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Rectilinear slit conformal mappings

Dedicated to Professor Yukio Kusunoki on his 70th birthday

By

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1. Introduction

Let G be a region in the extended complex plane and $\widehat{G}$ be the Kerékjártó-Stoïlow compactification. Take a real-valued function $\widehat{\varphi}$ on $\widehat{G}-G$ which we call an angle assignment. Our purpose is to give a rectilinear slit conformal mapping on G such that it maps each boundary component p to a slit which lies on a line of inclination $\widehat{\varphi}(p)$ to the positive real axis, where a slit may be a point. This rectilinear slit mapping is said to achieve the angle assignment $\widehat{\varphi}$. Koebe [KP] showed the following. In the case G is a finitely connected domain, there exists a unique rectilinear slit mapping with a normalization which achieves an arbitrary given angle assignment. On the other hand, in the case G has a countable number of boundary components, there exist angle assignments which are not achieved (cf. [W]). If G has an uncountable number of boundary components, even parallel slit mapping with a normalization is not always unique as a region whose boundary consists of parallel slits of positive measure. We follow the suggestion of B. Rodin [ABB] and assume that the angle assignment $\widehat{\varphi}$ is continuous. In this paper, we assume further additional conditions about $\widehat{\varphi}$ and by Shiba's theorem [S] argue the uniqueness and existence of rectilinear slit mapping. Since the normalized rectilinear slit mapping with extremal crossing module, which F. Weening [W] gave, has boundary behavior as in Shiba's theorem, we can prove the uniqueness.

We are grateful to Dr. Frederick Weening.

2. Notation and Preliminaries

Let $\Lambda=\Lambda(G)$ be the real Hilbert space of square integrable complex differentials whose inner product is given by

$$\langle\omega,\ \sigma\rangle = \text{real part of} \iint_G \omega\wedge{}^*\bar{\sigma} = \Re(\omega,\ \sigma),$$

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where $^*\sigma$ denotes the conjugate differential of σ and $\bar{\sigma}$ denotes the complex conjugate of σ. Let Λ_{eo} be the completion of the class consisting of differentials of complex-valued C^∞-functions with compact support and Λ_h be the space of harmonic differentials. We know the following orthogonal decomposition;

$$\Lambda = \Lambda_h \dot{+} \Lambda_{eo} \dot{+} {}^*\Lambda_{eo}.$$

We use the following subspaces:
$\Gamma_{eo} = \{\omega \in \Lambda_{eo} : \omega \text{ is a real differential}\}$,
$\Gamma_h = \{\omega \in \Lambda_h : \omega \text{ is a real differential}\}$,
$\Gamma_{he} = \{\omega \in \Gamma_h : \omega$ is exact i.e. there exists a harmonic function w such that $dw = \omega\}$,
$\Gamma_{hm} = \{\sigma \in \Gamma_h : \langle \sigma, {}^*\omega \rangle = 0 \text{ for any } \omega \in \Gamma_{he}\}$.
We know $\Gamma_{hm} \subset \Gamma_{he}$ and $\Gamma_{hm} = \overline{B\Gamma_{hm}}$, $\Gamma_{he} = \overline{B\Gamma_{he}}$, where $B\Gamma_{hm} = \{du \in \Gamma_{hm} : u$ is bounded$\}$, $B\Gamma_{he} = \{dv \in \Gamma_{he} : v \text{ is bounded}\}$, and $\overline{B\Gamma_x}$ denotes the completion of the class $B\Gamma_x$. The class Λ_{eo} coincides with $\Gamma_{eo} \dot{+} i\Gamma_{eo}$ and set $\Lambda_{he} = \Gamma_{he} \dot{+} i\Gamma_{he}$, $\Lambda_{hm} = \Gamma_{hm} \dot{+} i\Gamma_{hm}$.

Now we assume that

$$(*)\begin{cases} \hat{\varphi} \textit{ has a continuous extension } \varphi \textit{ to } \widehat{G} \textit{ which satisfies} \\ \qquad d\varphi \in \Gamma_{hm} \dot{+} \Gamma_{eo} \text{ on } G. \end{cases}$$

Put $\Phi = \exp(i\varphi)$. For a bounded harmonic function v which satisfies $dv \in B\Gamma_{he}$, $d(\Phi v)$ belongs to $\Lambda_{he} \dot{+} \Lambda_{eo}$. We can write it uniquely as follows;

$$d(\Phi v) = dv(\Phi) + dv_0(\Phi),\ dv(\Phi) \in \Lambda_{he},\ dv_0(\Phi) \in \Lambda_{eo}.$$

Set

$$B\Gamma_{\Phi he} = \{dv(\Phi) \in \Lambda_{he} : v \text{ satisfies } dv \in B\Gamma_{he}\},$$
$$B\Gamma_{\Phi hm} = \{dv(\Phi) \in \Lambda_{he} : v \text{ satisfies } dv \in B\Gamma_{hm}\}.$$

We have the following.

Lemma 1. *Let $v(\Phi)$ and $u(\Phi)$ satisfy $dv(\Phi), du(\Phi) \in B\Gamma_{\Phi he}$. Then* $\langle dv(\Phi), {}^*du(\Phi) \rangle = \langle dv, {}^*du \rangle$.

Proof. We have

$$\begin{aligned} \langle dv(\Phi), {}^*du(\Phi) \rangle &= \langle d(v(\Phi) + v_0(\Phi)), {}^*d(u(\Phi) + u_0(\Phi)) \rangle \\ &= \langle d(\Phi v), {}^*d(\Phi u) \rangle = -\Re \iint_G d(\Phi v) \wedge \overline{d(\Phi u)} \\ &= -\Re \lim_{m\to\infty} \int_{\partial G_m} \Phi v (\bar{\Phi} du + u d\bar{\Phi}) \\ &= -\Re \lim_{m\to\infty} \int_{\partial G_m} (v du + (-i) v u d\varphi) \\ &= \langle dv, {}^*du \rangle, \end{aligned}$$

where $\{G_m\}$ is a regular exhaustion of G.

Lemma 2. *The subspace $B\Gamma_{\Phi he}$ is orthogonal to ${}^*B\Gamma_{\Phi hm}$.*

Proof. By Lemma 1

$$\langle dv(\Phi), {}^*du(\Phi)\rangle = \langle dv, {}^*du\rangle = 0,$$

because Γ_{he} is orthogonal to ${}^*\Gamma_{hm}$.

Set $\Lambda_x = \overline{B\Gamma_{\Phi he} + iB\Gamma_{\Phi hm}}$. We have

Lemma 3. *The subspace Λ_x is orthogonal to $i^*\Lambda_x$.*

Proof. It is sufficient to show that

$$\langle d(v(\Phi) + iu(\Phi)), i^*d(v_1(\Phi) + iu_1(\Phi))\rangle = 0$$

for $dv(\Phi), dv_1(\Phi) \in B\Gamma_{\Phi he}$ and $du(\Phi), du_1(\Phi) \in B\Gamma_{\Phi hm}$. By Lemma 2, we have

$$\langle dv(\Phi), -{}^*du_1(\Phi)\rangle = \langle du(\Phi), {}^*dv_1(\Phi)\rangle = 0.$$

From the proof of Lemma 1 and the assumption $(*)$, it follows that

$$\langle dv(\Phi), i^*dv_1(\Phi)\rangle = -\langle d(vv_1), {}^*d\varphi\rangle = 0$$

and

$$\langle idu(\Phi), -{}^*du_1(\Phi)\rangle = -\langle d(uu_1), {}^*d\varphi\rangle = 0.$$

Therefore we get the conclusion

$$\langle dv(\Phi) + idu(\Phi), i^*(dv_1(\Phi) + idu_1(\Phi))\rangle = 0.$$

We remark the following.

Lemma 4. *If dw belongs to $B\Gamma_{hm} + iB\Gamma_{hm}$, then*

$$dw \in (B\Gamma_{\Phi hm} + iB\Gamma_{\Phi he}) \cap (B\Gamma_{\Phi he} + iB\Gamma_{\Phi hm}).$$

Proof. Since Λ_{he} is orthogonal to ${}^*\Lambda_{hm}$,

$$\langle d\Phi v, {}^*dw\rangle = \langle dv(\Phi), {}^*dw\rangle = 0, \text{ for } dv \in B\Gamma_{he},$$
$$\langle id\Phi u, {}^*dw\rangle = \langle idu(\Phi), {}^*dw\rangle = 0, \text{ for } du \in B\Gamma_{hm}.$$

From the proof of Lemma 1 and the assumption $(*)$, we have

$$\langle dv(\Phi), {}^*dw\rangle = \langle dv(\Phi), {}^*d(\Phi\bar{\Phi}w)\rangle = \langle dv, {}^*d(\bar{\Phi}w)\rangle.$$

Similarly, we have

$$\langle idu(\Phi), {}^*dw\rangle = \langle idu, {}^*d(\bar{\Phi}w)\rangle = \langle du, -i^*d(\bar{\Phi}w)\rangle.$$

Since

$$\langle dv(\Phi), {}^*dw\rangle = \langle idu(\Phi), {}^*dw\rangle = 0,$$

we have

$$\langle dv, {}^*d(\bar{\Phi}w)\rangle = \langle du, -i^*d(\bar{\Phi}w)\rangle = 0.$$

It follows that

$$\Re^*d(\bar{\Phi}w) \in {}^*\Gamma_{hm} + {}^*\Gamma_{eo},$$

and

$$\Im^*d(\bar{\Phi}w) \in {}^*\Gamma_{he} + {}^*\Gamma_{eo}.$$

Remarking that w is bounded, we can write

$$\bar{\Phi}w = s + it + p,$$

where $ds \in B\Gamma_{hm}$, $dt \in B\Gamma_{he}$, $dp \in \Lambda_{eo}$ and p is a bounded Dirichlet potential (cf. [CC]). Since p and Φp vanish on the harmonic boundary of Royden compactification of G, Φp is a Dirichlet potential and $d(\Phi p) \in \Lambda_{eo}$ (cf. [CC]). It follows that

$$\begin{aligned} dw &= d(\Phi s + i\Phi t + \Phi p) \\ &= d(s(\Phi) + s_0(\Phi)) + id(t(\Phi) + t_0(\Phi)) + d(\Phi p), \end{aligned}$$

and

$$dw = ds(\Phi) + idt(\Phi) \in B\Gamma_{\Phi hm} + iB\Gamma_{\Phi he}.$$

Since idw also belongs to $B\Gamma_{hm} + iB\Gamma_{hm}$, we know $dw \in B\Gamma_{\Phi he} + iB\Gamma_{\Phi hm}$.

3. Uniqueness of rectilinear slit mappings

We say that a meromorphic function f has the Λ_x*—behavior* if df coincides with an element in $\Lambda_x + \Lambda_{eo}$ on a neighborhood of the ideal boundary. By Shiba's argument we have the following.

Proposition 1. *Let meromorphic functions f_1, f_2 have the Λ_x—behavior and the same singularities i.e. $f_1 - f_2$ is analytic. Then $f_1 - f_2$ is constant.*

Proof. The function $f_1 - f_2$ has also the Λ_x*—behavior* and no singularities. It follows that $d(f_1 - f_2) \in \Lambda_x$. Since $d(f_1 - f_2)$ is analytic, $d(f_1 - f_2) = i^*d(f_1 - f_2) \in \Lambda_x \cap i^*\Lambda_x = \{0\}$. Hence $f_1 - f_2$ is constant.

Here we remark the following. When a harmonic function u satisfies $du \in \Gamma_{he}$, u has a quasi-continuous extension to the Kuramochi compactification of G (cf. [CC]). If $du \in \Gamma_{hm}$, it takes a constant value quasi-everywhere on each Kerékjártó-Stoïlow boundary-component (cf. [KY]). By these facts, it is sufficient that φ is assumed to be quasi-continuous in the assumption ($*$).

Assume that G contains ∞. A conformal mapping f achieving an angle assignment is called a normalized rectilinear slit mapping if it has the following expansion;

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \text{ in a neighborhood of } \{\infty\}.$$

Theorem 1. *Assume that the region G has the following property: every harmonic function u ($du \in \Gamma_{he}$) whose quasi-continuous extension to the Kuramochi compactification of G takes a constant value quasi-everywhere on every Kerékjártó-Stoïlow boundary-component satisfies $du \in \Gamma_{hm}$. There is at most one normalized rectilinear slit mapping which achieves the angle assignment $\widehat{\varphi}$ ($d\varphi \in \Gamma_{hm}$).*

Proof. If a conformal mapping f on G maps each boundary component p to a slit which lies on a line of inclination $\widehat{\varphi}(p)$, the imaginary part of $\overline{\Phi}f$ takes a constant value on each Kerékjártó-Stoïlow boundary-component. By the assumption, $d(\overline{\Phi}f)$ coincides with an element of $B\Gamma_{he} + iB\Gamma_{hm} + \Lambda_{eo}$ in a neighborhood of the ideal boundary. It follows that df coincides with an element of $\Lambda_x + \Lambda_{eo}$ in a neighborhood of the ideal boundary and so f has the Λ_x $-$*behavior.* Therefore, by Proposition 1, we conclude the uniqueness.

If G has at most a countable number of Kerékjátró-Stoïlow boundary components, it has the property in the Theorem (cf. [MF]). We have the following.

Corollary 1. *When G has a countable number of boundary components, there is at most one normalized rectilinear slit mapping which achieves the angle assignment $\widehat{\varphi}$ ($d\varphi \in \Gamma_{hm}$).*

Remark. In this case G is of countable connectivity, F. Weening [W] showed, by using the argument principle, that there is at most one normalized rectilinear slit mapping which achieves an arbitrary angle assignment.

4. The existence of rectilinear quasi-slit mappings

In this section, we assume that the angle assignment $\widehat{\varphi}$ satisfies an additional condition;

$$(**)\begin{cases} \text{there exists a positive constant } M \text{ such that} \\ \qquad \langle d(\overline{\Phi}w), d(\overline{\Phi}w)\rangle \leq M\langle dw, dw\rangle \\ \qquad \langle d(\Phi w), d(\Phi w)\rangle \leq M\langle dw, dw\rangle, \\ \text{where } dw \in B\Lambda_{he} \text{ and for a fixed point } a \in G, \\ \qquad w(a) = 0. \end{cases}$$

We have the following.

Theorem 2.

$$\Lambda_x \dotplus i^*\Lambda_x = \Lambda_h.$$

Proof. Suppose that an $\omega \in \Lambda_h$ is orthogonal to $\Lambda_x \dotplus i^*\Lambda_x$. By the orthogonal decomposition:

$$\Lambda_h = \Lambda_{hm} \dotplus (\Gamma_{he} \cap {}^*\Gamma_{he} \dotplus i(\Gamma_{he} \cap {}^*\Gamma_{he})) \dotplus {}^*\Lambda_{hm},$$

we can write ω as

$$\omega = dw_1 + {}^*dw_2 + {}^*dw_3,$$

where $dw_1, dw_3 \in \Lambda_{hm}$, and $dw_2 \in (\Gamma_{he} \cap {}^*\Gamma_{he} \dotplus i(\Gamma_{he} \cap {}^*\Gamma_{he}))$. There exist sequences $\{dw_{1n}\}, \{dw_{3n}\} \subset B\Gamma_{hm} \dotplus iB\Gamma_{hm}$ which converge to dw_1, dw_3 in the Dirichlet norm, respectively. By Lemma 4, $dw_{1n} + {}^*dw_{3n}$ belongs to $B\Gamma_{\Phi he} + iB\Gamma_{\Phi hm} + i^*(B\Gamma_{\Phi he} + iB\Gamma_{\Phi hm}) \subset \Lambda_x \dotplus i^*\Lambda_x$. It follows that

$$0 = \lim_{n\to\infty} \langle dw_{1n} + {}^*dw_{3n}, \omega\rangle = \langle dw_1 + {}^*dw_3, dw_1 + {}^*dw_3\rangle.$$

Hence $dw_1 = {}^*dw_3 = 0$ and $\omega = {}^*dw_2$. By the supposition for $dv \in B\Gamma_{he}$, $du \in B\Gamma_{hm}$,

$$\langle d(\Phi v), {}^*dw_2\rangle = \langle id(\Phi u), {}^*dw_2\rangle = 0.$$

If w_2 is bounded, from Lemma 1

$$\langle dv, {}^*d(\bar{\Phi} w_2)\rangle = \langle d(\Phi v), {}^*dw_2\rangle = 0$$

and

$$\langle du, -i^*d(\bar{\Phi} w_2)\rangle = \langle id(\Phi u), {}^*dw_2\rangle = 0.$$

It follows that

$$\Re^*d(\bar{\Phi} w_2) \in {}^*\Gamma_{hm} + {}^*\Gamma_{eo} \text{ and } \Im^*d(\bar{\Phi} w_2) \in {}^*\Gamma_{he} + {}^*\Gamma_{eo}.$$

We have $d(\bar{\Phi} w_2) \in B\Gamma_{hm} + iB\Gamma_{he} + \Lambda_{eo}$ and $dw_2 \in B\Gamma_{\Phi hm} + iB\Gamma_{\Phi he}$. Hence ${}^*dw_2 \in i^*\Lambda_x$ and

$$0 = \langle {}^*dw_2, \omega\rangle = \langle {}^*dw_2, {}^*dw_2\rangle.$$

Therefore $\omega = 0$.

For unbounded w_2 $(w_2(a) = 0)$, take a sequence $\{dw_{2n}\} \subset B\Gamma_{he} + iB\Gamma_{he}$ which converges to dw_2 in the Dirichlet norm. We may assume that $\{w_{2n}\}$ $(w_{2n}(a) = 0)$ also converges to w_2. Since $\{dw_{2n}\}$ is a Cauchy sequence, by the condition $(**)$ $d(\bar{\Phi} w_{2n})$ is also a Cauchy sequence and converges to an element $ds \in \Lambda_{he} + \Lambda_{eo}$. Since $\bar{\Phi} w_{2n}$ converges to $\bar{\Phi} w_2$, we can choose $s = \bar{\Phi} w_2$. It follows that from Lemma 1, for $dv \in B\Gamma_{he}$

$$0 = \langle d(\Phi v), {}^*dw_2\rangle = \lim_{n\to\infty} \langle d(\Phi v), {}^*dw_{2n}\rangle$$

$$= \lim_{n\to\infty} \langle dv, {}^*d(\bar{\Phi} w_{2n})\rangle = \langle dv, {}^*ds\rangle.$$

Hence $\Re ds \in \Gamma_{hm} + \Gamma_{eo}$ and $ds \in \Gamma_{hm} + \Gamma_{eo} + i(\Gamma_{he} + \Gamma_{eo})$. Write $s = s' + s_0'$, where $ds' \in \Gamma_{hm} + i\Gamma_{he}$ and s_0' is a Dirichlet potential. There exists a sequence $\{ds_n\} \subset B\Gamma_{hm} + iB\Gamma_{he}$ such that $s_n(a) = 0$ and $\{s_n\}$ converges to $s' - s'(a)$.

Then by $(**)$ $\{d(\Phi s_n)\}$ converges in the Dirichlet norm. Since $d(s_n(\Phi)) \in i\Lambda_x$, $\{d(s_n(\Phi))\}$ converges to an element in $i\Lambda_x$. Then $\{\Phi s_n\}$ converges to $\Phi(s' - s'(a))$ and $\Phi(s' - s'(a)) + \Phi s_0' + s'(a)\Phi = w_2$. Remarking $\Phi s_0'$ is an Wiener potential (cf. [CC]), we know $dw_2 \in i\Lambda_x$. Therefore ${}^*dw_2 \in i{}^*\Lambda_x$ and $dw_2 = 0$.

Now assume that $G \ni \infty$ and $G \supset \{z : |z| > M > 0\}$. The following is the simplest case of Shiba's theorem in [S].

Theorem 3. *Under the assumptions* $(*)$ *and* $(**)$, *there exists a unique meromorphic function f which has the Λ_x-behavior and has a simple pole only at $\{\infty\}$, where f is normalized as follows;*

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \text{ in a neighborhood of } \{\infty\}.$$

Proof. Take a positive constant M such that $\{z : |z| > M\} \subset G$, and a C^∞-function ρ whose support is contained in $\{z : |z| > M\}$ and $\rho(z) = 1$ on $\{z : |z| > M_1\}$ $(M_1 > M)$. Consider a differential $\frac{1}{2}(d(z\rho) - i{}^*d(z\rho))$. It is a C^∞-differential whose support is contained in $\{z : M \leq |z| \leq M_1\}$. It has the following representation;

$$\frac{1}{2}(d(z\rho) - i{}^*d(z\rho)) = \omega_1 + i{}^*\omega_2 + \omega_0 + i{}^*\tau_0,\ \omega_1, \omega_2 \in \Lambda_x,\ \omega_0, \tau_0 \in \Lambda_{eo}.$$

Then $\sigma = \frac{1}{2}d(z\rho) - \omega_1 - \omega_0 = i({}^*\omega_2 + {}^*\tau_0 + \frac{1}{2}{}^*d(z\rho))$ is closed and coclosed on $G - \{\infty\}$, hence harmonic there. It follows that $\sigma + i{}^*\sigma$ is meromorphic and coincides with an element in $\Lambda_x + \Lambda_{eo}$, because

$$\sigma + i{}^*\sigma = d(z\rho) - \omega_1 - \omega_0 + \omega_2 + \tau_0.$$

Since $\sigma + i{}^*\sigma$ is exact, there exists a meromorphic function f such that $df = \sigma + i{}^*\sigma$. The f has the Λ_x-*behavior* and a simple pole only at $\{\infty\}$. As for the uniqueness, the same argument as in the previous section can be applied.

Remark. If f has the Λ_x-*behavior*, by the remark in the previous section, the imaginary part of $\bar{\Phi} f$ takes a constant value on every boundary component except for a set of Kuramochi capacity 0. So we call this f a rectilinear quasi-slit mapping.

Next we consider a particular case. In an arbitrary region, assume that $\hat{\varphi}$ is continuous and takes a finite number of values. Then there exists a finite number of disjoint closed boundary neighborhoods $\{V_i\}$ such that $\hat{\varphi}$ takes a constant value on the boundary part of V_i and the complement of their union is relatively compact. Then $\hat{\varphi}$ has a continuous extension φ to $\hat{G}$ such that φ is constant on V_i and a C^∞-function on G. It is clear that $d\varphi \in \Gamma_{hm} + \Gamma_{eo}$ and the assumption $(*)$ is satisfied. Note that for a harmonic function w ($dw \in B\Gamma_{he}$, $w(a) = 0$),

$$\begin{aligned}\langle d(\bar{\Phi}w), d(\bar{\Phi}w)\rangle &= \iint_G d(\bar{\Phi}w) \wedge \overline{{}^*d(\bar{\Phi}w)} \\ &= \iint_G (\bar{\Phi}dw + wd\bar{\Phi}) \wedge (\Phi^*dw + w^*d\Phi) \\ &= \iint_G dw \wedge {}^*dw + \iint_G w^2 d\varphi \wedge {}^*d\varphi \\ &\quad + i\left(\iint_G wdw \wedge {}^*d\varphi - \iint_G wd\varphi \wedge {}^*dw\right) \\ &= \langle dw, dw\rangle + \iint_G w^2 d\varphi \wedge {}^*d\varphi.\end{aligned}$$

We know that w^2 has a harmonic majorant W and there exists a constant K_1 which satisfies $W(a) \leq K_1 \langle dw, dw\rangle$, where K_1 is independent of w. Then there exists a constant K such that $W(z) \leq K\langle dw, dw\rangle$ on $G - \cup V_i$. Hence we have

$$\begin{aligned}\iint_G w^2 d\varphi \wedge {}^*d\varphi \leq K\langle dw, dw\rangle \iint_G d\varphi \wedge {}^*d\varphi \\ = K\langle d\varphi, d\varphi\rangle \langle dw, dw\rangle.\end{aligned}$$

Similary we have

$$\langle d(\Phi w), d(\Phi w) \leq M\langle dw, dw\rangle.$$

Hence the assumption $(**)$ is satisfied for $M = 1 + K\langle d\varphi, d\varphi\rangle$. Therefore there exists uniquely the normalized rectilinear quasi-slit mapping with the Λ_x*−behavior.* In this case the behavior coincides with the one of those treated by K. Matsui [MK].

On the other hand, F. Weening [W] showed that there exists a normalized conformal mapping f which achieves the angle assignment and has a property called extremal crossing module. He conjectured the uniqueness. Take a square $S_j = \{e^{i\theta_j}w ; |\Re w| < M, |\Im w| < M\}$ which contains $f(\cup V_i)$ and the complement of the image of f, where $\varphi(z) = \theta_j$ on V_j. Let $\Xi_j = \{\gamma$ is a chain of locally rectifiable arcs in S_j which are allowed to cross $\cup_{i \neq j} f(V_i)$, joins the sides of S_j at inclination $\theta_j + \frac{\pi}{2}\}$, the ρ-length of Ξ_j

$$L_\rho(\Xi_j) = \inf_{\gamma \in \Xi_j}\left\{\sum_{k=1}^{n}\int_{\gamma_k}\rho(w)|dw| + \sum_{k=1}^{n-1}|\gamma_{k+1}(0) - \gamma_k(1)|\right\},$$

where $\gamma_{k+1}(0)$ is the starting point and $\gamma_k(1)$ is the ending point, the ρ-area of Ξ_j with respect to $\cup_{i\neq j} f(V_i)$,

$$A_\rho(f(G), \cup_{i\neq j} f(V_i)) = \iint \rho(u+iv)^2 du dv + \text{area of } \cup_{i\neq j} f(V_i),$$

and the crossing module of the Ξ_j:

$$\mu(\Xi_j) = \inf_\rho \frac{A_\rho(f(G), \cup_{i\neq j} f(V_i))}{L_\rho(\Xi_j)^2} \leq 1.$$

The property of extremal crossing module means $\mu(\Xi_j) = 1$.

Theorem 4. *In an arbitrary region, if $\hat{\varphi}$ is continuous and takes a finite number of values, there exists uniquely the normalized rectilinear slit mapping which achieves the angle assignment $\hat{\varphi}$ and satisfies the property of extremal crossing module.*

Proof. As F. Weening showed, there exists a normalized rectilinear conformal mapping f which achieves the angle assignment and satisfies the property of extremal crossing module. For a real-valued C^∞-function h whose support is contained in the closure of $f(V_j)$, put $\rho_t |dw| = |dw + t\Phi dh|$, where t is a real number. We have

$$\begin{aligned}
&\sum_{k=1}^{n} \int_{\gamma_k} \rho_t(w)|dw| + \sum_{k=1}^{n-1} |\gamma_{k+1}(0) - \gamma_k(1)| \\
&\geq \sum_{k=1}^{n} \Re \int_{\gamma_k} \exp(-i\theta_j)(dw + t\Phi dh) + \sum_{k=1}^{n-1} |\gamma_{k+1}(0) - \gamma_k(1)| \geq 2M, \\
&2A_{\rho_t}(f(G), \cup_{i\neq j} f(V_i)) \\
&= \iint (dw + t\Phi dh) \wedge {}^*\overline{(dw + t\Phi dh)} \\
&= \iint dw \wedge {}^*\overline{dw} + 2t\Re \iint \Phi dh \wedge {}^*\overline{dw} + t^2 \iint dh \wedge {}^*\overline{dh}.
\end{aligned}$$

If $\Re \int\int \Phi dh \wedge {}^*\overline{dw} \neq 0$, for a certain small t,

$$A_{\rho_t}(f(G), \cup_{i\neq j} f(V_i)) < 4M^2.$$

Then $\mu(\Xi_j) < 1$. Thus

$$\begin{aligned}
0 &= \Re \iint_{S_j} \Phi dh \wedge {}^*\overline{dw} = \Re \iint_{S_j} dh \wedge {}^*\overline{d(\bar{\Phi} w)} \\
&= \Re \iint_{\exp(-i\theta_j)S_j} (h_\xi d\xi + h_\eta d\eta) \wedge i(d\xi - id\eta) \\
&= \iint_{\exp(-i\theta_j)S_j} (h_\xi d\xi + h_\eta d\eta) \wedge d\eta = \iint_{\exp(-i\theta_j)S_j} h_\xi d\xi d\eta,
\end{aligned}$$

where $\xi + i\eta = w \exp(-i\theta_j)$. It follows that

$$0 = \Re \iint dh \wedge \overline{{}^*d(\bar{\Phi} f)}$$
$$= \langle dh, d(\bar{\Phi} f) \rangle,$$

and $\Re d(\bar{\Phi} f)$ coincides with an element in ${}^*\Gamma_{hm} + {}^*\Gamma_{eo}$. Hence $\Im d(\bar{\Phi} f)$ coincides with an element in $B\Gamma_{hm} + \Gamma_{eo}$. Therefore f has the Λ_x-*behavior* and so the uniqueness which is conjectured in [W] follows.

Remark. If the univalence of the normalized rectilinear quasi-slit mapping with the Λ_x-*behavior* is shown, applying Koebe's lemma (cf. [SO]) on each V_j;

$$\iint_{\exp(-i\theta_j)S_j} h_\xi d\xi d\eta = 0,$$

we can show that it maps every component to a segment precisely and achieves the angle assignment.

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