

## Euler Classes and complete intersections

*Dedicated to Professor R. Sridharan on his 60th Birthday*

By

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### Introduction

In ([Mu2], Theorem 3.7) Murthy proved the following

**Theorem 1.** *Let  $A$  be a reduced affine algebra of dimension  $n$  over an algebraically closed field  $F$  with  $F^n K_0(A)$  torsion free. Suppose  $P$  is a projective  $A$ -module of rank  $n$ . Let  $f: P \rightarrow I$  be a surjection where  $I \subseteq A$  is a local complete intersection of height  $n$ . Assume that  $[A/I] = 0$  in  $K_0(A)$ . Then there exists a surjection from  $P$  to  $A$ . (i.e. If the top Chern class of  $P$  vanishes, then  $P$  has a unimodular element.)*

A relative version of Theorem 1 was proved by Mandal and Murthy ([MM], unpublished):

**Theorem 2.** *Let  $A$  be a reduced affine algebra of dimension  $n$  over an algebraically closed field  $F$  with  $F^n K_0(A)$  torsion free. Let  $P$  be a projective  $A$ -module of rank  $n$ . Suppose  $f: P \rightarrow I_1$  is a surjective map where  $I_1 \subseteq A$  is a local complete intersection of height  $n$ . Assume that  $I_2 \subseteq A$  is a local complete intersection of height  $n$ , satisfying the property that  $[A/I_1] = [A/I_2]$  in  $K_0(A)$ . Then there exists a surjection  $g: P \rightarrow I_2$ .*

We note that Theorem 2 implies Theorem 1.

The theorems proved in this paper were motivated by a conjectural formulation of Theorem 1 in the case when  $A$  is a noetherian ring with  $\dim A = n$ . Roughly one wants to prove the following.

**Conjecture.** Let  $A$  be a noetherian ring with  $\dim A = n$ . Let  $P$  be a projective  $A$ -module with  $\text{rank } P = n$ . Suppose that the " $n^{\text{th}}$  Euler class of  $P$ " vanishes, then  $P$  has a unimodular element.

We must of course define what one means by the  $n^{\text{th}}$  Euler class of  $P$ . In Section 1 we define an Euler Class group. The conjectural version of Theorem 2 is stated in Section 1, Question D.

The idea of Section 1, is to use the concept of homotopy of sections of projective modules to define an Euler Class group. We only give a sketch and refer the reader to Section 1 for details.

Let  $A$  be a noetherian ring with  $\dim A = n$ . Let  $P$  be a projective  $A$ -module with  $\text{rank } P = n$ . Suppose that we are given a surjective map  $f: P \rightarrow I_1$ , where  $I_1 \subseteq A$  is an ideal of height  $n$ . Let  $I_2 \subseteq A$  be an ideal of height  $n$ . Under certain conditions, we want to say that there exists a surjection  $g: P \rightarrow I_2$ .

If such a  $g$  exists, then we can construct a homotopy  $\psi: P[t] \rightarrow I$  such that

- 1)  $\dim A[t]/I = 1$
- 2)  $\psi(0) = f, \psi(1) = g$ .

Thus we are led to asking when such a homotopy exists. In this connection we ask the following (see Question B in Section 1):

**QUESTION:** Let  $A$  be a noetherian ring with  $\dim A = n$ . Let  $I \subseteq A[t]$  be an ideal such that  $\dim A[t]/I = 1$ . Let  $P$  be a projective  $A$ -module of rank  $n$ . Suppose  $f: P \rightarrow I(0)$  is a surjective map. Assume that there exists a surjection  $\varphi: P[t]/IP[t] \rightarrow I/I^2$  such that  $\varphi(0) = f \bmod I(0)^2$ . Does there exist a surjection  $\psi: P[t] \rightarrow I$  such that  $\psi$  lifts  $\varphi$  and  $\psi(0) = f$ ?

This question has been answered in the affirmative in the case when  $I$  contains a monic polynomial and  $n \geq 3$  in [Ma] (see Theorem 2.1, [Ma]). Bhatwadekar, Mohan Kumar and Srinivas (unpublished) have shown that the answer to the above question is negative in general. In their counterexample the ring  $A$  is normal but not regular. One expects that the answer to the above Question is “Yes” if  $A$  is regular and  $\dim A \geq 3$ .

In Section 1 we point out the connection between the above question and a group to evaluate Euler Classes. In Section 2 we answer a particular case of the above question in the affirmative (see Theorem 2.3). We use Theorem 2.3 in Section 3 to prove the following addition and subtraction principles which are related to Theorem 2 of this introduction.

**Theorem 3.** *Let  $A$  be a noetherian ring with  $\dim A = n \geq 3$ . Let  $I_1$  and  $I_2$  be two comaximal ideals of height  $n$  in  $A$  such that  $I_1$  is generated by  $n$  elements. Suppose  $P$  is a projective  $A$ -module of rank  $n$  with trivial determinant. Assume that there exists a surjective map  $f: P \rightarrow I_2$ . Then there also exists a surjection  $g: P \rightarrow I_1 \cap I_2$ .*

This theorem is proved in Theorem 3.2.

We also prove (see Theorems 3.5 and 3.14) the following subtraction principle.

**Theorem 4.** *Let  $A$  be an affine algebra over a field  $F$  with  $\dim A = n \geq 3$ . Let  $I_1$  and  $I_2$  be two co-maximal ideals of height  $n$  in  $A$  such that  $I_1$  is generated by  $n$  elements. Suppose  $P$  is a projective  $A$ -module of rank  $n$  having trivial determinant. Let  $f: P \rightarrow I_1 \cap I_2$  be a surjective map. Then there is a surjection  $g: P \rightarrow I_2$  in the following cases.*

1.  $I_1$  is a maximal ideal corresponding to an  $F$ -rational point of  $\text{Spec } A$ .
2.  $I_1$  is the intersection of finitely many maximal ideals  $m_1, m_2, \dots$  where the ideals  $m_i$ , satisfy the property that  $A/m_i$  is quadratically closed for every  $i$ .

We recall that a field  $k$  is quadratically closed if every element of  $k$  is a square. For example algebraically closed fields are quadratically closed.

We now state the previously known results dealing with Theorems 3 and 4 in chronological order.

Theorems 3 and 4 were proved by Mohan Kumar in the case where  $A$  is a reduced affine algebra over an algebraically closed field or a reduced finitely generated algebra over  $\mathbf{Z}$ ,  $P = A^n$  and  $I_1, I_2$  are local complete intersections of height  $n$ . Mohan Kumar also proved Theorem 4 in the case when  $I_2 = A$  under the same conditions (cf. [MK] Theorem 1, Theorem 2, Corollary 1).

Theorem 4 was proved by Murthy ([Mu2] Theorem 1.3) in the case when  $A$  is a reduced affine algebra over an algebraically closed field or a reduced finitely generated algebra over  $\mathbf{Z}$ , and  $I_1, I_2$  are local complete intersections of height  $n$ .

Theorem 3 follows from unpublished results of Mandal and Murthy ([MM]) in the case when  $A$  is a reduced affine algebra over an algebraically closed field with  $F^n K_0(A)$  torsion free.

Theorems 3 and 4 were proved in [RS1] in the case when  $I_2 = A$  (cf. [RS1] Theorems 1, 2 and 5). The two dimensional analogues of Theorems 3 and 4 follows from results of [RS2]. In view of this, we assume after the preliminaries that the dimension of  $A \geq 3$ .

Using Theorems 3 and 4 we classify in Section 4, those real affine quadric hypersurfaces of dimension  $n$ , over which any projective module of rank  $n$  has a unimodular element. This includes some examples of Murthy ([Mu2] 3.10).

In Section 5, we prove the following subtraction principle.

**Theorem.** *Let  $A$  be a noetherian ring such that  $\dim A = n$  is even. Let  $I_1$  and  $I_2$  be two comaximal ideals of height  $n$  in  $A$  such that  $I_1/I_1^2$  and  $I_2/I_2^2$  are free  $A/I_1$  and  $A/I_2$  modules of rank  $n$ . Suppose that  $I_1$  and  $I_1 \cap I_2$  are generated by  $n$  elements. Then there exists a stably free  $A$ -module  $P$  of rank  $n$  mapping surjectively onto  $I_2$ .*

In Section 0, we state some preliminaries.

In this paper all rings considered are assumed to be commutative, noetherian and to have identity elements. All modules considered are assumed to be finitely generated. All mappings considered are either  $A$ -linear or ring homomorphisms (depending on the context).

We close this introduction with a few words about the notation. Generally  $m, m_i$  will be maximal ideals. Let  $A$  be a noetherian ring with  $\dim A = n$ . A maximal ideal  $m$  of  $A$  is said to be *regular* if  $A_m$  is a regular local ring of dimension  $n$ . By  $e_i$  we mean the element  $(0, 0, \dots, 0, 1, 0, \dots, 0)$  where the 1 is

in the  $i^{\text{th}}$  place. If  $I \subseteq A[t]$  is an ideal, and  $s \in A$ , let  $I(s) = \{f(s) | f \in I\}$ . For all other unexplained notation and definitions we refer to [Ba].

## §0. Some preliminaries

In this section, we state some known results and recall some standard definitions. These will be used in later sections.

**Definition 0.1.** Let  $A$  be a commutative ring. A row  $(a_0, a_1, \dots, a_{n+1}) \in A^{n+2}$  is said to be unimodular if there exist  $b_0, b_1, \dots, b_{n+1}$  in  $A$  such that  $\sum_{i=0}^{n+1} a_i b_i = 1$ .

**Definition 0.2.** Let  $A$  be a noetherian ring. Let  $P$  be a projective  $A$ -module. An element  $p \in P$  is said to be *unimodular* if there exists a linear map  $f: P \rightarrow A$  such that  $f(p) = 1$ .

We now state a theorem of Serre.

**Theorem 0.3.** Let  $A$  be a noetherian ring with  $\dim A = d$ . Then any projective  $A$ -module having rank  $> d$  has a unimodular element.

For a proof of this theorem we refer to ([Ba], Page 173). From Theorem 0.3, we deduce

**Corollary 0.4.** Let  $A$  be a noetherian ring with  $\dim A = 1$ . Then any projective  $A$ -module having trivial determinant is free.

We state a theorem which was proved by Swan-Towber and independently by Suslin. We refer to [Sw-To] for a proof of this theorem.

**Theorem 0.5.** Let  $(a, b, c) \in A^3$  be a unimodular row. Then there is a matrix in  $SL_3(A)$  having  $(a^2, b, c)$  as its first row.

## §1. Nori's group to evaluate Euler Classes

The contents of this section are due to Nori. We thank Nori for giving us permission to include this section in our paper.

**1.1. Definition of a group.** Let  $A$  be a noetherian ring with  $\dim A = n$  and  $S$  be the set of pairs  $(m, k)$ , where  $m$  is a regular maximal ideal of  $A$  and  $k: A/m \rightarrow \wedge^n m/m^2$  is an isomorphism. Let  $G$  be the free abelian group generated by  $S$ .

Suppose  $P$  is a projective  $A$ -module of rank  $n$  having trivial determinant. Assume that  $f: P \rightarrow J$  is a surjective map with  $J = \bigcap m_i$ , where the  $m_i$  are regular maximal ideals of  $A$ . Let  $i': A \rightarrow \wedge^n P$  be an isomorphism. We can associate to the pair  $(f, i')$  an element of  $G$  in the following manner. Let bar denote reduction modulo  $J$ . We consider the following sequence of isomorphisms

$$A/J \xrightarrow{\bar{i}'} \wedge^n P/J P \xrightarrow{\wedge^n \bar{f}} \wedge^n J/J^2$$

The composite isomorphism gives rise in a natural way to an element of  $G$ .

**1.2. Relations given by curves.** Let as before  $A$  be a noetherian ring with  $\dim A = n$ . Let  $I \subseteq A[t]$  be an ideal which satisfies the following properties.

1.  $\dim A[t]/I = 1$
2.  $I/I^2$  is generated by  $n$  elements
3.  $\wedge^n I/I^2$  is isomorphic to  $A[t]/I$
4.  $I(0) = \bigcap_1^r m_i = J$  and  $I(1) = \bigcap_1^s m'_j = J'$ , where the  $m_i$  and  $m'_j$  are regular maximal ideals of  $A$ . (Here  $I(0)$ ,  $I(1)$  denote the specialisations of  $I$  at 0 and 1).

Any isomorphism  $k[t]: A[t]/I \rightarrow \wedge^n I/I^2$  gives rise, when we specialize at  $t = 0$  and  $t = 1$  to two elements  $g_0$  and  $g_1 \in G$ . Suppose that there exists a projective  $A$ -module  $P$  of rank  $n$  and having trivial determinant and a surjection  $f: P \rightarrow J$ , and an isomorphism  $i': A \rightarrow \wedge^n P$  such that the element of  $G$  associated to the pair  $(f, i')$  is  $g_0$ . We now pose the following question.

**QUESTION A:** Does there exist a surjection  $g: P \rightarrow J'$  such that the element of  $G$  associated to  $(g, i')$  is  $g_1$ ?

**1.3. A Question of Nori.** In order to answer question A the following question was posed by Nori:

**QUESTION B:** Let  $A$  be a noetherian ring with  $\dim A = n \geq 3$ . Let  $I \subseteq A[t]$  be an ideal such that  $\dim A[t]/I = 1$ . Let  $P$  be a projective  $A$ -module of rank  $n$ . Suppose  $f: P \rightarrow I(0)$  is a surjective map. Assume that there exists a surjection  $\varphi: P[t]/IP[t] \rightarrow I/I^2$  such that  $\varphi(0) = f \bmod I(0)^2$ . Does there exist a surjection  $\psi: P[t] \rightarrow I$  such that  $\psi$  lifts  $\varphi$  and  $\psi(0) = f$ ?

**Theorem.** *An affirmative answer to Question B implies an affirmative answer to Question A.*

*Proof.* Let  $h: (A[t]/I)^n \rightarrow I/I^2$  be any surjection. We may assume by altering by a suitable automorphism of  $(A[t]/I)^n$  that

$$\wedge^n h(e_1 \wedge e_2 \dots \wedge e_n) = k[t](1).$$

Since  $P[t]/IP[t]$  is a projective module of trivial determinant over  $A[t]/I$  (which has dimension 1), by Corollary 0.4,  $P[t]/IP[t]$  is a free  $A[t]/I$  module of rank  $n$ . We choose an isomorphism  $\ell: P[t]/IP[t] \rightarrow (A[t]/I)^n$  such that  $\wedge^n \ell(i'(1)) = e_1 \wedge e_2 \dots \wedge e_n$ . We can do so by choosing any isomorphism and altering it by a suitable automorphism of  $(A[t]/I)^n$ . There exists an element  $k' \in GL_n(A/J)$  such that  $h(0) \circ k' \circ \ell(0) = f \bmod J^2$ . By the choice of our isomorphisms, it follows using exterior powers that  $k' \in SL_n(A/J)$ . We have a surjection  $j: A[t]/I \rightarrow A/J$  that sends any polynomial to its constant term. Since  $k' \in SL_n(A/J)$  and  $A/J$  is semilocal,  $k' \in E_n(A/J)$ . We lift  $k'$  via  $j$  to an element  $k \in SL_n(A[t]/I)$ . The surjection  $\varphi = hk\ell: P[t]/IP[t] \rightarrow I/I^2$  satisfies the property that  $\varphi(0) = f \bmod J^2$ . If Question B has an affirmative answer, then there exists a surjection  $\psi: P[t] \rightarrow I$  such that  $\psi$  lifts  $\varphi$  and  $\psi(0) = f$ . We set  $g = \psi(1)$ . Then  $g$  satisfies the property required of it by Question A. Thus Question A has an affirmative answer.

Therefore we are led to Question B which has been answered affirmatively by Mandal ([Ma] Theorem 2.1) if  $I$  contains a monic polynomial. Bhatwadekar, Mohan Kumar and Srinivas have constructed a counter example to Question B if one does not assume  $I$  contains a monic polynomial (unpublished). However the ring  $A$  in their example is normal but not regular. This leads to the following natural question.

QUESTION C: Is the answer to Question B Yes if  $A$  is regular?

We do not know the answer to Question C.

In this paper we answer Question B in the affirmative (cf. Section 2, theorem 2.3) in some particular cases, which arise naturally when one tries to prove Theorem 3 and Theorem 4 of the introduction.

**1.4. Nori's group to evaluate Euler Classes.** Let  $H$  be the subgroup of  $G$  generated by the set of all  $g_0 - g_1$ , where  $g_0$  and  $g_1$  are obtained in 1.2, by running through the set of all ideals  $I$  satisfying the conditions of 1.2. We define  $G/H$  to be a group to evaluate Euler Classes.

Let  $P$  be a projective  $A$  module of rank  $n$  having trivial determinant. Let  $f: P \rightarrow J$  be a surjective map where  $J = \bigcap_{i=1}^r m_i$ . If we choose an isomorphism  $i': A \rightarrow \wedge^n P$  then we obtain an element  $\bar{g}_0 \in G/H$  associated to the pair  $(f, i')$  which we call the  $n$ th Euler class of  $P$ . Generalizing Question B, one can ask.

QUESTION D: Suppose there exist  $\lambda'_1, \lambda'_2, \dots, \lambda'_s \in G$  such that the element  $g_0 - (\lambda'_1 + \lambda'_2 + \dots + \lambda'_s) \in H$ . Does there exist a surjection  $g: P \rightarrow \bigcap_1^s m'_j$  such that  $\lambda'_1 + \lambda'_2 + \dots + \lambda'_s$  is the element of  $G$  associated to  $(g, i')$ ?

**Remark.** Question D is the conjectural version of Theorem 2 stated in the Introduction. One can similarly formulate a conjectural version of Theorem 1. We do not know the complete answer to any of the Questions A–D. However the formalism of this section will be used often in the proofs of the theorems of the later sections.

## §2. Homotopy theorems

Let  $A$  be a commutative noetherian ring. Let  $P$  be a projective  $A$ -module and  $R = A[t]$ . By a homotopy of sections, we mean an  $R$ -linear map  $\psi: P \otimes_A A[t] \rightarrow R$ .

In [Ma] a question of M. V. Nori about homotopy of sections of projective modules was considered. In this section, we prove a variant of Thm. 2.1 of [Ma]. This is used frequently and is the main ingredient in the proofs of the theorems in the next section. We first recall some notation.

**2.1. Notation.** Let  $A$  be a commutative ring and  $R = A[t]$ .

- 1) For an ideal  $I \subseteq R$ , let  $I(0) = \{f(0) \mid f \in I\}$
- 2) For an  $A$ -module  $M$ , let  $M[t] = M \otimes_A A[t]$ . If  $f: M[t] \rightarrow R$  is a homotopy of sections, and  $s$  is any element of  $A$ , let  $f(s)$  be the specialization of  $f$  at  $t = s$ .

The following is essentially a restatement of Theorem 2.1 of [Ma].

**2.2. Theorem.** *Let  $R = A[t]$ , where  $A$  is a commutative noetherian ring. Let  $I \subseteq R$  be an ideal which contains a monic polynomial. Suppose  $P$  is a projective  $A$ -module with  $\text{rank } P \geq \dim R/I + 2$  and  $f: P \rightarrow I(0)$  is a surjective map. Assume that there is a surjection  $\varphi: P[t]/IP[t] \rightarrow I/I^2$  such that  $\varphi(0) = f \text{ mod } I(0)^2$ . Then there is a surjection  $\psi: P[t] \rightarrow I$ , such that  $\psi$  lifts  $\varphi$  and  $\psi(0) = f$ .*

*Further if  $K$  is the kernel of  $\psi$  and  $u \in I \cap A$ , then  $K_u$  is an extended projective module.*

*Proof.* Only the last assertion is new. It follows from the proof of Theorem 2.1 of [Ma], that  $K_u$  is a locally extended projective module and is therefore globally extended by [Q].

The following theorem is a variant of the above theorem (2.2) and is the main result of this section.

**2.3. Theorem.** *Let  $A$  be a commutative noetherian ring and  $R = A[t]$ . Suppose  $I = I' \cap I''$  is the intersection of two ideals  $I'$  and  $I''$  in  $R$  such that*

1.  $I'$  contains a monic polynomial,
2.  $I'' = I''(0)R$  is an extended ideal and
3.  $I' + I'' = R$ .

*Suppose  $P$  is a projective  $A$ -module of rank  $r \geq \dim R/I' + 2$  and  $f: P \rightarrow I(0)$  and  $\varphi: P[t]/I'P[t] \rightarrow I'/I'^2$  are two surjective linear maps such that  $\varphi(0) = f \text{ mod } I'(0)^2$ . Then there is a surjective map  $\psi: P[t] \rightarrow I$  such that  $\psi(0) = f$ .*

*Proof.* Let  $J' = I' \cap A$ . Since  $I'$  has a monic polynomial and  $I''$  is extended, it follows that  $J' + I''(0) = A$ . (see [La], Chapter 3, Section 1). We choose  $s_1$  in  $J'$  and  $s_2$  in  $I''(0)$  such that  $s_1 + s_2 = 1$ . By Theorem (2.2) above, there is a surjection  $\psi_1: P_{s_2}[t] \rightarrow I'_{s_2} = I_{s_2}$  such that  $\psi_1(0) = f_{s_2}$ . Further, by the last assertion of (2.2), if  $K'_1$  is the kernel of  $\psi_1$ , then  $K'_{1s_1}$  is an extended projective module.

Let  $\psi_2 = f_{s_1} \otimes A[t]: P_{s_1}[t] \rightarrow I'_{s_1} = I_{s_1}$ , let  $K_1 = \text{Ker } \psi_{1s_1} = K'_{1s_1}$  and  $K_2 = \text{ker } \psi_{2s_2}$ .

We have exact sequences

$$\begin{aligned} 0 \rightarrow K_1 \rightarrow P_{s_1s_2}[t] &\xrightarrow{\psi_{1s_2}} I''_{s_1s_2} = A_{s_1s_2}[t] \rightarrow 0 \\ 0 \rightarrow K_2 \rightarrow P_{s_1s_2}[t] &\xrightarrow{\psi_{2s_2}} I''_{s_1s_2} = A_{s_1s_2}[t] \rightarrow 0 \end{aligned}$$

Let bar denote reduction “modulo  $t$ ”. Since  $\psi_1(0) = f_{s_2}$  and  $\psi_2(0) = f_{s_1}$  and  $K_1$  and  $K_2$  are extended projective modules, there is an isomorphism  $\alpha_0: \bar{K}_1 \rightarrow \bar{K}_2$  such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{K}_1 & \longrightarrow & P_{s_1s_2} & \xrightarrow{\bar{\psi}_1} & I(0)_{s_1s_2} = A_{s_1s_2} \longrightarrow 0 \\ & & \downarrow \alpha_0 & & \downarrow Id & & \downarrow Id \\ 0 & \longrightarrow & \bar{K}_2 & \longrightarrow & P_{s_1s_2} & \xrightarrow{\bar{\psi}_2} & I(0)_{s_1s_2} = A_{s_1s_2} \longrightarrow 0 \end{array}$$

commutes.

We can find isomorphisms  $\alpha: K_1 \rightarrow K_2$  and  $\beta: P_{s_1s_2}[t] \rightarrow P_{s_1s_2}[t]$  such that  $\bar{\alpha} = \alpha_0$  and  $\bar{\beta} = Id$  and such that the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_1 & \longrightarrow & P_{s_1s_2}[t] & \longrightarrow & I_{s_1s_2} \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow Id \\
 0 & \longrightarrow & K_2 & \longrightarrow & P_{s_1s_2}[t] & \longrightarrow & I_{s_1s_2} \longrightarrow 0
 \end{array}$$

commutes.

Since  $\bar{\beta} = Id$ , by Quillen's lemma (See, for example, [Ma], Lemma 1.2)  $\beta = \beta_2\beta_1^{-1}$  with  $\beta_2 \in \text{Aut } P_{s_1}[t]$  and  $\beta_1 \in \text{Aut } P_{s_2}[t]$ , with  $\bar{\beta}_1 = \bar{\beta}_2 = Id$ . Hence  $(\psi_2\beta_2)_{s_2} = (\psi_1\beta_1)_{s_1}$ .

By patching  $P_{s_1}[t]$  and  $P_{s_2}[t]$  via the isomorphism  $\beta^{-1}: P_{s_1s_2}[t] \rightarrow P_{s_1s_2}[t]$ , we get a projective  $A[t]$ -module  $P'$ . Let  $\eta_1: P' \rightarrow I$  be the map got by patching  $\psi_1: P_{s_2}[t] \rightarrow I_{s_2}$  and  $\psi_2: P_{s_1}[t] \rightarrow I_{s_1}$  and let  $\eta_2: P[t] \rightarrow P'$  be the map got by patching  $\beta_1: P_{s_2}[t] \rightarrow P_{s_2}[t]$  and  $\beta_2: P_{s_1}[t] \rightarrow P_{s_1}[t]$ .

It follows that  $\eta_2$  is an isomorphism and that  $\psi = \eta_1\eta_2: P[t] \rightarrow I$  is surjective. Since  $\bar{\beta}_1 = Id$  and  $\bar{\beta}_2 = Id$ , it follows that  $\psi_{s_2}(0) = f_{s_2}$  and  $\psi_{s_1}(0) = f_{s_1}$ . Therefore  $\psi(0) = f$ . This completes the proof of Theorem (2.3).

The following is an interesting extension of Theorem (2.2).

**2.4. Theorem.** *Let  $A$  be a regular ring, essentially of finite type over a field  $k$  and  $R = A[t]$  be the polynomial ring. Let  $I = I' \cap I''$  be the intersection of two ideals  $I'$  and  $I''$  in  $R$  such that*

1.  $I'$  contains a monic polynomial,
2.  $I'' = I''(0)R$  is a extended reduced ideal of height  $r \geq 3$  such that  $A/I''(0)$  is regular,
3.  $I' + I'' = R$ .

*Suppose  $P$  is a projective  $A$ -module of rank  $r \geq \dim R/I' + 2$  and suppose  $f: P \rightarrow I(0)$  and  $\varphi: P[t] \rightarrow I/I^2$  are two surjective linear maps such that  $\varphi$  modulo  $(t, I) \equiv f$  modulo  $I(0)^2$ .*

*Then there is a surjective map  $\psi: P[t] \rightarrow I$  such that  $\psi$  lifts  $\varphi$  and  $\psi(0) = f$ .*

*Proof.* It is essentially similar to the proof of (2.3) and we only outline the difference. We pick  $s_1, s_2$  as in (2.3). By (2.2) there is a surjective map  $\psi_1: P_{s_2}[t] \rightarrow I_{s_2}$  such that  $\psi_1(0) = f_{s_2}$  and  $\psi_1$  lifts  $\varphi_{s_2}$ . By ([Ma], Theorem (2.3)), there is a surjective map  $\psi_2: P_{s_1}[t] \rightarrow I_{s_1}$  such that  $\psi_2(0) = f_{s_1}$  and  $\psi_2$  lifts  $\varphi_{s_1}$ . Since  $A$  is regular, and essentially of finite type over a field  $k$ , by Lindel's theorem ([Li]),  $\ker(\psi_1)_{s_1}$  and  $\ker(\psi_2)_{s_2}$  are extended projective modules. The rest of the proof is similar to that of (2.3). To see that  $\psi: P[t] \rightarrow I$  lifts  $\varphi$ , note that  $\beta_i \equiv Id$  modulo  $I$  for  $i = 1, 2$ .

### §3. Some addition and subtraction principles

In this section, we prove the addition and subtraction principles stated in the introduction. We begin with a key lemma which is proved in ([RS1], Lemma 3).

**Lemma 3.1** (Nori). *Let  $A$  be a noetherian ring with  $\dim A = n$ . Let  $I \subseteq A$  be an ideal of height  $n$  which is generated by  $n$  elements  $a_1, a_2, \dots, a_n$ . Suppose that  $J$  is an ideal of height  $n$  in  $A$  such that  $I + J = A$ . Then, we can find a matrix  $C$  belonging to  $E_n[A]$  such that  $[a_1, a_2, \dots, a_n]C^T = [c_1, c_2, \dots, c_n]$ , where  $c_1, c_2, \dots, c_n$  are a set of generators of  $I$  satisfying the following properties*

1.  $\dim A/(c_1, c_2, \dots, c_{n-1}) = 1$ ,
2.  $(c_1, c_2, \dots, c_{n-1}) + J = A$

We now prove the following

**Theorem 3.2** (Addition Principle). *Let  $A$  be a noetherian ring with  $\dim A = n \geq 3$ . Let  $I_1$  and  $I_2$  be two comaximal ideals of height  $n$  in  $A$  such that  $I_1$  is generated by  $n$  elements. Let  $P$  be a projective  $A$ -module of rank  $n$  with trivial determinant. Suppose that there exists a surjective map  $f: P \rightarrow I_2$ . Then there also exists a surjective map  $g$  from  $P$  to  $I_1 \cap I_2$ .*

*Proof.* By Lemma 3.1, we may choose a set of generators  $c_1, c_2, \dots, c_n$  of  $I_1$  such that

1.  $\dim A/(c_1, c_2, \dots, c_{n-1}) = 1$
2.  $(c_1, c_2, \dots, c_{n-1}) + I_2 = A$

Let  $I' = (c_1, c_2, \dots, c_{n-1}, t - 1)$ ,  $I'' = I_2 A[t]$  and  $I = I' \cap I''$ . The module  $P[t]/I'P[t]$  is a projective module of trivial determinant over the ring  $A[t]/I'$  (which has dimension 1 by (1)). By Corollary 0.4,  $P[t]/I'P[t]$  is free. We choose an isomorphism  $\ell: P[t]/I'P[t] \rightarrow (A[t]/I')^n$ . Composing  $\ell$  with the surjection  $h: (A[t]/I')^n \rightarrow I'/I'^2$  (which sends  $e_i$  to  $c_i$ ,  $1 \leq i \leq n - 1$  and  $e_n$  to  $t - 1$ ), we obtain a surjection  $\varphi = h\ell: P[t]/I'P[t] \rightarrow I'/I'^2$ . Since  $I'(0) = A$ ,  $\varphi(0) = f \bmod I'(0)^2$ . By Theorem 2.3, we obtain a surjection  $\psi: P[t] \rightarrow I$ . Specializing  $\psi$  at  $t = 1 + c_n$ , we obtain a surjection from  $P$  to  $I_1 \cap I_2$ . This completes the proof of (3.2).

Setting  $I_2 = A$  in the above theorem, we obtain the following corollary which was proved in [RS1]. We remark that the proof of 3.2 works even when  $I_2 = A$ .

**Corollary 3.3** ([RS1], Theorem). *Let  $A$  be a commutative noetherian ring with  $\dim A = n \geq 3$ . Let  $J$  be an ideal of height  $n$  which is generated by  $n$  elements. Suppose  $P$  is a projective  $A$ -module of rank  $n$  with trivial determinant. Suppose further that  $P$  has a unimodular element. Then there exists a surjective map  $g$  from  $P$  to  $J$ .*

**Corollary 3.4.** *Let  $A$  be a commutative noetherian ring with  $\dim A = n \geq 3$ . Let  $I_1$  and  $I_2$  be two comaximal ideals of height  $n$  in  $A$ , which are both generated by  $n$  elements. Then so is  $I_1 \cap I_2$ .*

Corollary 3.4 was proved in [RS1] (see [RS1] Theorem 4). The two dimensional analogue of Corollary 3.4 is proved in [RS2] (see [RS2] Theorem 2.3).

**Theorem 3.5.** *Let  $A$  be an affine algebra over a field  $F$  with  $\dim A = n \geq 3$ . Let  $I_1 \subseteq A$  be a maximal ideal which corresponds to a  $F$ -rational point of  $\text{Spec } A$ . Let  $I_2 \subseteq A$  be an ideal of height  $n$  which is comaximal with  $I_1$ . Suppose  $P$  is a projective  $A$ -module of rank  $n$  with trivial determinant. Suppose further that there exists a surjective map  $f: P \rightarrow I_1 \cap I_2$ . Then there also exists surjective map  $g$  from  $P$  to  $I_2$ .*

*Proof.* By Lemma 3.1, we choose a set of generators  $c_1, c_2, \dots, c_n$  of  $I_1$  such that

1.  $\dim A/(c_1, c_2, \dots, c_{n-1}) = 1$ ,
2.  $(c_1, c_2, \dots, c_{n-1}) + I_2 = A$ .

Let  $I' = (c_1, c_2, \dots, c_{n-1}, t - c_n)$  and  $I'' = I_2 A[t]$ . As in Theorem 3.2, we have an isomorphism  $\ell: P[t]/I'P[t] \rightarrow [A[t]/I']^n$ , and a surjection  $h: (A[t]/I')^n \rightarrow I'/I'^2$ . We therefore obtain a surjection  $\varphi = h\ell$  from  $P[t]/I'P[t] \rightarrow I'/I'^2$ . There exists an element  $k' \in GL_n(A/I_1)$  such that  $h(0) \circ k' \circ \ell(0) = f \bmod I(0)^2 = I_1^2$ . We have a map  $j: A[t]/I' \rightarrow A/I_1$  which sends any polynomial to its constant term. Since  $A/I_1 \simeq F$  and  $A$  is an affine algebra over  $F$ , we see easily that the map  $j'$  induced by  $j$  from  $GL_n(A[t]/I')$  to  $GL_n(A/I_1)$  is a surjection. We lift  $k'$  via  $j'$  to an element  $k$  belonging to  $GL_n(A[t]/I')$ . The surjection  $\varphi = hk\ell: P[t]/I'P[t] \rightarrow I'/I'^2$  satisfies the property that  $\varphi(0) = f \bmod I_1^2$ . Applying Theorem 2.3, we obtain a surjection  $\psi: P[t] \rightarrow I$ , where  $I = I' \cap I''$ . Specializing  $\psi$  at  $t = 1 + c_n$ , we obtain a surjection from  $P$  to  $I_2$ .

In order to prove the next theorem, we need some lemmas.

**Lemma 3.6.** *Let  $B$  be a finitely generated algebra over  $\mathbf{Z}$  with  $\dim B = 2$ . Let  $J \subseteq B$  be an ideal which is generated by two elements  $a_1, a_2$ . Let  $u \in B$  be a unit modulo  $J$ . Then there exists a matrix  $C \in M_2(B)$  with  $\det(C) = u$  modulo  $J$  such that  $[a_1, a_2]C^T = [c_1, c_2]$  and  $(c_1, c_2) = J$ .*

*Proof.* We choose  $v \in B$  such that  $uv = 1 \bmod J$ . Let  $f: B^3 \rightarrow J$  be defined as follows:  $f(1, 0, 0) = 0$ ,  $f(0, 1, 0) = a_1$ ,  $f(0, 0, 1) = a_2$ . The element  $(v, a_2, -a_1) \in \ker f$  and is unimodular, since  $v$  is a unit mod  $J$ . Since  $B$  is a finitely generated algebra over  $\mathbf{Z}$  with  $\dim B = 2$ , by a theorem of Vaserstein ([Su-Va], Corollary 18.1),  $(v, a_2, -a_1)$  is completable to a matrix

$$D = \begin{pmatrix} v & a_2 & -a_1 \\ c & \lambda_{11} & \lambda_{12} \\ d & \lambda_{21} & \lambda_{22} \end{pmatrix} \in SL_3(B)$$

Since  $D \in SL_3(B)$ , the rows of  $D$  generate  $B^3$ , therefore the elements  $f(v, a_2, -a_1)$ ,  $f(c, \lambda_{11}, \lambda_{12})$ ,  $f(d, \lambda_{21}, \lambda_{22})$  generate  $J$ . Hence the elements  $d_1 = \lambda_{11}a_1 + \lambda_{12}a_2$  and  $d_2 = \lambda_{21}a_1 + \lambda_{22}a_2$  generate  $J$ . The matrix  $C = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$  satisfies the required properties.

Before we state the next remark, we need a well known lemma whose proof is included for completeness.

**Lemma 3.7.** *Let  $A$  be a commutative ring with 2 invertible in  $A$ . Let  $u \in A$  be a unit, such that  $u$  is a square modulo the nil radical of  $A$ . Then  $u$  is a square in  $A$ .*

*Proof.* Let  $u + \lambda = w^2$ , where  $\lambda \in A$  is nilpotent. Then,  $u(1 + \lambda u^{-1}) = w^2$ . Since  $\lambda u^{-1}$  is nilpotent and  $\frac{1}{2} \in A$ ,  $1 + \lambda u^{-1} = y^2$ , for some  $y \in A$  which is a unit (cf. [La], Lemma 2.5, Chapter 6). Thus  $u = (wy^{-1})^2$ .

**Remark 3.8.** All that is used in the proof of 3.6 is that the unimodular row  $(v, a_2, -a_1)$  is completable. This is true (by the Swan-Towber, Suslin Theorem) if  $v$  is a square mod  $(a_1, a_2)$ . This will be so if either

- a)  $A/(a_1, a_2)$  is a product of quadratically closed fields or
- b) if  $A$  is an affine algebra over a field  $F$ ,  $\text{Char } F \neq 2$  and  $A/\sqrt{(a_1, a_2)}$  is a product of quadratically closed fields (using Lemma 3.7).

Using Remark 3.8 and the proof of Lemma 3.6 one can prove the following.

**Lemma 3.9.** *Let  $B$  be an affine algebra over a field  $F$ . Let  $J \subseteq B$  be an ideal which is generated by two elements  $a_1, a_2$ , satisfying the property that  $\dim B/J = 0$ . Let  $u \in B$  be a unit modulo  $J$ . Assume either that*

- a)  $\text{Char } F \neq 2$ ,  $B/\sqrt{J}$  is a product of quadratically closed fields or
- b)  $\text{Char } F = 2$ ,  $B/J$  is a product of quadratically closed fields.

*Then there exists a matrix  $C \in M_2(B)$ , with  $\det C = u$  modulo  $J$  such that  $[a_1, a_2]C^T = [c_1, c_2]$  and  $(c_1, c_2) = J$ .*

We now prove a higher dimensional analogue of Lemma 3.6.

**Lemma 3.10.** *Let  $A$  be a finitely generated algebra over  $\mathbf{Z}$  with  $\dim A = n \geq 2$ . Let  $J \subseteq B$  be an ideal of height  $n$  which is generated by  $n$  elements  $a_1, a_2, \dots, a_n$ . Suppose  $u \in A$  is a unit modulo  $J$ . Then there exists a matrix  $C \in M_n(A)$  with  $\det(C) = u$  modulo  $J$ , such that  $[a_1, a_2, \dots, a_n]C^T = [c_1, c_2, \dots, c_n]$  and  $(c_1, c_2, \dots, c_n) = J$ .*

*Proof.* By multiplying the vector  $[a_1, a_2, \dots, a_n]$  by an elementary matrix and using standard stability arguments ([La], Lemma 3.4, Chapter 3) we may assume that the  $a_1, a_2, \dots, a_n$  satisfy the property that  $\dim A/(a_1, a_2, \dots, a_{n-2}) = 2$ . (Note that we use the fact that an elementary matrix has determinant 1). Let  $B = A/(a_1, a_2, \dots, a_{n-2})$ . Let bar denote reduction modulo the ideal  $(a_1, a_2, \dots, a_{n-2})$ . By Lemma 3.6, one can choose a matrix  $D \in M_2(B)$  such that

1.  $\det(D) = u$  modulo  $J$ ,
2. If  $[\bar{a}_{n-1}, \bar{a}_n]D^T = [\bar{c}_{n-1}, \bar{c}_n]$ , then  $a_1, a_2, \dots, a_{n-2}, c_{n-1}, c_n$  generate  $J$ . One sets  $c_1 = a_1, c_2 = a_2, \dots, c_{n-2} = a_{n-2}$  and verifies easily that  $c_1, c_2, \dots, c_n$  satisfy the required properties.

Similarly, one can prove the following higher dimensional analogue of 3.10.

**Lemma 3.11.** Let  $A$  be an affine algebra over a field  $F$  with  $\dim A = n \geq 2$ . Let  $J \subseteq A$  be an ideal of height  $n$  which is generated by  $n$  elements  $a_1, a_2, \dots, a_n$ . Let  $u \in A$  be such that  $u$  is a unit modulo  $J$ . Assume that either

1.  $\text{Char } F \neq 2$ ,  $A/\sqrt{J}$  is a product of quadratically closed fields or
2.  $\text{Char } F = 2$ ,  $A/J$  is a product of quadratically closed fields.

Then there exists a matrix  $C \in M_n[A]$  with  $\det(C) = u$  modulo  $J$  such  $[a_1, a_2, \dots, a_n]C^T = [c_1, c_2, \dots, c_n]$  and  $(c_1, c_2, \dots, c_n) = J$ .

We summarize the results from 3.6 to 3.11 for later use as follows.

**Corollary 3.12.** Let  $A$  be a noetherian ring with  $\dim A = n$ . Let  $J \subseteq A$  be an ideal of height  $n$ , which is generated by  $n$  elements. Assume that  $J/J^2$  is a free  $A/J$  module of rank  $n$ .

Further assume that either

1.  $A$  is a finitely generated algebra over  $\mathbf{Z}$ ,
2.  $A$  is an affine algebra over a field  $F$  ( $\text{Char } F \neq 2$ ) and  $A/\sqrt{J}$  is a product of quadratically closed fields or
3.  $A$  is an affine algebra over a field  $F$ ,  $\text{Char } F = 2$  and  $A/J$  is a product of quadratically closed fields

Let  $h: A/J \rightarrow \wedge^n J/J^2$  be any isomorphism. Then there exists a set of generators  $c_1, c_2, \dots, c_n$  of  $J$  such that  $h(\bar{1}) = \bar{c}_1 \wedge \bar{c}_2 \wedge \dots \wedge \bar{c}_n$  (bar denotes reduction modulo  $J$ ).

*Proof.* We choose any set of generators  $a_1, a_2, \dots, a_n$  of  $J$ . We see that  $h(\bar{1}) = \bar{u}(\bar{a}_1 \wedge \bar{a}_2 \wedge \dots \wedge \bar{a}_n)$  where  $u \in A$  is a unit mod  $J$ . By 3.6, 3.9, 3.10 and 3.11,  $\bar{u}(\bar{a}_1 \wedge \dots \wedge \bar{a}_n) = \bar{c}_1 \wedge \dots \wedge \bar{c}_n$  for some set of generators  $c_1, c_2, \dots, c_n$  of  $J$ .

**Remark 3.13.** The  $c_1, c_2, \dots, c_n$  in Corollary 3.12 are not unique. For example one can choose any  $c_1, \dots, c_n$  which satisfy the requirements of 3.12 and multiply the vector  $[c_1, \dots, c_n]$  by a matrix of determinant 1, to obtain another set of generators of  $J$  satisfying the required property.

We now prove the subtraction principle that was stated in the introduction.

**Theorem 3.14.** Let  $A$  be a noetherian ring with  $\dim A = n \geq 3$ . Let  $I_1$  and  $I_2$  be two comaximal ideals of height  $n$  in  $A$ . Assume further that  $I_1$  is generated by  $n$  elements and that  $I_1/I_1^2$  is a free  $A/I_1$  module of rank  $n$ . Let  $P$  be a projective  $A$ -module of rank  $n$  and having trivial determinant. Suppose that  $f: P \rightarrow I_1 \cap I_2$  is a surjective map. Then there exists a surjection  $g: P \rightarrow I_2$  in the following cases

1.  $A$  is a finitely generated algebra over  $\mathbf{Z}$ ,
2.  $A$  is an affine algebra over a field  $F$ ,  $\text{char } F \neq 2$  and  $A/\sqrt{I_1}$  is a product of quadratically closed fields.
3.  $A$  is an affine algebra over a field  $F$ ,  $\text{char } F = 2$  and  $A/I_1$  is a product of quadratically closed fields.

*Proof.* We choose an isomorphism  $i: A \rightarrow \wedge^n P$ . Let  $\bar{f} = f \otimes A/I_1: P/I_1 P \rightarrow$

$I_1/I_1^2$ . We then have the following isomorphisms

$$A/I_1 \xrightarrow{\bar{i}} \wedge^n P/I_1 P \xrightarrow{\wedge^n \bar{f}} \wedge^n I_1/I_1^2$$

Let  $f_1 = \wedge^n \bar{f} \circ \bar{i}$ . By Corollary 3.12, there exist a set of generators  $c_1, c_2, \dots, c_n$  of  $I_1$  such that  $f_1(\bar{1}) = \bar{c}_1 \wedge \bar{c}_2 \cdots \wedge \bar{c}_n$  (where bar denotes reduction modulo  $I_1$ ). Further, by multiplying the vector  $[c_1, c_2, \dots, c_n]$  by an elementary matrix, one may assume by Lemma 3.1 and Remark 3.13 that

1.  $\dim A/(c_1, c_2, \dots, c_{n-1}) = 1$
2.  $(c_1, c_2, \dots, c_{n-1}) + I_2 = A$

Let  $I' = (c_1, c_2, \dots, c_{n-1}, t + c_n)$ ,  $I'' = I_2 A[t]$ , and  $I = I' \cap I''$ . We choose an isomorphism  $\ell: P[t]/I'P[t] \rightarrow (A[t]/I')^n$ , such that  $\wedge^n \ell(i(1)) = e_1 \wedge e_2 \cdots \wedge e_n$ . We can choose such an isomorphism  $\ell$  by choosing any isomorphism and altering it by a suitable automorphism of  $(A[t]/I')^n$ . Composing  $\ell$  with the surjection  $h: (A[t]/I')^n \rightarrow I'/I'^2$  which sends  $e_i$  to  $c_i$ ,  $1 \leq i \leq n - 1$  and  $e_n$  to  $t + c_n$ , we obtain a surjection  $\varphi' = h\ell: P[t]/I'P[t] \rightarrow I'/I'^2$ . There exists an element  $k' \in GL_n(A/I_1)$  such that  $h(0) \circ k' \circ \ell(0) = f \bmod I_1^2$ . From the way we have chosen various isomorphisms it follows (using exterior powers), that  $k' \in SL_n(A/I_1)$ . We have a map  $j: A[t]/I' \rightarrow A/I_1$  which sends any polynomial to its constant term. Since  $A/I_1$  is semi local,  $SL_n(A/I_1) = E_n(A/I_1)$  and we can lift  $k'$  via  $j$  to an element  $k \in SL_n(A[t]/I')$ . The surjection  $\varphi = hk\ell: P[t]/I'P[t] \rightarrow I'/I'^2$  satisfies the property that  $\varphi(0) = f \bmod I_1^2$ . By Theorem 2.3 there exists a surjection  $\psi: P[t] \rightarrow I$ . Specializing  $\psi$  at  $t = 1 - c_n$ , we obtain a surjection  $g: P \rightarrow I_2$ .

**Example 3.15.** Let  $A = \mathbf{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$  be the coordinate ring of the real two sphere. Let  $P = A^3/(x, y, z)$ . We define a surjection  $f$  from  $P$  to the ideal generated by  $y$  and  $z$  in  $A$  as follows:  $f(e_1) = 0$ ,  $f(e_2) = z$ ,  $f(e_3) = -y$ . It is known from topology that  $P$  is not free (i.e. there does not exist a surjection from  $P$  to  $A$ ). This example shows that Theorem 3.14 is not valid (when  $I_2 = A$ ) if we do not assume that either of the conditions 1, 2, or 3 hold. One can similarly construct examples in higher dimensions using even dimensional spheres.

**Corollary 3.16** ([MK], Theorem 2). *Let  $A$  be a reduced affine algebra over an algebraically closed field  $F$  or a finitely generated algebra over  $\mathbf{Z}$  with  $\dim A = n$ . Let  $I_1$  and  $I_2$  be two comaximal ideals of height  $n$  in  $A$ , which are local complete intersections. Suppose  $I_1$  and  $I_1 \cap I_2$  are both generated by  $n$  elements. Then so is  $I_2$ .*

**Remark 3.17.** One expects Corollary 3.16 to be true when  $A$  is any noetherian ring of dimension  $n$ . We refer to the last section for results in this direction.

#### §4. Projective modules over real quadric hypersurfaces

In this section, we apply the addition and subtraction principles that we have proved to answer the following question.

QUESTION: Let  $A$  be the coordinate ring of a real quadric hypersurface with  $\dim A = n$ . Let  $P$  be a projective  $A$ -module of rank  $n$ . When does  $P$  have a unimodular element?

Let  $A = \mathbf{R}[X_1, X_2, \dots, X_{n+1}]/(\sum_{i=1}^{n+1} a_i X_i^2 - b)$  where  $a_i, b \in \mathbf{R}$ . Let  $P$  be a projective  $A$ -module of rank  $n$ . We want to say when  $P$  has a unimodular element. The case  $n = 1$  is classical.

The case  $n = 2$  is also understood in view of results of Murthy (cf. [SW2], Corollary 16.2).

We consider the case  $n > 2$ . We first restate the subtraction principle in the form that it will be used in this section.

**Theorem 4.1.** *Let  $A$  be an affine algebra over  $\mathbf{R}$  with  $\dim A = n$ . Let  $I_1 \subseteq A$  be a maximal ideal of height  $n$  which is generated by  $n$  elements and  $I_2$  be an ideal of height  $n$  in  $A$  which is the intersection of finitely many maximal ideals. Assume further that  $I_1 + I_2 = A$ . Let  $P$  be a projective  $A$ -module of rank  $n$  having trivial determinant. Suppose that there is a surjection  $f: P \rightarrow I_1 \cap I_2$ . Then there also exists a surjection  $g: P \rightarrow I_2$ .*

*Proof.* The residue field  $A/I_1$  is isomorphic to  $\mathbf{R}$  or  $\mathbf{C}$ . The Theorem now follows from Theorem 3.5 and Theorem 3.14.

The following lemma was proved by Swan (cf. [SW3], Lemma 6.2) in the case where  $A$  is the coordinate ring of the real 2 sphere. The proof we give is the same.

**Lemma 4.2.** *Let  $A = \mathbf{R}[X_1, X_2, \dots, X_{n+1}]/(\sum_{i=1}^{n+1} a_i X_i^2 - b)$  be as above. Let  $m \subseteq A$  be a maximal ideal such that  $A/m \simeq \mathbf{C}$ . Then  $m$  is generated by  $n$  elements.*

*Proof.* Let  $f: A \rightarrow \mathbf{C}$  be a surjective homomorphism such that  $\ker f = m$ . We assume without loss of generality that  $f(x_1) \in \mathbf{C} - \mathbf{R}$ . Let  $c_2, c_3, \dots, c_{n+1} \in \mathbf{R}$  be chosen so that  $f(x_2 + c_2 x_1), f(x_3 + c_3 x_1), \dots, f(x_{n+1} + c_{n+1} x_1)$  belong to  $\mathbf{R}$ . Let  $f(x_i + c_i x_1) = d_i$ . Then the elements  $x_i + c_i x_1 - d_i$ ,  $2 \leq i \leq n + 1$ , are in the kernel of  $f$ . These elements generate the kernel, for, if we go modulo these elements in  $A$ , we obtain a two dimensional vector space over  $\mathbf{R}$ . Now, comparing dimensions, we see that these elements generate  $m = \ker f$ .

**Lemma 4.3.** *Let  $A = \mathbf{R}[X_1, X_2, \dots, X_{n+1}]/(\sum_{i=1}^{n+1} a_i X_i^2 - b)$  be as above (where  $n > 2$ ). Then  $\text{Pic } A = 0$ .*

*Proof.* If  $b = 0$ , then  $A$  is graded and hence  $\text{Pic } A = 0$  by ([Mu1], Lemma 5.1). If  $b \neq 0$ ,  $\text{Pic } A = 0$  by ([SW2], Theorem 9.2).

We now answer the question stated in the beginning of the section. The first two examples are due to Murthy. However the proofs we give are different (cf. [Mu2], Examples 3.10).

**Example 1** (Murthy). Let  $A = \mathbf{R}[X_1, X_2, \dots, X_{n+1}]/(\sum_{i=1}^{n+1} X_i^2 + 1)$ ,  $n > 2$ . Then any projective  $A$ -module of rank  $n$  has a unimodular element.

*Proof.* Let  $P$  be a projective  $A$ -module with rank  $P = n$ . By Swan's Bertini Theorem ([SW1], Theorem 1.3, 1.4), we can choose a surjection  $s: P \rightarrow I$ , where  $I \subset A$  is the intersection of finitely many maximal ideals  $m_i$ . By Lemma 4.3,  $P$  has trivial determinant. Since  $A/m_i \simeq \mathbf{C}$ , by Lemma 4.2,  $m_i$  is generated by  $n$  elements. Applying Theorem 4.1 repeatedly, we see that there exists a surjection  $s: P \rightarrow A$ . Therefore  $P$  has a unimodular element.

**Example 2.** Let  $A = \mathbf{R}[X_1, X_2, \dots, X_{n+1}]/(\sum_{i=1}^{n+1} X_i^2)$ ,  $n > 2$ . Let  $P$  be a projective  $A$ -module of rank  $n$ . Then  $P$  has a unimodular element.

*Proof.* The proof is similar to that of Example 1. We choose a surjection  $s: P \rightarrow I$ , such that  $I = \bigcap_{i=1}^k m_i$ , where  $A/m_i \simeq \mathbf{C}$  for all  $i$ . We can choose such a surjection by Swan's Bertini theorem, since  $A$  has only one real maximal ideal.

We thank Nori for pointing out the following example. It is due to Barge and Ojanguren ([BO]) when  $n = 2$ .

**Example 3.** Let  $A = \mathbf{R}[X_1, X_2, \dots, X_{n+1}]/(\sum_{i=1}^{n+1} X_i^2 - 1)$ ,  $n > 2$ . Let  $X = S^n$  (the  $n$  sphere). Let  $P$  be a projective  $A$ -module of rank  $n$ . Then  $P$  has a unimodular element if and only if  $P \otimes_A C(X)$  has a unimodular element (where  $C(X)$  is the ring of real valued continuous functions on  $X$ ).

*Proof.* If  $P$  has a unimodular element, then clearly  $P \otimes_A C(X)$  has a unimodular element. Suppose conversely that  $P \otimes_A C(X)$  has a unimodular element. Then, one can show using Bertini arguments (as is done in [RS3]), that there exists a surjection  $s: P \rightarrow I$  with  $I = \bigcap_{i=1}^k m_i$ ,  $A/m_i \simeq \mathbf{C}$  for all  $i$ . Now proceeding as in Example 1, we see that  $P$  has a unimodular element.

**Example 4.** Let  $A = \mathbf{R}[X_1, X_2, \dots, X_{n+1}]/(X_1X_2 + \sum_{i=3}^{n+1} a_iX_i^2)$  or  $A = \mathbf{R}[X_1, X_2, \dots, X_{n+1}]/(X_1X_2 + \sum_{i=3}^{n+1} a_iX_i^2 - b)$  where  $b \neq 0$ . Let  $P$  be a projective  $A$ -module with rank  $P = n$ . Then  $P$  has a unimodular element.

*Proof.* We only prove the case when  $A = \mathbf{R}[X_1, X_2, \dots, X_{n+1}]/(X_1X_2 + \sum_{i=3}^{n+1} a_iX_i^2)$ . The other case is similar. We first show that all real maximal ideals of  $A$  which lie outside a closed set  $Y$  such that  $\dim Y = n - 1$  are generated by  $n$  elements. Let  $Y = V(X_1X_2)$ . Let  $m \subset A$  be such that  $X_1X_2 \notin m$  and  $A/m \simeq \mathbf{R}$ . Then  $m = (X_1 - \lambda_1, X_2 - \lambda_2, \dots, X_{n+1}, -\lambda_{n+1})$ , where  $\lambda_i \in \mathbf{R}$  and  $\lambda_1 \neq 0, \lambda_2 \neq 0$ . If we go modulo  $X_3 - \lambda_3, \dots, X_{n+1} - \lambda_{n+1}$  in  $A$ , we obtain a principal ideal domain. Hence after going modulo these  $n - 1$  elements, the image  $\bar{m}$  of  $m$  is principal. Hence  $m$  is generated by  $n$  elements. By Swan's Bertini theorem, we can choose a surjection  $s: P \rightarrow I$  such that  $I = \bigcap_{i=1}^k m_i$ , and none of the  $m_i$  contains  $X_1X_2$ . Now using Theorem 4.1 and proceeding as in example 1, we see that  $P$  has a unimodular element.

**§5. Subtraction principles and sections of stably free modules**

The aim of this section is to prove the following:

**Theorem 5.1.** *Let  $A$  be a noetherian ring such that  $\dim A = n$  is even. Let*

$I_1$  and  $I_2$  be two comaximal ideals of height  $n$  in  $A$  such that  $I_1/I_1^2$  (respectively  $I_2/I_2^2$ ) is a free  $A/I_1$  (respectively  $A/I_2$ ) module of rank  $n$ . Suppose that  $I_1$  and  $I_1 \cap I_2$  are generated by  $n$  elements. Then there exists a surjection from a stably free  $A$ -module  $P$  of rank  $n$  onto  $I_2$ .

In order to prove this theorem, we need a variant of Theorem 2.3 whose proof is the same as that of Theorem 2.3.

**Theorem 5.2.** *Let  $A$  be a commutative noetherian ring and  $R = A[t]$ . Suppose  $I = I' \cap I''$  is the intersection of two ideals  $I'$  and  $I''$  in  $R$  such that*

1.  $I'$  contains a monic polynomial,
2.  $I'' = I''(0)R$  is an extended ideal,
3.  $I' + I'' = R$ .

*Suppose  $P$  is a projective  $A$ -module of rank  $r \geq \dim R/I + 2$  and  $f: P \rightarrow I(0)$ ,  $\varphi: P[t]/IP[t] \rightarrow I/I^2$  are two surjective linear maps, such that  $\varphi(0) = f \bmod I(0)^2$ . Then there is a surjective map  $\psi: P[t] \rightarrow I$  such that  $\psi(0) = f$ .*

We now turn to the proof of Theorem 5.1.

*Proof of Theorem 5.1.* We only prove the case  $n = 4$  so that the notation is simple. The proof in the general case is similar. By Lemma 3.1, we may choose a set of generators  $c_1, c_2, c_3, c_4$  of  $I_1$  such that

- a)  $\dim A/(c_1, c_2, c_3) = 1$ .
- b)  $(c_1, c_2, c_3) + I_2 = A$

Let  $I = (c_1, c_2, c_3, t - c_4) \cap I_2 A[t]$ . We note that  $I(0) = I_1 \cap I_2$ . We choose as usual a surjection  $h[t]: (A[t]/I)^4 \rightarrow I/I^2$ . Specializing at  $t = 0$ , we obtain a surjection  $h(0): (A/I(0))^4 \rightarrow I(0)/I(0)^2$ . We choose any set of generators  $d_1, d_2, d_3, d_4$  of  $I(0) = I_1 \cap I_2$ . Taking exterior powers, we obtain an isomorphism

$$\wedge^4 h(0): A/I(0) \rightarrow \wedge^4 I(0)/I(0)^2.$$

Let  $\wedge^4 h(0)$  send  $e_1 \wedge e_2 \wedge e_3 \wedge e_4$  to  $\bar{v}(\bar{d}_1 \wedge \bar{d}_2 \wedge \bar{d}_3 \wedge \bar{d}_4)$  (bar denotes reduction modulo  $I(0)$ ), where  $v \in A$  is such that  $\bar{v} \in A/I(0)$  is a unit. We choose an element  $u \in A$  such that  $uv = 1 \bmod I(0)$ . Let  $P = A^5/(u, d_1, d_2, d_3, d_4)$ . Since  $uv = 1 \bmod I(0)$ , there exist  $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in A$  such that

$$uv + \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 + \lambda_4 d_4 = 1.$$

There is a surjection  $f: P \rightarrow I(0)$  which sends  $e_1$  to 0,  $e_2$  to  $-d_1$ ,  $e_3$  to  $d_2$ ,  $e_4$  to  $-d_3$ ,  $e_5$  to  $d_4$ . (It is here that the assumption that  $n$  is even is used. The rest of the proof will use the fact that  $n$  is even). It is easy to verify the element

$$w = ve_2 \wedge e_3 \wedge e_4 \wedge e_5 + \lambda_1 e_3 \wedge e_4 \wedge e_5 \wedge e_1 + \lambda_2 e_4 \wedge e_5 \wedge e_1 \wedge e_2 + \dots$$

generates  $\wedge^4 P$ . Let  $i: A \rightarrow \wedge^4 P$  be the isomorphism which sends 1 to  $w$ . We have the following sequence of maps

$$\begin{aligned} A/I(0) &\xrightarrow{\bar{i}} \wedge^4 P/I(0)P \xrightarrow{\wedge^4 \bar{f}} \wedge^4 I(0)/I(0)^2 \\ \wedge^4 f \circ i(\bar{1}) &= \bar{v} \bar{d}_1 \wedge \bar{d}_2 \wedge \bar{d}_3 \wedge \bar{d}_4 \end{aligned}$$

We choose an isomorphism  $\ell: P[t]/IP[t] \rightarrow (A[t]/I)^4$  such that

$$\wedge^4 \ell(i(\bar{1})) = e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

We can do so by choosing any isomorphism and then altering it by an automorphism of  $(A[t]/I)^4$ . The map  $\varphi' = h\ell$  is a surjection  $P[t]/IP[t] \rightarrow I/I^2$ . By the choice of our isomorphisms it follows (using exterior powers) that there exists  $k' \in SL_n(A/I(0))$  such that

$$h(0) \circ k' \circ \ell(0) = f \bmod I(0)^2$$

We have a surjection  $j: A[t]/I \rightarrow A/I(0)$ , which sends any polynomial to its constant coefficient. We lift  $k'$  via  $j$  to an element  $k \in SL_n(A[t]/I)$ . We can do this as  $SL_n(A/I(0)) = E_n(A/I(0))$  (note that  $A/I(0)$  is semilocal). The surjection  $\varphi = hk\ell: P[t]/IP[t] \rightarrow I/I^2$  satisfies the property that  $\varphi(0) = f \bmod I(0)^2$ . Applying Theorem 5.2, we obtain a surjection  $\psi: P[t] \rightarrow I$ . Specializing  $\psi$  at  $t = 1 + c_n$ , we obtain a surjection from  $P$  to  $I_2$ , with  $P$  stably free of rank  $n$ .

**Acknowledgement.** We thank Madhav Nori for generously sharing many ideas. Section 1 of this paper is based entirely on his suggestions. Also the key Lemma 3.1 came up during discussions with him. These arguments have been extremely crucial for this paper. We also thank Ravi Rao for constant encouragement.

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