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On ill-posedness and a Tikhonov regularization for a multidimensional inverse hyperbolic problem

By

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§1. Introduction

We consider an initial/boundary value problem for a hyperbolic equation:

$$(1.1)\qquad \begin{cases} u''(x,t) = \Delta u(x,t) + \lambda(t) f(x) & (x \in \Omega,\ 0 < t < T) \\ u(x,0) = u'(x,0) = 0 & (x \in \Omega) \\ u(x,t) = 0 & (x \in \partial\Omega,\ 0 < t < T). \end{cases}$$

Here $r \geq 2$ and $\Omega \subset \mathbf{R}^r$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$, and we set $u'(x,t) = \frac{\partial u}{\partial t}(x,t)$, $u''(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$, and Δ is the Laplacian. Henceforth we always assume

$$(1.2)\qquad \lambda(0) \neq 0, \quad \lambda \in C^1[0, T].$$

Let $L^2(\Omega)$ be the space of all real-valued square integrable functions with the inner probuct $(\cdot,\cdot)_{L^2(\Omega)}$ and the norm $\|\cdot\|_{L^2(\Omega)}$. Let us denote the Sobolev space of order $s > 0$ by $H^s(\Omega)$ (e. g. Lions and Magenes [13]). Under the assumption (1.2), for any $f \in L^2(\Omega)$, there exists a unique solution $u = u(f)$ to (1.1) such that

$$u = u(f) \in C^1([0, T]; H_0^1(\Omega)) \cap C^2([0, T]; L^2(\Omega))$$

and

$$\frac{\partial u(f)}{\partial n} \in H^1(0, T; L^2(\partial\Omega))$$

(Lasiecka, Lions and Triggiani [10, Theorem 2.1] and the argument in §4 of Yamamoto [24]).

The term $\lambda(t) f(x)$ is considered an external force causing a vibration. We assume that λ is a known non-zero C^1-function and is independent of the space variable x, and $f \in L^2(\Omega)$ is unknown. We discuss

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Inverse problem. Determine f from

$$\frac{\partial u(f)}{\partial n}(x, t) \quad (x \in \partial\Omega,\ 0<t<T).$$

Here we set

$$\frac{\partial u}{\partial n}(x) = \sum_{i=1}^{r} \nu_i(x) \frac{\partial u}{\partial x_i}(x) \qquad (x \in \partial\Omega)$$

where $\nu(x) = (\nu_1(x), \cdots, \nu_r(x))$ is the outward unit normal to $\partial\Omega$ at x.

More precisely, in this inverse problem, we are required to discuss
(I) (Uniqueness) Is the correspondence

$$f \longleftrightarrow \frac{\partial u(f)}{\partial n}(x, t) \qquad (x \in \partial\Omega, 0<t<T)$$

one to one? That is, does $\frac{\partial u(f)}{\partial n}(x, t) = 0$ $(x \in \partial\Omega,\ 0<t<T)$ imply $f(x) = 0$ $(x \in \Omega)$?

(II) (Continutity) Estimate f by $\frac{\partial u(f)}{\partial n}(x, t)$ with appropriate norms.

(III) (Existence of f realizing given boundary data) Does $f \in L^2(\Omega)$ exist such that

$$\frac{\partial u(f)}{\partial n}(x, t) = y(x, t) \quad (x \in \partial\Omega,\ 0<t<T)$$

for a given y? In other words, can we characterize

$$\left\{ \frac{\partial u(f)}{\partial n} ; f \in L^2(\Omega) \right\}?$$

(IV) (Reconstruction) Give a reconstruction formula of f in terms of $\frac{\partial u(f)}{\partial n}$.

Throughout this paper, as the set of unknown f's, we take $L^2(\Omega)$, although another choice such as $H^s(\Omega)$ is possible. Then we notice that firstly we have to choose a space of data $\frac{\partial u}{\partial n}$'s and its topology in order to discuss "(II) Continuity". For theoretical discussions of our inverse problem, it is desirable to search for the space Y_0 and the topology of $\frac{\partial u}{\partial n}$'s which guarantee that the map

$$f \longrightarrow \frac{\partial u(f)}{\partial n}$$

from $L^2(\Omega)$ to Y_0, is bijective and the inverse map is continuous on Y_0. As such a result, by Theorem 1 in Yamamoto [24], we have

Theorem 0. *We set*

(1.3) $$d=\sup\{|x_1-x_2|;\ x_1, x_2\in\Omega\}=\text{the diameter of } \Omega.$$

We assume

(1.4) $$T>d.$$

(1) (*Uniqueness*) *If the solution to $u(f)$ to* (1.1) *satisfies*

$$\frac{\partial u(f)}{\partial n}(x, t)=0 \qquad (x\in\partial\Omega,\ 0<t<T),$$

then $f(x)=0$ for almost all $x\in\Omega$.
(2) (*Continuity*) *There exists a constant $C_1=C_1(\Omega, T)$ such that*

(1.5) $$C_1^{-1}\left\|\frac{\partial u(f)}{\partial n}\right\|_{H^1(0,T;L^2(\partial\Omega))}\le\|f\|_{L^2(\Omega)}\le C_1\left\|\frac{\partial u(f)}{\partial n}\right\|_{H^1(0,T;L^2(\partial\Omega))}$$

for any $f\in L^2(\Omega)$.

Here and henceforth we set

$$H^1(0, T; L^2(\partial\Omega))=\{u\in L^2(0, T; L^2(\partial\Omega));\ u'\in L^2(0, T; L^2(\partial\Omega))\},$$
$$(u, v)_{H^1(0,T;L^2(\partial\Omega))}=\int_0^T\int_{\partial\Omega}(u(x, t)v(x, t)+u'(x, t)v'(x, t))dS_x dt$$

and

$$\|u\|_{H^1(0,T;L^2(\partial\Omega))}=(u, u)^{1/2}_{H^1(0,T;L^2(\partial\Omega))}$$

for $u, v\in H^1(0, T; L^2(\partial\Omega))$.

Remark. Yamamoto [24] gives also characterization of the range

$$\left\{\frac{\partial u(f)}{\partial n};\ f\in L^2(\Omega)\right\}\subset H^1(0, T; L^2(\partial\Omega))$$

and a representation formula of f in terms of $\dfrac{\partial u(f)}{\partial n}$.

For the uniqueness and the continuity, the restriction like (1.4) is necessary because of the finiteness of propagation of waves. That is, T is greater than the diameter of Ω in which the wave propagates at the speed 1 according to (1.1).

This theorem means that we have to observe the time derivative of $\dfrac{\partial u}{\partial n}$ as well as $\dfrac{\partial u}{\partial n}$ itself for stable construction of $f\in L^2(\Omega)$. However, from a practical point of view, the observation of the time derivative is not desirable, and frequently we are obliged to construct $f\in L^2(\Omega)$ only on the basis of $\dfrac{\partial u}{\partial n}$ itself which is polluted with L^2-errors. As is seen by Theorem 1 in §2, if we choose

$L^2(0, T; L^2(\partial\Omega))$, not $H^1(0, T; L^2(\partial\Omega))$, as the range space of G, then the map

$$(1.6)\qquad Gf=\frac{\partial u(f)}{\partial n}: L^2(\Omega)\rightarrow L^2(0, T; L^2(\partial\Omega))$$

is compact, so that G^{-1} cannot be continuous from $L^2(0, T; L^2(\partial\Omega))$ to $L^2(\Omega)$. Thus the problem of determining $f\in L^2(\Omega)$ from $\frac{\partial u(f)}{\partial n}\in L^2(0, T; L^2(\partial\Omega))$ is *ill-posed* in the sense of Hadamard. For stable construction of solutions of the ill-posed problem, we have to apply regularization techniques such as Tikhonov's regularization (Baumeister [1], Groestch [6], Hofmann [7], Tikhonov and Arsenin [20]). In this paper, we will apply Tikhonov's regularization to the equation $y=Gf$ for $y\in L^2(0, T; L^2(\partial\Omega))$.

The proposes of this paper are

(1) to discuss the asymptotic behaviour of the singular values of G, which gives information about the degree of ill-posedness.

(2) to discuss convergence rates of the regularizing scheme for finding approximate solutions of $y=Gf$ with a given $y\in L^2(0, T; L^2(\partial\Omega))$.

This paper is composed of ten sections and two appendices.

§2: we present the compactness of the operator G from $L^2(\Omega)$ to $L^2(0, T; L^2(\partial\Omega))$ (Theorem 1) and the asymptotic behaviour of the singular values of G (Theorem 2).

§3: we derive conditional stability estimates of $\|f\|_{L^2(\Omega)}$ by $\left\|\frac{\partial u(f)}{\partial n}\right\|_{L^2(0,T;L^2(\partial\Omega))}$ with a-priori boundedness on f's (Theorem 3).

§4: we treat a Tikhonov regularization on the basis of Theorem 3 and derive convergence rates of regularized solutions (Theorem 4).

§5: we discuss a discretization in the Tikhonov regularization in §4 and convergence rates of discretized regularized solutions (Theorem 5).

§6: we prove Theorem 1.

§7: we prove Theorem 2.

§8: we prove Theorem 3.

§9: we prove Theorem 4.

§10: we prove Theorem 5.

§2. Ill-posedness of the inverse problem and the singular values

Let us recall that the operator G is defined by (1.6). First we show

Theorem 1. *The operator G is compact from $L^2(\Omega)$ to $L^2(0, T; L^2(\partial\Omega))$.*

Since $GL^2(\Omega)$ is infinitely dimensional (e.g. [24, Theorem 2]), our inverse problem of solving

$$(2.1)\qquad y=Gf$$

for a given $y\in L^2(0, T; L^2(\partial\Omega))$, is ill-posed (e.g. Theorem 2.6 (p. 20) in Baumeister [1]). Moreover by Theorem 1, there exist a sequence $\{\sigma_n\}_{n\geq 1}$ of real numbers and $\{\phi_n\}_{n\geq 1}\subset L^2(\Omega)$ and $\{\psi_n\}_{n\geq 1}\subset L^2(0, T; L^2(\partial\Omega))$ such that

$$\sigma_1\geq\sigma_2\geq\cdots\cdots, \qquad \lim_{n\to\infty}\sigma_n=0, \tag{2.2}$$

$$(\phi_n, \phi_m)_{L^2(\Omega)}=\delta_{mn}$$

$$(\psi_m, \psi_n)_{L^2(0,T;L^2(\partial\Omega))}\equiv\int_0^T\int_{\partial\Omega}\psi_m(x, t)\psi_n(x, t)dS_x dt = \delta_{mn} \qquad (n, m\geq 1) \tag{2.3}$$

and

$$G\phi_n=\sigma_n\psi_n \qquad G^*\psi_n=\sigma_n\phi_n \qquad (n\geq 1) \tag{2.4}$$

(e.g. Baumeister [1], Groetsch [6], Hofmann [7]). Here and henceforth we set

$$\delta_{mn}=\begin{cases}1, & \text{if } n=m,\\ 0, & \text{if } n\neq m\end{cases}$$

and G^* is the adjoint operator $G: L^2(\Omega)\to L^2(0, T; L^2(\partial\Omega))$.

The real numbers σ_n $(n\geq 1)$ and the system $\{\sigma_n, \phi_n, \psi_n\}_{n\geq 1}$ are called respectively the *singular values* and a *singular system* of G. The singular system of G is useful for studying the ill-posed problem $y=Gf$, and in particular, the asymptotic behaviour of the singular values specifies the degree of ill-posedness (e.g. [1], Colton and Kress [4], [6]). As for the asymptotic behaviour, we have

Theorem 2. *We assume* (1.2) *and* (1.4). *There exists a constant* $C_2=C_2(\Omega,T)>0$ *such that*

$$\sigma_n\leq C_2 n^{-\frac{1}{4(r-1)}}, \tag{2.5}$$

provided that we renumber σ_n $(n\geq 1)$ *if necessary. Here we recall that* r *is the spatial dimension.*

This theorem gives an upper bound of the singular values. So far we cannot determine the order of the asymptotic behaviour of σ_n. However, the upper bound gives a lower estimate of the degree of ill-posedness of our inverse problem. That is, we consider perturbations around $f_0\in L^2(\Omega)$ and deviations of y by them. More precisely, we assume $y_0=Gf_0$ for some $f_0\in L^2(\Omega)$. Let data y_0 be perturbed in the direction of ψ_n; $y_0\Rightarrow y_{\varepsilon,n}=y_0+\varepsilon\psi_n$ with $\varepsilon\in\mathbf{R}$. Then by (2.4) the corresponding solution $f_{\varepsilon,n}$ is

$$f_0+\frac{\varepsilon}{\sigma_n}\phi_n$$

and by Theorem 2 we have

$$\frac{\|f_0-f_{\varepsilon,n}\|_{L^2(\Omega)}}{\|y_0-y_{\varepsilon,n}\|_{L^2(0,T;L^2(\partial\Omega))}}=\sigma_n^{-1}\geq C_2^{-1}n^{\frac{1}{4(r-1)}}. \tag{2.6}$$

This means that errors $\|f_0-f_\varepsilon\|_{L^2(\Omega)}$ in solutions may arbitrarily become large with a lower bound (2.6), no matter how small errors $\varepsilon=\|y_0-y_\varepsilon\|_{L^2(0,T;L^2(\partial\Omega))}$ in data may be.

Theorem 2 can be restated by the degree of ill-posedness of our problem $y=Gf$ (e.g. Definition 2.42 (p.31) in [7]): The degree of ill-posedness is greater that or equal to $\frac{1}{4(r-1)}$.

Thus some techniques such as the regularization are necessary for stably solving $y=Gf$ on the basis of data polluted by errors. In §4 and §5, we will discuss a regularization method by Tikhonov.

Remark. We can consider the ill-posedness of our inverse problem in terms of a factorization of G into the ill-posed part and the well-posed part. We define an operator $\tilde{G}$ form $L^2(\Omega)$ to $L^2(0, T; H^{\frac{1}{2}}(\partial\Omega)) \cap H^1(0, T; L^2(\partial\Omega))$ by $\tilde{G}f=\frac{\partial u(f)}{\partial n}$. Moreover let I be the embedding of $L^2(0, T, H^{\frac{1}{2}}(\partial\Omega)) \cap H^1(0, T; L^2(\partial\Omega))$ into $L^2(\partial\Omega\times(0, T))$. Then we can factorize $G: L^2(\Omega)\rightarrow L^2(0, T; (\partial\Omega))$ into the ill-posed factor I and the well-posed factor $\tilde{G}$: $G=I\tilde{G}$. In fact, as is proved in Lemma 2 in §6, both $\tilde{G}$ and $\tilde{G}^{-1}|_{\mathscr{R}(\tilde{G})}$ are continuous, while I is a compact operator. Therefore the ill-posedness in solving $y=Gf$ comes mainly from I. With respect to such factorization, we can refer to Theorem 5.2 in Hofmann [8], where mainly nonlinear ill-posed problems are discussed in the case where the linearized well-posed part has not only a continuous inverse, but also is a surjective (cf. Lemma 2.46 (p.33) in [7]).

§3. Conditional stability with a-priori information

It follows from Theorem 1 in §2 that G^{-1} is not continuous from $L^2(0, T; L^2(\partial\Omega))$ to $L^2(\Omega)$. However if we can assume a-priori information on unknown f's so that f's can be restricted to a compact set $\mathscr{U}$ in $L^2(\Omega)$, then we can restore the stability of G^{-1}. In fact, by a well-known theorem (e.g. Lavrent'ev, Romanov and Shishat·skiĭ [11, p. 28]), the restriction G to the set $\mathscr{U}$ has a continuous inverse G^{-1}: $f_n, f\in\mathscr{U}$ $(n\ge 1)$ and $\|Gf_n-Gf\|_{L^2(0,T;L^2(\partial\Omega))}\rightarrow 0$ as $n\rightarrow\infty$ imply $\|f_n-f\|_{L^2(\Omega)}\rightarrow 0$. The rate of the convergence $\|f_n-f\|_{L^2(\Omega)}\rightarrow 0$, depends on the choice of the admissible set $\mathscr{U}$ of f's and from the general theorem, we cannot, in general, specify the order of continuity of the restricted inverse. On the other hand, for discussions of convergence rates in Tikhonov's regularization treated in §4 and §5, it is necessary to determine the order of continuity. In this paper, adopting spaces of fractional order defined by the elliptic operator (see below) as admissible sets of f's, we will give rates of continuity of restricted inverses.

For defining admissible sets, we will introduce an operator and notations. Let A be the operator in $L^2(\Omega)$ defined by

$$(3.1)\qquad \begin{aligned}&(Au)(x) = -\Delta u(x) \quad (x \in \Omega),\\ &\mathscr{D}(A) = \{u \in H^2(\Omega);\ u|_{\partial\Omega} = 0\} = H^2(\Omega) \cap H_0^1(\Omega).\end{aligned}$$

Then for any $\alpha \in \boldsymbol{R}$, we can define the fractional power A^α (e. g. Pazy [17]) and for $0 \leq \alpha \leq 1$, by Fujiwara [5], we see:

$$(3.2)\qquad \begin{aligned}&\mathscr{D}(A^\alpha) = H^{2\alpha}(\Omega), \quad 0 \leqq \alpha < \frac{1}{4},\\ &\mathscr{D}(A^\alpha) = H_0^{2\alpha}(\Omega), \quad \frac{1}{4} < \alpha \leq 1,\ \alpha \neq \frac{3}{4}\\ &\mathscr{D}(A^{\frac{1}{4}}) \subset H^{\frac{1}{2}}(\Omega), \qquad \mathscr{D}(A^{\frac{3}{4}}) \subset H^{\frac{3}{2}}(\Omega),\end{aligned}$$

and there exists a constant $C = C(\Omega, \alpha) > 0$ such that

$$(3.3)\qquad \|f\|_{H^{2\alpha}(\Omega)} \leq C\|A^\alpha f\|_{L^2(\Omega)} \qquad (f \in \mathscr{D}(A^\alpha))$$

for $\alpha \geq 0$ and that

$$(3.4)\qquad C^{-1}\|A^\alpha f\|_{L^2(\Omega)} \leq \|f\|_{H^{2\alpha}(\Omega)} \leq C\|A^\alpha f\|_{L^2(\Omega)} \qquad (f \in \mathscr{D}(A^\alpha))$$

for $\alpha \in [0, 1]$, $\alpha \neq \frac{1}{4}, \frac{3}{4}$. In particular, for $m \in \boldsymbol{N}$, we have $\mathscr{D}(A^\alpha) = \{f \in H^{2m}(\Omega);\ A^j f|_{\partial\Omega} = 0\ (0 \leq j \leq m-1)\}$. For given $\alpha \geq 0$ and $M > 0$, we set

$$(3.5)\qquad \mathscr{U}_{M,\alpha} = \{f \in \mathscr{D}(A^\alpha);\ \|A^\alpha f\|_{L^2(\Omega)} \leq M\}$$

as an admissible set of f's.

Our answer to specification of order of the restricted inverse on $G\mathscr{U}_{M,\alpha}$ is

Theorem 3. *We assume* (1.2), (1.4) *and*

$$(3.6)\qquad \lambda \in C^\infty[0, T].$$

Let $\alpha \geq 0$ *and* $M > 0$ *be arbitrarily given. Then there exists a constant* $C_3 = C_3(\Omega, T, \lambda, \alpha) > 0$ *such that*

$$(3.7)\qquad \|f - g\|_{L^2(\Omega)} \leq C_3 M^{\frac{1}{2\alpha+1}} \left\| \frac{\partial u(f)}{\partial n} - \frac{\partial u(g)}{\partial n} \right\|_{L^2(0,T;L^2(\partial\Omega))}^{\frac{2\alpha}{2\alpha+1}}$$

for any $f, g \in \mathscr{U}_{M,\alpha}$.

This theorem asserts that the restricted inverse is Hölder continuous with exponent $\frac{2\alpha}{2\alpha+1}$ which increases as the degree α of a-priori regularity on f increases.

If we do not assume any a-priori regularity on f and g, that is, $\alpha = 0$, then the estimate (3.7) is trivial.

Remark. In a similar inverse problem for a parabolic equation, if we adopt norms such as $\|\cdot\|_{H^1(0,T;L^2(\partial\Omega))}$ for boundary data, we can merely obtain a much weaker conditional stability estimate, namely, a logarithmic continuity even with a similar a-priori information (Yamamoto [22], [23]). The difference comes from the smoothing property in the parabolic equation. Under an assumption on λ, we can specify the norm for boundary data admitting an equivalent stablility estimate for the $L^2(\Omega)$-norm of f (Vu Kim Tuan and Yamamoto [21]).

§4. Convergence rates of regularized solutions by a Tikhonov method

Let us proceed to stably solving

$$y=Gf \tag{4.1}$$

with respect to $f\in L^2(\Omega)$ for a given $y\in GL^2(\Omega)\subset L^2(0, T; L^2(\partial\Omega))$ around exact data. That is, we consider the following reconstruction problem of f_0 from inexact data y: Let $y_0\in GL^2(\Omega)$, that is, let a solution f_0 to (4.1) exist. Moreover, as available data of y_0, we can observe only y_ε which is polluted by L^2-errors:

$$\|y_\varepsilon-y_0\|_{L^2(0,T;L^2(\partial\Omega))}\leq\varepsilon. \tag{4.2}$$

This is very usual in practical applications. Then we are required to search for approximate solution $\tilde{f}_\varepsilon$ such that

$$\|\tilde{f}_\varepsilon-f_0\|_{L^2(\Omega)}\to 0 \text{ as } \varepsilon\downarrow 0.$$

Moreover it is very desirable to specify the convergence rate of $\|\tilde{f}_\varepsilon-f_0\|_{L^2(\Omega)}$. Here it should be noticed that we do not know whether or not $y_\varepsilon\in GL^2(\Omega)$, so that the equation $y_\varepsilon=G\tilde{f}_\varepsilon$ does not necessarily possess a solution $\tilde{f}_\varepsilon$. Even if we can choose $y_{\varepsilon_n}(n\geq 1)$ such that $y_{\varepsilon_n}\in GL^2(\Omega)$ and $\|y_{\varepsilon_n}-y_0\|_{L^2(0,T;L^2(\partial\Omega))}\to 0$ as $n\to\infty$, by Theorem 1 we can not always conclude that $\|\tilde{f}_{\varepsilon_n}-f_0\|_{L^2(\Omega)}\to 0$.

For overcoming these difficulties, various regularization techniques are proposed. In this section, we discuss a regularized method with a-priori information (e.g. Baumeister [Chapter 6, 1], Natterer [15], [16], Tikhonov and Arsenin [20, Chapter 2]) and derive convergence-rates of regularized solutions toward f_0. We assume that a-priori information of f_0 is available:

$$f_0\in\mathcal{D}(A^\alpha),\ \|A^\alpha f_0\|_{L^2(\Omega)}\leq E \tag{4.3}$$

with some constants $\alpha>0$ and $E>0$.
The Tikhonov regularization method which we discuss, is formulated as follows.

Minimize

$$F(\varepsilon, E, \alpha, y_\varepsilon)(f)\equiv\|Gf-y_\varepsilon\|^2_{L^2(0,T;L^2(\partial\Omega))}+\frac{\varepsilon^2}{E^2}\|A^\alpha f\|^2_{L^2(\Omega)}. \tag{4.4}$$

Remark. It is possible to discuss convergence rates of regularized solutions in another regularization (e.g. Groetsch [6, Chapter 3]) which requires no a-priori information of higher regularity of an exact solution. In a forthcoming paper, we will treat such a regularization on the basis of the factorization mentioned in Remark in §2.

The regularization adopted here is with term prescribing higher regularity (i.e. $\|A^{\alpha}f\|^2_{L^2(\Omega)}$), and a similar method is discussed in relation with the Radon transformation (Natterer [15], [16]). For his argument, detailed information of the degree of ill-posedness (e.g. (1.1) in [15]) is necessary, while in our case, it is difficult to obtain such information. Thus we must modify his approach for our inverse problem.

First we show existence and stability of the minimizer of (4.4).

Proposition 1. *Let $\alpha>0$, $\varepsilon>0$ and $E>0$ be given.*
(1) *For given $y\in L^2(0, T; L^2(\partial\Omega))$, there exists a unique minimizer $f=f(y)\in\mathfrak{D}(A^{\alpha})$ of $F(\varepsilon, E, \alpha, y)$.*
(2) *For $y_1, y_2\in L^2(0, T; L^2(\partial\Omega))$, there exists a constant $C=C\left(\alpha, \frac{\varepsilon}{E}\right)>0$ such that*

$$\|f(y_1)-f(y_2)\|_{L^2(\Omega)}\le C\|y_1-y_2\|_{L^2(0,T;L^2(\partial\Omega))}. \tag{4.5}$$

The proof of this proposition follows, for example, from Theorem 3.7 (p. 35) in [1], and for convenience, we will prove it in Appendix I.

This proposition means that for fixed $\varepsilon>0$, we can construct the minimizer f_ε stably against deviations of y_ε provided that the error bound of y_ε's is ε, that is, y_ε's satisfy (4.2). Furthermore we can show that $\lim_{\varepsilon\downarrow 0} f_\varepsilon=f_0$ (e.g. [1]), but so far we do not know the rate of convergence. With our formulation (4.4) with (4.3), we have

Theorem 4. *Let $f_\varepsilon=f(y_\varepsilon)$ be a unique minimizer of the regularized problem (4.4) for $y_\varepsilon\in L^2(0, T; L^2(\partial\Omega))$ satisfying (4.2). Let us assume (4.3). Then there exists a constant $C_3=C_3(\Omega, T, \lambda, \alpha)>0$ such that*

$$\|f_\varepsilon-f_0\|_{L^2(\Omega)}\le C_3E^{\frac{1}{2\alpha+1}}\varepsilon^{\frac{2\alpha}{2\alpha+1}}. \tag{4.6}$$

Here C_3 is independent of E and ε.

The larger $\alpha>0$ we choose for a-priori information on regularity of f_0, the better the convergence rate of regularized solutions toward f_0 is. However the exponent of convergence rate can not exceed 1.

§5. Discretization in the Tikhonov method

In this section, we discuss the problem of minimizing (4.4) in a discretized version, which is a modification of an approach in Natterer [15], Baumeister [1, pp. 109-111] and establish a convergence result of discretized regularized solutions toward the exact solution. We do not intend that this section is devoted to a complete description concerning discretization, and we do not mention concrete discretizations in $L^2(\Omega)$ and $L^2(0, T; L^2(\partial\Omega))$.

We assume (4.3) for some $\alpha>0$ and $E>0$. Let $\{X_h\}_{h>0}$ be a family of subspaces of $L^2(\Omega)$ such that the following properties hold.

$$X_h\subset\mathscr{D}(A^\alpha),\ \dim X_h<\infty. \tag{5.1}$$

There exist a linear operator I_h: $\mathscr{D}(A^\alpha)\to X_h$ and $\beta>0$ such that

$$\|f-I_hf\|_{L^2(\Omega)}\le Ch^\beta\|A^\alpha f\|_{L^2(\Omega)} \qquad (f\in\mathscr{D}(A^\alpha)) \tag{5.2}$$

and

$$\|A^\alpha I_hf\|_{L^2(\Omega)}\le C\|A^\alpha f\|_{L^2(\Omega)} \qquad (f\in\mathscr{D}(A^\alpha)). \tag{5.3}$$

Here $C>0$ is a constant independent of h.

Example for $X_h\,(h>0)$. Let

$$\Omega\subset\mathbf{R}^2 \tag{5.4}$$

be a bounded domain with smooth boundary $\partial\Omega$ and for $h>0$ let $T_h=\{K\}$ be a triangulation of Ω such that

$$h_K=\text{the diameter of } K\in T_h$$

$$\rho_K=\text{the diameter of the circle inscribed in } K$$

$$h=\max_{K\in T_h} h_K$$

$$\Omega_h=\bigcup_{K\in T_h} K$$

(e. g. Ciarlet [3], Johnson [9], Raviart and Thomas [18]). We assume that there exists a constant $\tau>0$ independent of $h>0$ such that

$$\frac{\rho_K}{h_K}\ge\tau \tag{5.5}$$

for any $K\in T_h$. Moreover we set

$$X_h=\{v\in C^0(\overline{\Omega});\ v|_K \text{ is linear for any } K\in T_h \text{ and } v|_{\Omega\setminus\Omega_h}=0\}. \tag{5.6}$$

For $f\in\mathscr{D}(A^{\frac{1}{2}})$, we define $I_hf\in X_h$ by

$$(I_hf)(x)=\begin{cases} u(x), & \text{if } x \text{ is a node which does not belong to } \partial\Omega_h \\ 0, & \text{if } x \text{ is a node} \in\partial\Omega_h. \end{cases} \tag{5.7}$$

Then we can prove

Lemma 1. *We assume* (5.5). *Then X_h and I_h given by* (5.6) *and* (5.7) *satisfy* (5.1) - (5.3) *for $\alpha=\frac{1}{2}$ and $\beta=1$.*

For completeness, we will prove this lemma in Appendix II.

Let us return to the general framework (5.1) - (5.3). The discretized problem is to determine the minimizer $f_{\varepsilon,h}$ of

$$\|Gf_h - y_\varepsilon\|^2_{L^2(0,T;L^2(\partial\Omega))} + \frac{\varepsilon^2}{E^2}\|A^\alpha f_h\|^2_{L^2(\Omega)} \tag{5.8}$$

in $X_h \subset \mathcal{D}(A^\alpha)$. Similarly to Proposition 1 in §4, we see that there exists a unique minimizer $f_{\varepsilon,h} \in X_h$ for a given y_ε.

Our next problem is concerning the convergence rate of $f_{\varepsilon,h}$ toward f_0 as $h \downarrow 0$ and $\varepsilon \downarrow 0$.

Theorem 5. *Let us assume* (4.3) *for the exact solution* f_0. *If for a given* $\gamma > 0$, *we choose*

$$h = O(\varepsilon^\gamma), \tag{5.9}$$

then there exists a constant $C_4 = C_4(\Omega, T, \lambda, \alpha, \beta) > 0$ *independent of* ε, h, E *and the choice of* y_ε *such that*

$$\|f_{\varepsilon,h} - f_0\|_{L^2(\Omega)} \le \begin{cases} C_4(E+1)^{\frac{2\alpha+2}{2\alpha+1}}\varepsilon^{\frac{2\alpha}{2\alpha+1}}, & \text{if } \gamma \ge \frac{1}{\beta} \\ C_4(E+1)^{\frac{2\alpha+2}{2\alpha+1}}\varepsilon^{\beta\gamma - \frac{1}{2\alpha+1}}, & \text{if } \frac{1}{2\alpha+1}\frac{1}{\beta} < \gamma \le \frac{1}{\beta}. \end{cases} \tag{5.10}$$

If $\gamma \le \frac{1}{2\alpha+1}\frac{1}{\beta}$, then $\beta\gamma - \frac{1}{2\alpha+1} \le 0$, namely, (5.10) is meaningless for the convergence of discretized regularized solutions. The optimal choice of γ's is seen to be $\gamma \ge \frac{1}{\beta}$. The fastest rate guaranteed by this theorem is $\frac{2\alpha}{2\alpha+1}$ and for the rate, $\gamma \ge \frac{1}{\beta}$ should be satisfied. Moreover we cannot improve the rate $\frac{2\alpha}{2\alpha+1}$, even though we choose finer discretization $\left(\text{i. e. } \gamma > \frac{1}{\beta}\right)$, provided that α and β are fixed. The exponent α corresponds to a-priori information on regularity of the exact solution and, in general, the greater α is, the greater β we can take in the finite element method (e. g. Johnson [9], Raviart and Thomas [18]). Therefore the greater β we can take, the less γ (i. e. the coarser h) gives the fastest rate $\frac{2\alpha}{2\alpha+1}$. Furthermore the greater α is, the greater the fastest rate $\frac{2\alpha}{2\alpha+1}$ is, but cannot exceed 1.

In the example, from Lemma 1, we can rewrite (5.10) as follows: if $h = O(\varepsilon^\gamma)$, then

$$\|f_{\varepsilon,h} - f\|_{L^2(\Omega)} \le \begin{cases} C_4(E+1)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}, & \text{if } \gamma \ge 1 \\ C_4(E+1)^{\frac{3}{2}}\varepsilon^{\gamma - \frac{1}{2}}, & \text{if } \frac{1}{2} < \gamma \le 1, \end{cases}$$

under (5.4)-(5.7).

§6. Proof of Theorem 1

First we introduce Sobolev spaces on $\partial\Omega\times(0, T)$. For $r, s\geq 0$, we set

$$H^{r,s}(\partial\Omega\times(0, T))=L^2(0, T; H^r(\partial\Omega))\cap H^s(0, T; L^2(\partial\Omega)) \tag{6.1}$$

(e. g. Lions and Magenes [13, Vol. II, pp. 6-8]), and this is a Hilbert space with the inner product $(\cdot,\cdot)_{r,s}$ and the norm $\|\cdot\|_{r,s}$:

$$\begin{aligned}(u, v)_{r,s}&=(u, v)_{L^2(0,T;H^r(\partial\Omega))}+(u, v)_{H^s(0,T;L^2(\partial\Omega))}\\ \|u\|_{r,s}&=(\|u\|^2_{L^2(0,T;H^r(\partial\Omega))}+\|u\|^2_{H^s(0,T;L^2(\partial\Omega))})^{\frac{1}{2}}.\end{aligned} \tag{6.2}$$

In particular, we can rewrite $H^{0,0}(\partial\Omega\times(0; T))=L^2(0, T, L^2(\partial\Omega))$. In order to prove Theorem 1, it is sufficient to prove

Lemma 2. *There exists a constant $C=C(\Omega, T)>0$ independent* of $f\in L^2(\Omega)$ *such that*

$$C^{-1}\|f\|_{L^2(\Omega)}\leq\|Gf\|_{\frac{1}{2},1}\leq C\|f\|_{L^2(\Omega)} \tag{6.3}$$

for any $f\in L^2(\Omega)$.

Let Lemma 2 be proved. Then since the embedding $H^{\frac{1}{2}}(\partial\Omega)\rightarrow L^2(\partial\Omega)$ is compact, the embedding $H^{\frac{1}{2},1}(\partial\Omega\times(0, T))=L^2(0, T; H^{\frac{1}{2}}(\partial\Omega))\cap H^1(0,T;L^2(\partial\Omega))\rightarrow H^{0,0}(\partial\Omega\times(0, T))$ is compact (Theorem 2.1 (p. 271) in Temam [19]). Therefore the operator G is compact from $L^2(\Omega)$ to $L^2(0, T; L^2(\partial\Omega))$.

Now we proceed to

Proof of Lemma 2. The left inequality is easily seen from (1.5) in Theorem 0, by $\|y\|_{H^1(0,T;L^2(\partial\Omega))}\leq\|y\|_{\frac{1}{2},1}$. For the right inequality, we will proceed as follows. Since $\lambda\in C^1[0, T]$, we can apply Theorem 2.1 (pp. 95-96) in Lions and Magenes [13, Vol. II], so that we obtain

$$\|u(f)\|_{L^2(0,T;L^2(\partial\Omega))}\leq C\|f\|_{L^2(\Omega)}.$$

Here $C=C(\Omega, T)>0$ is a constant which is independent of f. By the trace theorem (e. g. Theorem 9.4 (pp. 41-42) in [13, Vol. I]), we get

$$\begin{aligned}\|Gf\|_{L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))}&=\left\|\frac{\partial u(f)}{\partial n}\right\|_{L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))}\\ &\leq C\|u(f)\|_{L^2(0,T;H^2(\Omega))}\leq C\|f\|_{L^2(\Omega)}\end{aligned} \tag{6.4}$$

for any $f\in L^2(\Omega)$.

Next we have to prove

(6.5)
$$\left\| \frac{\partial}{\partial t} \frac{\partial u(f)}{\partial n} \right\|_{L^2(0,T;L^2(\partial\Omega))} \leq C\|f\|_{L^2(\Omega)}$$

for any $f \in L^2(\Omega)$.

Let z be the weak solution to

$$\left\{ \begin{array}{ll} z''(x, t) = \Delta z(x, t) + \lambda'(t) f(x) & (x \in \Omega, t>0) \\ z(x, 0) = 0, z'(x, 0) = \lambda(0) f(x) & (x \in \Omega) \\ z(x, t) = 0 & (x \in \partial\Omega, t>0). \end{array} \right\}$$

Since $\lambda' f \in L^1(0, T; L^2(\partial\Omega))$ and $\lambda(0) f \in L^2(\Omega)$, it follows from Lemma 3.6 (p. 39) and Théorème 4.1 (p. 44) in Lions [12] that

(6.6)
$$z \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$$

and

(6.7)
$$\begin{aligned} \left\| \frac{\partial z}{\partial n} \right\|_{L^2(0,T;L^2(\partial\Omega))} &\leq C(\|\lambda' f\|_{L^1(0,T;L^2(\partial\Omega))} + \|\lambda(0) f\|_{L^2(\Omega)}) \\ &\leq C\|f\|_{L^2(\Omega)} \end{aligned}$$

for any $f \in L^2(\Omega)$. By (6.6) we can set

$$\tilde{u}(x, t) = \int_0^t z(x, s) ds \quad (x \in \Omega, 0<t<T),$$

so that $\tilde{u} \in C^1([0, T]; H_0^1(\Omega)) \cap C^2([0, T]; L^2(\Omega))$ and $\tilde{u}$ satisfies (1.1). By uniqueness of weak solution (Lemme 3.6 (p.39) in [12]), we get

$$u(f)(x, t) = \tilde{u}(x, t) \qquad (x \in \Omega, 0<t<T)$$

and

$$z(x, t) = u(f)'(x, t) \qquad (x \in \Omega, 0<t<T).$$

Therefore we can prove (6.5) by using (6.7). Combining (6.4) with (6.5), we complete the proof of Lemma 2.

§7. Proof of Theorem 2

We will divide the proof into the following four steps.

FIRST STEP. We factorize G: $L^2(\Omega) \to L^2(0, T; L^2(\partial\Omega))$ as follows. Let I be the embedding of $H^{\frac{1}{2},1}(\partial\Omega \times (0, T))$ into $L^2(0, T; L^2(\partial\Omega))$, and $\tilde{G}$ be the operator from $L^2(\Omega)$ to $H^{\frac{1}{2},1}(\partial\Omega \times (0, T))$ such that

$$\tilde{G} f = G f \qquad (f \in L^2(\Omega)).$$

Then we have

(7.1)
$$G f = I \tilde{G} f \qquad (f \in L^2(\Omega))$$

and we can rewrite (6.3) as

$$C^{-1}\|\tilde{G}f\|_{\frac{1}{2},1}\le\|f\|_{L^2(\Omega)}\le C\|\tilde{G}f\|_{\frac{1}{2},1} \tag{7.2}$$

for any $f\in L^2(\Omega)$. Here we set $C=C(\Omega,T)>0$.

SECOND STEP. The operator I is compact from $H^{\frac{1}{2},1}(\partial\Omega\times(0,T))$ to $L^2(0,T;L^2(\partial\Omega))$, so that I has a singular system $\{\tau_n,\zeta_n,\theta_n\}_{n\ge1}$:

$$\tau_1\ge\tau_2\ge\cdots\ge0,\quad \lim_{n\to\infty}\tau_n=0 \tag{7.3}$$

$$\begin{aligned}&\zeta_n\in H^{\frac{1}{2},1}(\partial\Omega\times(0,T)),\quad (\zeta_n,\zeta_m)_{\frac{1}{2},1}=\delta_{mn}\\ &\theta_n\in H^{0,0}(\partial\Omega\times(0,T)),\quad (\theta_n,\theta_m)_{0,0}=\delta_{mn},\quad (m,n\ge1)\end{aligned} \tag{7.4}$$

$$I\zeta_n=\tau_n\theta_n,\qquad I^*\theta_n=\tau_n\zeta_n\qquad (n\ge1) \tag{7.5}$$

$$y=\sum_{n=1}^{\infty}(y,\zeta_n)_{\frac{1}{2},1}\zeta_n,\quad y\in H^{\frac{1}{2},1}(\partial\Omega\times(0,T)) \tag{7.6}$$

$$z=\sum_{n=1}^{\infty}(z,\theta_n)_{0,0}\theta_n,\quad z\in H^{0,0}(\partial\Omega\times(0,T))=L^2(0,T;L^2(\partial\Omega)). \tag{7.7}$$

The series in (7.6) and (7.7) are convergent respectively in $H^{\frac{1}{2},1}(\partial\Omega\times(0,T))$ and $H^{0,0}(\partial\Omega\times(0,T))$. Since it is not easy to directly estimate the singular values themselves of G, we will do via the singular values $\tau_n\,(n\ge1)$ of I. To this end, we will show

Lemma 3. *Let $\sigma_n\,(n\ge1)$ be the singular values of G. Then there exists a constant $C=C(\Omega,T)>0$ such that*

$$\sigma_n\le C\tau_n\qquad (n\ge1). \tag{7.8}$$

Proof of Lemma 3. By (7.1) and the min-max principle (e.g. Theorem 4.19 (p. 65) in [1], Lemma 2.44 (p. 32) in [7]), we have

$$\sigma_{n+1}=\inf\left\{\sup\left\{\frac{\|I\tilde{G}f\|_{0,0}}{\|f\|_{L^2(\Omega)}};f\perp f_k\,(1\le k\le n),f\ne0\right\};f_1,\cdots f_n\in L^2(\Omega)\right\}. \tag{7.9}$$

Here and henceforth we write $f\perp g$ when $(f,g)_{L^2(\Omega)}=0$. Therefore by the left inequality in (7.2), we get

$$\begin{aligned}\sigma_{n+1}&=\inf\left\{\text{suf}\left\{\frac{\|I\tilde{G}f\|_{0,0}\|\tilde{G}f\|_{\frac{1}{2},1}}{\|\tilde{G}f\|_{\frac{1}{2},1}\|f\|_{L^2(\Omega)}};f\perp f_k\,(1\le k\le n),f\ne0\right\};f_1,\cdots,f_n\in L^2(\Omega)\right\}\\ &\le C\inf\left\{\sup\left\{\frac{\|I\tilde{G}f\|_{0,0}}{\|\tilde{G}f\|_{\frac{1}{2},1}};f\perp f_k\,(1\le k\le n),f\ne0\right\};f_1,\cdots,f_n\in L^2(\Omega)\right\}\\ &=C\inf\left\{\sup\left\{\frac{\|Iy\|_{0,0}}{\|y\|_{\frac{1}{2},1}};\ y\perp(\tilde{G}^*)^{-1}f_k\,(1\le k\le n),\ y\ne0,\ y\in\mathcal{R}(\tilde{G})\right\};\right.\end{aligned}$$

$$f_1, \cdots, f_n \in L^2(\Omega)\Big\}.$$

At the last equality we note that $(\widetilde{G}^*)^{-1}$ exists. Moreover

$$(\widetilde{G}^*)^{-1}L^2(\Omega) = H^{\frac{1}{2},1}(\partial\Omega\times(0, T)) \tag{7.10}$$

since $\widetilde{G}$ is injective and $\widetilde{G}L^2(\Omega)$ can be proved to be closed in $H^{\frac{1}{2},1}(\partial\Omega\times(0, T))$ by (7.2). Consequently we get

$$\begin{aligned}\sigma_{n+1} &\le C \inf\Big\{\sup\Big\{\frac{\|Iy\|_{0,0}}{\|y\|_{\frac{1}{2},1}}; y\perp y_k(1\le k\le n), y\ne 0, y\in\mathcal{R}(G)\Big\};\\ &\quad y_1, \cdots, y_n\in H^{\frac{1}{2},1}(\partial\Omega\times(0, T))\Big\}\\ &\le C \inf\Big\{\sup\Big\{\frac{\|Iy\|_{0,0}}{\|y\|_{\frac{1}{2},1}}; y\perp y_k(1\le k\le n), y\ne 0\Big\};\\ &\quad y_1, \cdots, y_n\in H^{\frac{1}{2},1}(\partial\Omega\times(0, T))\Big\}\\ &= C\tau_{n+1}\end{aligned}$$

for $n\ge 1$. At the last equality, we use the min-max principle for the embedding I: $H^{\frac{1}{2},1}(\partial\Omega\times(0; T))\to L^2(0, T; L^2(\partial\Omega))$. Thus the proof of Lemma 3 is complete.

Remark. Since $\mathcal{R}(G)\ne H^{\frac{1}{2},1}(\partial\Omega\times(0, T))$, we do not know whether we can get a reverse estimate $\tau_n\le C'\sigma_n\,(n\ge 1)$.

THIRD STEP. By Lemma 3, for the proof of Theorem 2, it is sufficient to discuss the asymptotic behaviour of τ_n $(n\ge 1)$. In this step, we construct a singular system of I. For this, let us introduce notations and an operator. Let $\Delta_{\partial\Omega}$ be the Laplace-Beltrami operator on $\partial\Omega$. There exist countable eigenvalues of $-\Delta_{\partial\Omega}$ and we number them repeatedly according to their multiplicities:

$$0=\mu_0\le\mu_1\le\cdots\to\infty \tag{7.11}$$

(e.g. Minakshisundaram and Pleijel [14]). Let η_n be the eigenfunction of $-\Delta_{\partial\Omega}$ for $\mu_n\,(n\ge 0)$ such that

$$(\eta_n, \eta_m)_{L^2(\Omega)}=\delta_{mn} \qquad (m, n\ge 0). \tag{7.12}$$

Furthermore we define $e_n\in L^2(0, T)$ $(n\ge 0)$ by

$$e_n(t)=\begin{cases}\dfrac{\sqrt{2}}{\sqrt{T}}\cos\dfrac{n\pi t}{T}, & n\ge 1\\ \dfrac{1}{\sqrt{T}}, & n=0.\end{cases} \tag{7.13}$$

Then as is direclty seen, we have

(7.14) $$(e_n, e_m)_{L^2(0,T)}=\delta_{mn} \qquad (m, n\geq 0).$$

The purpose in this step is to prove the following proposition.

Proposition 2. *We set*

(7.15) $$\sigma_{mn}=\left(\frac{n^2\pi^2}{T^2}+\mu_m^{\frac{1}{2}}\right)^{-\frac{1}{2}} \qquad (m, n\geq 0).$$

The operator I has a singular system $\{\sigma_{nm}, \sigma_{nm}e_n(t)\eta_m(x), e_n(t)\eta_m(x)\}_{n,m\geq 0}$.

This proposition is proved by Lemmata 4-6.

Lemma 4. *The system $\{e_n(t)\ \eta_m(x)\}_{n,m\geq 0}$ is an orthonormal basis in $H^{0,0}(\partial\Omega\times(0,T))=L^2(0,T;L^2(\partial\Omega))$.*

Lemma 5. *We set*

(7.16) $$\langle u, v\rangle_{\frac{1}{2},1}=\sum_{n,m=0}^{\infty}\sigma_{nm}^{-2}(u, e_n\eta_m)_{0,0}(v, e_n\eta_m)_{0,0}$$

provided that the right hand side is convergent. Then $<\cdot,\cdot>_{\frac{1}{2},1}$ is an inner product defining the norm in $H^{\frac{1}{2},1}(\partial\Omega\times(0,T))$ provided that a constant multiplier is neglected.

Lemma 6. *The system $\{\sigma_{nm}e_n(t)\eta_m(x)\}_{n,m\geq 0}$ is an orthonormal basis in $H^{\frac{1}{2},1}(\partial\Omega\times(0,T))$.*

Let Lemmata 4-6 be proved. Then we can complete the proof of Proposition 2 as follows. For the proof, by Lemmata 4 and 6, it is sufficient to verify

(7.17) $$\begin{aligned} &I(\sigma_{nm}e_n\eta_m)=\sigma_{nm}\times e_n\eta_m \\ &I^*(e_n\eta_m)=\sigma_{nm}\times\sigma_{nm}e_n\eta_m \qquad (m, n\geq 0). \end{aligned}$$

The first equality in (7.17) is trivial. Let us verify the second. Since $<\cdot,\cdot>_{\frac{1}{2},1}$ is an inner product in $H^{\frac{1}{2},1}(\partial\Omega\times(0,T))$ by Lemma 5 and $(Iu, e_n\eta_m)_{0,0}=<u, I^*e_n\eta_m>_{\frac{1}{2},1}$ for any $u\in H^{\frac{1}{2},1}(\partial\Omega\times(0,T))$, we see

$$(u, e_n\eta_m)_{0,0}=(u, \sigma_{nm}^{-2}I^*e_n\eta_m)_{0,0}$$

for any $u\in H^{\frac{1}{2},1}(\partial\Omega\times(0,T))$, which is the second inequality in (7.17).

Now we proceed to proofs of Lemmata 4-6.

Proof of Lemma 4. The orthonormality is directly seen from (7.12) and (7.14). Assume that $y\in H^{0,0}(\partial\Omega\times(0,T))$ satisfies $(y, e_n\eta_m)_{0,0}=0\ (m, n\geq 0)$. We have to prove that $y=0$. For this, we have

$$0=(y, e_n\eta_m)_{0,0}=\int_0^T(y(\cdot,t), \eta_m)_{L^2(\partial\Omega)}e_n(t)dt \quad (m, n\geq 0).$$

By the completeness of $\{e_n\}_{n\geq 0}$ in $L^2(0, T)$, this implies $(y(\cdot, t), \eta_m)_{L^2(\partial\Omega)} = 0$ $(m \geq 0)$ for almost all $t \in [0, T]$. Since $\{\eta_m\}_{m\geq 0}$ is an orthonormal basis in $L^2(\partial\Omega)$ (e. g. Chavel [2]), we reach $y(x, t) = 0$ for almost all $x \in \partial\Omega$, $t \in [0, T]$. Thus the proof of Lemma 4 is complete.

Proof of Lemma 5. We note

$$(u, v)_{H^{\frac{1}{2}}(\partial\Omega)} = \sum_{m=0}^{\infty} \mu_m^{\frac{1}{2}} (u, \eta_m)_{L^2(\partial\Omega)}$$

provided that a constant multiplier is neglected (e. g. [13, p. 37, Vol. I]). Let $u \in H^{\frac{1}{2},1}(\partial\Omega \times (0, T))$. We have

$$\begin{aligned}
\|u\|^2_{\frac{1}{2},1} &= \|u\|^2_{L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;L^2(\partial\Omega))} \\
&= \int_0^T \sum_{m=0}^{\infty} \mu_m^{\frac{1}{2}} (u(\cdot, t), \eta_m)^2_{L^2(\partial\Omega)} dt \\
&\quad + \int_0^T \sum_{n,m=0}^{\infty} \left(\frac{\partial u}{\partial t}(\cdot, t), \eta_m\right)^2_{L^2(\partial\Omega)} dt \\
&= \sum_{n,m=0}^{\infty} \mu_m^{\frac{1}{2}} |((u, \eta_m)_{L^2(\partial\Omega)}, e_n)_{L^2(0,T)}|^2 \\
&\quad + \sum_{n,m=0}^{\infty} \left(\left(\frac{\partial u}{\partial t}(\cdot, t), \eta_m\right)_{L^2(\partial\Omega)}, \frac{\sqrt{2}}{\sqrt{T}} \sin\frac{n\pi t}{T}\right)^2_{L^2(0,T)} \\
&= \sum_{n,m=0}^{\infty} \mu_m^{\frac{1}{2}} (u, e_n\eta_m)^2_{0,0} \\
&\quad + \sum_{n,m=0}^{\infty} \left(\int_0^T \frac{\sqrt{2}}{\sqrt{T}} \sin\frac{n\pi t}{T} \left(\frac{\partial u}{\partial t}(\cdot, t), \eta_m\right)_{L^2(\partial\Omega)} dt\right)^2.
\end{aligned}$$

At the first equality and the second term in the third equality, we use respectively

$$\|\phi\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = \sum_{m=0}^{\infty} \mu_m^{\frac{1}{2}} (\phi, \eta_m)^2_{L^2(\partial\Omega)}$$

and

$$\sum_{n=0}^{\infty} \left(g, \frac{\sqrt{2}}{\sqrt{T}} \sin\frac{n\pi t}{T}\right)^2_{L^2(0,T)} = \|g\|^2_{L^2(0,T)}$$

for any $g(t) = \left(\frac{\partial u}{\partial t}(\cdot, t), \eta_m\right)_{L^2(\Omega)} \in L^2(0, T)$. Moreover by integration by parts, we see

$$\int_0^T \frac{\sqrt{2}}{\sqrt{T}} \sin\frac{n\pi t}{T}\left(\frac{\partial u}{\partial t}(\cdot, t), \eta_m\right)_{L^2(\partial\Omega)} dt$$
$$= \frac{n\pi}{T}((u, \eta_m)_{L^2(\partial\Omega)}, e_n)_{L^2(0,T)}$$
$$= \frac{n\pi}{T}(u, e_n\eta_m)_{0,0}.$$

Therefore we get

$$\|u\|^2_{\frac{1}{2},1} = \sum_{n,m=0}^{\infty} \left(\mu_m^{\frac{1}{2}} + \frac{n^2\pi^2}{T^2}\right)(u, e_n\eta_m)^2_{0,0}$$
$$= \sum_{n,m=0}^{\infty} \sigma_{nm}^{-2}(u, e_n\eta_m)^2_{0,0}.$$

Thus the proof of Lemma 5 is complete.

Proof of Lemma 6. The orthonormality of the system in $H^{\frac{1}{2},1}(\partial\Omega\times(0, T))$ follows from Lemmata 4 and 5. Noting that $\langle u, \sigma_{nm}e_n\eta_m\rangle_{\frac{1}{2},1} = \sigma_{nm}^{-1}(u, e_n\eta_m)_{0,0}$, we can readily get that $\{\sigma_{nm}e_n\eta_m\}_{m,n\geq 0}$ is a basis in $H^{\frac{1}{2},1}(\partial\Omega\times(0, T))$.

FOURTH STEP. We renumber the singular values σ_{nm} $(m, n\geq 0)$ with double suffix which is given by (7.15). We define the rearrangement $\{\tau_l\}_{l\geq 1}$ of $\{\sigma_{nm}\}_{m,n\geq 0}$ by

$$\tau_l = \begin{cases} \sigma_{l-k^2-1,k}, & \text{if } k^2+1\leq l\leq k^2+k+1 \\ \sigma_{k,(k+1)^2-l}, & \text{if } k^2+k+1\leq l\leq (k+1)^2, \end{cases} \tag{7.18}$$

for $k\geq 0$. In other words,

$$\begin{aligned} &\tau_{k^2+1+n} = \sigma_{nk} \quad (0\leq n\leq k) \\ &\tau_{(k+1)^2-m} = \sigma_{km} \quad (0\leq m\leq k). \end{aligned} \tag{7.18}$$

We will derive the asymptotic behaviour of the τ_l $(l\geq 1)$. For a given $l\in\boldsymbol{N}$, we fix $k\in\boldsymbol{N}$ such that $k^2+1\leq l\leq (k+1)^2$. Then by (7.15) we have

$$\sigma_{kk}\leq\tau_l\leq\max\{\sigma_{0k}, \sigma_{k0}\},$$

namely,

$$\left(\frac{k^2\pi^2}{T^2}+\mu_k^{\frac{1}{2}}\right)^{-\frac{1}{2}} \leq \tau_l \leq \max\left\{\frac{T}{\pi k}, \frac{1}{\mu_k^{\frac{1}{4}}}\right\}. \tag{7.19}$$

By [14], we have

$$\mu_m = c_0 m^{\frac{2}{r-1}} + o(m^{\frac{2}{r-1}}) \qquad \text{as } m\to\infty. \tag{7.20}$$

Here $r \geq 2$ is the spatial dimension and $c_0 = c_0(\Omega) > 0$ is a constant. Therefore for sufficiently large $k \in \mathbf{N}$, we obtain

$$\begin{aligned}\tau_l &\leq \max\left\{\frac{T}{\pi k}, \frac{1}{c_0 k^{\frac{1}{2(r-1)}}(1+o(1))}\right\} \\ &\leq C\left(\frac{1}{k} + \frac{1}{k^{\frac{1}{2(r-1)}}}\right)\end{aligned}$$

where $C = C(\Omega, T) > 0$ is a constant. Since $\frac{1}{2(r-1)} < 1$, we have $\frac{1}{k^{\frac{1}{2(r-1)}}} > \frac{1}{k}$, so that

$$\frac{1}{k} + \frac{1}{k^{\frac{1}{2(r-1)}}} < \frac{2}{k^{\frac{1}{2(r-1)}}}.$$

Therefore $\tau_l \leq \frac{C}{k^{\frac{1}{2(r-1)}}}$. Noting that $k \geq \sqrt{l} - 1$ by $(k+1)^2 \geq l$, we get $\tau_l \leq \frac{C}{l^{\frac{1}{4(r-1)}}}$. Therefore by Lemma 3, we can reach the conclusion of Theorem 2.

§8. Proof of Theorem 3

For the proof, we will prove

Proposition 3. *Let $\alpha \geq 0$ be arbitrarily given. Then there exists a constant $C = C(\Omega, T, \lambda, \alpha) > 0$ such that if $f \in \mathcal{D}(A^\alpha)$, then*

$$\left\| \frac{\partial u(f)}{\partial n} \right\|_{2\alpha + \frac{1}{2}, 2\alpha + 1} \leq C \| A^\alpha f \|_{L^2(\Omega)}.$$

Here we recall (6.1) and (6.2).

Proof of Proposition 3. We will prove the proposition in the following four steps.

FIRST STEP. In this step, we will prove: if

$$f \in \mathcal{D}(A^m) \tag{8.1}$$

for $m \in \mathbf{N} \cup \{0\}$, then $v_m \equiv A^m u(f)$ satisfies

$$\begin{aligned} &v_m \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \\ &v_m'' \in L^2(0, T; L^2(\Omega)) \end{aligned} \tag{8.2}$$

and

$$\begin{cases} v_m''(x, t) = \Delta v_m(x, t) + \lambda(t) A^m f(x) & (x \in \Omega, t > 0) \\ v_m(x, 0) = v_m'(x, 0) = 0 & (x \in \Omega) \\ v_m(x, t) = 0 & (x \in \partial\Omega, t > 0). \end{cases} \tag{8.3}$$

We prove this assertion as follows. Let $\widetilde{v_m}$ be the weak solution to (8.3). That is, we have

(8.4) $$(\widetilde{v_m}''(\cdot, t), \phi)_{L^2(\Omega)} + (\nabla \widetilde{v_m}(\cdot, t), \nabla \phi)_{L^2(\Omega)} = (\lambda(t) A^m f, \phi)_{L^2(\Omega)}$$

for any $t>0$ and any $\phi \in H_0^1(\Omega)$, and

(8.5) $$\widetilde{v_m}(x, 0) = \widetilde{v_m}'(x, 0) = 0 \qquad (x \in \Omega).$$

By Lemme 3.6 (p. 39) in [12] and Theorem 2.1 (pp.95-96) in [13, Vol. II], we see that $\widetilde{v_m}$ satisfies (8.2). Therefore by uniqueness of weak solutions, it is sufficient to prove that $A^{-m}\widetilde{v_m}$ is the weak solution to (1.1).
First it is clear that $A^{-m}\widetilde{v_m}(x, 0) = 0$. Since $\widetilde{v_m} \in C^1([0, T]; L^2(\Omega))$ and A^{-m} is bounded from $L^2(\Omega)$ to itself, we have $(A^{-m}\widetilde{v_m})'(x, 0) = A^{-m}\widetilde{v_m}'(x, 0)$. Therefore $(A^{-m}\widetilde{v_m})'(x, 0) = 0\ (x \in \Omega)$ follows from $\widetilde{v_m}'(x, 0) = 0$. Firstly we have to prove

(8.6) $$\begin{aligned} &((A^{-m}\widetilde{v_m})''(\cdot, t), \phi)_{L^2(\Omega)} + (\nabla(A^{-m}\widetilde{v_m})(\cdot, t), \nabla\phi)_{L^2(\Omega)} \\ &= (\lambda(t) f, \phi)_{L^2(\Omega)} \end{aligned}$$

for any $t>0$ and any $\phi \in H_0^1(\Omega)$. To this end, we will prove

(8.7) $$(A^{-m}\widetilde{v_m})''(x, t) = A^{-m}\widetilde{v_m}''(x, t) \qquad (x \in \Omega,\ 0<t<T).$$

In fact, since $\widetilde{v_m}'' \in L^2(0, T; L^2(\Omega))$, we have

$$\int_0^T (\widetilde{v_m}''(\cdot, t), \widetilde{\eta})_{L^2(\Omega)} \psi(t)\, dt = \int_0^T (v_m(\cdot, t), \widetilde{\eta})_{L^2(\Omega)} \psi''(t)\, dt$$

for any $\widetilde{\eta} \in L^2(\Omega)$ and $\psi \in C_0^\infty(0, T)$. Setting $\widetilde{\eta} = A^{-m}\eta$ for any $\eta \in L^2(\Omega)$, and using $(A^{-m})^* = (A^*)^{-m} = A^{-m}$, we obtain

$$\int_0^T (A^{-m}\widetilde{v_m}''(\cdot, t), \eta)_{L^2(\Omega)} \psi(t)\, dt = \int_0^T (A^{-m}\widetilde{v_m}(\cdot, t), \eta)_{L^2(\Omega)} \psi''(t)\, dt$$

for any $\eta \in L^2(\Omega)$. Therefore we get

$$\begin{aligned} (A^{-m}\widetilde{v_m}''(\cdot, t), \eta)_{L^2(\Omega)} &= \frac{d^2}{dt^2}(A^{-m}\widetilde{v_m}(\cdot, t), \eta)_{L^2(\Omega)} \\ &= ((A^{-m}\widetilde{v_m})''(\cdot, t), \eta)_{L^2(\Omega)} \end{aligned}$$

for any $\eta \in L^2(\Omega)$, which is (8.7). Let us return to the proof of (8.6). By (8.7) and $(A^{-m})^* = A^{-m}$, we have

(8.8) $$\begin{aligned} &((A^{-m}\widetilde{v_m})''(\cdot, t), \phi)_{L^2(\Omega)} = (A^{-m}\widetilde{v_m}''(\cdot, t), \phi)_{L^2(\Omega)} \\ &= (\widetilde{v_m}''(\cdot, t), \phi)_{L^2(\Omega)}. \end{aligned}$$

On the other hand, since $H_0^1(\Omega) = \mathcal{D}(A^{\frac{1}{2}})$ (e.g. Fujiwara [5]), we have $(\nabla\phi, \nabla\psi)_{L^2(\Omega)} = (A^{\frac{1}{2}}\phi, A^{\frac{1}{2}}\psi)_{L^2(\Omega)}$ for $\phi, \psi \in H_0^1(\Omega)$. Consequently

$$(8.9)\qquad \begin{aligned}&(\nabla(A^{-m}\widetilde{v_m})(\cdot,t),\ \nabla\phi)_{L^2(\Omega)}=(A^{-m}A^{\frac{1}{2}}\widetilde{v_m}(\cdot,t),\ A^{\frac{1}{2}}\phi)_{L^2(\Omega)}\\&=(A^{\frac{1}{2}}\widetilde{v_m}(\cdot,t),\ A^{\frac{1}{2}}(A^{-m}\phi))_{L^2(\Omega)}=(\nabla\widetilde{v_m}(\cdot,t),\ \nabla(A^{-m}\phi))_{L^2(\Omega)}.\end{aligned}$$

We apply (8.8) and (8.9), so that we obtain

$$\begin{aligned}&((A^{-m}\widetilde{v_m})''(\cdot,t),\ \phi)_{L^2(\Omega)}+(\nabla(A^{-m}\widetilde{v_m})(\cdot,t),\ \nabla\phi)_{L^2(\Omega)}\\&=(\widetilde{v_m}''(\cdot,t),\ A^{-m}\phi)_{L^2(\Omega)}+(\nabla\widetilde{v_m}(\cdot,t),\ \nabla(A^{-m}\phi))_{L^2(\Omega)}\\&=(\lambda(t)A^mf,\ A^{-m}\phi)_{L^2(\Omega)}.\end{aligned}$$

At the last equality, we can use (8.4) by $A^{-m}\phi\in H_0^1(\Omega)$. That is, we have proved (8.6), namely, that $A^{-m}\widetilde{v_m}$ is the weak solution to (1.1).

SECOND STEP. In this step, we prove: if (8.1) holds, then $u_m\equiv\dfrac{\partial^{2m+1}}{\partial t^{2m+1}}u(f)$ satisfies

$$(8.10)\qquad \begin{aligned}&u_m\in C^0([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))\\&u_m''\in C^0([0,T];H^{-1}(\Omega))\end{aligned}$$

and

$$(8.11)\qquad \left\{\begin{aligned}&U''(x,t)=\Delta U(x,t)+\lambda^{(2m+1)}(t)f(x) && (x\in\Omega,\ t>0)\\&U(x,0)=\sum_{k=0}^{m-1}\lambda^{(2m-1-2k)}(0)(\Delta^kf)(x) && (x\in\Omega)\\&U'(x,0)=\sum_{k=0}^{m}\lambda^{(2m-2k)}(0)(\Delta^kf)(x) && (x\in\Omega)\\&U(x,t)=0 && (x\in\partial\Omega,\ 0<t<T).\end{aligned}\right\}$$

Here and henceforth we set $H^{-1}(\Omega)=(H_0^1(\Omega))'$: the dual of $H_0^1(\Omega)$ and $\lambda^{(k)}(t)=\dfrac{d^k\lambda}{dt^k}(t)$. For the proof, we prove

Lemma 7. *Let*

$$(8.12)\qquad \begin{aligned}&\chi\in C^\infty[0,T],\quad f\in\mathcal{D}(A)\\&a\in\mathcal{D}(A),\quad \Delta a\in H_0^1(\Omega),\quad b\in\mathcal{D}(A).\end{aligned}$$

Let w be the weak solution to

$$(8.13)\qquad \left\{\begin{aligned}&w''(x,t)=\Delta w(x,t)+\chi(t)f(x) && (x\in\Omega,\ 0<t<T)\\&w(x,0)=a(x),\ w'(x,0)=b(x) && (x\in\Omega)\\&w(x,t)=0 && (x\in\partial\Omega,\ 0<t<T).\end{aligned}\right\}$$

Then $v(x, t) \equiv \frac{\partial^2}{\partial t^2} w(x, t)$ *satisfies*

$$v \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]: L^2(\Omega)) \quad (8.14)$$
$$v'' \in C^0([0, T]; H^{-1}(\Omega))$$

and v *is the weak solution to*

$$(8.15) \quad \begin{cases} v''(x, t) = \Delta v(x, t) + \chi^{(2)}(t) f(x) & (x \in \Omega, 0 < t < T) \\ v(x, 0) = \Delta a(x) + \chi(0) f(x) & (x \in \Omega) \\ v'(x, 0) = \Delta b(x) + \chi'(0) f(x) & (x \in \Omega) \\ v(x, t) = 0 & (x \in \partial\Omega, 0 < t < T). \end{cases}$$

Proof of Lemma 7. Let z be the weak solution to (8.15). Since $z(\cdot,0) \in H_0^1(\Omega)$ and $z'(\cdot,0) \in L^2(\Omega)$ by (8.12), it follows from Theorem 8.2 (p.275) in [13, Vol. I] that

$$z \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \quad (8.16)$$
$$z'' \in C^0([0, T]; H^{-1}(\Omega)).$$

We set

$$(8.17) \quad \tilde{w}(x, t) = \int_0^t (t-s) z(x, s) ds + tb(x) + a(x) \quad (x \in \Omega, 0 < t < T).$$

This $\tilde{w}$ is proved to be a weak solution to (8.13). In fact, since z is the weak solution to (8.15), we have

$$(8.18) \quad \begin{aligned} &(\nabla z(\cdot, t), \nabla \phi)_{L^2(\Omega)} \\ &= -(z''(\cdot, t), \phi)_{L^2(\Omega)} + (\chi^{(2)}(t) f, \phi)_{L^2(\Omega)} \end{aligned}$$

for any $t > 0$ and $\phi \in H_0^1(\Omega)$, and

$$(8.19) \quad \begin{aligned} &z(x, 0) = \Delta a(x) + \chi(0) f(x) \\ &z'(x, 0) = \Delta b(x) + \chi'(0) f(x) \quad (x \in \Omega) \end{aligned}.$$

Therefore for any $\phi \in H_0^1(\Omega)$, we get

$$\begin{aligned} &(\tilde{w}''(\cdot, t), \phi)_{L^2(\Omega)} + (\nabla \tilde{w}(\cdot, t), \nabla \phi)_{L^2(\Omega)} \\ &= (z(\cdot, t), \phi)_{L^2(\Omega)} + \int_0^t (t-s)(\nabla z(\cdot, s), \nabla \phi)_{L^2(\Omega)} ds \\ &+ (t \nabla b + \nabla a, \nabla \phi)_{L^2(\Omega)} \quad (\text{by } (8.17)) \\ &= (z(\cdot, t), \phi)_{L^2(\Omega)} - \int_0^t (t-s)(z''(\cdot, s), \phi)_{L^2(\Omega)} ds \\ &+ \int_0^t (t-s)(\chi^{(2)}(s) f, \phi)_{L^2(\Omega)} ds \\ &+ (t \nabla b + \nabla a, \nabla \phi)_{L^2(\Omega)} \quad (\text{by } (8.18)). \end{aligned}$$

On the other hand, carrying ont integrations by parts, we obtain

$$-\int_0^t (t-s)\,(z''(\cdot, s), \phi)_{L^2(\Omega)}ds$$
$$=t(z'(\cdot, 0), \phi)_{L^2(\Omega)}+(z(\cdot, 0), \phi)_{L^2(\Omega)}-(z(\cdot, t), \phi)_{L^2(\Omega)}$$

and

$$\int_0^t (t-s)\,(\chi^{(2)}(s)f, \phi)_{L^2(\Omega)}ds$$
$$=-t(\chi'(0)f, \phi)_{L^2(\Omega)}-(\chi(0)f, \phi)_{L^2(\Omega)}+(\chi(t)f, \phi)_{L^2(\Omega)}.$$

Moreover, by Green's formula, (8.12) and $\phi\in H_0^1(\Omega)$, we have

$$(t\nabla b+\nabla a, \nabla\phi)_{L^2(\Omega)}=-(t\Delta b+\Delta a, \phi)_{L^2(\Omega)}.$$

Therefore for any $t>0$ and $\phi\in H_0^1(\Omega)$, we get

$$\begin{aligned}&(\widetilde{w}''(\cdot, t), \phi)_{L^2(\Omega)}+(\nabla\widetilde{w}(\cdot, t), \nabla\phi)_{L^2(\Omega)}\\&=t(z'(\cdot,0)-\chi'(0)f-\Delta b, \phi)_{L^2(\Omega)}+(z(\cdot,0)-\chi(0)f-\Delta a, \phi)_{L^2(\Omega)}\\&+(\chi(t)f, \phi)_{L^2(\Omega)}\\&=(\chi(t)f, \phi)_{L^2(\Omega)} \qquad \text{(by (8.19))}.\end{aligned}$$

Therefore $\widetilde{w}$ is the weak solution to (8.13), so that by uniqueness of weak solutions we get $\widetilde{w}(x, t)=w(x, t)$ $(x\in\Omega, 0<t<T)$ and

$$z(x, t)=v(x, t)=\frac{\partial^2}{\partial t^2}w(x, t) \qquad (x\in\Omega, 0<t<T).$$

Consequently (8.16) implies the conclusion of Lemma 7.

Now we proceed to the proof of (8.10) and (8.11) by induction. Let $m=1$, namely, $f\in\mathfrak{D}(A)$. From the proof of Lemma 2 in §6 and Theorem 8.2 (p. 275) in [13, Vol. I], we see that $u_0=\frac{\partial u(f)}{\partial t}$ satisfies

$$(8.20)\qquad \begin{aligned}&u_0\in C^0([0, T]; H_0^1(\Omega))\cap C^1([0, T]; L^2(\Omega))\\&u_0''\in C^0([0, T]; H^{-1}(\Omega))\end{aligned}$$

and (8.13) with $\chi(t)=\lambda'(t)$ $(0\le t\le T)$, $a(x)=0$, $b(x)=\lambda(0)f(x)$ $(x\in\Omega)$. Since (8.12) is satisfied by $f\in\mathfrak{D}(A)$, we can apply Lemma 7, so that $u_1\equiv\frac{\partial^2 u_0}{\partial t^2}$ $=\frac{\partial^3 u(f)}{\partial t^3}$ satisfies (8.14) and

$$\left\{\begin{aligned}&u_1''(x, t)=\Delta u_1(x, t)+\lambda^{(3)}(t)f(x) && (x\in\Omega, 0<t<T)\\&u_1(x, 0)=\lambda'(0)f(x) && (x\in\Omega)\\&u_1'(x, 0)=\lambda(0)\Delta f(x)+\lambda^{(2)}(0)f(x) && (x\in\Omega)\\&u_1(x, t)=0 && (x\in\Omega, 0<t<T),\end{aligned}\right.$$

which is (8.11) with $m=1$, that is, we see that (8.10) and (8.11) hold for $m=1$.

Next assume that (8.10) and (8.11) hold if $f\in\mathscr{D}(A^l)$. Then we have to prove that (8.10) and (8.11) hold for $f\in\mathscr{D}(A^{l+1})$. By the assumption, u_l satisfies (8.13) where $\chi(t)=\lambda^{(2l+1)}(t)$ $(0\le t\le T)$, $a(x)=\sum_{k=0}^{l-1}\lambda^{(2l-1-2k)}(0)(\Delta^k f)(x)$ and $b(x)=\sum_{k=0}^{l}\lambda^{(2l-2k)}(0)(\Delta^k f)(x)$ $(x\in\Omega)$. It follows from $f\in\mathscr{D}(A^{l+1})$ that $a\in\mathscr{D}(A)$, $\Delta a\in\mathscr{D}(A)\subset H_0^1(\Omega)$ and $b\in\mathscr{D}(A)$. Consequently we can apply Lemma 7, so that $u_{l+1}=\dfrac{\partial^2 u_l}{\partial t^2}$ satisfies (8.10) and

$$\begin{cases} u''_{l+1}(x,t)=\Delta u_{l+1}(x,t)+\lambda^{(2l+3)}(t)f(x) & (x\in\Omega,\ 0<t<T)\\ u_{l+1}(x,0)=\Delta\Big(\sum_{k=0}^{l-1}\lambda^{(2l-1-2k)}(0)\Delta^k f\Big)(x)+\lambda^{(2l+1)}(0)f(x) & (x\in\Omega)\\ u'_{l+1}(x,0)=\Delta\Big(\sum_{k=0}^{l}\lambda^{(2l-2k)}(0)\Delta^k f\Big)(x)+\lambda^{(2l+2)}(0)f(x) & (x\in\Omega)\\ u_{l+1}(x,t)=0 & (x\in\partial\Omega,\ 0<t<T), \end{cases}$$

which is (8.11) with $m=l+1$. Thus by induction, we have proved (8.10) and (8.11) if (8.1) holds for any $m\ge 1$.

THIRD STEP. In this step, we prove: if (8.1) holds for $m\ge 0$, then

$$\left\|\frac{\partial u(f)}{\partial n}\right\|_{2m+\frac{1}{2},2m+1}\le C\|A^m f\|_{L^2(\Omega)}. \tag{8.21}$$

Here $C=C(\Omega,T,\lambda,m)>0$ is independent of f.

For $m=0$, the estimate (8.21) is nothing but Lemma 2 in §6. Therefore let us prove (8.21) for $m\ge 1$. First, as is shown in First Step, $A^m u(f)$ satisfies (8.3). Consequently by Theorem 2.1 (pp. 95-96) in [13, Vol. II], we get

$$\|A^m u(f)\|_{L^2(0,T;H^2(\Omega))}\le C\|A^m f\|_{L^2(\Omega)} \tag{8.22}$$

with $C=C(\Omega,T,\lambda,m)>0$. Since $A^m u(f)(\cdot,t)\in\mathscr{D}(A)$ for almost all $t\in[0,T]$, we see that $\|u(f)(\cdot,t)\|_{H^{2m+2}(\Omega)}\le C(\Omega,m)\|A^m u(f)(\cdot,t)\|_{L^2(\Omega)}$, so that (8.22) implies

$$\|u(f)\|_{L^2(0,T;H^{2m+2}(\Omega))}\le C\|A^m f\|_{L^2(\Omega)}.$$

By the trace theorem (e.g. pp. 41-42 in [13, Vol. I]), we reach

$$\left\|\frac{\partial u(f)}{\partial n}\right\|_{L^2(0,T;H^{2m+\frac{1}{2}}(\partial\Omega))}\le C\|A^m f\|_{L^2(\Omega)}. \tag{8.23}$$

Second, from Second Step, $u_m=\dfrac{\partial^{2m+1}u(f)}{\partial t^{2m+1}}$ satisfies (8.11), we apply

Théorème 4.1 (p. 44) in [12] to get

$$\left\|\frac{\partial}{\partial n}\frac{\partial^{2m+1}u(f)}{\partial t^{2m+1}}\right\|_{L^2(0,T;L^2(\partial\Omega))}$$

$$\leq C(\Omega, T)\left(\left\|\sum_{k=0}^{m-1}\lambda^{(2m-1-2k)}(0)\,\nabla(\Delta^k f)\right\|_{L^2(\Omega)}+\left\|\sum_{k=0}^{m}\lambda^{(2m-2k)}(0)\,\Delta^k f\right\|_{L^2(\Omega)}\right.$$

$$\left.+\|\lambda^{(2m+1)}f\|_{L^1(0,T;L^2(\partial\Omega))}\right)$$

$$\leq C(\Omega, T)\,(C(\Omega, \lambda, m)\|A^m f\|_{L^2(\Omega)}+C(\Omega, \lambda, m)\|A^m f\|_{L^2(\Omega)}+C(\lambda, m)\|f\|_{L^2(\Omega)})$$

(by $\Delta^k f\in\mathscr{D}(A): 0\leq k\leq m-1$)

$$\equiv C(\Omega, T, \lambda, m)\|A^m f\|_{L^2(\Omega)}.$$

Consequently we obtain

$$\left\|\frac{\partial^{2m+1}}{\partial t^{2m+1}}\frac{\partial u(f)}{\partial n}\right\|_{L^2(0,T;L^2(\partial\Omega))}\leq C(\Omega, T, \lambda, m)\|A^m f\|_{L^2(\Omega)},$$

with which we combine (8.23), so that we reach (8.21).

FOURTH STEP. By the interpolation theorem, we will complete the proof of Proposition 3. The estimate (8.21) implies Proposition 3 for any $m\in\mathbf{N}\cup\{0\}$. Let $\alpha\geq 0$ be not an integer. We take the $m\in\mathbf{N}\cup\{0\}$ such that $m<\alpha<m+1$. Since the map $f\rightarrow\frac{\partial u(f)}{\partial n}$ is bounded from $\mathscr{D}(A^m)$ to $H^{2m+\frac{1}{2},2m+1}(\partial\Omega\times(0, T))$ and from $\mathscr{D}(A^{m+1})$ to $H^{2m+2+\frac{1}{2},2m+3}(\partial\Omega\times(0, T))$ by (8.21), it follows from Theorem 5.1 (p. 27) in [13, Vol. I] that the map is bounded from $[\mathscr{D}(A^{m+1}), \mathscr{D}(A^m)]_{1-\alpha+m}$ to

$$[H^{2m+2+\frac{1}{2},2m+3}(\partial\Omega\times(0, T)), H^{2m+\frac{1}{2},2m+1}(\partial\Omega\times(0, T))]_{1-\alpha+m}.$$

Noting that $A_m\equiv A|_{\mathscr{D}(A^m)}$ is positive self-adjoint operator in the Hilbert space $Y\equiv\mathscr{D}(A^m)$ with the inner product $(u, v)_m=(A^m u, A^m v)_{L^2(\Omega)}$ and $\mathscr{D}(A_m)=\mathscr{D}(A^{m+1})$, by the interpolation theorem (e.g. [13, Vol. I]), we get

$$[\mathscr{D}(A^{m+1}), \mathscr{D}(A^m)]_{1-\alpha+m}=[\mathscr{D}(A^m), Y]_{1-\alpha+m}=\mathscr{D}(A_m^{\alpha-m})=\mathscr{D}(A^\alpha).$$

Moreover by [13, pp. 6-8, Vol. II], we see

$$[H^{2m+2+\frac{1}{2},2m+3}(\partial\Omega\times(0, T)), H^{2m+\frac{1}{2},2m+1}(\partial\Omega\times(0, T))]_{1-\alpha+m}$$
$$=H^{2\alpha+\frac{1}{2},2\alpha+1}(\partial\Omega\times(0, T)).$$

Thus the proof of Proposition 3 is complete.

Now we are ready to finish the proof of Theorem 3. By the interpolation inequality (e.g. Proposition 2.3 (p. 19) in [13, Vol. I]) and the interpolation result (pp. 6-8 in [13, Vol. II]), we see that for each $\theta \in [0, 1]$

$$\|y\|_{(1-\theta)(2\alpha+\frac{1}{2}),(1-\alpha)(2\alpha+1)} \leq C(\alpha, \theta) \|y\|^{1-\theta}_{2\alpha+\frac{1}{2},2\alpha+1} \|y\|^{\theta}_{0,0}$$

for $y \in H^{2\alpha+\frac{1}{2},2\alpha+1}(\partial\Omega \times (0, T))$. Let $\theta = \dfrac{2\alpha}{2\alpha+1}$. This value is the maximal $\theta \in [0, 1]$ such that $(1-\theta)\left(2\alpha+\frac{1}{2}\right) \geq 0$ and $(1-\theta)(2\alpha+1) \geq 1$, that is, $H^{(1-\theta)(2\alpha+\frac{1}{2}),(1-\theta)(2\alpha+1)}(\partial\Omega \times (0, T)) \subset H^{0,1}(\partial\Omega \times (0, T))$.

Consequently we have

$$\|y\|_{0,1} \leq C(\alpha) \|y\|^{\frac{1}{2\alpha+1}}_{2\alpha+\frac{1}{2},2\alpha+1} \|y\|^{\frac{2\alpha}{2\alpha+1}}_{0,0}$$

for any $y \in H^{2\alpha+\frac{1}{2},2\alpha+1}(\partial\Omega \times (0, T))$. Let $f, g \in \mathcal{U}_{M,\alpha}$. Then by Proposition 3, we get $\dfrac{\partial u(f)}{\partial n}, \dfrac{\partial u(g)}{\partial n} \in H^{2\alpha+\frac{1}{2},2\alpha+1}(\partial\Omega \times (0, T))$ and

$$\left\| \frac{\partial u(f)}{\partial n} - \frac{\partial u(g)}{\partial n} \right\|_{2\alpha+\frac{1}{2},2\alpha+1} \leq C(\alpha) \|A^\alpha f - A^\alpha g\|_{L^2(\Omega)} \leq 2MC(\Omega, T, \lambda, \alpha).$$

Therefore we reach

$$\begin{aligned} &\left\| \frac{\partial u(f)}{\partial n} - \frac{\partial u(g)}{\partial n} \right\|_{0,1} \\ &\leq C(\alpha)\,(2MC(\Omega, T, \lambda, \alpha))^{\frac{1}{2\alpha+1}} \left\| \frac{\partial u(f)}{\partial n} - \frac{\partial u(g)}{\partial n} \right\|^{\frac{2\alpha}{2\alpha+1}}_{0,0} \\ &\equiv C(\Omega, T, \lambda, \alpha) M^{\frac{1}{2\alpha+1}} \left\| \frac{\partial u(f)}{\partial n} - \frac{\partial u(g)}{\partial n} \right\|^{\frac{2\alpha}{2\alpha+1}}_{0,0} \end{aligned}$$

with which we combine (1.5) in Theorem 0 in §1 and we can complete the proof of Theorem 3.

§9. Proof of Theorem 4

We set

(9.1) $\quad \omega_\alpha(\varepsilon, E) = \sup \{\|f\|_{L^2(\Omega)};\ \|Gf\|_{0,0} \leq \varepsilon, f \in \mathfrak{D}(A^\alpha), \|A^\alpha f\|_{L^2(\Omega)} \leq E\}.$

Then by Theorem 3.4 (p. 32) in [1], we get

$$\|f_\varepsilon - f_0\|_{L^2(\Omega)} \leq 2\sqrt{2}\,\omega_\alpha(\varepsilon, E)$$

on the assumption (4.3). On the other hand, by Theorem 3, we see

(9.2) $\quad \omega_\alpha(\varepsilon, E) \leq C_3(\Omega, T, \lambda, \alpha) E^{\frac{1}{2\alpha+1}} \varepsilon^{\frac{2\alpha}{2\alpha+1}} \qquad (\varepsilon > 0, E > 0).$

Therefore we obtain

$$\|f_\varepsilon - f_0\|_{L^2(\Omega)} \le C_3(\alpha) E^{\frac{1}{2\alpha+1}} \varepsilon^{\frac{2\alpha}{2\alpha+1}}$$

which is the condition of Theorem 4.

§10. Proof of Theorem 5

First we show

Lemma 8. *Let H_1 and H_2 be Hilbert spaces and Λ be a positive self-adjoint operator from H_1 to H_1 such that Λ^{-1} is compact from H_1 to H_1. For a fixed $\alpha > 0$, let H_h $(h>0)$ be subspaces of H_1 such that the following properties hold*:

(10.1) $$H_h \subset \mathfrak{D}(\Lambda^\alpha), \ \dim H_h < \infty.$$

There exist linear operator Π_h: $\mathfrak{D}(\Lambda^\alpha) \to H_h$ and $\beta > 0$ such that

(10.2) $$\|f - \Pi_h f\|_{H_1} \le C h^\beta \|\Lambda^\alpha f\|_{H_1}$$

and

(10.3) $$\|\Lambda^\alpha \Pi_h f\|_{H_1} \le C \|\Lambda^\alpha f\|_{H_1}$$

for all $f \in \mathfrak{D}(\Lambda^\alpha)$. Here $C>0$ is a constant independent of f.

Let K be a bounded linear operator from H_1 to H_2 with $\mathfrak{D}(K) = H_1$. We set

(10.4) $$\rho_\alpha(\varepsilon, E) = \sup\{\|f\|_{H_1}: \|Kf\|_{H_2} \le \varepsilon, f \in \mathfrak{D}(\Lambda^\alpha), \|\Lambda^\alpha f\|_{H_1} \le E\}.$$

Then

(i) *For an arbitrarily fixed $y \in H_2$ and $\varepsilon > 0$, $h > 0$, there exists a unique minimizer $f_{\varepsilon,h} = f_{\varepsilon,h}(y)$ of*

(10.5) $$F_\varepsilon(f) = \|Kf - y\|^2_{H_2} + \frac{\varepsilon^2}{E^2}\|\Lambda^\alpha f\|^2_{H_1}$$

over $f \in H_h$.

(ii) *We assume that*

(10.6) $$Kf_0 = y_0, \quad f_0 \in \mathfrak{D}(\Lambda^\alpha), \quad \|\Lambda^\alpha f_0\|_{H_1} \le E$$

and

(10.7) $$\|y_\varepsilon - y_0\|_{H_2} \le \varepsilon, \ y_\varepsilon \in H_2.$$

Then there exists a constant $C>0$ such that

(10.8) $$\|f_0 - f_{\varepsilon,h}\|_{H_1} \le \rho_\alpha\left(C(h^\beta E + \varepsilon), \ C\left(\frac{h^\beta E^2}{\varepsilon} + E\right)\right).$$

Here $C>0$ is independent of E and the choice of y_ε.

In (10.4), since $\{f\in\mathfrak{D}(\Lambda^{\alpha});\ \|\Lambda^{\alpha}f\|_{H_1}\leq E\}$ is a compact set in H_1, we see $\rho_{\alpha}(\varepsilon, E)<\infty$ for any $\varepsilon>0$ and $E>0$ (e.g. [1, pp. 21-26]).

Remark. This lemma is a variant of Natterer [15] and Theorem 6.6 (p. 109) in [1], where it is assumed that there exists $a>0$ such that the norms $\|Kf\|_{H_2}$ and $\|\Lambda^{-a}f\|_{H_1}$ are equivalent for any $f\in H_1$. In our case and similar inverse problems for a parabolic equation (e.g. Yamamoto [22, 23]), however, it is difficult to get such equivalence and so we modify Natterer's result for our purpose in terms of the degree of continuty $\rho_{\alpha}(\varepsilon, E)$.

Proof of Lemma 8. The part (i) is a standard result in the regularization (e.g. Theorem 3.7 (p. 35) in [1]) and proved similarly to Proposition 1 in §4.

We can prove the part (ii) similarly to [15] or Theorem 6.16 (p. 109) in [1], but for completeness, we give the proof. Since $f_{\varepsilon,h}$ is the minimizer, we have

$$
\begin{aligned}
&\|Kf_{\varepsilon,h}-y_{\varepsilon}\|_{H_2}^2+\frac{\varepsilon^2}{E^2}\|\Lambda^{\alpha}f_{\varepsilon,h}\|_{H_1}^2\\
&\leq\|K\Pi_h f_0-y_{\varepsilon}\|_{H_2}^2+\frac{\varepsilon^2}{E^2}\|\Lambda^{\alpha}\Pi_h f_0\|_{H_1}^2\\
&\leq(\|K(\Pi_h f_0-f_0)\|_{H_2}+\|Kf_0-y_{\varepsilon}\|_{H_2})^2\\
&+\frac{\varepsilon^2}{E^2}C\|\Lambda^{\alpha}f_0\|_{H_1}^2 \qquad \text{(by (10.3))}\\
&\leq(Ch^{\beta}\|\Lambda^{\alpha}f_0\|_{H_1}+\varepsilon)^2+C\varepsilon^2\\
&\text{(by (10.6), (10.7), (10.2) and the boundedness of } K)\\
&\leq C(h^{2\beta}E^2+\varepsilon^2) \qquad \text{(by (10.7))}.
\end{aligned}
\tag{10.9}
$$

Here and henceforth $C>0$ is a constant independent of ε, h, E and y_{ε}. Therefore by (10.9) we get

$$
\begin{aligned}
&\|Kf_{\varepsilon,h}-Kf_0\|_{H_2}\leq\|Kf_{\varepsilon,h}-y_{\varepsilon}\|+\|y_{\varepsilon}-Kf_0\|_{H_2}\\
&\leq C'(C(h^{2\beta}E^2+\varepsilon^2))^{\frac{1}{2}}+\varepsilon \qquad \text{(by (10.7))}\\
&\leq C(h^{\beta}E+\varepsilon),
\end{aligned}
\tag{10.10}
$$

and

$$\|\Lambda^{\alpha}f_{\varepsilon,h}\|_{H1}\leq C\left(\frac{h^{\beta}E^2}{\varepsilon}+E\right).$$

Without loss of generality, we may assume that $C\left(\frac{h^{\beta}E^2}{\varepsilon}+E\right)\geq E$, so that

$$\|\Lambda^{\alpha}f_{\varepsilon,h}\|_{H_1},\ \|\Lambda^{\alpha}f_0\|_{H_1}\leq C\left(\frac{h^{\beta}E^2}{\varepsilon}+E\right)$$

by (10.7). Consequently applying (10.10) in (10.4), we see (10.8). Thus the proof of (ii) of Lemma 8 is complete.

Now we proceed to the proof of Theorem 5. By (5.1) - (5.3), in Lemma 8, setting $H_1=L^2(\Omega)$, $H_2=L^2(0, T; L^2(\partial\Omega))$, $\Lambda=A$ defined by (3.1), $K=G$, $H_h=X_h\,(h>0)$, $\Pi_h=I_h$, we can apply (10.8) in our inverse problem. Therefore we obtain

$$\|f_0-f_{\varepsilon,h}\|_{L^2(\Omega)}\le\rho_\alpha\Bigl(C\,(h^\beta E+\varepsilon),\ C\Bigl(\frac{h^\beta E^2}{\varepsilon}+E\Bigr)\Bigr).$$

On the other hand, by Theorem 3 and the definition (10.4) of ρ_α, we have an estimate

$$\rho_\alpha(\varepsilon, E)\le C_3E^{\frac{1}{2\alpha+1}}\varepsilon^{\frac{2\alpha}{2\alpha+1}},$$

where $C_3=C_3(\Omega, T, \lambda, \alpha)>0$. Therefore we get

$$\|f_0-f_{\varepsilon,h}\|_{L^2(\Omega)}\le C\Bigl(\frac{h^\beta E^2}{\varepsilon}+E\Bigr)^{\frac{1}{2\alpha+1}}(h^\beta E+\varepsilon)^{\frac{2\alpha}{2\alpha+1}}.$$

Here and henceforth $C=C(\Omega, T, \lambda, \alpha)>0$ is a constant independent of ε, h, E and the choice of y_ε. Then we get

$$\begin{aligned}
&\|f_0-f_{\varepsilon,h}\|^{2\alpha+1}_{L^2(\Omega)}\\
&\le C\,(h^\beta\varepsilon^{-1}E^2+E)\,(h^{2\alpha\beta}E^{2\alpha}+\varepsilon^{2\alpha})\\
&=C\,(h^{2\alpha\beta+\beta}\varepsilon^{-1}E^{2\alpha+2}+h^\beta\varepsilon^{2\alpha-1}E^2+h^{2\alpha\beta}E^{2\alpha+1}+E\varepsilon^{2\alpha})\\
&\le C\,(E+1)^{2\alpha+2}(\varepsilon^{2\alpha\beta\gamma+\beta\gamma-1}+\varepsilon^{2\alpha+\beta\gamma-1}+\varepsilon^{2\alpha\beta\gamma}+\varepsilon^{2\alpha}).
\end{aligned}$$

Moreover we have

$$\min\{2\alpha\beta\gamma+\beta\gamma-1,\ 2\alpha+\beta\gamma-1,\ 2\alpha\beta\gamma,\ 2\alpha\}=2\alpha,\qquad \text{if } \gamma\ge\frac{1}{\beta}$$

and

$$\min\{2\alpha\beta\gamma+\beta\gamma-1,\ 2\alpha+\beta\gamma-1,\ 2\alpha\beta\gamma,\ 2\alpha\}=2\alpha\beta\gamma+\beta\gamma-1,\ \text{if } \frac{1}{\beta(2\alpha+1)}<\gamma\le\frac{1}{\beta},$$

so that we reach

$$\|f_0-f_{\varepsilon,h}\|^{2\alpha+1}_{L^2(\Omega)}\le\begin{cases} C\,(E+1)^{2\alpha+2}\varepsilon^{2\alpha} & \text{if } \gamma\ge\frac{1}{\beta}\\ C\,(E+1)^{2\alpha+2}\varepsilon^{2\alpha\beta\gamma+\beta\gamma-1}, & \text{if } \frac{1}{\beta(2\alpha+1)}<\gamma\le\frac{1}{\beta},\end{cases}$$

which is (5.10), the conclusion of Theorem 5.

Appendix I. Proof of Proposition 1

For simplicity, we set $\beta=\frac{\varepsilon^2}{E^2}>0$, because we fix the ratio $\frac{\varepsilon}{E}$ in the proposition. Since it follows from (1.5) that $\|Gf\|_{L^2(0,T;L^2(\partial\Omega))}\leq\|f\|_{L^2(\Omega)}$, and the norm

$$(\|Gf\|^2_{L^2(0,T;L^2(\partial\Omega))}+\|A^\alpha f\|^2_{L^2(\Omega)})^{\frac{1}{2}}$$

is equivalent to the norm $\|A^\alpha f\|_{L^2(\Omega)}$ for $f\in\mathcal{D}(A^\alpha)$, we can apply a result in the regularization (e.g. Theorem 3.7 (p. 35) in Baumeister [1]), so that we see (1)-(3):

(1) There exists a unique minimizer $f(y)\in\mathcal{D}(A^\alpha)$ of $F(\varepsilon, E, \alpha, y)$ for any $y\in L^2(0, T; L^2(\partial\Omega))$.

(2) There exists a constant $C(\alpha, \beta)>0$ such that

$$\|(G^*G+\beta(A^\alpha)^*A^\alpha)^{-1}y\|_{L^2(\Omega)}\leq C(\alpha, \beta)\|y\|_{L^2(0,T;L^2(\partial\Omega))}$$

for $y\in L^2(0, T; L^2(\partial\Omega))$. Here and henceforth G^* and $(A^\alpha)^*$ are adjoint operators respectively of bounded operators G: $L^2(\Omega)\rightarrow L^2(0, T; L^2(\partial\Omega))$ and A^α: $\mathcal{D}(A^\alpha)\rightarrow L^2(\Omega)$.

(3) $f=(G^*G+\beta(A^\alpha)^*A^\alpha)^{-1}G^*y$.

Thus the parts (i) and (ii) of Proposition 1 follow.

Appendix II. Proof of Lemma 1

Since $X_h\subset\mathcal{D}(A^{\frac{1}{2}})=H_0^1(\Omega)$ ([5]), by the definitions (5.6) and (5.7) of X_h and I_h, we see (5.1). Next we have to verify (5.2) and (5.3). By a result on interpolation with piecewise linear functions (e.g. Johnson [9, Chapter 4]) for any $f\in H_0^1(\Omega)=\mathcal{D}(A^{\frac{1}{2}})$, there exists a constant $C=C(\tau)>0$ independent of h such that

(1) $$\|f-I_hf\|_{L^2(\Omega_h)}\leq Ch\|f\|_{H^1(\Omega_h)}.$$

On the other hand, for example, by the estimate (5.2-18) (p. 118) in Raviart and Thomas [18], we have

(2) $$\begin{aligned}\|f\|_{L^2(\Omega\backslash\Omega_h)}&\leq C(\tau)(h\|f\|_{L^2(\partial\Omega)}+h^2\|f\|_{H^1(\Omega\backslash\Omega_h)})\\&\leq C(\tau)h^2\|f\|_{H_0^1(\Omega)}.\end{aligned}$$

At the last inequality, we note that $\|f\|_{L^2(\partial\Omega)}=0$ by $f\in H_0^1(\Omega)$. By the definition (5.6) and (5.7) of I_hf, we have $I_hf|_{\Omega\backslash\Omega_h}=0$ $(f\in H_0^1(\Omega))$. Therefore by (1) and (2), we see

$$\begin{aligned}\|f-I_hf\|_{L^2(\Omega)}&\leq\|f-I_hf\|_{L^2(\Omega_h)}+\|f-I_hf\|_{L^2(\Omega\backslash\Omega_h)}\\&\leq Ch(1+h)\|A^{\frac{1}{2}}f\|_{L^2(\Omega)}\\&\leq Ch(1+d)\|A^{\frac{1}{2}}f\|_{L^2(\Omega)}.\end{aligned}$$

Here we recall that $d=\sup\{|x_1-x_2|; x_1, x_2\in\Omega\}$. Thus (5.2) is verified with $\alpha=\frac{1}{2}$ and $\beta=1$.

Finally we have to verify (5.3). For example, by the estimate (4.23) (p. 91) in Johnson [9], we obtain $\|f-I_hf\|_{H^1(\Omega_h)}\leq C(\tau)\|f\|_{H^1(\Omega_h)}$, namely,

$$\|I_hf\|_{H^1(\Omega_h)}\leq (C(\tau)+1)\|f\|_{H^1(\Omega_h)}, \tag{3}$$

by the triangle inequality. Let $f\in H_0^1(\Omega)=\mathcal{D}(A^{\frac{1}{2}})$. Since I_hf is piecewise linear in Ω and $I_hf|_{\Omega\setminus\Omega_h}=0$, we have $\|I_hf\|_{H^1(\Omega)}\leq\|I_hf\|_{H^1(\Omega_h)}$. Thus (3) implies (5.3) with $\alpha=\frac{1}{2}$.

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