4]} a^{\mu_3} e_{pq}^{(rs)} & \text{if } r, s \text{ odd}, \end{cases}$$

$$f_{3,\pm}(a) = \begin{cases} \displaystyle\sum_{p+q \equiv r+s,\, p+q \geq 0} \beta_{pq} \frac{[2s; (r + s - p - q)/4]}{[2r - 4; (r + s - p - q)/4]} a^{\mu_3} (e_{pq}^{(rs)} \pm e_{-p,-q}^{(rs)}) & \text{if } r, s \text{ even} \\ \displaystyle\sum_{p+q \equiv r+s-2,\, p+q \geq 0} \beta_{pq} \frac{[2s - 2; (r + s - 2 - p - q)/4]}{[2r - 6; (r + s - 2 - p - q)/4]} a^{\mu_3} (e_{pq}^{(rs)} \pm e_{-p,-q}^{(rs)}) & \text{if } r, s \text{ odd}, \end{cases}$$

$$f_{4,0}(a) = \begin{cases} \displaystyle\sum_{p+q \equiv r+s-2} \beta_{pq} \frac{[2s - 2; (r + s - 2 - p - q)/4][r - s + p + q - 4; (s - q)/2]}{[6s - 2; (r + 3s - 2 - p - 3q)/4]} a^{\mu_4} e_{pq}^{(rs)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } r, s \text{ even} \\ \displaystyle\sum_{p+q \equiv r+s} \beta_{pq} \frac{[2s; (r + s - p - q)/4][r - s + p + q - 4; (s - q)/2]}{[6s; (r + 3s - p - 3q)/4]} a^{\mu_4} e_{pq}^{(rs)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } r, s \text{ odd}, \end{cases}$$

$$f_{4,\pm}(a) = \begin{cases} \displaystyle\sum_{p+q\equiv r+s,\, p+q\geq 0} \beta_{pq} \frac{[2s;(r+s-p-q)/4][r-s+p+q-4;(s-q)/2]}{[6s;(r+3s-p-3q)/4]} \\ \quad a^{\mu_4}(e^{(rs)}_{pq} \pm e^{(rs)}_{-p,-q}) & \text{if } r, s \text{ even} \\ \displaystyle\sum_{p+q\equiv r+s-2,\, p+q\geq 0} \beta_{pq} \frac{[2s-2;(r+s-2-p-q)/4][r-s+p+q-4;(s-q)/2]}{[6s-2;(r+3s-2-p-3q)/4]} \\ \quad a^{\mu_4}(e^{(rs)}_{pq} \pm e^{(rs)}_{-p,-q}) & \text{if } r, s \text{ odd,} \end{cases}$$

$$f_{5,\pm}(a) = a^{\mu_5}(e^{(rs)}_{rs} \pm e^{(rs)}_{-r,-s}).$$

Extend these $f_{j,*}$'s to G by

$$f_{j,*}(kan) = \tau_\lambda(k) f_{j,*}(a) \qquad (k \in K, a \in A, n \in N).$$

Then, $f_{j,*}$'s give functions in $C^\infty_{\tau_\lambda}(G)$. The explicit form of $\operatorname{Ker} \mathscr{D}_{\lambda,1_N}$ is as in the following lemma.

Lemma 4.2. *Suppose that λ is far from the walls, then the above $f_{j,*}$'s form a basis of* $\operatorname{Ker} \mathscr{D}_{\lambda,1_N}$ *and the subspace $\mathbf{C}f_{j,*}$ of* $\operatorname{Ker} \mathscr{D}_{\lambda,1_N}$ *is MA-invariant. As MA-modules*

$$\mathbf{C}f_{j,*} \simeq \sigma_{\varepsilon_1,\varepsilon_2} \otimes e^\mu$$

with ε_1, ε_2, μ listed in the following table.

$(j, *)$	ε_1	ε_2	μ	$(j, *)$	ε_1	ε_2	μ
$(1, 0)$	$(-1)^{(r-s)/2}$	$(-1)^{(r+s)/2}$	μ_1	$(1, 1)$	$(-1)^{(r-s+2)/2}$	$(-1)^{(r+s)/2}$	μ_1
$(2, 0)$	$(-1)^r$	$(-1)^{(r-s)/2}$	μ_2	$(2, +)$	$(-1)^{r+1}$	$(-1)^{(r+s+2)/2}$	μ_2
$(2, -)$	$(-1)^{r+1}$	$(-1)^{(r+s)/2}$	μ_2	$(3, 0)$	$(-1)^{(r+s+2)/2}$	$(-1)^{r+1}$	μ_3
$(3, +)$	$(-1)^{(r+s)/2}$	$(-1)^r$	μ_3	$(3, -)$	$(-1)^{(r+s)/2}$	$(-1)^{r+1}$	μ_3
$(4, 0)$	$(-1)^{(r+s+2)/2}$	$(-1)^{(r-s)/2}$	μ_4	$(4, +)$	$(-1)^{(r+s)/2}$	$(-1)^r$	μ_4
$(4, -)$	$(-1)^{(r+s)/2}$	$(-1)^{r+1}$	μ_4	$(5, +)$	$(-1)^{(r-s)/2}$	1	μ_5
$(5, -)$	$(-1)^{(r-s)/2}$	-1	μ_5				

4.5. Proof of Lemma 4.2. Here we give a proof of Lemma 4.2.

Keep (4.2) and (4.3) in §4.1 in mind and solve equations (3.3)–(3.6) with respect to $L_{\tilde{H}_j} c_{pq}$'s. Then we obtain

$$L_{\tilde{H}_1}c_{pq} = \frac{1}{4}\sqrt{\frac{(r+2+p)(r+4+p)(r+6+p)(s+2-q)}{(r-4-p)(r-2-p)(r-p)(s+q)}}$$
$$\times (r-s-p-q-4)c_{p+6,q-2} + \frac{1}{4}(r+3s-p-q+8)c_{pq} \tag{4.21}$$

for $p = -r, -r+2, \dots, r-6$; $q = -s+2, -s+4, \dots, s$,

$$L_{\tilde{H}_1}c_{pq} = \frac{1}{4}\sqrt{\frac{(r+2-p)(r+4-p)(r+6-p)(s+2+q)}{(r-4+p)(r-2+p)(r+p)(s-q)}}$$
$$\times (r-s+p+q-4)c_{p-6,q+2} + \frac{1}{4}(r+3s+p+q+8)c_{pq} \tag{4.22}$$

for $p = -r+6, -r+8, \dots, r$; $q = -s, -s+2, \dots, s-2$,

$$L_{\tilde{H}_2}c_{pq} = \frac{3}{4}\sqrt{\frac{(r+2+p)(r+4+p)}{(r-2-p)(r-p)}}\cdot(r-s-p-q-4)c_{p+4,q} - \frac{1}{2}(p+3q)c_{pq}$$
$$-\frac{1}{4}\sqrt{\frac{(r+2-p)(s+2+q)}{(r+p)(s-q)}}\cdot(r+3s+p-3q)c_{p-2,q+2}, \tag{4.23}$$

for $p = -r+2, -r+4, \dots, r-4$; $q = -s, -s+2, \dots, s-2$,

$$L_{\tilde{H}_2}c_{pq} = \frac{3}{4}\sqrt{\frac{(r+2-p)(r+4-p)}{(r-2+p)(r+p)}}\cdot(r-s+p+q-4)c_{p-4,q} + \frac{1}{2}(p+3q)c_{pq}$$
$$-\frac{1}{4}\sqrt{\frac{(r+2+p)(s+2-q)}{(r-p)(s+q)}}\cdot(r+3s-p+3q)c_{p+2,q-2}, \tag{4.24}$$

for $p = -r+4, -r+6, \dots, r-2$; $q = -s+2, -s+4, \dots, s$.

In order to obtain eigenfunctions of the differential operator $L_{\tilde{H}_1}$, we define γ_{pq}, φ_{pq} and ψ_{pq} as

$$\gamma_{pq} = \sqrt{\frac{(r+2+p)(r+4+p)(r+6+p)(s+2-q)}{(r-4-p)(r-2-p)(r-p)(s+q)}},$$

$$\varphi_{pq} = c_{pq} + \gamma_{pq}c_{p+6,q-2},$$

$$\psi_{pq} = (r-s+p+q)c_{pq} - \gamma_{pq}(r-s-p-q-4)c_{p+6,q-2},$$

then (4.21) and (4.22) imply that φ_{pq} and ψ_{pq} satisfy the equations

$$L_{\tilde{H}_1}\varphi_{pq} = \tfrac{1}{2}(r+s+4)\varphi_{pq}, \tag{4.25}$$

$$L_{\tilde{H}_1}\psi_{pq} = (s+3)\psi_{pq}, \tag{4.26}$$

for $p = -r, -r+2, \dots, r-6$; $q = -s+2, -s+4, \dots, s$.

Now put $\mathbf{I} = \{(r, s), (r-2, s), (r-4, s), (-r, -s), (-r+2, -s), (-r+4, -s)\}$ and $\tilde{c}_{pq}(x_1, x_2) = c_{pq}(\exp(x_1\tilde{H}_1 + x_2\tilde{H}_2))$ for $x_1, x_2 \in \mathbf{R}$. Then using (4.25) and (4.26), and calculating as in the argument deriving (4.13) in the proof of Lemma

4.1, we see that for $(p, q) \notin \mathrm{I}$, $\tilde{c}_{pq}$ is of the following form:

$$\tilde{c}_{pq}(x_1, x_2) = u_{pq}(x_2)\exp(-(s+3)x_1) + v_{pq}(x_2)\exp(-\tfrac{1}{2}(r+s+4)x_1), \tag{4.27}$$

where u_{pq}'s and u_{pq}'s are smooth functions on $\mathbf{R}$ satistying

$$u_{pq} + \gamma_{pq}u_{p+6,q-2} = 0, \tag{4.28}$$

$$(r - s + p + q)v_{pq} - \gamma_{pq}(r - s - p - q - 4)v_{p+6,q-2} = 0, \tag{4.29}$$

for $p = -r, \ -r+2, \ \dots, \ r-6$; $q = -s+2, \ -s+4, \ \dots, \ s$.

As a next setp, we determine the functions u_{pq}, v_{pq} in (4.27). By means of (4.23), (4.24), (4.28) and (4.29), we can deduce the following relations.

(i) If c_{pq}, $c_{p-2,q+2}$ and $c_{p+4,q}$ can be expressed in the form of (4.27), then

$$u'_{pq} = \frac{1}{2}(p+3q)u_{pq} + \frac{1}{2}\sqrt{\frac{(r+2-p)(s+2+q)}{(r+p)(s-q)}}\cdot(2r-p-3q-6)u_{p-2,q+2}, \tag{4.30}$$

$$v'_{pq} = \frac{1}{2}(p+3q)v_{pq} - \frac{1}{2}\sqrt{\frac{(r+2-p)(s+2+q)}{(r+p)(s-q)}}\cdot(r-3s+p+3q)v_{p-2,q+2}. \tag{4.31}$$

(ii) If c_{pq}, $c_{p+2,q-2}$ and $c_{p-4,q}$ can be expressed in the form of (4.27), then

$$u'_{pq} = -\frac{1}{2}(p+3q)u_{pq} + \frac{1}{2}\sqrt{\frac{(r+2+p)(s+2-q)}{(r-p)(s+q)}}\cdot(2r+p+3q-6)u_{p+2,q-2}, \tag{4.32}$$

$$v'_{pq} = -\frac{1}{2}(p+3q)v_{pq} - \frac{1}{2}\sqrt{\frac{(r+2+p)(s+2-q)}{(r-p)(s+q)}}\cdot(r-3s-p-3q)v_{p+2,q-2}. \tag{4.33}$$

From these four equations, we can derive the following differential equations for u_{pq} or v_{pq}:

$$u''_{pq} + 2u'_{pq} - (r-1)(r-3)u_{pq} = 0, \tag{4.34}$$

$$4v''_{pq} + 8v'_{pq} - (r-3s-4)(r-3s)v_{pq} = 0, \tag{4.35}$$

for $(p, q) \in \mathrm{I}'$, with

$$\mathrm{I}' = \{(p,q) \mid -r+4 \le p \le r-6 \quad \text{and} \quad -s+2 \le q \le s\}$$
$$\cup \{(p,q) \mid -r+6 \le p \le r-4 \quad \text{and} \quad -s \le q \le s-2\}.$$

Define linear forms μ_j $(1 \le j \le 4)$ as in §4.4 and solve equations (4.34) and (4.35). Then we see that c_{pq} $((p, q) \in \mathrm{I}')$ is a linear combination of a^{μ_j} $(1 \le j \le 4)$. By using (4.21)–(4.24), we can conclude that

(∗) c_{pq} is expressed as a linear combination of a^{μ_j} $(1 \le j \le 4)$ if $(p, q) \notin \mathrm{I}$.

Now we determine the form of c_{pq} with $(p, q) \in \mathrm{I}$. First we consider $c_{r-4,s}$. By eliminating c_{rs} from the equations $B^{(21)}_{r-1,s-1} = 0$ and $B^{(11)}_{r-1,s-1} = 0$ in §3.4, we get

$$\sqrt{2s}(6L_{\tilde{H}_1} - r - 3s - 20)c_{r-4,s} = \sqrt{r-1}(2L_{\tilde{H}_2} + r + 3s - 8)c_{r-2,s-2}. \tag{4.36}$$

Combining this equation with the relation $B^{(30)}_{r-7,s+1} = 0$, we see that $c_{r-4,s}$ is also a linear combination of a^{μ_j} $(1 \leq j \leq 4)$ if $r \neq 3s - 2$.

In case $r = 3s - 2$, from the equations $B^{(30)}_{r-7,s+1} = 0$ and $B^{(30)}_{r-5,s-1} = 0$, we can deduce that $c_{r-4,s}(a)$ $(a \in A)$ is expressed as a linear combination of a^{μ_j} $(1 \leq j \leq 4)$, $x_1 a^{\mu_2}$ and $x_2 a^{\mu_2}$ with $a = \exp(x_1 \tilde{H}_1 + x_2 \tilde{H}_2)$. Write $c_{r-4,s}$ as a linear combination of those terms and carry it into each of the equations (4.36), $B^{(30)}_{r-7,s+1} = 0$, (4.21) and (4.24), then we find that

(**1) $c_{r-4,s}$ is a linear combination of a^{μ_j} $(1 \leq j \leq 4)$ even if $r = 3s - 2$.

By similar arguments, one can see that

(**2) $c_{-r+4,-s}$, $c_{r-2,s}$ and $c_{-r+2,-s}$ are also expressed as linear combinations of a^{μ_j} $(1 \leq j \leq 4)$.

In the calculation of $c_{r-2,s}$ in (**2), we may use four equations $B^{(30)}_{r-5,s+1} = 0$, $B^{(11)}_{r+1,s-1} = 0$, (4.21), (4.24), instead of (4.36), $B^{(30)}_{r-7,s+1} = 0$, (4.21) and (4.24) in the case of $c_{r-4,s}$.

For c_{rs} and $c_{-r,-s}$, we find that they are linear combinations of a^{μ_j} $(1 \leq j \leq 5)$, by solving the equations $B^{(30)}_{r-3,s+1} = 0$, $B^{(21)}_{r-1,s-1} = 0$, $B^{(30)}_{-r+3,-s-1} = 0$ and $B^{(21)}_{-r+1,-s+1} = 0$. Here μ_5 is defined as in the last subsection. By virtue of this fact, (*), (**1) and (**2), $f \in \operatorname{Ker} \mathscr{D}_{\lambda, 1_N}$ can be written in the form:

$$f(a) = \sum_{p,q,j} \alpha^{(j)}_{pq} a^{\mu_j} e^{(rs)}_{pq} + \sum_{j=1}^{5} (\alpha^{(j)}_{rs} a^{\mu_j} e^{(rs)}_{rs} + \alpha^{(j)}_{-r,-s} a^{\mu_j} e^{(rs)}_{-r,-s}), \tag{4.37}$$

for $a \in A$ with $\alpha^{(j)}_{pq} \in \mathbf{C}$. For the first term in the right-hand side of the above equation, the sum is taken for $(p, q) \neq (r, s)$, $(-r, -s)$ and $j = 1, 2, 3, 4$.

Finally we compute the coefficients $\alpha^{(j)}_{pq}$ in (4.37). In general, by using $B^{(30)}_{p,\pm(s+1)} = 0$, we may express $\alpha^{(j)}_{p-4,s}$ (resp. $\alpha^{(j)}_{p+4,-s}$) in terms of $\alpha^{(j)}_{ps}$ (resp. $\alpha^{(j)}_{p,-s}$) and describe $\alpha^{(j)}_{p,\pm s}$ by $\alpha^{(j)}_{\pm r,\pm s}$ and $\alpha^{(j)}_{\pm(r-2),\pm s}$. By (4.30)–(4.33), we can derive a formula giving a relation between $\alpha^{(j)}_{pq}$ and $\alpha^{(j)}_{p+2,q-2}$, and describe $\alpha^{(j)}_{p+2,q-2}$ in terms of $\alpha^{(j)}_{pq}$ for almost all (p, q). For exceptional (p, q)'s, we can use other relations such as (4.22).

Calculating the coefficients $\alpha^{(j)}_{pq}$ by means of the above strategy, we see that the functions $f_{j,*}$ defined in §4.4 actually form a basis for $\operatorname{Ker} \mathscr{D}_{\lambda, 1_N}$. For the MA-module structure of $\operatorname{Ker} \mathscr{D}_{\lambda, 1_N}$, by similar arguments to those in the last part of the proof of Lemma 4.1, we see that the subspace $\mathbf{C}f_{j,*}$ is an MA-module described in the lemma.

Now the proof is completed.

4.6. Solutions for the equation $\mathscr{D}_{\lambda, 1_N} f = 0$: Δ^+_{III}-case. Here we will specify $\operatorname{Ker} \mathscr{D}_{\lambda, 1_N}$ in case $\Delta^+ = \Delta^+_{III}$. We use the symbols $\equiv$ and $[\cdot, \cdot]$ in the same meaning as in §4.4. First define linear forms μ_1 and μ_2 on $\mathfrak{a}$ through

$$\mu_1(\tilde{H}_1) = -(r-s+2)/2, \qquad \mu_1(\tilde{H}_2) = (r+3s-2)/2,$$
$$\mu_2(\tilde{H}_1) = -(r-s+2)/2, \qquad \mu_2(\tilde{H}_2) = -(r+3s+2)/2,$$

and set

$$\beta_{pq} = \begin{cases} \sqrt{\binom{r}{\frac{1}{2}(r-p)}\binom{s}{\frac{1}{2}(s-q)}} & \text{if } p+q \equiv r+s \\ \sqrt{\frac{1}{r}\binom{r}{\frac{1}{2}(r-p)}\binom{s}{\frac{1}{2}(s-q)}} & \text{if } p+q \equiv r+s-2. \end{cases}$$

For $a \in A$, put $f_{ij}(a)$ $(i = 1, 2; j = 0, 1)$ as follows:

$$f_{10}(a) = \sum_{p+q \equiv r+s} (-1)^{(r+s-p-q)/4} \beta_{pq} a^{\mu_1} e_{pq}^{(rs)},$$

$$f_{11}(a) = \sum_{p+q \equiv r+s-2} (-1)^{(r+s-2-p-q)/4} \beta_{pq} a^{\mu_1} e_{pq}^{(rs)},$$

$$f_{20}(a) = \sum_{p+q \equiv r+s} (-1)^{(s-q)/2} \beta_{pq} \frac{[r+3s-2-p-3q; (r+3s-p-3q)/4]}{[2r+6s-2; (r+3s-p-3q)/4]} a^{\mu_2} e_{pq}^{(rs)},$$

$$f_{21}(a) = \sum_{p+q \equiv r+s-2} (-1)^{(s-q)/2} \beta_{pq} \frac{[r+3s-2-p-3q; (r+3s-p-3q)/4]}{[2r+6s-4; (r+3s-2-p-3q)/4]} a^{\mu_2} e_{pq}^{(rs)}.$$

Extend these f_{ij}'s to G as

$$f_{ij}(kan) = \tau_\lambda(k) f_{ij}(a) \qquad (k \in K, a \in A, n \in N),$$

then f_{ij}'s give elements of $C^\infty_{\tau_\lambda}(G)$. The structure of $\operatorname{Ker} \mathscr{D}_{\lambda, 1_N}$ is given in the following lemma.

Lemma 4.3. *Suppose that λ is far from the walls, then the functions f_{ij} defined above belong to* $\operatorname{Ker} \mathscr{D}_{\lambda, 1_N}$ *and form its basis. Moreover the subspace* $\mathbf{C} f_{ij}$ *of* $\operatorname{Ker} \mathscr{D}_{\lambda, 1_N}$ *is MA-invariant and*

$$\mathbf{C} f_{ij} \simeq \sigma_{\varepsilon_1, \varepsilon_2} \otimes e^\mu \qquad \text{as } MA\text{-modules}$$

with $\varepsilon_1, \varepsilon_2$, μ listed in the following table.

(i,j)	ε_1	ε_2	μ	(i,j)	ε_1	ε_2	μ
$(1,0)$	$(-1)^{(r-s)/2}$	$(-1)^{(r+s)/2}$	μ_1	$(1,1)$	$(-1)^{\frac{r-s}{2}+1}$	$(-1)^{(r+s)/2}$	μ_1
$(2,0)$	$(-1)^{(r-s)/2}$	$(-1)^r$	μ_2	$(2,1)$	$(-1)^{\frac{r-s}{2}+1}$	$(-1)^{r+1}$	μ_2

4.7. Proof of Lemma 4.3. In this subsection, we give a proof of Lemma 4.3. Keeping (4.2) and (4.3) in mind, we can deduce the following equations (4.38)–(4.41) from equations (3.7)–(3.10) in §3.5.

$$L_{\tilde{H}_1} c_{pq} = \frac{1}{2}\sqrt{\frac{(r+2+p)(r+4+p)(r+6+p)}{(r-4-p)(r-2-p)(r-p)}}$$

$$\times \sqrt{(s+2-q)(s+q)}c_{p+6,q-2} + \frac{1}{2}(r+q+2)c_{pq} \tag{4.38}$$

for $p = -r, -r+2, \ldots, r-6$; $q = -s, -s+2, \ldots, s$,

$$L_{\tilde{H}_1} c_{pq} = \frac{1}{2}\sqrt{\frac{(r+2-p)(r+4-p)(r+6-p)}{(r-4+p)(r-2+p)(r+p)}}$$

$$\times \sqrt{(s-q)(s+2+q)}c_{p-6,q+2} + \frac{1}{2}(r-q+2)c_{pq} \tag{4.39}$$

for $p = -r+6, -r+8, \ldots, r$; $q = -s, -s+2, \ldots, s$,

$$L_{\tilde{H}_2} c_{pq} = \frac{1}{2}\sqrt{\frac{r+2-p}{r+p}\cdot(r+4-p)(r-2+p)}c_{p-4,q} + \frac{1}{2}(p+3q)c_{pq}$$

$$-\frac{3}{2}\sqrt{\frac{r+2+p}{r+p}\cdot(s+2-q)(s+q)}c_{p+2,q-2} \tag{4.40}$$

for $p = -r+4, -r+6, \ldots, r-2$; $q = -s, -s+2, \ldots, s$,

$$L_{\tilde{H}_2} c_{pq} = \frac{1}{2}\sqrt{\frac{r+2+p}{r-p}\cdot(r+4+p)(r-2-p)}c_{p+4,q} - \frac{1}{2}(p+3q)c_{pq}$$

$$-\frac{3}{2}\sqrt{\frac{r+2-p}{r+p}\cdot(s+2+q)(s-q)}c_{p-2,q+2} \tag{4.41}$$

for $p = -r+2, -r+4, \ldots, r-4$; $q = -s, -s+2, \ldots, s$.

First, as in the proof of Lemma 4.2, we find eigenfunctions of $L_{\tilde{H}_1}$. For this purpose, define γ_p, φ_{pq}, ψ_{pq} as

$$\gamma_p = \sqrt{\frac{(r+2+p)(r+4+p)(r+6+p)}{(r-4-p)(r-2-p)(r+p)}},$$

$$\varphi_{pq} = \sqrt{s+q}\,c_{pq} + \gamma_p\sqrt{s+2-q}\,c_{p+6,q-2},$$

$$\psi_{pq} = \sqrt{s+2-q}\,c_{pq} - \gamma_p\sqrt{s+q}\,c_{p+6,q-2},$$

for $p = -r, -r+2, \ldots, r-6$; $q = -s+2, -s+4, \ldots, s$. By (4.38) and (4.39), it can be seen that φ_{pq} and ψ_{pq} satisfy the equations

$$L_{\tilde{H}_1}\varphi_{pq} = \tfrac{1}{2}(r+s+4)\varphi_{pq}, \tag{4.42}$$

$$L_{\tilde{H}_1}\psi_{pq} = \tfrac{1}{2}(r-s+2)\psi_{pq}. \tag{4.43}$$

Let I, I′ and $\tilde{c}_{pq}$ be as in the proof of Lemma 4.2. Then, by (4.42) and (4.43), we see that if $(p, q) \notin \mathrm{I}$, $\tilde{c}_{pq}$ is of the following form

$$\tilde{c}_{pq}(x_1, x_2) = u_{pq}(x_2)\exp(-\tfrac{1}{2}(r+s+4)x_1) + v_{pq}(x_2)\exp(-\tfrac{1}{2}(r-s+2)x_1), \tag{4.44}$$

with smooth functions u_{pq}, v_{pq} on $\mathbf{R}$. By means of the equations $C^{(2)}_{p,q,j}=0$, for $(p,q,j) = (\pm(r-1), \pm(s-2), 1), (\pm(r-1), \pm s, 1), (\pm(r-3), \pm s, -1), (\pm(r-3), \pm s, 1)$ and $C^{(3)}_{\pm(r-3),s,1}=0$ in §3.5, we see that $\tilde{c}_{pq}$ is expressed as in (4.44) for all p, q. Using (4.40)–(4.43) again, for the functions u_{pq}, v_{pq} in (4.44), we obtain the following equations:

$$\begin{cases} u'_{pq} = -\dfrac{1}{2}\sqrt{\dfrac{(r+2+p)(s+q)}{(r-p)(s+2-q)}}\cdot(r-3s+p+3q-8)u_{p+2,q-2} - \dfrac{1}{2}(p+3q)u_{pq} \\ v'_{pq} = \dfrac{1}{2}\sqrt{\dfrac{(r+2+p)(s+2-q)}{(r-p)(s+q)}}\cdot(r+3s+p+3q-2)v_{p+2,q-2} - \dfrac{1}{2}(p+3q)v_{pq} \end{cases} \tag{4.45}$$

for $p=-r+4, -r+6, \ldots, r-2$; $q=-s+2, -s+4, \ldots, s-2$.

$$\begin{cases} u'_{pq} = -\dfrac{1}{2}\sqrt{\dfrac{(r+2-p)(s-q)}{(r+p)(s+2+q)}}\cdot(r-3s-p-3q-8)u_{p-2,q+2} + \dfrac{1}{2}(p+3q)u_{pq} \\ v'_{pq} = \dfrac{1}{2}\sqrt{\dfrac{(r+2-p)(s+2+q)}{(r+p)(s-q)}}\cdot(r+3s-p-3q-2)v_{p-2,q+2} + \dfrac{1}{2}(p+3q)v_{pq} \end{cases} \tag{4.46}$$

for $p=-r+2, -r+4, \ldots, r-4$; $q=-s, -s+2, \ldots, s-2$.

As in the case of Lemma 4.2, derive second order differential equations for u_{pq} or v_{pq} from (4.45) and (4.46), and solve them. Then we see that if $(p,q)\notin \mathrm{I}'$, $c_{pq}(a)$ is a linear combination of a^{μ_j} $(1\le j\le 4)$, where μ_j's are defined as in the following table.

μ	$\mu(\tilde{H}_1)$	$\mu(\tilde{H}_2)$
μ_1	$-\frac{1}{2}(r-s+2)$	$\frac{1}{2}(r+3s-2)$
μ_2	$-\frac{1}{2}(r-s+2)$	$-\frac{1}{2}(r+3s+2)$
μ_3	$-\frac{1}{2}(r+s+4)$	$\frac{1}{2}(r-3s-8)$
μ_4	$-\frac{1}{2}(r+s+4)$	$-\frac{1}{2}(r-3s-4)$

By the equations $C^{(2)}_{\pm(r-1),\pm(s-2),1}=0$ etc. used above, we find that

(**) $c_{pq}(a)$ is a linear combination of a^{μ_j} $(1\le j\le 4)$ for all p and q.

According to (**), write c_{pq}'s as a sum of a^{μ_j}'s with complex coefficients and calculate the coefficients as in the proof of Lemma 4.2. Then we can conclude that f_{ij}'s defined in §4.6 form a basis of $\mathrm{Ker}\,\mathscr{D}_{\lambda,1_N}$. In the calculation, we may use (4.45), (4.46) and $C^{(3)}_{p,\pm s,\pm 1}=0$ mainly. Note that the terms containing a^{μ_3} or a^{μ_4} always vanish.

The MA-module structure of $\mathrm{Ker}\,\mathscr{D}_{\lambda,1_N}$ is determined in a similar way as in the proof of Lemma 4.1, and we obtain the results stated in Lemma 4.3. This completes the proof.

5. Main result

Applying Theorem 1.3 to Lemmas 4.1–4.3, we can determine, into which principal series representations, a given discrete series can be embedded as a $(\mathfrak{g}, K)$-module. Before stating the results, we prepare some symbols.

Define subsets Ξ_J of Ξ, $J = I, II, III$, as

$$\Xi_J = \{\Lambda \in \Xi \mid \Delta^+ = \Delta_J^+\},$$

where Δ^+ is taken in such a way that Λ is Δ^+-dominant. For $\Lambda \in \Xi_I$, let Λ_J be the unique element in $\Xi_J \cap W \cdot \Lambda$, and set $\pi_J = \pi_{\Lambda_J}$. Then π_I, π_{II} and π_{III} are all the discrete series with the same infinitesimal character $\chi_{[\Lambda]}$ defined in §1.5. Let W' be the Weyl group of Ψ and $\tilde{s}_1$ (resp. $\tilde{s}_2$) the reflection with respect to λ_1 (resp. λ_2). The unique Ψ^+-dominant element in the W'-orbit of $\Lambda \circ u^{-1}$ is denoted by $\tilde{\Lambda}$. We have $\tilde{\Lambda} = (\tilde{s}_2 \tilde{s}_1)^2 (\Lambda \circ u^{-1})$. Recall that $r' = \Lambda(H_{10})$ and $s' = \Lambda(H_{32})$.

We can rewrite the results in the last section by means of Theorem 1.3 and obtain the following theorem describing the embeddings completely.

Theorem 5.1. *Let P be the minimal parabolic subgroup of G defined in §2.2 and assume that $\Lambda \in \Xi_I$. Then for any J, ε_1, ε_2, and $\mu \in \mathfrak{a}^*$, we have*

$$\dim \operatorname{Hom}_{(\mathfrak{g}, K)}(\pi_J, \operatorname{Ind}_P^G(\sigma_{\varepsilon_1, \varepsilon_2} \otimes e^{\mu} \otimes 1_N)) \leq 1.$$

The equality holds if and only if

$$\mu = \tilde{s} \cdot \tilde{\Lambda} \quad \text{and} \quad (\varepsilon_1, \varepsilon_2) \in S_\Lambda(J, \tilde{s}) \quad \text{with an} \quad \tilde{s} \in W'(J),$$

where $W'(J)$ and $S_\Lambda(J, \tilde{s})$ are subsets of W' and $\{\pm 1\} \times \{\pm 1\}$ defined respectively as follows:

$$W'(I) = \{\tilde{s}_1, \tilde{s}_2 \tilde{s}_1\},$$

$$W'(II) = \{1, \tilde{s}_1, \tilde{s}_2, \tilde{s}_1 \tilde{s}_2, \tilde{s}_2 \tilde{s}_1\},$$

$$W'(III) = \{\tilde{s}_2, \tilde{s}_1 \tilde{s}_2\},$$

$$S_\Lambda(II, 1) = \{((-1)^{(r'+s')/2}, (-1)^{(r'-s'+2)/2}), ((-1)^{(r'+s'+2)/2}, \pm 1)\},$$

$$S_\Lambda(J, \tilde{s}_1) = \begin{cases} \{((-1)^{r'+1}, (-1)^{(r'+s')/2})\} & \text{for } J = I \\ \{((-1)^{(r'+s')/2}, (-1)^{r'+1}), ((-1)^{(r'+s'+2)/2}, \pm 1)\} & \text{for } J = II, \end{cases}$$

$$S_\Lambda(J, \tilde{s}_2) = \begin{cases} \{((-1)^{r'}, (-1)^{(r'-s'+2)/2}), ((-1)^{r'+1}, \pm 1)\} & \text{for } J = II \\ \{((-1)^{(r'-s')/2}, (-1)^{r'+1}), ((-1)^{(r'-s'+2)/2}, (-1)^{r'})\} & \text{for } J = III, \end{cases}$$

$$S_\Lambda(J, \tilde{s}_1 \tilde{s}_2) = \{(\pm 1), (-1)^{(r'+s'+2)/2})\} \qquad \text{for } J = II, III,$$

$$S_\Lambda(J, \tilde{s}_2 \tilde{s}_1) = \{((-1)^{(r'-s'+2)/2}, \pm 1)\} \qquad \text{for } J = I, II.$$

Remarks. (1) The number of the embeddings of π_J into principal series is three, thirteen or four according as $J = I$, II or III.

(2) Note that $W'(I)$, $W'(III) \subset W'(II)$, $S_\Lambda(I, \tilde{s}_1) \subset S_\Lambda(II, \tilde{s}_1)$ and that $S_\Lambda(III, \tilde{s}_2) \subset S_\Lambda(II, \tilde{s}_2)$. Then we find that the discrete series π_{II} is embedded into all the possible principal series into which π_J ($J = I, II, III$) can be embedded.

Proof of Theorem 5.1. For the case where the Blattner parameter λ of π_Λ is "far from the walls", apply Theorem 1.3 to the results of Lemmas 4.1–4.3, then straightforward calculations give the statements in the above theorem. In that argument, we note that every discrete series representation of G is self-contragredient.

Next we consider the "not far from the walls" case. We keep to the notations in § 1.5. Since $\mathfrak{g}_0 = \mathrm{Lie}(G)$ is a normal real form of $\mathfrak{g}$, the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{l}$ and the positive system Ψ^+ is taken for $\Delta_{\mathfrak{a}}^+$. Put $\tilde{\rho} = \frac{1}{2}\sum_{\lambda \in \Psi^+} \lambda$, then $\tilde{\rho}$ is Ψ^+-dominant and $\mathfrak{g}$-action on $F_{\tilde{\rho}}$ can be lifted to a G-action. The same facts hold for the linear form $4\tilde{\rho}$.

The infinitesimal character of $\pi_J = \pi_{\Lambda_J}$ with respect to $\mathfrak{h}$ is $\chi_{[\tilde{\Lambda}]}$. For $\sigma \in \hat{M}$ and $\mu \in \mathfrak{a}^*$, the principal series $\mathrm{Ind}_P^G(\sigma \otimes e^\mu \otimes 1_N)$ has infinitesimal character $\chi_{[\mu]}$. Therefore, π_J is embedded into $\mathrm{Ind}_P^G(\sigma \otimes e^\mu \otimes 1_N)$ only if $\mu = \tilde{s}\cdot\tilde{\Lambda}$ for $\tilde{s} \in W'$.

For $\sigma \in \hat{M}$ and $\tilde{s} \in W'$, we find that

$$ {}^{L_P}\psi_{[\tilde{s}\cdot\tilde{\Lambda}]}^{[\tilde{s}\cdot(\tilde{\Lambda}+4\tilde{\rho})]}(\sigma \otimes e^{\tilde{s}\cdot(\tilde{\Lambda}+4\tilde{\rho})}) \simeq \sigma \otimes e^{\tilde{s}\cdot\tilde{\Lambda}}, \tag{5.1} $$

by noting that M acts trivially on the space spanned by a nonzero highest weight vector of $F_{4\tilde{s}\cdot\tilde{\rho}}$. Together with Proposition 1.1–1.3, relation (5.1) yields

$$ \begin{aligned} &\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_J, \mathrm{Ind}_P^G(\sigma \otimes e^{\tilde{s}\cdot\tilde{\Lambda}} \otimes 1_N)) \\ &\quad\simeq \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_J, \mathrm{Ind}_P^G({}^{L_P}\psi_{[\tilde{s}\cdot\tilde{\Lambda}]}^{[\tilde{s}\cdot(\tilde{\Lambda}+4\tilde{\rho})]}(\sigma \otimes e^{\tilde{s}\cdot(\tilde{\Lambda}+4\tilde{\rho})}) \otimes 1_N)) \\ &\quad\simeq \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_J, {}^G\psi_{[\tilde{\Lambda}]}^{[\tilde{\Lambda}+4\tilde{\rho}]}(\mathrm{Ind}_P^G(\sigma \otimes e^{\tilde{s}\cdot(\tilde{\Lambda}+4\tilde{\rho})} \otimes 1_N))) \\ &\quad\simeq \mathrm{Hom}_{(\mathfrak{g},K)}({}^G\varphi_{[\tilde{\Lambda}+4\tilde{\rho}]}^{[\tilde{\Lambda}]}(\pi_J), (\mathrm{Ind}_P^G(\sigma \otimes e^{\tilde{s}\cdot(\tilde{\Lambda}+4\tilde{\rho})} \otimes 1_N)) \\ &\quad\simeq \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_{\Lambda_J+4\rho_J}, \mathrm{Ind}_P^G(\sigma \otimes e^{\tilde{s}\cdot(\tilde{\Lambda}+4\tilde{\rho})} \otimes 1_N)), \end{aligned} \tag{5.2} $$

where $\rho_J = \frac{1}{2}\sum_{\alpha \in \Delta_J^+} \alpha$.

Let λ_J be the Blattner parameter of the discrete series π_J, then the discrete series $\pi_{\Lambda_J+4\rho_J}$ has Blattner parameter $\lambda_J + 4\rho_J$ satisfying the condition for being "far from the walls" described in § 3. So, relations (5.1) and (5.2) and the result for "far from the walls" case show that for $\sigma_{\varepsilon_1,\varepsilon_2} \in \hat{M}$ and $\tilde{s} \in W'$,

$$ \begin{gathered} \dim \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_J, \mathrm{Ind}_P^G(\sigma_{\varepsilon_1,\varepsilon_2} \otimes e^{\tilde{s}\cdot\tilde{\Lambda}} \otimes 1_N)) = 1 \\ \Leftrightarrow \dim \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_{\Lambda_J+4\rho_J}, \mathrm{Ind}_P^G(\sigma_{\varepsilon_1,\varepsilon_2} \otimes e^{\tilde{s}\cdot(\tilde{\Lambda}+4\tilde{\rho})} \otimes 1_N)) = 1 \\ \Leftrightarrow \tilde{s}\cdot(\tilde{\Lambda} + 4\tilde{\rho}) = \tilde{s}'\cdot(\Lambda_J + 4\rho_J)^\sim \quad \text{and} \quad (\varepsilon_1, \varepsilon_2) \in S_{\Lambda+4\rho_I}(J, \tilde{s}') \quad \text{with an} \quad \tilde{s}' \in W'(J), \end{gathered} \tag{5.3} $$

where $S_{\Lambda+4\rho_I}(J, \tilde{s})$ and $W'(J)$ are defined as in the theorem. Note that $(\Lambda + 4\rho_I)_J = \Lambda_J + 4\rho_J$, that $\Lambda(H_{10}) \equiv (\Lambda + 4\rho_I)(H_{10})(\text{mod } 4)$ and that $\Lambda(H_{32}) \equiv (\Lambda + 4\rho_I)(H_{32})(\text{mod } 4)$. Then one gets $S_{\Lambda+4\rho_I}(J, \tilde{s}') = S_\Lambda(J, \tilde{s}')$. Since $(\Lambda_J + 4\rho_J)^\sim = \tilde{\Lambda} + 4\tilde{\rho}$ is regular, the equation $\tilde{s}\cdot(\tilde{\Lambda} + 4\tilde{\rho}) = \tilde{s}'\cdot(\Lambda_J + 4\rho_J)^\sim$ implies $\tilde{s} = \tilde{s}'$. Then the statements of the theorem immediately follow from (5.3). Thus the proof is completed.

6. Knapp-Wallach's embeddings

6.1. Szegö mapping $\overline{S_{\lambda,\nu}}$. In their paper [5], A. W. Knapp and N. R. Wallach gave certain $(\mathfrak{g}, K)$-homomorphisms of some principal series repsentations onto discrete series for semisimple Lie groups. Here we apply their result to the group G of type G_2 and in the next subsection compare it with our Theorem 5.1.

Let $\mathfrak{g}$ be a complex simple Lie algebra of type G_2 as before. For $\Lambda \in \Xi_c$, define Δ^+, Δ_c^+, Δ_n^+, and λ as in §3. We introduce the notion of "fundamental sequence".

Definition 6.1. A sequence (β_1, β_2) of positive noncompact roots is said to be *fundamental* if it satisfies the following two conditions.

(i) The root β_1 is a simple root in Δ^+, and β_1 and β_2 are strongly orthogonal.

(ii) For $\gamma \in \Delta_n^+$, define $\beta(\gamma)$ as the first β_j in (β_1, β_2) such that γ is not strongly orthogonal to β_j. Then one of the following (ii-a) and (ii-b) holds:

(ii-a) $|\beta(\gamma)| \geq |\gamma|$,

(ii-b) $|\beta(\gamma)| < |\gamma|$ and $\gamma - 3\beta(\gamma) \in \Delta$.

Note that the existence of $\beta(\gamma)$ in (ii) is assured since $\mathbf{R}\beta_1 + \mathbf{R}\beta_2 = \sqrt{-1}\mathfrak{t}_0^*$. For the definition of fundamental sequences in case of a general semisimple $\mathfrak{g}$ and the existence of fundamental sequences, see [5].

For a fundamental sequence (β_1, β_2), put

$$\tilde{H}_{\Lambda,1} = E_{\beta_1} + E_{-\beta_1},$$

$$\tilde{H}_{\Lambda,2} = E_{\beta_2} + E_{-\beta_2},$$

$$(\mathfrak{a}_\Lambda)_0 = \mathbf{R}\tilde{H}_{\Lambda,1} + \mathbf{R}\tilde{H}_{\Lambda,2},$$

where $E_\beta(\beta \in \Delta)$ is the root vector defined in §2. By Definition 6.1, $(\mathfrak{a}_\Lambda)_0$ is a maximal abelian subspace of $\mathfrak{p}_0$. Equip $(\mathfrak{a}_\Lambda)_0^*$ with the lexicographic order with respect to the ordered basis $(\tilde{H}_{\Lambda,1}, \tilde{H}_{\Lambda,2})$. Let Ψ_Λ be the system of the restricted roots of $\mathfrak{g}_0$ relative to $(\mathfrak{a}_\Lambda)_0$, and $(\Psi_\Lambda)^+$ the set of all positive elements in Ψ_Λ. Take a Lie subalgebra $(\mathfrak{n}_\Lambda)_0$ of $\mathfrak{g}_0$ as

$$(\mathfrak{n}_\Lambda)_0 = \sum_{\mu \in (\Psi_\Lambda)^+} (\mathfrak{g}_\mu)_0,$$

where $(\mathfrak{g}_\mu)_0 = \{X \in \mathfrak{g}_0 \mid [H, X] = \mu(H)X (\forall H \in \mathfrak{a}_\Lambda)\}$. Then we have an Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus (\mathfrak{a}_\Lambda)_0 \oplus (\mathfrak{n}_\Lambda)_0$ of $\mathfrak{g}_0$ and the corresponding decomposition $G = KA_\Lambda N_\Lambda$ of G.

Let M_Λ be the centralizer of A_Λ in K and τ_λ the irreducible representation of K with highest weight λ. The representation space of τ_λ is denoted by V_λ. Take a nonzero highest weight vector φ_λ of V_λ and define U_λ to be the M_Λ-cyclic subspace of V_λ generated by φ_λ. Then we have a representation σ_λ of M_Λ on U_λ defined by

$$\sigma_\lambda(m) = \tau_\lambda(m)|_{U_\lambda} \qquad (m \in M_\Lambda).$$

The representation σ_λ is not always irreducible, and let

$$\sigma_\lambda \simeq \bigoplus_j \sigma_\lambda^{(j)}, \qquad U_\lambda = \bigoplus_j U_\lambda^{(j)} \tag{6.1}$$

be an irreducible decomposition of $(\sigma_\lambda, U_\lambda)$.

Now we introduce two function spaces $C^\infty_{\tau_\lambda}(G)$ and $C^\infty_{\sigma_\lambda}(K)$ as follows:

$$C^\infty_{\tau_\lambda}(G) = \{f\colon G \overset{C^\infty}{\to} V_\lambda \mid f(kg) = \tau_k(k)f(g) \quad (\forall (k, g) \in K \times G)\},$$
$$C^\infty_{\sigma_\lambda}(K) = \{f\colon K \overset{C^\infty}{\to} U_\lambda \mid f(mk) = \sigma_\lambda(m)f(k) \quad (\forall (m, k) \in M_\Lambda \times K)\},$$

The definition of $C^\infty_{\tau_\lambda}(G)$ is the same as that in § 1.3. For a linear form ν on $\mathfrak{a}_\Lambda$, we extend each function f in $C^\infty_{\sigma_\lambda}(K)$ to G in the following way:

$$f(nak) = a^\nu f(k) \qquad (n \in N_\Lambda, a \in A_\Lambda, k \in K).$$

Then G acts on these function spaces by the right translation:

$$(g \cdot f)(x) = f(xg) \qquad (g, x \in G).$$

We denote this representation of G on $C^\infty_{\sigma_\lambda}(K)$ by $W(\sigma_\lambda, \nu)$. For each $\sigma_\lambda^{(j)}$ in (6.1), the G-representation $W(\sigma_\lambda^{(j)}, \nu)$ is defined in the same way. Note that the representation $W(\sigma_\lambda^{(j)}, \nu)$ is equivalent to the principal series $\mathrm{Ind}^G_{P_\Lambda}(\sigma_\lambda^{(j)} \otimes e^{\nu-\rho^+} \otimes 1_{N_\Lambda})$ induced from the minimal parabolic subgroup $P_\Lambda = M_\Lambda A_\Lambda N_\Lambda$, where $\rho^+ = \frac{1}{2}\sum_{\mu \in (\Psi_\Lambda)^+} \mu$. In the following, $\mathrm{Ind}^G_P(\sigma_{\varepsilon_1,\varepsilon_2} \otimes e^{\nu-\rho_P} \otimes 1_N)(\sigma_{\varepsilon_1,\varepsilon_2} \in \hat{M}, \nu \in \mathfrak{a}^*)$ is denoted by $W_0(\sigma_{\varepsilon_1,\varepsilon_2}, \nu)$. Here P is the minimal parabolic subgroup defined in § 2.2. For the definition of $\mathrm{Ind}^G_P(\sigma_{\varepsilon_1,\varepsilon_2} \otimes e^{\nu-\rho_P} \otimes 1_N)$, see § 1.3. Moreover $W(\sigma_\lambda, \nu)^0$, $W_0(\sigma_{\varepsilon_1,\varepsilon_2}, \nu)^0$ etc. stand for the $(\mathfrak{g}, K)$-modules of all K-finite vectors in $W(\sigma_\lambda, \nu)$, $W_0(\sigma_{\varepsilon_1,\varepsilon_2}, \nu)$ etc. respectively.

We write an element g of G as $g = \kappa(g)e^{H(g)}n(g)$ with $\kappa(g) \in K$, $H(g) \in (\mathfrak{a}_\Lambda)_0$, $n(g) \in N_\Lambda$, and for a linear form ν on $\mathfrak{a}_\Lambda$, define a function $S_{\lambda,\nu}\colon G \to \mathrm{End}(V_\lambda)$ by

$$S_{\lambda,\nu}(g) = \exp(-\nu(H(g)))\tau_\lambda(\kappa(g)),$$

and put $\overline{S_{\lambda,\nu}}\colon C^\infty_{\sigma_\lambda}(K) \to C^\infty_{\tau_\lambda}(G)$ as

$$(\overline{S_{\lambda,\nu}}f)(g) = \int_K S_{\lambda,\nu}(gk^{-1})f(k)\,dk \qquad (g \in G).$$

This definition of $\overline{S_{\lambda,\nu}}$ is equivalent to that of the operator S in [5].

According to [5], we define a linear form $\nu(\lambda)$ on $\mathfrak{a}_\Lambda$ as

$$\nu(\lambda)(\tilde{H}_{\Lambda,j}) = \lambda(H_{\beta_j}) + 2n_j,$$

where $n_j = |\{\gamma \in \Delta_n^+ \mid \beta(\gamma) = \beta_j \text{ and } \beta_j + \gamma \in \Delta\}|$. Then the following result is shown in [5].

Theorem 6.1 (cf. [5, Theorem A]). *Let notations be as above, then the mapping*

$$\overline{S_{\lambda,\nu(\lambda)}}|_{W(\sigma_\lambda^{(j)}, 2\rho^+ - \nu(\lambda))^0}: W(\sigma_\lambda^{(j)}, 2\rho^+ - \nu(\lambda))^0 \to (\mathrm{Ker}\, \mathscr{D}_\lambda)^0 \tag{6.2}$$

gives a nonzero $(\mathfrak{g}, K)$*-homomorphism and the image* $\overline{S_{\lambda,\nu(\lambda)}} W(\sigma_\lambda^{(j)}, 2\rho^+ - \nu(\lambda))^0$ *is isomorphic to the discrete series* $(\mathfrak{g}, K)$*-module for* π_Λ.

Moreover, if the Blattner parameter λ *of* π_Λ *is far from the walls, the mapping in* (6.2) *is surjective.*

Note that $(\mathrm{Ker}\, \mathscr{D}_\lambda)^0$ realizes the $(\mathfrak{g}, K)$-module of discrete series π_Λ provided that λ is far from the walls.

6.2. Comparison with Theorem 5.1. According as $\Lambda \in \Xi_J$ with $J = I$, II or III, possible fundamental sequence (β_1, β_2) and the data $\mathfrak{a}_\Lambda$, M_Λ, $\sigma_\lambda \simeq \oplus_j \sigma_\lambda^{(j)} \nu_\lambda$ are described explicitly as follows:

Case I: $\Delta^+ = \Delta_I^+$

fundamental sequence: $(\alpha_2, 2\alpha_1 + \alpha_2)$,

$\mathfrak{a}_\Lambda = \mathfrak{a}$, $M_\Lambda = M$,

$\sigma_\lambda \simeq \sigma_{\varepsilon_\lambda,1} \oplus \sigma_{\varepsilon_\lambda,-1}$ with $\varepsilon_\lambda = (-1)^{(r-s)/2}$,

$\nu(\lambda) = \tilde{s}_2 \tilde{s}_1 \tilde{\Lambda} + \rho_P$.

Case II: $\Delta^+ = \Delta_{II}^+$

fundamental sequence: $(-\alpha_2, 2\alpha_1 + \alpha_2)$,

$\mathfrak{a}_\Lambda = \mathfrak{a}$, $M_\Lambda = M$,

$\sigma_\lambda \simeq \sigma_{\varepsilon_\lambda,1} \oplus \sigma_{\varepsilon_\lambda,-1}$ with $\varepsilon_\lambda = (-1)^{(r-s)/2}$,

$\nu(\lambda) = \tilde{s}_2 \tilde{s}_1 \tilde{\Lambda} + \rho_P$.

Case III: $\Delta^+ = \Delta_{III}^+$

fundamental sequence: $(-(\alpha_1 + \alpha_2), 3\alpha_1 + \alpha_2)$,

$\mathfrak{a}_\Lambda = \mathrm{Ad}(m_0)\mathfrak{a}$, $M_\Lambda = m_0 M m_0^{-1}$,

with $m_0 = \left(\frac{1}{2}\begin{pmatrix} -1+\sqrt{-1} & 1+\sqrt{-1} \\ -1+\sqrt{-1} & -1-\sqrt{-1} \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+\sqrt{-1} & 1-\sqrt{-1} \\ -1-\sqrt{-1} & 1-\sqrt{-1} \end{pmatrix}\right)^{\ddagger}$,

$\sigma_\lambda \simeq m_0 \cdot \sigma_{1,\varepsilon'_\lambda} \oplus m_0 \cdot \sigma_{-1,\varepsilon'_\lambda}$ with $\varepsilon'_\lambda = (-1)^{(r+s)/2}$,

$\nu(\lambda) = m_0 \cdot (\tilde{s}_1 \tilde{s}_2 \tilde{\Lambda} + \rho_P)$.

Here $m_0 \cdot \mu(H) = \mu(\mathrm{Ad}(m_0)^{-1}H)$ for $\mu \in \mathfrak{a}^*$ and $H \in \mathfrak{a}_A$, $m_0 \cdot \sigma(m) = \sigma(m_0^{-1}mm_0)$ for a representation σ of M and $m \in m_0Mm_0^{-1}$. Note that for the case of Δ_{III}^+, the isomorphism

$$\mathrm{Ind}_{m_0Pm_0^{-1}}^G(m_0 \cdot \sigma \otimes e^{m_0 \cdot \mu} \otimes 1_N) \simeq \mathrm{Ind}_P^G(\sigma \otimes e^{\mu} \otimes 1_N) \qquad (\sigma \in \hat{M}, \mu \in \mathfrak{a}^*),$$

implies that

$$W(m_0 \cdot \sigma, m_0 \cdot \mu) \simeq W_0(\sigma, \mu) \qquad (\sigma \in \hat{M}, \mu \in \mathfrak{a}^*).$$

Since $W_0(\sigma, 2\rho_P - \mu)^* \simeq W_0(\sigma, \mu)$ for $\sigma \in \hat{M}$ and $\mu \in \mathfrak{a}^*$, the above remarks and Theorem 6.1 imply that π_Λ can be embedded into $\mathrm{Ind}_P^G(\sigma_{\varepsilon_1, \varepsilon_2} \otimes e^{\mu} \otimes 1_N)$ with parameters ε_1, ε_2 and μ listed below:

- **Case I.** $\Delta^+ = \Delta_I^+ : \mu = \tilde{s}_2\tilde{s}_1\tilde{\Lambda}$, $(\varepsilon_1, \varepsilon_2) \in S_{\Lambda_l}(I, \tilde{s}_2\tilde{s}_1)$,
- **Case II.** $\Delta^+ = \Delta_{II}^+ : \mu = \tilde{s}_2\tilde{s}_1\tilde{\Lambda}$, $(\varepsilon_1, \varepsilon_2) \in S_{\Lambda_l}(II, \tilde{s}_2\tilde{s}_1)$,
- **Case III.** $\Delta^+ = \Delta_{III}^+ : \mu = \tilde{s}_1\tilde{s}_2\tilde{\Lambda}$, $(\varepsilon_1, \varepsilon_2) \in S_{\Lambda_l}(III, \tilde{s}_1\tilde{s}_2)$,

These two embeddings for the case of Δ_I^+ (resp. Δ_{II}^+, Δ_{III}^+) appear in the three (resp. thirteen, four) embeddings determined in Theorem 5.1.

Department of Mathematics,
Kyoto University

Department of Mathematics,
Hokkaido University

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