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# Homotopy-commutativity in rotation groups

By

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## 1. Introduction

Assume $G$ is a topological group and $S$, $S'$ are subspaces of $G$, each of which contains the unit as its base point. There is the commutator map $c$ from $S \wedge S'$ to $G$ which maps $(x, y) \in S \wedge S'$ to $xyx^{-1}y^{-1} \in G$. We say $S$ and $S'$ homotopy-commute in $G$ if $c$ is null homotopic.

In this paper, we describe the homotopy-commutativity of the case $G = SO(n+m-1)$, $S = SO(n)$ and $S' = SO(m)$ where $n, m > 1$. Here we use the usual embeddings

$$SO(1) \subset SO(2) \subset SO(3) \subset \cdots .$$

Trivially $SO(n)$ and $SO(m)$ homotopy-commute in $SO(n+m)$. And it is known that if $n+m > 4$, $SO(n)$ and $SO(m)$ do not homotopy-commute in $SO(n+m-2)$. (See [1] and [2].) But the homotopy-commutativity in $SO(n+m-1)$ has not been solved exactly.

We shall say a pair $(n, m)$ is irregular if $SO(n)$ and $SO(m)$ homotopy-commute in $SO(n+m-1)$, and regular if they do not. In [1] the following problem is proposed; "when is $(n, m)$ irregular?", and the next theorem is showed.

**Theorem 1.1** (James and Thomas). *Let $n+m \neq 4, 8$. If $n$ or $m$ is even or if $d(n) = d(m)$ then $(n, m)$ is regular, where $d(q)$, for $q \geq 2$, denotes the greatest power of 2 which devides $q-1$.*

In this paper we shall prove the more strict result as showed in the next theorem.

**Theorem 1.2.** *If $n$ or $m$ is even or if* $\binom{n+m-2}{n-1} \equiv 0 \bmod 2$ *then $(n, m)$ is regular.*

We identify $\mathbf{RP}^{k-1} \overset{i_k}{\hookrightarrow} SO(k)$ by the following way. Let $i'_k\colon \mathbf{RP}^{k-1} \to O(k)$ be the map which attaches a line $l \in \mathbf{RP}^{k-1}$ with $i'_k(l) \in O(k)$ defined by

$$i'_k(l)(v) = v - 2(v, e)e,$$

where $e$ is a unit vector of $l$ and $v \in \mathbf{R}^k$. And let $i_k(l) = i'_k(l_0)^{-1} \cdot i'_k(l)$ where $l_0$ is the base point of $\mathbf{RP}^{k-1}$. Then $i_k$ preserves the base points.

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Theorem 1.2 follows from the next theorem.

**Theorem 1.3.** *Let $n$ and $m$ be odd. $\mathbf{RP}^{n-1} \subset SO(n)$ and $\mathbf{RP}^{m-1} \subset SO(m)$ homotopy-commute in $SO(n+m-1)$ if and only if*

$$\binom{n+m-2}{n-1} \equiv 1 \bmod 2.$$

Let $\mathbf{SO}$ be $\lim_{\to}(SO(1) \subset SO(2) \subset SO(3) \subset \cdots)$ and consider the fibration $SO(n+m-1) \to \mathbf{SO} \to \mathbf{SO}/SO(n+m-1)$. Then we have a sequence of spaces

$$\cdots \to \Omega\mathbf{SO} \xrightarrow{\Omega p} \Omega(\mathbf{SO}/SO(n+m-1)) \xrightarrow{\delta} SO(n+m-1) \xrightarrow{i} \mathbf{SO} \xrightarrow{p} \mathbf{SO}/SO(n+m-1).$$

We can see $i \circ c|_{\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1}} \simeq *: \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \mathbf{SO}$. This means there exists $\lambda: \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega\mathbf{SO}/SO(n+m-1)$ such that $\delta \circ \lambda = c|_{\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1}}$. The construction of $\lambda$ and the cohomology map $\lambda^*$ are argued in §2. We describe about lifts of $\lambda$ in §3 and finally, in §4, we determine when a lift of $\lambda$ exists, which means when $c|_{\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1}} \simeq *$.

## 2. Lift $\lambda$ of $c$

**Definition.** A sequence of spaces $X_i$ and continuous maps $f_i$

$$\cdots \to X_{i+1} \xrightarrow{f_i} X_i \to \cdots \xrightarrow{f_0} X_0$$

is called a fibration sequence if, for any $i \geq 0$, there exists a fibration $Y_i^{(2)} \xrightarrow{j_i} Y_i^{(1)} \xrightarrow{\pi_i} Y_i^{(0)}$, homotopy equivalence maps $\psi_i^{(k)}: X_{i+k} \to Y_i^{(k)}$ $(k = 0, 1, 2)$, and the following diagram commutes upto homotopy.

$$\begin{array}{ccccc} X_{i+2} & \xrightarrow{f_{i+1}} & X_{i+1} & \xrightarrow{f_i} & X_i \\ \simeq \downarrow \psi_i^{(2)} & & \simeq \downarrow \psi_i^{(1)} & & \simeq \downarrow \psi_i^{(0)} \\ Y_i^{(2)} & \xrightarrow{j_i} & Y_i^{(1)} & \xrightarrow{\pi_i} & Y_i^{(0)} \end{array}$$

For example, given a fibration $F \to E \to B$, there is a fibration sequence

$$\cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B.$$

Consider the fibration $\mathbf{SO} \to \mathbf{SO}/SO(n+m-1)$ with the fibre $SO(n+m-1)$. Then we have a fibration sequence.

$$\cdots \to \Omega\mathbf{SO} \xrightarrow{\Omega p} \Omega(\mathbf{SO}/SO(n+m-1)) \xrightarrow{\delta} SO(n+m-1) \xrightarrow{i} \mathbf{SP} \xrightarrow{p} \mathbf{SO}/SO(n+m-1)$$

Obviously $i \circ c: SO(n) \wedge SO(m) \to \mathbf{SO}$ is null homotopic. This means there exists a lift of $c$, that is, a map $\lambda: \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega(\mathbf{SO}/SO(n+m-1))$ such that $\delta \circ \lambda \simeq c$.

In R. Bott [3] it is showed that the following map $\lambda_0: SO(n) \wedge SO(m) \to \Omega(\mathbf{SO}/SO(n+m-1))$ is a lift of $c$.

Recall the fibration $SO(k-1) \to SO(k) \xrightarrow{p_k} S^{k-1}$. Define $h$ as $h = \Sigma(p_n \wedge p_m)$:

$\Sigma(SO(n) \wedge SO(m)) \to \Sigma(S^{n-1} \wedge S^{m-1}) \simeq S^{n+m-1}$. Then ad $h$ is a lift of $c$ in the following fibration sequence. (See [5].)

$$\begin{array}{ccccccc}\cdots \to \Omega SO(n+m) \to \Omega S^{n+m-1} & \to & SO(n+m-1) & \to & SO(n+m) & \to & S^{n+m-1} \\ & \nwarrow^{\text{ad } h} & \uparrow c & & & & \\ & & SO(n) \wedge SO(m) & & & & \end{array}$$

The fibration $SO(n+m) \to S^{n+m-1}$ is the restriction of $\mathbf{SO} \to \mathbf{SO}/SO(n+m-1)$ to $S^{n+m-1} = SO(n+m)/SO(n+m-1) \overset{j}{\hookrightarrow} \mathbf{SO}/SO(n+m-1)$. Therefore we define $\lambda_0$ as $\Omega j \circ \text{ad } h$. Refer to the commutative diagram below.

The rest of this section is devoted to the computation of the cohomology map of $\lambda$. And throughout this paper we use $\mathbf{Z}/2\mathbf{Z}$ as the coefficient ring of cohomology unless mentioned.

$$\begin{array}{ccccc} & & & & \Omega\mathbf{SO} \\ & & & & \downarrow \Omega p \\ & & \Omega S^{n+m-1} & \xrightarrow{\Omega j} & \Omega(\mathbf{SO}/SO(n+m-1)) \\ & \nearrow^{\text{ad } h} & \downarrow & & \downarrow \delta \\ SO(n) \wedge SO(m) & \longrightarrow & SO(n+m-1) & = & SO(n+m-1) \\ \uparrow & & \downarrow & & \downarrow i \\ \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} & & SO(n+m) & \longrightarrow & \mathbf{SO} \\ & & \downarrow & & \downarrow p \\ & & S^{n+m-1} & \xrightarrow{j} & \mathbf{SO}/SO(n+m-1) \end{array}$$

First we refer to the cohomology rings of spaces which are used in this paper, that is,

$$\mathrm{H}^*(\Omega_0\mathbf{SO}) = \mathbf{Z}/2\mathbf{Z}[\alpha_2, \alpha_4, \alpha_6, \ldots]/(\alpha_{4k} - \alpha_{2k}^2),$$

$$\mathrm{H}^*(\Omega(\mathbf{SO}/SO(n+m-1))) = \mathbf{Z}/2\mathbf{Z}[\alpha'_{n+m-2}, \alpha'_{n+m}, \ldots]/(\alpha'_{4k} - \alpha'^2_{2k}),$$

$$\mathrm{H}^*(SO(k)) = \Delta(x_1, \ldots, x_{k-1}),$$

$$\mathrm{H}^*(SO(k)/SO(k-1)) = \Delta(x'_{k-l}, \ldots, x'_{k-1}),$$

where $\deg(\alpha_{2i}) = 2i$, $\deg(\alpha'_{2i}) = 2i$, $\deg(x_i) = i$. And also

$$\Omega p^*(\alpha'_k) = \alpha_k.$$

**Lemma 2.4.** $\lambda_0^*(\alpha'_{n+m-2}) = x_{n-1} \otimes x_{m-1}$.

*Proof.* Consider the fibration $p_k\colon SO(k) \to S^{k-1}$ with the fibre $SO(k-1)$. Let

$c_i$ be the generator of $\mathrm{H}^i(S^i)$. Then $p_k^*(c_{k-1}) = x_{k-1}$. Thus we have

$$h^*(c_{n+m-1}) = \Sigma(p_n \wedge p_m)^*(\Sigma c_{n-1} \otimes c_{m-1})$$
$$= \Sigma(x_{n-1} \otimes x_{m-1}).$$

Hence $(\mathrm{ad}\ h)^*(\sigma c_{n+m-1}) = x_{n-1} \otimes x_{m-1}$, where $\sigma$ is the cohomology suspension $\sigma$: $\mathrm{H}^{*+1}(X) \to \mathrm{H}^*(\Omega X)$.

On the other hand, $j^*(x_{n+m-1}) = c_{n+m-1}$ means

$$(\Omega j)^*(\alpha'_{n+m-2}) = (\Omega j)^*(\sigma x'_{n+m-1})$$
$$= \sigma c_{n+m-1}.$$

Therefore it follows that

$$\lambda_0^*(\alpha'_{n+m-2}) = (\mathrm{ad}\ h)^*(\Omega j)^*(\alpha'_{n+m-2})$$
$$= x_{n-1} \otimes x_{m-1}.$$

Q.E.D.

Now let $\lambda = \lambda_0 \circ (i_m \wedge i_n)$: $\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega(\mathbf{SO}/SO(n+m-1))$ and in the following we use $c$ as the commutator map from $\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1}$ to $SO(n+m-1)$. Easily we have $i_k^*(x_{k-1}) = \tau^{k-1}$ where $\tau$ means the generator of $H^1(\mathbf{RP}^{k-1})$. (See Whitehead [4].) Thus

$$\lambda^*(\alpha'_{n+m-2}) = (i_m \wedge i_n)^* \circ \lambda_0^*(\alpha'_{n+m-2})$$
$$= \tau^{n-1} \otimes \tau^{m-1}.$$

## 3. Lift of $\lambda$ and homotopy commutativity

In this section we prove the next theorem.

**Theorem 3.5.** *Let n, m be odd.*

1. $c \simeq *$ *if and only if there exists a lift of* $\lambda$, *that is, a map* $x$: $\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega_0(\mathbf{SO})$ *such that* $\lambda = \Omega p \circ x$.
2. $c \simeq *$ *if and only if there exists* $x$: $\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega_0(\mathbf{SO})$ *such that* $x^*(\alpha_{n+m-2}) \simeq \tau^{n-1} \otimes \tau^{m-1}$.

*Proof.* 1. The sequence

$$\cdots \to \Omega_0(\mathbf{SO}) \xrightarrow{\Omega p} \Omega(\mathbf{SO}/SO(n+m-1)) \xrightarrow{\delta} SO(n+m-1)$$

is a fibration sequence and $\lambda$ is a lift of $c$. Therefore the statement follows.

2. By the first statement it is sufficient to prove that $x$ is a lift of $\lambda$ if and only if $x^*(\alpha_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1}$. We need the following lemma.

**Lemma 3.6.** *Let n and m be odd. Then*

$$\pi_i(\mathbf{SO}/SO(n+m-1)) = \begin{cases} 0 & i \leq n+m-2 \\ \mathbf{Z}/2\mathbf{Z} & i = n+m-1 \end{cases}$$

*Proof.* Consider the fibration

$$SO(n+m+1)/SO(n+m-1) \to \mathbf{SO}/SO(n+m-1) \to \mathbf{SO}/SO(n+m+1)$$

and see the homotopy exact sequence. Remark that $\pi_i(\mathbf{SO}/SO(2k+1)) = 0$ for $i \leq 2k$ and we obtain

$$\pi_{n+m-1}(\mathbf{SO}/SO(n+m-1)) = \pi_{n+m-1}(SO(n+m+1)/SO(n+m-1)).$$

It is known that $\pi_{n+m-1}(SO(n+m+1)/SO(n+m-1)) = \mathbf{Z}/2\mathbf{Z}$ provided $n+m-1$ is odd. Hence we obtained the statement. Q.E.D.

By Lemma 3.6 it follows that

$$\pi_i(\Omega(\mathbf{SO}/SO(n+m-1))) = \begin{cases} 0 & i \leq n+m-3 \\ \mathbf{Z}/2\mathbf{Z} & i = n+m-2. \end{cases}$$

Now add cells $e_i$ $(i \geq 1)$ to $\Omega(\mathbf{SO}/SO(n+m-1))$ so that $\pi_k(\Omega(\mathbf{SO}/SO(n+m-1))$ vanishes for $k \geq n+m-1$, where $\dim e_i \geq n+m$. We shall call the obtained space $K$, that is,

$$\Omega(\mathbf{SO}/SO(n+m-1)) \hookrightarrow \Omega(\mathbf{SO}/SO(n+m-1)) \cup e_1 \cup e_2 \cup \cdots = K \tag{1}$$

and

$$\pi_i(K) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & i = n+m-2 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Thus $K$ is an Eilenberg-Maclane space $K(\mathbf{Z}/2\mathbf{Z}; n+m-2)$. Let $\gamma$ denote the inclusion map from $\Omega(\mathbf{SO}/SO(n+m-1))$ to $K$. Here

$$\gamma_*\colon \pi_{n+m-2}(\Omega(\mathbf{SO}/SO(n+m-1))) \to \pi_{n+m-2}(K)$$

is not a 0-map. This means that by the isomorphism

$$\Omega(\mathbf{SO}/SO(n+m-1)), K] \cong \mathrm{H}^{n+m-2}(\Omega(\mathbf{SO}/SO(n+m-1)))$$

$\gamma$ corresponds to $\alpha'_{n+m-2}$, that is, $\gamma^* u = \alpha'_{n+m-2}$ where $u$ is the generator of $\mathrm{H}^{n+m-2}(K)$.

On the other hand, (1) and (2) imply that $\gamma_*\colon \pi_i(\Omega(\mathbf{SO}/SO(n+m-2))) \to \pi_i(K)$ is isomorphic for $i \leq n+m-2$ and epic for $i \geq n+m-1$. Then by Whitehead's theorem

$$\begin{aligned}[\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1}, \Omega(\mathbf{SO}/SO(n+m-1))] &\cong [\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1}, K] \\ &\cong \mathrm{H}^{n+m-2}(\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1}).\end{aligned}$$

Thus maps $f$ and $g\colon \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega(\mathbf{SO}/SO(n+m-2))$ are homotopic if and only if $f^*(\alpha'_{n+m+-2}) = g^*(\alpha'_{n+m-2})$.

Now we assume $x\colon \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega(\mathbf{SO}/SO(n+m-1))$ satisfies that $x^*(\alpha_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1}$. Then

$$(\Omega p \circ x)^*(\alpha'_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1}.$$

By §2 $\lambda^*(\alpha'_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1}$. Thus we obtain $\Omega p \circ x \simeq \lambda$ and $x$ is a lift of $\lambda$.

The inverse is trivial and the proof of theorem 3.5 is finished.

## 4. Existence of lift of $\lambda$

In this section we prove the next theorem which completes the proof of Theorem 1.3.

**Theorem 4.7.** *Let $n$ and $m$ be odd. There exists a map $x$: $\mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega_0(\mathbf{SO})$ such that $x^*(\alpha_{n+m-2}) = \tau^{n+1} \otimes \tau^{m-1}$ if and only if*

$$\binom{n+m-2}{n-1} \equiv 1 \bmod 2.$$

*Proof.* First consider

$$\theta := (r_1 - 1) \hat{\otimes} (r_1 - 1) \hat{\otimes} (r_\infty - 1) \hat{\otimes} (r_\infty - 1) \in \widetilde{KO}(\Sigma^2(\mathbf{RP}^\infty \wedge \mathbf{RP}^\infty)).$$

Here $r_1$ is the Möbius line bundle over $S^1$ and $r_\infty$ is the canonical line bundle over $\mathbf{RP}^\infty$. Now we compute the total Stiefel Whitney class of $\theta$. We start from the next lemma.

**Lemma 4.8.** *Let $A = 1 + a_1 + a_2 + \cdots \in \mathrm{H}^{**}(\mathbf{RP}^\infty \times \mathbf{RP}^\infty)$ where $a_i \in \mathrm{H}^i(\mathbf{RP}^\infty \times \mathbf{RP}^\infty)$ and let $s_i \in \mathrm{H}^*(S^1 \times S^1 \times \mathbf{RP}^\infty \times \mathbf{RP}^\infty)$ $(i = 1, 2)$ be the pull back of the generator of $\mathrm{H}^1(S^1)$ by the canonical projection from $S^1 \times S^1 \times \mathbf{RP}^\infty \times \mathbf{RP}^\infty$ to the ith factor $S^1$. Then we have*

$$\frac{(A + s_1 + s_2)A}{(A + s_1)(A + s_2)} = \frac{A^2 + s_1 s_2}{A^2} \in \mathrm{H}^{**}(S^1 \times S^1 \times \mathbf{RP}^\infty \times \mathbf{RP}^\infty).$$

*Proof.* By direct computation, we see

$$\begin{aligned}\frac{(A + s_1 + s_2)A}{(A + s_1)(A + s_2)} &= \frac{\{(A + s_1)^2 + (A + s_1)s_2\}A}{(A + s_1)^2(A + s_2)} \\ &= \frac{(A^2 + s_2 A + s_1 s_2)A}{A^2(A + s_2)} \\ &= \frac{A(A + s_2)^2 + (A + s_2)s_1 s_2}{A(A + s_2)^2} \\ &= \frac{A^2 + s_1 s_2}{A^2}.\end{aligned}$$

Q.E.D.

Let $\pi$: $S^1 \times S^1 \times \mathbf{RP}^\infty \to \Sigma^2(\mathbf{RP}^\infty \wedge \mathbf{RP}^\infty)$ be the canonical projection and decompose $\pi^*\theta$ as

$$\pi^*\theta = r_1 \times r_1 \times r_\infty \times r_\infty + 1 \times 1 \times r_\infty \times r_\infty - 1 \times r_1 \times r_\infty \times r_\infty$$
$$- r_1 \times 1 \times r_\infty \times r_\infty - r_1 \times r_1 \times 1 \times r_\infty - 1 \times 1 \times 1 \times r_\infty$$
$$+ 1 \times r_1 \times 1 \times r_\infty + r_1 \times 1 \times 1 \times r_\infty - r_1 \times r_1 \times r_\infty \times 1$$
$$- 1 \times 1 \times r_\infty \times 1 + 1 \times r_1 \times r_\infty \times 1 + r_1 \times 1 \times r_\infty \times 1 + r_1 \times r_1 \times 1 \times 1$$
$$+ 1 \times 1 \times 1 \times 1 - 1 \times r_1 \times 1 \times 1 - r_1 \times 1 \times 1 \times 1.$$

Then the total Stiefel Whitney class $w(\pi^*\theta)$ of $\pi^*\theta$ is given by

$$w(\pi^*\theta) = \frac{(1 + \tau_1 + \tau_2 + s_1 + s_2)(1 + \tau_1 + \tau_2)}{(1 + \tau_1 + \tau_2 + s_1)(1 + \tau_1 + \tau_2 + s_2)} \cdot \frac{(1 + s_1 + s_2)}{(1 + s_1)(1 + s_2)}$$
$$\cdot \left\{ \frac{(1 + \tau_1 + s_1 + s_2)(1 + \tau_1)}{(1 + \tau_1 + s_1)(1 + \tau_1 + s_2)} \cdot \frac{(1 + \tau_2 + s_1 + s_2)(1 + \tau_2)}{(1 + \tau_2 + s_1)(1 + \tau_2 + s_2)} \right\}^{-1}.$$

Here $\tau_i$ $(i = 1, 2)$ is the pull back of the generator of the cohomology ring of the $i$th factor of $\mathbf{RP}^\infty \times \mathbf{RP}^\infty$. By the previous lemma, we obtain

$$w(\pi^*\theta) = \frac{1 + \tau_1^2 + \tau_2^2 + s_1 s_2}{1 + \tau_1^2 + \tau_2^2} \cdot (1 + s_1 s_2) \cdot \left( \frac{1 + \tau_1^2 + s_1 s_2}{1 + \tau_1^2} \right)^{-1} \cdot \left( \frac{1 + \tau_2^2 + s_1 s_2}{1 + \tau_2^2} \right)^{-1}$$
$$= \{1 + (1 + \tau_1^2 + \tau_2^2)^{-1} s_1 s_2\}(1 + s_1 s_2)\{1 + (1 + \tau_1^2)^{-1} s_1 s_2\}$$
$$\cdot \{1 + (1 + \tau_2^2)^{-1} s_1 s_2\}$$
$$= 1 + s_1 s_2\{(1 + \tau_1^2 + \tau_2^2)^{-1} + 1 + (1 + \tau_1^2)^{-1} + (1 + \tau_2^2)^{-1}\}$$
$$= 1 + s_1 s_2 \left\{ \sum_{i=0}^{\infty} (\tau_1^2 + \tau_2^2)^i + 1 + \sum_{i=0}^{\infty} \tau_1^{2i} + \sum_{i=0}^{\infty} \tau_2^{2i} \right\}$$
$$= 1 + s_1 s_2 \left\{ \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \binom{i}{j} \tau_1^{2j} \tau_2^{2i-2j} \right\}.$$

Therefore we see

$$w(\theta) = 1 + \Sigma^2 \left\{ \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \binom{i}{j} \tau^{2j} \otimes \tau^{2i-2j} \right\}.$$

Let $f$ be the classifying map of $\theta$, that is, the map

$$f\colon \Sigma^2(\mathbf{RP}^\infty \wedge \mathbf{RP}^\infty) \to \mathbf{BSO}$$

such that $f^*(\xi) = \theta$ where $\xi = \lim_{n\to\infty}(\xi_n - n)$ and $\xi_n$ is the universal $SO(n)$ vector bundle over $\mathbf{BSO}(n)$.

It is known that $\mathrm{H}^*(\mathbf{BSO}) = \mathbf{Z}/2\mathbf{Z}[w_1, w_2, \ldots]$ where $w_i$ is the $i$th Stiefel Whitney class. Let $\iota_k\colon \mathbf{RP}^k \to \mathbf{RP}^\infty$ be the inclusion map and let

$$x_0 := (\mathrm{ad}^2 f) \circ (\iota_{n-1} \wedge \iota_{m-1})\colon \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega\mathbf{SO}.$$

Then it follows that for $N \geq 1$

$$x_0^*(\alpha_{2N}) = (\iota_{n-1} \wedge \iota_{m-1})^*(\mathrm{ad}^2 f)^*\sigma^2 w_{2N+2}$$
$$= (\iota_{n-1} \wedge \iota_{m-1})^* \left( \sum_{j=1}^{N-1} \binom{2N}{2j} \tau^{2j} \otimes \tau^{2N-2j} \right).$$

Particularly $x_0^*(\alpha_{n+m-2}) = \binom{n+m-2}{n-1} \tau^{n-1} \otimes \tau^{m-1}$. Thus if $\binom{n+m-2}{n-1} \equiv 1$ then there exists $x_0\colon \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega\mathbf{SO}$ such that $x_0^*(\alpha_{n+m-2}) = \tau^{n-1} \otimes \tau^{m-1}$.

Now we shall prove the inverse, that is, prove that if $\binom{n+m-2}{n-1} \equiv 0 \bmod 2$ then $x^*(\alpha_{n+m-2}) = 0$ for any $x\colon \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega\mathbf{SO}$. Let $n = 2a + 1$, $m = 2b + 1$ where $a, b \in \mathbf{Z}$, $a, b \geq 1$. Moreover we set $a \leq b$.

Here we use the Steenrod's square operators $\mathrm{Sq}^i$. In $\mathrm{H}^*(\Omega_0\mathbf{SO})$, $\mathrm{Sq}^i$ acts as follows

$$\mathrm{Sq}^i(\alpha_{2j}) = \begin{cases} \binom{2j+1}{i} \alpha_{2j+i} & i \text{ is even} \\ 0 & i \text{ is odd.} \end{cases}$$

Let $x\colon \mathbf{RP}^{2a} \wedge \mathbf{RP}^{2b} \to \Omega_0\mathbf{SO}$ be an arbitrary map.

**Lemma 4.9.** *We set $a$, $b$, $x$ as above then*

$$x^*(\alpha_2) = 0 \quad \textit{and} \quad x^*(\alpha_6) = \tau^2 \otimes \tau^4 + \tau^4 \otimes \tau^2 \quad \textit{or} \quad 0.$$

*Proof.* Since $x^*(\alpha_2) \in \mathrm{H}^*(\mathbf{RP}^{2a} \wedge \mathbf{RP}^{2b})$, $x^*(\alpha_2) = \tau \otimes \tau$ or $0$. If $x^*(\alpha_2) = \tau \otimes \tau$, then we have

$$\mathrm{Sq}^1\, x^*(\alpha_2) = \tau^2 \otimes \tau + \tau \otimes \tau^2.$$

On the other hand,

$$\mathrm{Sq}^1\, x^*(\alpha_2) = x^*(\mathrm{Sq}^1\, \alpha_2) = 0.$$

Therefore $x^*(\alpha_2) = 0$.

Next we consider $x^*(\alpha_6)$. If $(a, b) = (1, 1)$ then $x^*(\alpha_6) = 0$, and if $(a, b) = (1, 2)$ we can see $x^*(\alpha_6) = \tau^2 \otimes \tau^4$ or $0$ as asserted. And otherwise, set

$$x^*(\alpha_6) = \rho_1 \tau \otimes \tau^5 + \rho_2 \tau^2 \otimes \tau^4 + \rho_3 \tau^3 \otimes \tau^3 + \rho_4 \tau^4 \otimes \tau^2 + \rho_5 \tau^5 \otimes \tau^1,$$

where $\rho_i \in \mathbf{Z}/2\mathbf{Z}$ and the statement follows the next two equations.

$$\mathrm{Sq}^1\, x^*(\alpha_6) = x^*(\mathrm{Sq}^1\, \alpha_6) = 0$$
$$\mathrm{Sq}^2\, x^*(\alpha_6) = x^*(\alpha_8) = x^*(\alpha_2)^4 = 0 \qquad \text{Q.E.D.}$$

Remark that if $2(a + b) = 2^d - 2$ for some $d \in \mathbf{N}$, then $\binom{2(a+b)}{2i} \equiv 1 \bmod 2$ for any $i \in \mathbf{Z}$ such that $0 \leq i \leq a + b$. And also when $2(a + b) = 2^d$ for some $d \in \mathbf{N}$,

$$\binom{2(a+b)}{2i} \equiv \begin{cases} 1 \bmod 2 & i = 0 \text{ or } a+b \\ 0 \bmod 2 & \text{otherwise.} \end{cases}$$

In this case

$$\begin{aligned} x^*(\alpha_{2(a+b)}) &= x^*(\alpha_{2^d}) \\ &= x^* \text{ (a power of } \alpha_2) \\ &= 0 \end{aligned}$$

as asserted. Hence we can assume that $2(a+b) \neq 2^k$ or $2^k - 2$ for any $k \in \mathbf{N}$.

Next we shall prove the next theorem.

**Theorem 4.10.** *Let a, b and x be as above. If* $x^*(\alpha_6) = 0$ *then* $x^*(\alpha_{2(a+b)}) = 0$.

*Proof.* Let $d$ be the number which satisfies

$$2^d < 2(a+b) < 2^{d+1} - 2 \qquad d \in \mathbf{N}. \quad (d \geq 3)$$

We distinguish between the following two cases.

I)

$$2^d < 2(a+b) < 3 \cdot 2^{d-1} - 2 \tag{3}$$

II)

$$3 \cdot 2^{d-1} - 2 \leq 2(a+b) < 2^{d+1} - 2 \tag{4}$$

**Lemma 4.11.** *Let a, b and x be as above. In any of the case* I) *and* II), *if* $x^*(\alpha_6) = 0$ *then one of the following holds.*

i) $x^*(\alpha_{2^k-2}) = 0$ *for* $3 \leq k \leq d-1$.

ii) $2a = 2^r - 2$ *for some* $r \in \mathbf{N}$, $r \leq d-1$ *and*

$$x^*(\alpha_{2^k-2}) = \begin{cases} 0 & 3 \leq k \leq r \\ \tau^{2^r-2} \otimes \tau^{2^k-2^r} & r+1 \leq k \leq d-1. \end{cases}$$

*Proof.* We use induction, that is, we prove the next two propositions.

a) If $x^*(\alpha_{2^{k-1}-2}) = 0$ and $4 \leq k \leq d-1$, then one of the followings holds.
   - $x^*(\alpha_{2^k-2}) = 0$.
   - $2a = 2^{k-1} - 2$ and $x^*(\alpha_{2^k-2}) = \tau^{2^{k-1}-2} \otimes \tau^{2^{k-1}}$.

b) If $2a = 2^r - 2$ and $x^*(\alpha_{2^{k-1}-2}) = \tau^{2^r-2} \otimes \tau^{2^{k-1}-2^r}$ and $r+2 \leq k \leq d-1$, then

$$x^*(\alpha_{2^k-2}) = \tau^{2^r-2} \otimes \tau^{2^k-2^r}.$$

First we assume $4 \leq k \leq d-1$ and $x^*(\alpha_{2^{k-1}-2}) = 0$ and prove a). Let

$$x^*(\alpha_{2^k-2}) = \sum_{i=s}^{t} \rho_i \tau^i \otimes \tau^{(2^k-2)-i},$$

where

$$s = \max\{1, (2^k - 2) - 2b\},$$

$$t = \min\{2^k - 3, 2a\},$$

$$\rho_i \in \mathbf{Z}/2\mathbf{Z}.$$

Since $\mathrm{Sq}^1(x^*(\alpha_{2^k-2})) = x^*(\mathrm{Sq}^1 \alpha_{2^k-2}) = 0$, we have that

$$\begin{aligned}
&\mathrm{Sq}^1\left(\sum_{i=s}^{t} \rho_i \tau^i \otimes \tau^{(2^k-2)-i}\right) \\
&= \sum_{s \le i \le t,\, i:\text{odd}} \rho_i(\tau^{i+1} \otimes \tau^{(2^k-2)-i} + \tau^i \otimes \tau^{(2^k-2)-i+1}) \\
&= \sum_{s \le i \le t,\, i:\text{odd}} \rho_i(\tau^i \otimes \tau^{(2^k-2)-i+1}) + \sum_{s+1 \le i \le t,\, i:\text{even}} \rho_{i-1}(\tau^i \otimes \tau^{(2^k-2)-i+1}) \\
&= 0.
\end{aligned}$$

Here, $\tau^i \otimes \tau^{(2^k-2)-i+1} \neq 0$ for $s + 1 \le i \le t$. Therefore

$$\rho_i = 0 \qquad \text{for } i\text{: odd}, \qquad s \le i \le t. \tag{5}$$

Next we use $\mathrm{Sq}^2$. By (5) we can set

$$x^*(\alpha_{2^k-2}) = \sum_{i=s'}^{t'} \rho_{2i}\tau^{2i} \otimes \tau^{(2^k-2)-2i},$$

$$\text{where} \quad s' = \max\left\{1, \frac{2^k-2}{2} - b\right\}$$

$$t' = \min\left\{\frac{2^k-4}{2}, a\right\}.$$

Since

$$\begin{aligned}
\mathrm{Sq}^2 x^*(\alpha_{2^k-2}) &= x^*(\mathrm{Sq}^2 \alpha_{2^k-2}) \\
&= x^*(\alpha_{2^k}) \\
&= x^*(\alpha_2^{2^{k-1}}) \\
&= 0,
\end{aligned}$$

we have

$$\begin{aligned}
&\mathrm{Sq}^2\left(\sum_{i=s'}^{t'} \rho_{2i}\tau^{2i} \otimes \tau^{(2^k-2)-2i}\right) \\
&= \sum_{s' \le 2j \le t'} \rho_{4j}\,\mathrm{Sq}^2(\tau^{4j} \otimes \tau^{(2^k-2)-4j}) + \sum_{s' \le 2j-1 \le t'} \rho_{4j-2}\,\mathrm{Sq}^2(\tau^{4j-2} \otimes \tau^{(2^k-2)-4j+2}) \\
&= \sum_{s' \le 2j \le t'} \rho_{4j}\tau^{4j} \otimes \tau^{2^k-4j} + \sum_{s' \le 2j-1 \le t'} \rho_{4j-2}\tau^{4j} \otimes \tau^{2^k-4j} = 0
\end{aligned} \tag{6}$$

Here $\tau^{4j} \otimes \tau^{2^k-4j} \neq 0$ for $s' + 1 \le 2j \le t'$. Thus

$$\rho_{4j} = \rho_{4j-2} \qquad \text{for } s' + 1 \le 2j \le t'. \tag{7}$$

Next we consider $\mathrm{Sq}^4$. Since

$$\begin{aligned}\mathrm{Sq}^4\, x^*(\alpha_{2^k-2}) &= x^*(\mathrm{Sq}^4\, \alpha_{2^k-2}) \\ &= x^*(\alpha_{2^k+2}) \\ &= x^*(\mathrm{Sq}^{2^{k-1}}\, \mathrm{Sq}^4\, \alpha_{2^{k-1}-2}) \\ &= \mathrm{Sq}^{2^{k-1}}\, \mathrm{Sq}^4\, x^*(\alpha_{2^{k-1}-2}) \\ &= 0,\end{aligned}$$

we have that

$$\begin{aligned}&\mathrm{Sq}^4\left(\sum_{i=s'}^{t'} \rho_{2i}\tau^{2i} \otimes \tau^{(2^k-2)-2i}\right) \\ &= \mathrm{Sq}^4\left(\sum_{s'\le 4j\le t'} \rho_{8j}\tau^{8j} \otimes \tau^{(2^k-2)-8j} + \sum_{s'\le 4j-1\le t'} \rho_{8j-2}\tau^{8j-2} \otimes \tau^{(2^k-2)-8j+2}\right. \\ &\quad \left. + \sum_{s'\le 4j-2\le t'} \rho_{8j-4}\tau^{8j-4} \otimes \tau^{(2^k-2)-8j+4} + \sum_{s'\le 4j+1\le t'} \rho_{8j+2}\tau^{8j+2} \otimes \tau^{(2^k-2)-8j-2}\right) \\ &= \sum_{s'\le 4j\le t'} \rho_{8j}\tau^{8j} \otimes \tau^{2^k+2-8j} + \sum_{s'\le 4j-1\le t'} \rho_{8j-2}\tau^{8j+2} \otimes \tau^{2^k-8j} \\ &\quad + \sum_{s'\le 4j-2\le t'} \rho_{2j-4}\tau^{8j} \otimes \tau^{2^k+2-8j} + \sum_{s'\le 4j+1\le t'} \rho_{8j+2}\tau^{8j+2} \otimes \tau^{2^k-8j} \\ &= 0. \end{aligned} \tag{8}$$

Thus

$$\begin{cases} \rho_{8j} = \rho_{8j-4} & \text{for } s'+2 \le 4j \le t' \\ \rho_{8j-2} = \rho_{8j+2} & \text{for } s'+1 \le 4j \le t'-1 \end{cases} \tag{9}$$

We set $A$ as the set $\{i \in \mathbf{N} \mid s' \le i \le t'\}$. (7) and (9) mean that

$$2i,\, 2i-1 \in A \quad \text{then} \quad \rho_{4i-2} = \rho_{4i}, \tag{10}$$

$$4i,\, 4i-2 \in A \quad \text{then} \quad \rho_{8i} \sim \rho_{8i-4}, \tag{11}$$

$$4i-1,\, 4i+1 \in A \quad \textit{then} \quad \rho_{8i-2} = \rho_{8i+2}. \tag{12}$$

Therefore, for $i \in A - \{s', t'-1, t'\}$, $\rho_{2i} = \rho_{2i+2}$. The reason is this: if $i$ is odd, it is trivial from (10); if $i = 4j$ for some $j$, $\rho_{8j} = \rho_{8j-2} = \rho_{8j+2}$; if $i = 4j$ for some $j$, $\rho_{8j-4} = \rho_{8j} = \rho_{8j-2}$.

We obtain that

$$\rho_{2s'+2} = \rho_{2s'+4} = \cdots = \rho_{2t'-2}.$$

Also, we see

$$2b \ge a + b > 2^{d-1} \quad \text{and} \quad \frac{2^k-2}{2} - b \le \frac{2^{d-1}-2}{2} - 2^{d-2} < 1 \tag{13}$$

and we have

$$s' = \max\left\{1, \frac{2^k-2}{2} - b\right\} = 1.$$

We see again (8) and look into the term of $\tau^2 \otimes \tau^{2^k}$, then we have that $\rho_2 = 0$ and from (10) $\rho_2 = \rho_4$. Hence we have

$$0 = \rho_2 = \rho_4 = \cdots = \rho_{2t'-2},$$

that is,

$$x^*(\alpha_{2^k-2}) = \rho_{2t'}\tau^{2t'} \otimes \tau^{(2^k-2)-2t'}. \tag{14}$$

If $2a \geq 2^k - 4$ then we have

$$t' = \min\left\{\frac{2^k-4}{2}, a\right\} = 2^{k-1} - 2$$

and from (10)

$$\rho_{2t'-2} = \rho_{2t'},$$

that is,

$$x^*(\alpha_{2^k-2}) = 0.$$

Therefore we can assume

$$2a < 2^k - 4, \tag{15}$$

that is, $t' = a$. Here if $2a = 2^{k-1} - 2$, then by (14) $x^*(\alpha_{2^k-2}) = \tau^{2^{k-1}-2} \otimes \tau^{2^{k-1}}$ or 0 as asserted. Hence what we have to prove is that if $2a \neq 2^{k-1} - 2$ then $\rho_{2t'} = 0$.

We set $p(2a)$ so that $2^{p(2a)}$ is the greatest power of 2 which devides $2a + 2$.

Let $p := p(2a)$. We remark that $p \leq k - 2$ since, if it were not, by (15) $2a = 2^{k-1} - 2$. Using $\mathrm{Sq}^{2^p}$, we see

$$\begin{aligned}\mathrm{Sq}^{2^p} x^*(\alpha_{2^k-2}) &= x^*(\alpha_{2^k+2^p-2}) \\ &= \mathrm{Sq}^{2^{k-1}} \mathrm{Sq}^{2^p} x^*(\alpha_{2^{k-1}-2}) \\ &= 0.\end{aligned}$$

Thus it follows that

$$\begin{aligned}\mathrm{Sq}^{2^p}(\rho_{2t'}\tau^{2a} \otimes \tau^{(2^k-2)-2a}) &= \rho_{2t'}\tau^{2a} \otimes \mathrm{Sq}^{2^p}\tau^{2^k-2-2a} \\ &= \rho_{2t'}\tau^{2a} \otimes \tau^{2^k+2^p-2-2a} \\ &= 0\end{aligned}$$

Here $\tau^{2a} \otimes \tau^{2^k+2^p-2-2a} \neq 0$ since by (3) and (4)

$$\begin{aligned} 2b &> 2^d - 2a \\ &\geq 2 \cdot 2^k - 2a \\ &> 2^k + 2^p - 2 - 2a. \end{aligned} \tag{16}$$

Thus $\rho_{2t'} = 0$, that is, $x^*(\alpha_{2^k-2}) = 0$ as asserted.

Next we shall prove b). Let $x^*(\alpha_{2^{k-1}-2}) = \tau^{2r-2} \otimes \tau^{2^{k-1}-2^r}$, $r + 2 \leq k \leq d - 1$ and $2a = 2^r - 2$. Then

$$\begin{aligned} \mathrm{Sq}^i\, x^*(\alpha_{2^{k-1}-2}) &= \tau^{2r-2} \otimes \mathrm{Sq}^i\,(\tau^{2^{k-1}-2^r}) \\ &= \binom{2^{k-1} - 2^r}{i} \tau^{2r-2} \otimes \tau^{2^{k-1}-2^r+i}. \end{aligned}$$

Here we remark that $r \geq 2$. For, if $r = 2$, by a) $x^*(\alpha_{2^i-2}) = 0$ for $3 \leq i \leq d - 1$. Thus $\mathrm{Sq}^4\, x^*(\alpha_{2^{k-1}-2}) = 0$ and we obtain

$$\begin{aligned} \mathrm{Sq}^1\,(x^*(\alpha_{2^k-2})) &= x^*(\mathrm{Sq}^1\, \alpha_{2^k-2}) = 0, \\ \mathrm{Sq}^2\,(x^*(\alpha_{2^k-2})) &= x^*(\alpha_2^{2^{k-1}}) = 0, \\ \mathrm{Sq}^4\,(x^*(\alpha_{2^k-2})) &= \mathrm{Sq}^{2^{k-1}}\, \mathrm{Sq}^4\, x^*(\alpha_{2^{k-1}-2}) = 0. \end{aligned}$$

Then it follows from the previous argument in a) that

$$x^*(\alpha_{2^k-2}) = \rho\tau^{2r-2} \otimes \tau^{2^k-2^r},$$

where $\rho \in \mathbf{Z}/2\mathbf{Z}$.

Next using $\mathrm{Sq}^{2^r}$, we have

$$\begin{aligned} \mathrm{Sq}^{2^r}\, x^*(\alpha_{2^k-2}) &= \rho\, \mathrm{Sq}^{2^r}\,(\tau^{2r-2} \otimes \tau^{2^k-2^r}) \\ &= \rho\tau^{2r-2} \otimes \tau^{2^k}, \end{aligned}$$

while

$$\begin{aligned} \mathrm{Sq}^{2^r}\, x^*(\alpha_{2^k-2}) &= x^*(\alpha_{2^k+2^r-2}) \\ &= x^*(\mathrm{Sq}^{2^{k-1}}\, \mathrm{Sq}^{2^r}\, \alpha_{2^{k-1}-2}) \\ &= \mathrm{Sq}^{2^{k-1}}\, \mathrm{Sq}^{2^r}\, x^*(\alpha_{2^{k-1}-2}) \\ &= \tau^{2r-2} \otimes \tau^{2^k}. \end{aligned}$$

Here $\tau^{2r-2} \otimes \tau^{2^k} \neq 0$ since

$$\begin{aligned} 2a &= 2^r - 2 \\ 2b &= 2(a + b) - 2a \\ &> 2^d - 2^r + 2 \\ &\geq 2^{d-1} \\ &\geq 2^k. \end{aligned} \tag{17}$$

Therefore $\rho = 1$ and

$$x^*(\alpha_{2^k-2}) = \tau^{2^r-2} \otimes \tau^{2^k-2^r}.$$

Thus lemma 4.11 is proved.

**Lemma 4.12.** *In the case* I) *if* $x^*(\alpha_6) = 0$ *then* $x^*(\alpha_{2(a+b)}) = 0$.

*Proof.* By Lemma 4.11

$$x^*(\alpha_{2^{d-1}-2}) = 0$$

or

$$2a = 2^r - 2 \quad \text{and} \quad x^*(\alpha_{2^{d-1}-2}) = \tau^{2^r-2} \otimes \tau^{2^{d-1}-2^r}.$$

Since

$$x^*(\alpha_{2(a+b)}) = \mathrm{Sq}^{2^{d-1}}\,\mathrm{Sq}^{2(a+b)-(2^d-2)}x^*(\alpha_{2^{d-1}-2}),$$

if $x^*(\alpha_{2(a+b)}) \neq 0$ then $x^*(\alpha_{2^{d-1}-2}) \neq 0$ and $2(a+b) \equiv -2 \bmod 2^r$. But if $2(a+b) \equiv -2 \bmod 2^r$ then

$$\binom{2(a+b)}{2a} = \binom{2(a+b)}{2^r-2} \equiv 1 \bmod 2.$$

Thus if $\binom{2(a+b)}{2a} \equiv 0 \bmod 2$ and $x^*(\alpha_6) = 0$ then

$$x^*(\alpha_{2(a+b)}) = 0. \qquad \text{Q.E.D.}$$

Now we consider the case II) we start from the next lemma.

**Lemma 4.13.** *Assume* $i + j = 2^d - 2$ *for some* $d \in \mathbf{N}$, $d > 3$, $i$ *and* $j$ *are even,* $i, j \geq 2$ *and*

$$i = \sum_{k=1}^{d-1} \varepsilon_k 2^k,$$

*where* $\varepsilon_k = 0$ *or* 1. *Then*

$$\mathrm{Sq}^{2^p}\,\tau^i \otimes \tau^j = \begin{cases} \tau^{i+2^p} \otimes \tau^j & \varepsilon_p = 1 \\ \tau^i \otimes \tau^{j+2^p} & \varepsilon_p = 0 \end{cases}$$

*for* $1 \leq p \leq d-1$ *where* $\tau^i \otimes \tau^j \in \mathrm{H}^{2^d-2}(\mathbf{RP}^\infty \wedge \mathbf{RP}^\infty)$.

*Proof.* We use induction. Let $\bar{\varepsilon}_k = 1 - \varepsilon_k$. Then $j = \sum_{k=1}^{d-1} \bar{\varepsilon}_k 2^k$.

The statement is true for $p = 1$. Let we assume that the statement is true for $\mathrm{Sq}^{2^{p-1}}$ and also $\varepsilon_{p-1} = 1$. Then

$$\mathrm{Sq}^{2^{p-1}}\tau^i\otimes\tau^j=\sum_{l=0}^{2^{p-1}}(\mathrm{Sq}^l\tau^i)\otimes(\mathrm{Sq}^{2^{p-1}-l}\tau^j)$$

$$=\sum_{l=0}^{2^{p-1}}\binom{i}{l}\binom{j}{2^{p-1}-l}\tau^{i+l}\otimes\tau^{j+2^{p-1}-l}$$

$$=\tau^{i+2^{p-1}}\otimes\tau^j,$$

that is,

$$\binom{i}{l}\binom{j}{2^{p-1}-l}=\begin{cases}0 & 0\le l\le 2^{p-1}-1\\ 1 & l=2^{p-1}.\end{cases}$$

Hence

$$\mathrm{Sq}^{2^p}\tau^i\otimes\tau^j=\sum_{l=0}^{2^p}\binom{i}{l}\binom{j}{2^p-l}\tau^{i+l}\otimes\tau^{j+2^p-l}$$

$$=\sum_{l=0}^{2^{p-1}}\binom{i}{l}\binom{j}{2^p-l}\tau^{i+l}\otimes\tau^{j+2^p-l}$$

$$+\sum_{l=0}^{2^{p-1}}\binom{i}{2^{p-1}+1}\binom{j}{2^{p-1}-l}\tau^{i+2^{p-1}+l}\otimes\tau^{j+2^{p-1}-l}$$

$$=\sum_{l=1}^{2^{p-1}}\binom{i}{l}\binom{\overline{\varepsilon_{p-1}}}{1}\binom{j}{2^{p-1}-l}\tau^{i+l}\otimes\tau^{j+2^p-l}$$

$$+\sum_{l=0}^{2^{p-1}-1}\binom{\varepsilon_{p-1}}{1}\binom{i}{l}\binom{j}{2^{p-1}-l}\tau^{i+2^{p-1}+l}\otimes\tau^{j+2^{p-1}-l}$$

$$+\binom{j}{2^p}\tau^i\otimes\tau^{j+2^p}+\binom{i}{2^p}\tau^{i+2^p}\otimes\tau^j$$

$$=\binom{\overline{\varepsilon_p}}{1}\tau^i\otimes\tau^{j+2^p}+\binom{\varepsilon_p}{1}\tau^{i+2^p}\otimes\tau^j$$

as asserted. And even if $\varepsilon_{p-1}=1$, it can be proved in the same manner.

Q.E.D.

**Lemma 4.14.** *Let* $b\ge a$. *In the case* II), *if* $x^*(\alpha_6)=0$, *then*

$$x^*(\alpha_{2^d-2})=\begin{cases}\rho\sum_{i=1}^{(2^d-4)/2}\tau^{2i}\otimes\tau^{(2^d-2)-2i}+\rho'\tau^{2^{d-1}-2}\otimes\tau^{2^{d-1}}\\ \qquad\textit{where}\quad 2a=2^{d-1}-2\quad\textit{if}\ \rho'=1\\ \textit{or}\\ \tau^{2^r-2}\otimes\tau^{2^d-2^r}\quad\textit{and}\quad 2a=2^r-2,\quad 3\le r\le d-2.\end{cases}$$

*Proof.* We start from the computation of $x^*(\alpha_{2^{d-1}-2})$. By lemma 4.11

$$x^*(\alpha_{2^{d-1}-2}) = \begin{cases} 0 \\ \text{or} \\ \tau^{2^r-2} \otimes \tau^{2^{d-1}-2^r} & \text{in this case } 2a = 2^r - 2, \quad 3 \le r \le d-1. \end{cases}$$

Next we consider $x^*(\alpha_{2^d-2})$. Since

$$\mathrm{Sq}^1\,(x^*(\alpha_{2^d-2})) = x^*(\mathrm{Sq}^1\,\alpha_{2^d-2}) = 0, \tag{18}$$

$$\mathrm{Sq}^2\,(x^*(\alpha_{2^d-2})) = x^*(\alpha_2^{2^{d-1}}) = 0, \tag{19}$$

$$\mathrm{Sq}^4\,(x^*(\alpha_{2^d-2})) = \mathrm{Sq}^{2^{d-1}}\,\mathrm{Sq}^4\,x^*(\alpha_{2^{d-1}-2}) = 0, \tag{20}$$

as in the proof of Lemma 4.11, we have

$$x^*(\alpha_{2^d-2}) = \rho\tau^{2s} \otimes \tau^{(2^d-2)-2s} + \rho' \sum_{i=s+1}^{t-1} \tau^{2i} \otimes \tau^{(2^d-2)-2i} + \rho''\tau^{2t} \otimes \tau^{(2^d-2)-2t},$$

where

$$s = \max\left\{1, \frac{2^d-2}{2} - b\right\},$$

$$t = \min\left\{\frac{2^d-4}{2}, a\right\}.$$

Firstly we assume $x^*(\alpha_{2^{d-1}-2}) = 0$. And we shall prove $\rho = \rho'$. If $s = 1$ then the equation $\mathrm{Sq}^2\,x^*(\alpha_{2^d-2}) = 0$ means $\rho = \rho'$. Thus we assume $s = \dfrac{2^d-2}{2} - b$, that is,

$$2b \le 2^d - 4. \tag{21}$$

Here we remark that by (4),

$$2b \ge a + b \tag{22}$$

$$> 2^{d-1} - 2 \tag{23}$$

Let $q := p(2b)$ then (21) and (23) mean $q \le d-2$. Also

$$\mathrm{Sq}^{2^q}\,x^*(\alpha_{2^d-2}) = \mathrm{Sq}^{2^{d-1}}\,\mathrm{Sq}^{2^q}\,x^*(\alpha_{2^{d-1}-2}) = 0.$$

Thus, by Lemma 4.13, compare tterm of $\tau^{(2^d-2)-2b+2^q} \otimes \tau^{2b}$ in $\mathrm{Sq}^{2^q}\,x^*(\alpha_{2^d-2})$ and we obtain

$$(\rho + \rho')\tau^{(2^d-2)-2b+2^q} \otimes \tau^{2b} = 0. \tag{24}$$

Here we remark that $(2^d-2) - 2b + 2^q \le 2a$ by (4). Thus (24) means $\rho' = \rho''$. Therefore

$$x^*(\alpha_{2^d-2}) = \rho' \sum_{i=s}^{t-1} \tau^{2i} \otimes \tau^{(2^d-2)-2i} + \rho''\tau^{2t} \otimes \tau^{(2^d-2)-2t}.$$

Next we consider the term $\rho'' \tau^{2t} \otimes \tau^{(2^d-2)-2t}$. If $2t = 2^d - 4$, then by the computation of $\mathrm{Sq}^2\, x^*(\alpha_{2^d-2})$ we have $\rho' = \rho''$ and $x^*(\alpha_{2^d-2}) = \sum_{i=s}^{t} \tau^{2i} \otimes \tau^{(2^d-2)-2i}$ or 0 as asserted. Thus we assume $2t = 2a$, that is,

$$2a < 2^d - 4 \tag{25}$$

Let $p := p(2a)$. Here from (25) $p \leq d - 1$. And $p = d - 1$ if and only if $2a = 2^{d-1} - 2$.

If $2a = 2^{d-1} - 2$ then

$$x^*(\alpha_{2^d-2}) = \rho' \sum_{i=1}^{(2^d-4)/2} \tau^{2i} \otimes \tau^{(2^d-2)-2i} + (\rho'' + \rho')\tau^{2^{d-1}-2} \otimes \tau^{2^{d-1}}.$$

If $p \leq d - 2$ then

$$\mathrm{Sq}^{2^p}\, x^*(\alpha_{2^d-2}) = \mathrm{Sq}^{2^{d-1}}\, \mathrm{Sq}^{2^p}\, x^*(\alpha_{2^{d-1}-2}) = 0. \tag{26}$$

By Lemma 4.13 look into the term of $\tau^{2a} \otimes \tau^{(2^d-2)-2a+2^p}$ of (26) and we obtain

$$(\rho' + \rho'')\tau^{2a} \otimes \tau^{(2^d-2)-2a+2^p} = 0. \tag{27}$$

Remark that by (4)

$$(2^d - 2) - 2a + 2^p \leq 2b.$$

Therefore $\rho' = \rho''$ and

$$x^*(\alpha_{2^d-2}) = \rho' \sum_{i=s}^{t} \tau^{2i} \otimes \tau^{(2^d-2)-2i}.$$

Secondly we assume $x^*(\alpha_{2^{d-1}-2}) = \tau^{2^r-2} \otimes \tau^{2^{d-1}-2^r}$ and $2a = 2^r - 2$ and observe $x^*(\alpha_{2^d-2})$ again. We reset

$$x^*(\alpha_{2^d-2}) = \rho\tau^{2s} \otimes \tau^{(2^d-2)-2s} + \rho' \sum_{i=s+1}^{t-1} \tau^{2i} \otimes \tau^{(2^d-2)-2i} + \rho''\tau^{2t} \otimes \tau^{(2^d-2)-2t},$$

where

$$s = \max\left\{1, \frac{2^d-2}{2} - b\right\},$$

$$t = \min\left\{\frac{2^d-4}{2}, a\right\}.$$

Then

$$2b = 2(a + b) - 2a \tag{28}$$

$$\geq (3 \cdot 2^{d-1} - 2) - (2^{d-1} - 2) \tag{29}$$

$$= 2^d \tag{30}$$

This means $s = 1$. Thus by the computation of $\mathrm{Sq}^2\, x^*(\alpha_{2^d-2})$ we have

$$\rho = \rho'$$

and also by the computation of $\mathrm{Sq}^4\, x^*(\alpha_{2^d-2})$ and by (30) we have

$$\rho = 0.$$

Therefore we obtain

$$x^*(\alpha_{2^d-2}) = \rho'' \tau^{2^r-2} \otimes \tau^{2^d-2^r}.$$

Finally we have obtained the following result

$$x^*(\alpha_{2^d-2}) = \begin{cases} \rho \sum_{i=1}^{(2^d-4)/2} \tau^{2i} \otimes \tau^{(2^d-2)-2i} + \rho' \tau^{2^{d-1}-2} \otimes \tau^{2^{d-1}} \\ \qquad \text{where} \quad 2a = 2^{d-1}-2 \quad \text{if } \rho' = 1 \\ \text{or} \\ \tau^{2^r-2} \otimes \tau^{2^d-2^r} \quad \text{and} \quad 2a = 2^r - 2, \quad 3 \le r \le d-2. \end{cases}$$

**Lemma 4.15.** *In the case* II) *if* $x^*(\alpha_6) = 0$ *then* $x^*(\alpha_{2(a+b)}) = 0$.

*Proof.* By (4)

$$x^*(\alpha_{2(a+b)}) = \mathrm{Sq}^{2(a+b)-(2^d-2)} x^*(\alpha_{2^d-2}).$$

And by Lemma 4.14 we shall prove that

$$\begin{cases} \mathrm{Sq}^{2(a+b)-(2^d-2)} \left(\sum_{i=1}^{(2^d-4)/2} \tau^{2i} \otimes \tau^{(2^d-2)-2i}\right) = 0 \\ \mathrm{Sq}^{2(a+b)-(2^d-2)} (\tau^{2^r-2} \otimes \tau^{2^d-2^r}) = 0 \quad \text{in case } a = 2^r - 2, \quad 3 \le r \le d-1. \end{cases}$$

Since

$$\sum_{i=1}^{(2^d-4)/2} \tau^{2i} \otimes \tau^{(2^d-2)-2i} = x_0^*(\alpha_{2^d-2}),$$

it follows that

$$\begin{aligned} \mathrm{Sq}^{2(a+b)-(2^d-2)} \left( \sum_{i=1}^{(2^d-4)/2} \tau^{2i} \otimes \tau^{(2^d-2)-2i} \right) &= \mathrm{Sq}^{2(a+b)-(2^d-2)} x_0^*(\alpha_{2^d-2}) \\ &= x_0^*(\alpha_{2(a+b)}) \\ &= \binom{2(a+b)}{2a} \tau^{2a} \otimes \tau^{2b} \\ &= 0. \end{aligned}$$

Also

$$\begin{aligned} &\mathrm{Sq}^{2(a+b)-(2^d-2)} (\tau^{2^r-2} \otimes \tau^{2^d-2^r}) \\ &\quad = \tau^{2^r-2} \otimes \binom{2^d-2^r}{2(a+b)-(2^d-2)} \tau^{2(a+b)-(2^d-2)} \\ &\quad = \begin{cases} \tau^{2^r-2} \otimes \tau^{2(a+b)-(2^r-2)} & \text{if } 2(a+b) \equiv -2 \bmod 2^r \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

But if $2(a+b) \equiv -2 \bmod 2^r$ then

$$\binom{2(a+b)}{2a} = \binom{2(a+b)}{2^r-2} \equiv 1 \bmod 2.$$

Thus if $\binom{2(a+b)}{2a} \equiv 0 \bmod 2$ then $x^*(\alpha_{2(a+b)}) = 0$. Q.E.D.

Now we shall finish the proof of Theorem 4.7. Let $x\colon \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega_0\mathbf{SO}$ be an arbitrary map, $n>1$, $m>1$ and $\binom{n+m-2}{n-1} \equiv 0 \bmod 2$. If $x^*(\alpha_6) = 0$ then by Lemma 4.12, Lemma 4.15 we obtain $x^*(\alpha_{n+m-2}) = 0$. Therefore we assume $x^*(\alpha_6) \neq 0$. The from Lemma 4.9

$$x^*(\alpha_6) = \tau^2 \otimes \tau^4 + \tau^4 \otimes \tau^2.$$

Let $x + x_0\colon \mathbf{RP}^{n-1} \wedge \mathbf{RP}^{m-1} \to \Omega_0\mathbf{SO}$ be a map which is contained in the homotopy class $[x] + [x_0]$. Since $\Omega_0\mathbf{SO}$ is an H-space and it is known that $\alpha_{2i} \in H^*(\Omega_0\mathbf{SO})$ are primitive elements,

$$(x + x_0)^*(\alpha_6) = 2(\tau^2 \otimes \tau^4 + \tau^4 \otimes \tau^2) = 0.$$

Therefore

$$(x + x_0)^*(\alpha_{n+m-2}) = 0,$$

while

$$\begin{aligned}(x + x_0)^*(\alpha_{n+m-2}) &= x^*(\alpha_{n+m-2}) + x_0^*(\alpha_{n+m-2}) \\ &= x^*(\alpha_{n+m-2}) + \binom{n+m-2}{n-1}\tau^{n-1} \otimes \tau^{m-1} \\ &= x^*(\alpha_{n+m-2}).\end{aligned}$$

Finally we obtained that $x^*(\alpha_{n+m-2}) = 0$ and Theorem 4.7 is proved.

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