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Adjoint action on homology mod 2 of E_8 on its loop space

By

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1. Introduction

Assume G is a compact, connected, simply connected Lie group. The space of free loops on G is called $LG(G)$ the free loop group of G, whose multiplication is defined as

$$\varphi \cdot \phi(t) = \varphi(t) \cdot \phi(t).$$

Let ΩG be the space of based loops on G, whose base point is the unit e. Then $LG(G)$ has ΩG as its normal subgroup and

$$LG(G) / \Omega G \cong G.$$

Identifying elements of G with constant maps from S^1 to G, $LG(G)$ is equal to the semidirect product of G and ΩG. Thus the mod p homology of $LG(G)$ is determined by the mod p homology of G and ΩG and the algebra structure of $\mathbf{H}_*(LG(G)\ ;\ \mathbf{Z}/p\mathbf{Z})$ depends on $\mathbf{H}_*(\mathrm{ad}\ ;\ \mathbf{Z}/p\mathbf{Z})$ where

$$\mathrm{ad} : G \times \Omega G \rightarrow \Omega G$$

is the adjoint map.

In [4] some properties of ad_* are studied and it is showed that $\mathrm{H}_*(\mathrm{ad}\ ;\ \mathbf{Z}/p\mathbf{Z})$ is equal to $\mathbf{H}_*(p_2\ ;\ \mathbf{Z}/p\mathbf{Z})$ where p_2 is the second projection if and only if $\mathbf{H}^*(G\ ;\ \mathbf{Z})$ is p-torsin free. For an exceptional Lie group G, $\mathbf{H}^*(G\ ;\ \mathbf{Z})$ has p-torsion when

$$\begin{array}{ll} G = G_2, F_4, E_6, E_7, E_8 & \text{for } p = 2, \\ G = F_4, E_6, E_7, E_8 & \text{for } p = 3, \\ G = E_8 & \text{for } p = 5. \end{array}$$

The case where $p=2$ and $G \neq E_8$ is discussed in [6] and the case of $p=3, 5$ is studied in [8, 7] respectively. In this paper we offer the result of the remained case, $(G, p) = (E_8, 2)$. The result is showed in Theorem 4. 1.

This paper is organixed as follows. In §2 we refer to the result of the algebra structure of $\mathbf{H}^*(G\ ;\ \mathbf{Z}/2\mathbf{Z})$ and $\mathbf{H}_*(\Omega G\ ;\ \mathbf{Z}/2\mathbf{Z})$ and the Hopf algebra

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structure and cohomology operations of them. And in §3 we introduce the adjoint action and observe its property. Finally in §4 the induced homomorphism from the adjoint action of E_8 is determined by using the result of E_7 and cohomology operations.

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2. $\mathbf{H}^*(G\ ;\ \mathbf{Z}/2\mathbf{Z})$ and $\mathbf{H}_*(\Omega G\ ;\ \mathbf{Z}/2\mathbf{Z})$

We refer to the result of [1] and [2] about $\mathrm{H}^*(G\ ;\ \mathbf{Z}/2\mathbf{Z})$ for $G=E_7$ and E_8.

Theorem 2. 1.

$$\mathrm{H}^*(E_7\ ;\ \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \wedge (x_{15}, x_{17}, x_{23}, x_{27})$$
$$\mathrm{H}^*(E_8\ ;\ \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \wedge (x_{17}, x_{23}, x_{27}, x_{29})$$

where x_i is a generator of degree i. Moreover there is a homomorphism

$$E_7 \to E_8$$

where induced homomorphism maps x_i in $\mathrm{H}^*(E_8\ ;\ \mathbf{Z}/2\mathbf{Z})$ *into x_i in* $\mathrm{H}^*(E_7\ ;\ \mathbf{Z}/2\mathbf{Z})$.

Theorem 2. 2. *The x_i's in Theorem 2. 1 can be chosen so as to satisfy*

$$x_5 = \mathrm{Sq}^2 x_3,$$
$$x_9 = \mathrm{Sq}^4 x_5,$$
$$\bar{\phi}(x_3) = \bar{\phi}(x_5) = \bar{\phi}(x_9) = 0$$

and the coproduct of x_{15} is

$$\bar{\phi}(x_{15}) = x_3{}^2 \otimes x_9 + x_5{}^2 \otimes x_5 + x_3{}^4 \otimes x_3.$$

The algebra structure of $\mathrm{H}_*(\Omega G\ ;\ \mathbf{Z}/2\mathbf{Z})$ can be determined as an application of the Eilenberg-Moore spectral sequence. And the Hopf algebra structures and the action of cohomology operations which acts on homology dually was determined by A. Kono and K. Kozima. See [5, 3] for detail.

Theorem 2. 3.

$$\mathrm{H}_*(\Omega E_7\ ;\ \mathbf{Z}/2\mathbf{Z}) = \wedge (b_2, b_4, b_8) \otimes \mathbf{Z}/2\mathbf{Z}[b_{10}, b_{14}, b_{16}, b_{18}, b_{22}, b_{26}, b_{34}]$$
$$\mathrm{H}_*(\Omega E_8\ ;\ \mathbf{Z}/2\mathbf{Z}) = \wedge (b_2, b_4, b_8, b_{14}) \otimes \mathbf{Z}/2\mathbf{Z}[b_{16}, b_{22}, b_{26}, b_{28}, b_{34}, b_{38}, b_{46}, b_{58}]$$

where b_i is a generator of degree i.

Theorem 2. 4. *The coproduct of* $\mathrm{H}_*(\Omega E_8\ ;\ \mathbf{Z}/2\mathbf{Z})$ *is given as*

$$\bar{\phi}(b_i) = 0 \textit{ for } i = 2, 14, 22, 26, 34, 38, 46, 58,$$

$$\bar{\phi}(b_4) = b_2 \otimes b_2,$$
$$\bar{\phi}(b_8) = b_2 \otimes b_2b_4 + b_4 \otimes b_4 + b_2b_4 \otimes b_2,$$
$$\bar{\phi}(b_{16}) = b_2 \otimes b_2b_4b_8 + b_4 \otimes b_4b_8 + b_2b_4 \otimes b_2b_8 + b_8 \otimes b_8 + b_2b_8 \otimes b_2b_4 + b_4b_8 \otimes b_4 + b_2b_4b_8 \otimes b_2,$$
$$\bar{\phi}(b_{28}) = b_{14} \otimes b_{14}.$$

3. Adjoint action

Let $\mathrm{Ad} : G \times G \to G$ and $\mathrm{ad} : G \times \Omega G \to \Omega G$ be the adjoint action of a Lie group G defined by $\mathrm{Ad}(g, h) = ghg^{-1}$ and $\mathrm{ad}(g, l)(t) = gl(t)g^{-1}$ *where* $g, h \in G$, $l \in \Omega G$ and $t \in [0, 1]$. These induce the homomorphisms

$$\mathrm{Ad}_* : \mathrm{H}_*(G ; \mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(G ; \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_*(G ; \mathbf{Z}/2\mathbf{Z})$$

and

$$\mathrm{ad}_* : \mathrm{H}_*(G ; \mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(\Omega G ; \mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_*(\Omega G ; \mathbf{Z}/2\mathbf{Z})$$

Put $y * y' = \mathrm{Ad}_*(y \otimes y')$ and $y * b = \mathrm{ad}_*(y \otimes b)$ where $y, y' \in \mathrm{H}_*(G ; \mathbf{Z}/2\mathbf{Z})$ and $b \in \mathrm{H}_*(\Omega G ; \mathbf{Z}/2\mathbf{Z})$. Following are the dual statement of the result in [4].

Theorem 3. 1. *For* $y, y', y'' \in \mathrm{H}_*(G ; \mathbf{Z}/2\mathbf{Z})$ *and* $b, b' \in \mathrm{H}_*(\Omega G ; \mathbf{Z}/2\mathbf{Z})$

(i) $1 * y = y$, $1 * \mathrm{b} = \mathrm{b}$.

(ii) $y * 1 = 0$, *if* $|y| > 0$, *whether* $1 \in \mathrm{H}_*(G ; \mathbf{Z}/2\mathbf{Z})$ *or* $1 \in \mathrm{H}_*(\Omega G ; \mathbf{Z}/2\mathbf{Z})$.

(iii) $(yy') * b = y * (y' * b)$.

(iv) $y * (bb') = \sum (y' * b)(y'' * b')$ *where* $\Delta_* y = \sum y' \otimes y''$.

(v) $\sigma(y * b) = y * \sigma(b)$ *where* σ *is the homology suspension.*

(vi) $\mathrm{Sq}^n_*(y * b) = \sum_i (\mathrm{Sq}^i_* y) * (\mathrm{Sq}^{n-i}_* b)$.

$\mathrm{Sq}^n_*(y * y') = \sum_i (\mathrm{Sq}^i_* y) * (\mathrm{Sq}^{n-i}_* y')$.

(vii) $\Delta_*(y * b) = (\Delta_* y) * (\Delta_* b)$
$= \sum (y' * b') \otimes (y'' * b'')$

where $\Delta_* y = \sum y' \otimes y''$ and $\Delta_* b = \sum b' \otimes b''$, *Also*

$$\bar{\Delta}_*(y * b) = (\Delta_* y) * (\bar{\Delta}_* b).$$

(viii) *If* b *is primitive then* $y * b$ *is primitive.*

Let $y_{2i} \in \mathrm{H}_*(G ; \mathbf{Z}/2\mathbf{Z})$ be the dual of x_i^2 for $i = 3, 5, 9, 15$ and y_{12}, y_{24}, y_{20} be the dual of x_3^4, x_3^8, x_5^4 respectively with respect to the monomial basis. Also in $\mathrm{H}_*(E_8 ; \mathbf{Z}/2\mathbf{Z})$ we put as

$$y^m = y_6^{m_1} y_{12}^{m_2} y_{24}^{m_3} y_{16}^{m_4} y_{20}^{m_5} y_{18}^{m_6} y_{30}^{m_7}$$

for $m = (m_1, m_2, \cdots, m_7) \in \mathbf{Z}/2\mathbf{Z}^7$. Then the. result of [4] implies the next theorem. See [6].

Theorem 3. 2. *We define a submodule A of* $H_*(G\ ;\ \mathbf{Z}/2\mathbf{Z})$ *as*

$$A = \wedge(y_6, y_{10}, y_{18}) \qquad \text{for } G = E_7$$
$$A = \langle y^m \text{ for all } m \in \mathbf{Z}/2\mathbf{Z}^7\rangle \qquad \text{for } G = E_8.$$

Then there exist a retraction $p : H_*(G\ ;\ \mathbf{Z}/2\mathbf{Z}) \to A$ *and the following diagram commutes.*

$$\begin{array}{ccc} H_*(G\ ;\ \mathbf{Z}/2\mathbf{Z}) \otimes H_*(\Omega G\ ;\ \mathbf{Z}/2\mathbf{Z}) & \xrightarrow{ad_*} & H_*(\Omega G\ ;\ \mathbf{Z}/2\mathbf{Z}) \\ \downarrow p \otimes 1 & \nearrow ad_* & \\ A \otimes H_*(\Omega G\ ;\ \mathbf{Z}/2\mathbf{Z}) & & \end{array}$$

Remark. 1. The submodule A has an algebra structure induced from that of $H_*(G\ ;\ \mathbf{Z}/2\mathbf{Z})$. When $G=E_7$, A is a commutative exterior algebra over $\mathbf{Z}/2\mathbf{Z}$. But when $G=E_8$, A is a non-commutative algebra over $\mathbf{Z}/2\mathbf{Z}$. In fact A is the dual of $\wedge(x_3^2, x_5^2, x_9^2)$ for $G=E_7$ and is the dual of $\mathbf{Z}/2\mathbf{Z}[x_3^2, x_5^2, x_9^2, x_{15}^2]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4)$ for $G = E_8$. Thus we can easily see that, for $G = E_8$, A is generated by $\{y_6, y_{12}, y_{24}, y_{10}, y_{20}, y_{18}\}$ as algebra and the fundamental relations are

$$y_{2i}^2 = 0 \text{ for } i = 3, 6, 12, 5, 10, 9,$$
$$[y_{2i}, y_{2j}] = 0 \text{ for } (i, j) \neq (6, 9), (9, 6), (5, 10), (10, 5), (3, 18), (18, 3)$$

and

$$[y_6, y_{24}] = [y_{10}, y_{20}] = [y_{12}, y_{18}]\ (= y_{30}).$$

Remark. 2. By Theorem 3. 1 (iv) and Theorem 3. 2 we see that for $b \in H_*(\Omega G\ ;\ \mathbf{Z}/2\mathbf{Z})$ and $i = 3, 5, 9$

$$y_{2i} * b^2 = (y_{2i} * b)b + (y_i * b)^2 + b(y_{2i} * b)$$

where y_i is the dual of x_i for $i = 3, 5, 9$ with respect to the monomial basis.

Remark. 3. By theorem 3. 1 and 3. 2, when $G=E_8$, if $y_i * b_j$ is determined for $i=6, 12, 24, 10, 20, 18$ and $b_j \in H_*(G\ ;\ \mathbf{Z}/2\mathbf{Z})$, then the map $H_*(\mathrm{ad}\ ;\ \mathbf{Z}/2\mathbf{Z})$ is determined completely.

4. Adjoint action on ΩE_8

The next theorem is the main result of this paper.

Theorem 4. 1. *For $j \in \{6, 12. 24, 10, 20, 18\}$ and $b_i \in H_*(\Omega E_8 ; \mathbf{Z}/2\mathbf{Z})$, $y_j * b_i$ is given by the following tables.*

b_j	$y_6 * b_j$	$y_{10} * b_j$	$y_{18} * b_j$
b_2	0	0	0
b_4	0	b_{14}	b_{22}
b_8	b_{14}	b_4b_{14}	$b_{26}+b_4b_{22}$
b_{14}	0	0	b_{16}^2
b_{16}	$b_{22}+b_8b_{14}$	$b_{26}+b_4b_8b_{14}$	$b_{34}+b_8b_{26}+b_4b_8b_{22}$
b_{22}	b_{14}^2	b_{16}^2	0
b_{26}	b_{16}^2	0	b_{22}^2
b_{28}	b_{34}	b_{38}	$b_{16}^2b_{14}+b_{46}$
b_{34}	0	b_{22}^2	b_{26}^2
b_{38}	b_{22}^2	0	b_{28}^2
b_{46}	b_{26}^2	b_{38}^2	b_{16}^4
b_{58}	b_{16}^4	b_{34}^2	b_{38}^2

b_j	$y_{12} * b_j$	$y_{20} * b_j$	$y_{24} * b_j$
b_2	b_{14}	b_{22}	b_{26}
b_4	b_2b_{14}	b_2b_{22}	$b_{28}+b_2b_{26}$
b_8	$b_2b_4b_{14}$	$b_{28}+b_2b_4b_{22}$	$b_4b_{28}+b_2b_4b_{26}$
b_{14}	0	b_{34}	b_{38}
b_{16}	$b_{28}+b_2b_4b_8b_{14}$	$b_8b_{28}+b_2b_4b_8b_{22}$	$b_4b_8b_{28}+b_2b_4b_8b_{26}$
b_{22}	b_{34}	0	b_{46}
b_{26}	b_{38}	b_{46}	0
b_{28}	0	0	b_{26}^2
b_{34}	0	0	b_{58}
b_{38}	0	b_{58}	0
b_{46}	b_{58}	0	0
b_{58}	0	0	0

Remark. The action of cohomology operations on, $H_*(\Omega E_8 ; \mathbf{Z}/2\mathbf{Z})$ is determined by A. Kono and K. Kozima in [3]. But we do not use them. We use the Hopf algebra structure of $H_*(\Omega E_8 ; \mathbf{Z}/2\mathbf{Z})$ and the result in $H_*(\Omega E_7 ; \mathbf{Z}/2\mathbf{Z})$.

Proof. In $H_*(\Omega E_7 ; \mathbf{Z}/2\mathbf{Z})$ $y_j * b_i$ and $Sq_*^{2^k} b_i$ are determined as follows. See Theorem 5. 11 in [6].

b_i	$y_6 * b_i$	$y_{10} * b_i$	$y_{18} * b_i$
b_2	0	0	b_{10}^2
b_4	b_{10}	b_{14}	$b_{22}+b_2b_{10}^2$
b_8	$b_{14}+b_4b_{10}$	$b_{18}+b_4b_{14}$	$b_{26}+b_4b_{22}+b_2b_4b_{10}^2$
b_{10}	0	b_{10}^2	b_{14}^2
b_{14}	b_{10}^2	0	b_{16}^2
b_{16}	$b_{22}+b_8b_{14}+b_4b_8b_{10}$	$b_{26}+b_8b_{18}+b_4b_8b_{14}$	$b_{34}+b_8b_{26}+b_4b_8b_{22}+b_2b_4b_8b_{10}^2$
b_{18}	0	b_{14}^2	b_{18}^2
b_{22}	b_{14}^2	b_{16}^2	b_{10}^4
b_{26}	b_{16}^2	b_{18}^2	b_{22}^2
b_{34}	b_{10}^4	b_{22}^2	b_{26}^2

b_i	$\mathrm{Sq}_*^2\, b_i$	$\mathrm{Sq}_*^4\, b_i$	$\mathrm{Sq}_*^8\, b_i$	$\mathrm{Sq}_*^{16}\, b_i$
b_4	b_2	—	—	—
b_8	b_2b_4	b_4	—	—
b_{10}	b_4^2	0	—	—
b_{14}	0	b_{10}	—	—
b_{16}	$b_{14}+b_2b_4b_8$	b_4b_8	b_8	—
b_{18}	0	0	b_{10}	—
b_{22}	b_{10}^2	0	b_{14}	—
b_{26}	0	b_{22}	b_{18}	—
b_{34}	b_{16}^2	0	0	b_{18}

By the naturality of adjoint action, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{H}_*(E_7\,;\,\mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(\Omega E_7\,;\,\mathbf{Z}/2\mathbf{Z}) & \xrightarrow{\mathrm{ad}_*} & \mathrm{H}_*(\Omega E_7\,;\,\mathbf{Z}/2\mathbf{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}_*(E_8\,;\,\mathbf{Z}/2\mathbf{Z}) \otimes \mathrm{H}_*(\Omega E_8\,;\,\mathbf{Z}/2\mathbf{Z}) & \xrightarrow{\mathrm{ad}_*} & \mathrm{H}_*(\Omega E_8\,;\,\mathbf{Z}/2\mathbf{Z}) \end{array}$$

Thus we can easily see that above tables remain true also in $\mathrm{H}_*(\Omega E_8\,;\,\mathbf{Z}/2\mathbf{Z})$ except for $y_j * b_{10}$ and $y_j * b_{18}$ by replacing b_{10}, b_{18}, by 0.

Also we can easily see that

$$\mathrm{Sq}_*^8\,\mathrm{Sq}_*^4\,\mathrm{Sq}_*^2\, b_{28} = \mathrm{Sq}_*^{14}\, b_{28} = b_{14} \neq 0.$$

This means $\mathrm{Sq}_*^2\, b_{28} = b_{26}$.

If b_i is primitive, $y_j * b_i$ is primitive. By (viii) of Theorem 3. 1, $y_j * b_i$ is primitive for

$$(i, j) \in \left\{ \begin{matrix} (10, 38), (12, 38), (12, 58), (20, 22), (20, 34), (20, 46), \\ (20, 58), (24, 26), (24, 38), (24, 46), (24, 58) \end{matrix} \right\}$$

Since primitive elements of these degrees are there in $H_*(\Omega E_8 ; \mathbf{Z}/2\mathbf{Z})$ these elements are 0.

Next we consider $y_{12} * b_2$. Because $y_{12} * b_2$ is primitive, it is b_{14} or 0. On the other hand, we have

$$\begin{aligned}\bar{\Delta}_*(y_{12} * b_4) &= (y_{12} * b_2) \otimes b_2 + (y_6 * b_2) \otimes (y_6 * b_2) + b_2 \otimes (y_{12} * b_2) \\ &= \bar{\Delta}_* (y_{12} * b_2)\, b_2.\end{aligned}$$

This means $y_{12} * b_4 = (y_{12} * b_2)\, b_2$ since there is no primitive element in $H_{16}(\Omega E_8 ; \mathbf{Z}/2\mathbf{Z})$. Therefore we have

$$Sq_*^2\, (y_{12} * b_4) = Sq_*^2\, (y_{12} * b_2)\, b_2 = 0,$$

while

$$Sq_*^2\, (y_{12} * b_4) = y_{16} * b_4 + y_{12} * b_2 = y_{14} + y_{12} * b_2.$$

Hence we obtain

$$\begin{aligned}y_{12} * b_2 &= b_{14}, \\ y_{12} * b_4 &= b_{14} b_2.\end{aligned}$$

In the same way we can easily show

$$\begin{aligned}y_{20} * b_2 &= b_{22}, \\ y_{20} * b_4 &= b_{22} b_2, \\ y_{24} * b_2 &= b_{26}, \\ y_{24} * b_4 &= b_{28} + b_{26} b_2.\end{aligned}$$

Since

$$\bar{\Delta}_*(y_{12} * b_8) = \Delta_*(y_{12}) * \bar{\Delta}_* b_8 = \bar{\Delta}_*(b_{14}\, b_4\, b_2)$$

and no primitive element is there in $H_{20}(\Omega E_8 ; \mathbf{Z}/2\mathbf{Z})$, we have

$$y_{12} * b_8 = b_{14} b_4 b_2,$$

In the similar way we can determine

$$y_{12} * b_{28},\ y_{20},\ b_{28},\ y_{20} * b_{28},\ y_{12} * b_{16},\ y_{20} * b_8,\ y_{20} * b_{16}$$

as in the table of Theorem.

Also as

$$\begin{aligned}\bar{\Delta}_*(y_{24} * b_8) &= \Delta_*\, y_{24} * \bar{\Delta}_*\, b_8 \\ &= \bar{\Delta}_*(b_{26} b_4 b_2 + b_{28} b_4)\end{aligned}$$

and the only primitive element in $H_{32}(\Omega E_8 ; \mathbf{Z}/2\mathbf{Z})$ is b_{16}^2, we can put

$$y_{24} * b_8 = b_{26} b_4 b_2 + b_4 b_{28} + p b_{16}^2 \tag{1}$$

where $\rho \in \mathbf{Z}/2\mathbf{Z}$. Applying Sq_*^4 to each side of (1), we have

$$Sq^4_*(y_{24} * b_8) = y_{20} * b_8 + y_{24} * b_4 = b_{22}b_4b_2 + b_{26}b_2,$$

while

$$Sq^4_*(y_{26}b_4b_2 + b_{28}b_4 + \rho b_{16}{}^2) = b_{22}b_4b_2 + b_{26}b_2 + \rho b_{14}^2.$$

Thus $\rho = 0$ and $y_{24} * b_8$ is determined. Now we can determine $y_{24} * b_{16}$ modulo primitive elements. Since no primitive elements is there in $H_{40}(\Omega E_8 ; \mathbf{Z}/2\mathbf{Z})$, we can determine $y_{24} * b_{16}$ as

$$y_{24} * b_{16} = b_{28}b_8b_4 + b_{26}b_8b_4b_2.$$

Since b_{14} is primitive, $y_{20} * b_{14} = b_{34}$ or 0. Also $Sq^2_*(y_{20} * b_{14}) = y_{18} * b_{14} = b_{16}{}^2$. This implies

$$y_{20} * b_{14} = b_{34}, \ Sq^2_* b_{34} = b_{16}{}^2.$$

In the similar way we apply Sq^2_* to $y_6 * b_{28}$, Sq^2_* to $y_{12} * b_{22}$, Sq^4_* to $y_{12} * b_{26}$ and Sq^2_* to $y_{20} * b_{26}$. and see that the followings are determined as the statement :

$$y_6 * b_{28}, \ y_{12} * b_{22}, \ y_{12} * b_{26}, \ y_{20} * b_{26}, \ Sq^4_* b_{38}, \ Sq^2_* b_{46}.$$

From the above result we can deduce that

$$Sq^8_* b_{46} = Sq^8_*(y_{20} * b_{26}) = y_{12} * b_{26} = b_{38}.$$

Also as $\bar{\Delta}_* Sq^4_* b_{28} = Sq^4_* b_{28} = 0$, we have $Sq^4_* b_{28} = 0$. In the similar way we have

$$Sq^{2^k}_* b_i = 0 \text{ for } (k, j) \in \left\{ \begin{array}{l} (3,28), (1,38), (3,38), (2,46), \\ (4,46), (2,58), (3,58), (4,58) \end{array} \right.$$

Using the above result we can compute $Sq^4_*(y_{18} * b_{38})$ as

$$Sq^4_* y_{18} * b_{38} = y_{18} * b_{34} = b_{26}^2,$$

while $y_{18} * b_{38} = b_{28}{}^2$ or 0. This implies $y_{18} * b_{38} = b_{28}^2$. In the similar manner, applying Sq^4_* to $y_{10} * b_{28}$, Sq^4_* to $y_{10} * b_{38}$, Sq^8_* to $y_6 * b_{46}$, Sq^2_* to $y_{12} * b_{34}$, Sq^4_* to $y_{24} * b_{14}$ and Sq^2_* to $y_{24} * b_{22}$, the followings are determined :

$$y_{10} * b_{28}, \ y_6 * b_{38}, \ y_6 * b_{46}, \ y_{12} * b_{34}, \ y_{24} * b_{14}, \ y_{24} * b_{22}$$

as in the table in Theorem.

Moreover by applying Sq^4_* to $y_{10} * b_{46}$, Sq^2_* to $y_{12} * b_{46}$ and Sq^2_* to $y_{20} * b_{38}$ we have that

$$y_{10} * b_{46} = b_{28}{}^2,$$
$$y_{12} * b_{46} = b_{58},$$
$$y_{20} * b_{38} = b_{58},$$

Since $y_{18}^2 * b_{28} = 0$, we can see

$$y_{18} * (y_{18} * b_{28}) = y_{18} * (b_{16}^2 b_{14} + b_{46}) = b_{16}^4 + y_{18} * b_{46} = 0.$$

Therefore $y_{18} * b_{46} = b_{16}^4$. In this way we compute $y_{12}^2 * b_2$, $y_{24}^2 * b_4$ to obtain

$$y_{12} * b_{14} = 0,$$
$$y_{24} * b_{28} = b_{26}^2.$$

Also we can compute $y_{24} * b_{34}$ as

$$y_{24} * b_{34} = y_{24} * (y_{20} * b_{14}) = y_{20} * (y_{24} * b_{14}) = y_{20} * b_{38} = b_{58}.$$

The rest we have to do is to determine $y_6 * b_{58}$, $y_{10} * b_{58}$ and $y_{18} * b_{58}$. By applying Sq_*^2 to $y_{20} * b_{38}$, we have

$$Sq_*^2\, b_{58} = Sq_*^2 (y_{20} * b_{38}) = y_{18} * b_{38} = b_{28}^2.$$

Thus by applying Sq_*^2 to $y_{12} * b_{58}$, it follows that

$$0 = Sq_*^2 (y_{12} * b_{58}) = y_{10} * b_{58} + y_{12} * b_{28}^2 = y_{10} * b_{58} + b_{34}^2.$$

Therefore $y_{10} * b_{58} = b_{34}^2$. We apply Sq_*^4 to $y_{10} * b_{58}$ and Sq_*^8 to $y_{18} * b_{58}$ to obtain

$$y_6 * b_{58} = b_{16}^4,$$
$$y_{18} * b_{58} = b_{38}^2.$$

Now we obtain the all entries of the tables in Theorem 4. 1.

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