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Limit theorems for local times of fractional Brownian motions and some other self-similar processes

By

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1. Introduction and Main Results

A real-valued stochastic process $X(t)$, $t \geq 0$, is called *self-similar with exponent H* (H-ss for brevity) if $X(ct) \stackrel{d}{=} c^H X(t)$ for each $c > 0$. It is called *of stationary increments* (si for brevity) if $X(t+b) - X(b) \stackrel{d}{=} X(t) - X(0)$ for each $b > 0$. The notation $\stackrel{d}{=}$ in the above means the finite-dimensional equivalence of two processes. In this paper, we consider the exponent $H: 0 < H < 1$, and thus $X(0) = 0$. One may refer to Maejima (1989) and Vervaat (1987) for intensive surveys on self-similar proccesses. We also assume that X is of continuous paths or of cadlag paths (cadlag = right-continuous with left-limits everywhere). Under a main assumption of "approximately independent increments", as we shall see in §2, almost every path $X(\cdot, \omega)$ has regular local times $L(t, x, \omega)$, $t \geq 0$ and $x \in \mathbf{R}$. We consider the following functionals :

$$
\begin{aligned}
F_0(t, \omega) &= L(t, 0, \omega), \\
F_1(t, \omega) &= \sup_x L(t, x, \omega), \\
F(\mu, t, \omega) &= \int_{-\infty}^{\infty} L(t, x, \omega)\, \mu(dx), \\
F(f, t, \omega) &= \int_{-\infty}^{\infty} L(t, x, \omega) f(x)\, dx = \int_0^t f(X(s, \omega))\, ds,
\end{aligned}
$$

where μ is a finite Borel measure on the line and f is a Lebesgue integrable function on the line. The above functionals, regarded as continuous-paths processes in t, have their own self-similarity. In this paper, we prove some limit theorems for the rescalings of these functionals. Let $\stackrel{w}{\rightarrow}$ denote the weak convergence in the law of the space of continuous functions, we have

Theorem 1. *As* $\lambda \rightarrow \infty$,

$$
\left\{ \frac{F(\mu, \lambda t)}{\lambda^{1-H}} \right\}_{t \geq 0} \stackrel{w}{\rightarrow} \left\{ \mu(\mathbf{R}) \cdot L(t, 0) \right\}_{t \geq 0},
$$

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$$\left\{\frac{F(f,\lambda t)}{\lambda^{1-H}}\right\}_{t\geq 0} \xrightarrow{w} \left\{\int_{-\infty}^{\infty} f(x)dx \cdot L(t,0)\right\}_{t\geq 0},$$

Let $D^{\gamma}_{\pm}$ denote the right- (resp. left-) handed fractional derivative of order γ (see §3), we have

Theorem 2. *Let $g(x)$ be a function with compact support and be Hölder continuous with order $\beta_0=\min(1, (1-H)/2H)$, and let γ : $0<\gamma<\beta_0$. As $\lambda\to\infty$,*

$$\left\{\frac{F(D^{\gamma}_{-}g,\lambda t)}{\lambda^{1-H(1+\gamma)}}\right\}_{t\geq 0} \xrightarrow{w} \left\{\int_{-\infty}^{\infty} g(x)\ dx \cdot D^{\gamma}_{+} L(t,\cdot)|_{x=0}\right\}_{t\geq 0}.$$

Under an additional assumption of "ergodicity" (see §2 also), we have the following pathwise asymptotics for $F_i(\cdot,\omega)$, $i=0, 1$. Let q denote the density of $X(1)$ at 0, which we assume that $q>0$ (The existence of q is a consequence of the conditions of Proposition 2.1 below).

Theorem 3. *Assume the ergodicity of the process. Then there exist finite positive constants $k_i (i=0, 1)$ such that, for almost sure ω,*

$$\lim_{T\to\infty} \frac{1}{\ln T}\int_1^T \frac{dt}{t}\frac{F_i(t,\omega)}{t^{1-H}} = k_i ;$$

moreover, $k_1 \geq k_0 = \dfrac{q}{1-H}$.

In case X is Brownian motion, in which $H=1/2$, Theorem 1 appeared in Darling-Kac (1957), Theorem 2 appeared in Yamada (1986), and Theorem 3 appeared in Brosamler (1973). The results of Darling-Kac and Yamada have been extended by various authors ; we mention Ikeda-Watanabe (1989, p146), Kasahara (1977), and Fitzsimmons-Getoor (1992). The result of Brosamler is recently related to a certain average-density property of some fractals, see Bedford-Fisher (1992). The most important case in our consideration is certainly the class of Gaussian ss si processes which are just *fractional Brownian motions of exponent H, i.e.* continuous-paths, mean-zero Gaussian processes with covariance function

$$EX(s)X(t) = |s|^{2H} + |t|^{2H} - |s-t|^{2H},$$

see Samorodnitsky-Taqqu (1994, Chapter 7). Our results in this paper hold for fractional Brownian motions; yet they also hold for some non-Gaussian case, including α-stable Lévy processes of index $\alpha > 1$ and some fractional stable processes, see the illustration at the end of §2.

The rest of this paper is organized as follows. In §2, we describe a H-ss si process by its canonical realization on the path space. We also introduce in §2 the "approximately independent increments" assumption for the process and discuss the consequent regularity of its local time. In §3, we prove Theorems 1 and 2, and in §4, we prove Theorem 3. In the final section §5, we extend our results to multi-parameter and multi-dimensional cases.

2. Self-similar processes and local times

It is convenient for us to consider the following canonical realization of a continuous-paths or cadlag-paths H-ss si process. Let the path space

$$\Omega = \{\omega : t \to \omega(t),\ t \geq 0\ ;\ \omega(0) = 0 \text{ and continuous}\},$$
$$\text{resp. } \{\omega : t \to \omega(t),\ t \geq 0\ ;\ \omega(0) = 0 \text{ and cadlag}\},$$
$$\mathfrak{F} = \text{the } \sigma\text{-algebra generated by cylinders.}$$

Note that $\mathfrak{F}$ is equal to Borel σ-algebra of the "local"-uniform topology (continuous case), resp., the "local"-Skorohod topology (cadlag case). Let P be a probability measure in $(\Omega, \mathfrak{F})$ such that scaling transformation

$$(\Delta_a\omega)(t) := \frac{\omega(at)}{a^H}$$

and the translation transformation

$$(T_b w)(t) := w(t+b) - w(b)$$

are both P-invariant (measure-preserving w.r.t. P) for all $a > 0$ and $b > 0$. This means equivalently that the canonical process $X(t, w) := w(t)$ is H-ss si. We always assume $H : 0 < H < 1$. Note that there corresponds a P-invariant flow $(\mathfrak{T}_{s+t} = \mathfrak{T}_s \circ \mathfrak{T}_t)$

$$\mathfrak{T}_s := \Delta_{exp(s)},\ -\infty < s < \infty.$$

We call X to be *ergodic* (*mixing*) if the flow $\mathfrak{T}_s$ is P-ergodic (P-mixing); this terminology appeared in Takashima (1989), in which several concrete examples were illustrated. Next, we call a process X, not necessarily in the canonical setting, to be (*locally*) *of approximately independent increments* (AII for brevity) on a compact time-interval $[a, b]$, if, for any integer $p \geq 2$, there exist positive δ_p, A_p and C_j, $j = 1, \cdots, p$, such that

$$\left| E\left[e^{i\sum_{j=1}^{p}\theta_j(X(t_j)-X(t_{j-1}))}\right]\right| \leq A_p \prod_{j=1}^{p}\left| E\left[e^{iC_j\theta_j(X(t_j)-X(t_{j-1}))}\right]\right|$$

for all $a \leq t_1 < \ldots < t_p \leq b$, $t_0 = 0$, with $t_p - t_1 < \delta_p$.

The terminology AII appeared in Nolan (1989) and Kôno-Shieh (1993). The following basic result can be proved in the same way as Kôno-Shieh (1993, §§4,5) with a little more refined manipulation. The arguments in Nolan (1988, 1989) are equally applicable; actually all the arguments in these works trace back to Berman (1973). For Gaussain or stable processes, the AII condition is essentially equal to the *local nondeterminism* of the processes. See Nolan (1989, Theorem 3.2).

Proposition 2.1. *Let X be an H-ss si process, $0<H<1$, which is AII on each compact $[a, b]$, $0<a<b<\infty$, and assume that the characteristic function $\phi(\theta)$ of $X(1)$ is decreasing in θ rapidly enough so that*

$$\int_{-\infty}^{\infty} |\theta|^{\epsilon} |\phi(\theta)| \, d\theta < \infty, \quad \text{for some } \epsilon > 2.$$

Then, there exists $\Omega_0 \subset \Omega$, $P(\Omega_0)=1$, such that, for each $\omega \in \Omega_0$ there exists $L(t, x) = L(t, x, \omega)$, $t \geq 0$ and $x \in \mathbf{R}$, which is jointly measurable in (t, x, ω) and the following holds for each $\omega \in \Omega_0$. We suppress the ω notation in the statements.

(i) *$(t, x) \to L(t, x)$ is jointly continuous.*

(ii) *For each x, $t \to L(t, x)$ is nondecreasing and is supported on $\{t : X(t) = x\}$.*

(iii) *The occupation-density formula holds, i.e.*

$$\int_0^t h(X(s))ds = \int_{\mathbf{R}} h(x) L(t, x) dx$$

for all $t \geq 0$ and all bounded or nonnegative Borel function $h(x)$.

(iv) *$L(t, x) = 0$ if $|x| > \sup\{|X(s)| : 0 \leq s \leq t\}$.*

(v) *Let $\beta : 0 < \beta < \beta_0 := \min(1, (1-H)/2H)$; for each $a > 1$, there is a positive finite $C_a = C_a(\omega)$, such that*

$$\sup_{1/a \leq t \leq a} |L(1/a, t, x) - L(1/a, t, y)| \leq C_a |x-y|^{\beta}, \ \forall \, x, y \in \mathbf{R},$$

where $L(s, t, x) := L(t, x) - L(s, x)$, $0 < s \leq t$.

(vi) *For $0 < s \leq t$,*

$$E\{[\sup_x L(s, t, x)]^p\} < \infty, \ \forall p : 1 \leq p < \infty.$$

Finally, if the AII assumption is true on each time-interval $[a, b]$ including $a=0$, then $L(t, x)$ itself satisfies the assertions (v) and (vi) (i.e. we do not need to be away from the initial time).

As usual, $L(t, x)$ in Proposition 2.1 is called the *local time of $X(\cdot)$ at x on the time-interval* $[0, t]$. The following scaling property of L is the key instrument for our limit theorems.

Proposition 2.2. *The Ω_0 in Proposition 2.1 is Δ_a-invariant : $\omega \in \Omega_0 \Rightarrow \Delta_a \omega \in \Omega_0$, $\forall a > 0$. Moreover,*

$$(2.1) \qquad L(t, x, \Delta_a \omega) = \frac{L(at, a^H x, \omega)}{a^{1-H}}, \text{ for all } t \geq 0, x \in \mathbf{R} \text{ and } \omega \in \Omega_0.$$

To proceed a proof of Proposition 2.1, we can mostly follow the arguments in the proof of Kôno-Shieh (1993, Theorem 5.2, with $p \uparrow \infty$ and $r = 0$ at there). To obtain the uniform Hölder continuity (v), which is not mentioned explicitly there, note that we may take any $\gamma < \delta := \beta_0$ in p.63 and then apply Kôno's lemma cited there. Note that the condition (5.11) in p.63 there is now satisfied, by our integrability assumption on the characteristic function $\phi(\theta)$. The arguments yield (i), (iii) & (v); in (v), we may consider $L(t, x)$ itself if our AII assumption includes the initial time $a = 0$. The (ii) & (iv) are consequences of (i) & (iii)(see Geman-Horowitz (1980, §6)); note that, although it is not mentioned explicitly in §6 there, the path continuity/cadlaguity is actually needed to ensure (ii) & (iv) (simply to consider a nowhere bounded measurable path, we cannot ensure anything about its path property even though we know the path has a regular local time; in Kôno-Shieh (1993, §4) such a mistake should also be corrected). Since the arguments leading to (v) are of Kolmogorov -criterion type, the last (vi) follows consequently. Moreover, the moment formulae for the local times given in Berman (1985, Lemma 2.1) also hold: for each x, $0 < t_1 < t_2$, and positive integer p,

$$E\{[L(t_1, t_2, x)]^p\} = \int_{t_1}^{t_2} \cdots \int_{t_1}^{t_2} P(s_1, \cdots, s_p, \overbrace{x, \cdots, x}^{p}) ds_1 \cdots ds_p, \tag{2.2}$$

where $P(s_1, \cdots, s_p, x_1, \cdots, x_p)$ denotes the joint density function of $X(s_1), \cdots, X(s_p)$ at $(x_1, \cdots, x_p)$. (The existence and the continuity of the joint density functions are consequences of our assumptions on the integrability of the characteristic function of $X(1)$ and the AII ; these two and the self-similarity together imply that the joint characteristic function of $X(t_1), \cdots, X(t_p)$ is also integrable over $\mathbf{R}^p$). The finiteness of the multiple integral in (2.2) is a consequence of AII assumption (*cf.* Geman-Horowitz (1980, §25)). We may include $t_1 = 0$ in (2.2) whenever the multiple integral in (2.2) is finite with $t_1 = 0$ there; in particular

$$E\{L(1, 0)\} = \frac{q}{1-H},$$

where q is the density of $X(1)$ at $x = 0$.

As for the proof of Proposition 2.2, elementary calculations on occupation-density for the path $\Delta_a \omega$ show that (2.1) holds for Lebesgue-a.e. x, for each $t > 0$. The equality extends to all (t, x) since both sides are known to be jointly continuous in (t, x).

Now, we illustrate specific examples. Propositons 2.1 and 2.2 are known to be applicable to the following.

- Brownian motion, $H = \frac{1}{2}$;
- α-atable Lévy process, $1 < \alpha < 2$ and $H = 1/\alpha$;
- fractional Brownian motion of exponent H, $0 < H < 1$, see Berman (1973);
- linear fractional α-stable process of exponent H, $1 < \alpha < 2$ and $1/\alpha < H < 1$, see Kôno-Shieh (1993);
- (real) harmonizable fractional α-stable process of exponent H, $1 \leq \alpha < 2$ and $0 < H < 1$, see Nolan (1989).

We remark that, except α-stable Lévy process which is of cadlag paths, other cases in the above illustration are of continuous paths. We also remark that, except harmonizable fractional stable process which is not ergodic (see Cambanis *et al.* (1987)), other cases in the above illustration are mixing (see Takashima (1989)).

3. The proof of Theorems 1 and 2

To prove Theorem 1, we consider the case $F(\mu, t)$ and the case $F(f, t)$ can be obtained parallelly. For $0 < t_1 < \cdots < t_k$, by the scaling property L (Proposition 2.2) and the P-invariance of Δ_λ (H-self-similarity),

$$\left\{\int_{x\in\mathbf{R}} \frac{L(\lambda t, x)\mu(dx)}{\lambda^{1-H}}\right\}_{t=t_1,\cdots,t_k} \stackrel{d}{=} \left\{\int_{x\in\mathbf{R}} L(t, \lambda^{-H}x)\mu(dx)\right\}_{t=t_1,\cdots,t_k}$$

from which the finite-dimensional convergence in Theorem 1 follows. As for the tightness of the scaled processes, we note, w.p.1, firstly that the integration in $F(\mu, t)$ is taken over $x \in Support\ L(t, \cdot)$, which is a compact subset of $\mathbf{R}$, and secondly that, for $t_1 < t_2$,

$$\frac{1}{\lambda^{1-H}}\left[F(\mu, \lambda t_2) - F(\mu, \lambda t_1)\right] = \frac{1}{\lambda^{1-H}}\int L(\lambda t_1, \lambda t_2, x)\mu(dx) \stackrel{d}{=} \int L(t_1, t_2, \lambda^{-H}x)\mu(dx)$$

The last integral tends to $\mu(\mathbf{R})L(t_1, t_2, 0)$ w.p.1 as λ tends to ∞. Thus, the general criterion for tightness in Billingsley (1968, Theorem 12.3) is applicable.

Remark. In case X is fractional Brownian motion, a weaker form of Theorem 1 is announced, without proof, in Kôno (1995) recently.

To prove Theorem 2, we mention that Yamada's Brownian result (1986) have been extended to the stable Lévy process case by Fitzsimmons-Getoor (1992). The arguments in the latter work can be adapted well to more general self-similar case, which we proceed as follows. Fix $\epsilon : 0 < \epsilon < 1$, by Proposition 2.1 (v), for each $t > \epsilon$, $x \to L_\epsilon(t, x) := L(\epsilon, t, x)$ is uniformly Hölder continuous in x with order as close as to $\beta_0 := \min(1, (1-H)/2H)$; thus for any $\gamma : 0 < \gamma < \beta_0$, the right-handed γ-derivative (more precisely, the fractional derivative of order γ) of $x \to L_\epsilon(t, x)$ is defined, for each fixed $t > \epsilon$, via

$$D_+^\gamma L(t, \cdot)|_x := \frac{1}{\Gamma(-\gamma)}\int_0^\infty \frac{L(t, x+y) - L(t, x)}{y^{1+\gamma}}dy.$$

In the above and in the below, we will suppress the ϵ from L_ϵ for the notation-al convenience. (We may allow $\epsilon=0$ if our AII assumption includes the initial time $a=0$). Note that $x \to L(t, x)$ is compactly supported (Proposition 2.1 (iv)), and thus we need only to consider $y=0+$ where the Hölder continuity of $x \to L(t, x)$ is applied. The $g(x)$ in Theorem 2 is a compactly supported function on $\mathbf{R}$ and is Hölder continuous with order β_0 ; thus, the left-handed γ-derivative of $g(x)$ can also be defined:

$$D_-^{\gamma} g(x) := \frac{1}{\Gamma(-\gamma)} \int_0^{\infty} \frac{g(x-y)-g(x)}{y^{1+\gamma}}\, dy.$$

The proof of Theorem 2 is based on the following identity

$$\int_{\mathbf{R}} D_-^{\gamma} g(x) L(t, x) dx = \int_{\mathbf{R}} g(x) D_+^{\gamma} L(t, \cdot)|_x \, dx,$$

which is the "switching identity" for the right-handed and the left-handed γ-derivatives, see F.-G. (1992, (2.15)) . Writing $H_t^x(\gamma+)$ instead of $D_+^{\gamma} L(t, \cdot)|_x$, by the scaling of L we have

$$H_t^x(\gamma+, \Delta_a \omega) = \frac{H_{at}^{a^H x}(\gamma+, \omega)}{a^{1-H(1+\gamma)}}, \; \omega \in \Omega_0.$$

Again, the P-invariance of Δ_λ then asserts that $H_{\lambda t}^x(\gamma+) \overset{d}{=} \lambda^{1-H(1+\gamma)} H_t^{\lambda^{-H} x}(\gamma+)$ as a random function of t, for each $\lambda > 0$ and each $x \in \mathbf{R}$. This proves the finite-dimensional convergence. As for tightness, by the fact that $x \to L(t, x)$ is Hölder continuous and compactly supported, we see that, for each $a > 1$ and β : $\gamma < \beta < \beta_0$.

$$\sup_{\epsilon < t \leq a} |H_t^x(\gamma+) - H_t^{x'}(\gamma+)| \leq C'_{\alpha, \beta, \gamma} |x - x'|^{\beta-\gamma}, \; \forall x, x' \in \mathbf{R}.$$

Thus, the tightness argument in the above proof of Theorem 1 still holds in the present case, note that we have assumed that $g(x)$ is of compact support.

Remark. There is an analogous result in case $f(x)$ is the Hilbert transform of $g(x)$. We omit this result here and refer to Yamada (1986), F.-G. (1992) for the parallel derivation.

4. The proof of Theorem 3 and a ratio ergodic property

Note that the assumption is that X is an H-ss si process, defined in the canonical setting, which is ergodic in the sense given in §2 and to which Propositions 2.1 and 2.2 are applicable. Our basic idea of proving Theorem 3 is to make use of the scaling low and the ergodicity. Let us begin with the setting:

Ω_0 = the full subspace of Ω (path space) in Proposition 2.1,
$\mathfrak{F}_0$ = the restriction of cylinder σ-algebra $\mathfrak{F}$ to Ω_0,
$\Omega' = \{ \nu : (t, x) \to \nu(t, x)$, $t \geq 0$ and $x \in \mathbf{R}$: $\nu(0, x) = 0$, jointly con-

tinuous in (t, x), and $\sup_x |\nu(t, x)| < \infty$ for each $t > 0\}$.

$\mathfrak{F}'$ = the cylinder σ-algebra on Ω'.

Define $L : \omega \in \Omega_0 \to L\omega \in \Omega'$ by

$$(L\omega)(t, x) = L(t, x, \omega).$$

We show that L is $(\mathfrak{F}_0, \mathfrak{F}')$-measurable. In fact, for each (t, x) and Borel $A \subset \mathbf{R}$

$$\{\omega \in \Omega_0 : (L\omega)(t, x) \in A\} = \{\omega : L(t, x, \omega) \in A\} \in \mathfrak{F}_0,$$

since it is stated in Proposition 2.1 that L is jointly measurable in (t, x, ω). This ensures $L^{-1}(S) \in \mathfrak{F}_0$ for each $\mathfrak{S} : \mathfrak{S}_{t,x} = \{\nu : \nu(t, x) \in A\}$, and hence for each cyclinder $\mathfrak{S} = \mathfrak{S}_{t_1, \cdots, t_k, x_1, \cdots, x_l} = \{\nu : \nu(t_i, x_j) \in A_{ij}\ \forall i = 1, \cdots, k$ and $j = 1, \cdots, l\}$. Now, we define the scaling transformation $\tilde{\Delta}_a$ on $(\Omega', \mathfrak{F}')$ by

$$(\tilde{\Delta}_a \nu)(t, x) = \frac{\nu(at, a^H x)}{a^{1-H}}.$$

Then, by Proposition 2.2, we have

$$(L(\Delta_a \omega))(t, x) = (\tilde{\Delta}_a (L\omega))(t, x), \ \forall (t, x).$$

This "homeomorphism" property, together with the ergodic assumption on P-measure, ensures that the image measure Q of P on $(\Omega', \mathfrak{F}')$ under the measurable mapping L is also ergodic. This "factor theorem" can be seen in Cornfeld-Fomin-Sinai (1980, p230). Set

$$\tilde{\mathfrak{T}}_s = \tilde{\Delta}_{exp(s)}, \quad -\infty < s < \infty.$$

Applying Birkhoff's individual ergodic theorem to $(\Omega', \mathfrak{F}', Q, \tilde{\mathfrak{T}}_s)$, we then have

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t G[\tilde{\mathfrak{T}}_s \nu] ds = E^Q[G] \quad Q\text{-}a.s.\ \nu$$

whenever $G[\nu]$, $\nu \in \Omega'$, is in $L^1(dQ)$. Pulling back to $(\Omega, \mathfrak{F}, P, \mathfrak{T}_x)$, we have

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t G[L(\mathfrak{T}_s \omega)] ds = E^P[G \circ L] \quad P\text{-}a.s.\ \omega.$$

To obtain the assertions of Theorem 3, we take $G[\nu] = \nu(1, 0)$ and $G[\nu] = \sup_x \nu(1, x)$ respectively; then we take a logarithmic change. Note that $k_0 = E^P[L(1, 0)]$ and $k_1 = E^P[\sup_x L(1, x)]$ are both finite positive and $k_0 = q/(1-H)$, by the moment formula for local times stated in §2. Note: It would be interesting if we can prove that $k_1 = k_0$, which is true in the Brownian case.

In the end, we state and prove the following ratio ergodic property of our process; unfortunately we can only prove the result in the weaker convergence in probability, not in the stronger a.s. convergence.

Proposition 4. 1. *Under the conditions of Theorem 3,*

$$\lim_{T\to\infty} \frac{F(\mu, T, \omega)}{L(T, 0, \omega)} = \mu(\mathbf{R}) \qquad \textit{in probability.}$$

Proof. Firstly, we note that, for P-a.s. $\omega \in \Omega_0$, $L(T, 0, \omega) > 0$ when T is sufficiently large. In fact, let $E = \{\omega \in \Omega_0 : L(t, 0, \omega) > 0$ for some $t = t(\omega) > 0\}$, then E is Δ_a-invariant : $\Delta_a(E) \subset E \ \forall a > 0$, which follows from Proposition 2.2. Thus $P(A) = 0$ or 1 by the ergodic assumption. By the moment formula for local times again,

$$E^P L(t, 0) = \frac{q \cdot t^{1-H}}{1-H} > 0, \text{ for each } t > 0.$$

Thus, $P(E) = 1$. Now, for fixed $\varepsilon > 0$ and $T > 0$, let

$$E_\varepsilon^T = \left\{\omega \in \Omega_0 : \left|1 - \frac{F(\mu, T, \omega)}{\mu(\mathbf{R}) \cdot L(T, 0, \omega)}\right| > \varepsilon\right\}$$
$$E_\varepsilon^{T,a} = \{\Delta_{a^{-1}}\omega, \omega \in E_\varepsilon^T\}.$$

Since $\Delta_{a^{-1}}$ is P-invariant, $P(E_\varepsilon^T) = P(E_\varepsilon^{T,a})$. Writing $\omega = \Delta_a(\Delta_{a^{-1}}\omega)$, by Proposition 2.2, we have

$$\frac{F(\mu, T, \Delta_a(\Delta_{a^{-1}}\omega))}{\mu(\mathbf{R}) \cdot L(T, 0, \Delta_a(\Delta_{a^{-1}}\omega))} = \frac{\int L(aT, a^H x, \Delta_{a^{-1}}\omega)\, \mu(dx)}{\mu(\mathbf{R}) \cdot L(aT, 0, \Delta_{a^{-1}}\omega)}.$$

Thus, with $a = T^{-1}$ in the above, we have

$$P(E_\varepsilon^T) = P\left\{\omega \in \Omega_0 : \left|1 - \frac{\int L(1, T^{-H}x, \omega)\, \mu(dx)}{\mu(\mathbf{R}) \cdot L(1, 0, \omega)}\right| > \varepsilon\right\},$$

which tends to 0 as $T \uparrow \infty$, by the continuity of $x \to L(1, x)$. This proves the assertion.

5. Multi-parameter and multi-dimensional extensions

For a real-valued N-parameter process $X(\mathbf{t})$, $\mathbf{t} \in \mathbf{R}_+^N$ and $N \geq 2$, the self-similarity is defined exactly as one-parameter case, namely $X(c\mathbf{t}) \stackrel{d}{=} c^H X(\mathbf{t})$ for all $c > 0$. For the stationary-increments property, it is assumed that $X(\mathbf{t})$ is invariant not only under translations but also under rotations; thus the distribution of $X(\mathbf{t}) - X(\mathbf{s})$ depends only on $\|\mathbf{t} - \mathbf{s}\|$ where $\|\cdot\|$ denotes the N-dimensional Euclidean norm. Now let $\mathbf{X} = (X_1, \cdots, X_d)$, $X_i = X_i(\mathbf{t})$ and $\mathbf{t} \in \mathbf{R}_+^N$, takes values in $\mathbf{R}^d$ so that $\mathbf{X}$ is an N-parameter, d-dimensional-valued vector field (we call it an (N, d) *field* for brevity). We assume that the components X_i are independent and each X_i is H_i-ss si (for si, it is in the above stronger sense). The exponents H_i are assumed to be $0 < H_i < 1$ and the sample functions $\mathbf{t} \to X_i(\mathbf{t})$ are assumed to be continuous. Thus, we have a continuous-paths (N,d) field $\mathbf{X}(\mathbf{t})$ which is of stationary increments (in the above stronger sense) and is *self-affine*, by which we mean

$$X(c\mathbf{t}) \stackrel{d}{=} (c^{H_1}X_1(\mathbf{t}), \cdots, c^{H_d}X_d(\mathbf{t})), \ \forall c > 0,$$

regarded as the equivalence in distribuation of two d-dimensional processes. The canonical setting for $\mathbf{X}$ is then to consider the path-space

$$\Omega = \{\omega = (\omega_1, \cdots, \omega_d) : \omega_j = \omega_j(\mathbf{t}), \mathbf{t} \in \mathbf{R}_+^N, \text{ continuous and } \omega_j(0) = 0\},$$

$$\mathfrak{F} = \text{cylinder } \sigma\text{-algebra},$$

and a probability measure P on $(\Omega, \mathfrak{F})$ such that the scaling and the strong-translation transformations

$$(\Delta_a\omega)(\mathbf{t}) := \left(\frac{\omega_1(a\mathbf{t})}{a^{H_1}}, \cdots, \frac{\omega_d(a\mathbf{t})}{a^{H_d}}\right), \quad a > 0$$

$$(\Delta_{\mathfrak{G}}\omega)(\mathbf{t}) := \omega(\mathfrak{G}\mathbf{t}), \ \mathfrak{G} : \text{rigid-body motion on } \mathbf{R}^N,$$

are both P-invariant and moreover that the component processes ω_j, $j = 1, \cdots, d$, are P-independent, we call $\mathbf{X}$ to be *ergodic* if any $A \in \mathfrak{F}$ which is Δ_a-invariant ($\Delta_a A \subset A \ \forall a > 0$) is of $P(A) = 0$ or 1. This definition indicates that, for each fixed $\mathbf{t}$, the one-parameter stationary process defined by $s \to \frac{\mathbf{X}(e^s\mathbf{t})}{e^{sH}}$, $s \in \mathbf{R}$, is ergodic. We assume that each X_i is AII on every compact box $J \subset \mathbf{R}^N$ bounded away from the origin (the definition of AII follows the previous one-parameter case, with some attentions on the ordering of the parameters $\mathbf{t}$, see Pitt (1978) and Nolan (1989)). Assume that

$$N > \sum_{i=1}^{d} H_i.$$

Let $f(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^d$, be a Lebesgue integrable function on $\mathbf{R}^d$ with nonzero $\int f(\boldsymbol{x})d\boldsymbol{x}$, and let the field $F(f, \mathbf{t})$, be defined by

$$F(f, \mathbf{t}) = \int_0^{t_1} \cdots \int_0^{t_N} f(X(\mathbf{s}))d\mathbf{s}, \ \mathbf{t} = (t_1, \cdots, t_N) \in \mathbf{R}_+^N.$$

The following two results can be proved in the same way as the one-parameter case in §3, 4.

Theorem 4. *As* $\lambda \to \infty$,

$$\left\{\frac{F(f, \lambda\mathbf{t})}{\lambda^{N-\sum_{i=1}^d H_i}}\right\}_t$$

converges in the law of the space of continuous N*-parameter* (N*-variable*) *functions to*

$$\left\{\int f(\mathbf{x})d\mathbf{x} \cdot L(\mathbf{t}, 0)\right\}_t.$$

Theorem 5. *Assume that* $\mathbf{X}$ *is ergodic, then almost surely*

$$\lim_{T\to\infty} \frac{1}{\ln T} \int_1^T \frac{dt}{t} \frac{\sup_x L((\overbrace{t, \cdots, t}^{N}), \mathbf{x})}{t^{N-\sum_{i=1}^d H_i}} = E \sup_{\mathbf{x}} L((\overbrace{1,\cdots,1}^{N}), \mathbf{x}) .$$

In the above, $L(\mathbf{t}, \mathbf{x})$, $\mathbf{t} \in \mathbf{R}_+^N$ and $\mathbf{x} \in \mathbf{R}^d$, denotes the local time of $\mathbf{X}(\cdot)$ at $\mathbf{x}$ on the box $\Pi_{i=1}^N [0, t_i]$, $\mathbf{t} = (t_1, \cdots, t_N)$. We remark that, for the (N, d) field $\mathbf{X}$, we have the extension of Proposition 2.1, when it is under suitable parameter-dimension modifications and we also have the following scaling property of $L(\mathbf{t}, \mathbf{x})$

$$L(\mathbf{t}, \mathbf{x}, \Delta_a\omega) = \frac{L(a\mathbf{t}, (a^{H_1}x_1, \cdots, a^{H_d}x_d), \omega)}{a^{N-\sum_{i=1}^d H_i}}.$$

The most important case in the above consideration is *fractional Brownian vector field*, in which each X_i is a mean-zero Gaussian field with covariance function

$$EX_i(\mathbf{s})\, X_i(\mathbf{t}) = \|\mathbf{s}\|^{2H_i} + \|\mathbf{t}\|^{2H_i} - \|\mathbf{t} - \mathbf{s}\|^{2H_i}.$$

The regularity of local times for this important vector field was studied by Pitt (1978); Theorems 4, 5 then show a certain limit behavior of this Gaussian local time.

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