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On May spectral sequences

Dedicated to Professor Teiichi Kobayashi on his 60th birthday

By

Mizuho HIKIDA

1. Introduction

In this paper, we generalize the algebraic Novikov spectral sequences. Then we define a homomorphism from the algebraic Novikov spectral sequence to the May spectral sequence and study conditions in which they are isomorphic.

In [3], we have indicated that the differentials of the generalized Adams spectral sequences can be calculated from those of the May spectral sequences. Moreover we can calculate the algebraic Novikov spectral sequences (c.f., Proposition 2.5). Hence the results of this paper can be applied to the calculation of the differentials of the generalized Adams spectral sequences. We may apply to the BP-and $E(n)$-Adams spectral sequences, and find the new elements of the stable homotopy groups of spheres and the $E(n)$-localization of spheres. These results will appear in the forthcomming papers.

Let F be a ring spectrum with unit τ^F: $S^0 \to F$, $S^0 \xrightarrow{\tau^F} F \xrightarrow{pr} \overline{F}$ the cofiber of τ^F and $\overline{F}^s = \overline{F} \wedge \cdots \wedge \overline{F}$ the s-fold smash product of $\overline{F}$. For any CW spectrum X, we have exact sequences

$$\cdots \xrightarrow{\partial^F} \pi_u(S^0 \wedge \overline{F}^s \wedge X) \xrightarrow{(\tau^F \wedge 1)*} \pi(F \wedge \overline{F}^s \wedge X) \xrightarrow{(pr \wedge 1)*} \pi(\overline{F} \wedge \overline{F}^s \wedge X) \xrightarrow{\partial^F} \pi_{u-1}(\overline{F}^s \wedge X) \to \cdots$$

and a filtration

$$\pi_u(X) \xleftarrow{\partial^F} \pi_{u+1}(\overline{F} \wedge X) \xleftarrow{\partial^F} \cdots \xleftarrow{\partial^F} \pi_{u+s}(\overline{F}^s \wedge X) \xleftarrow{\partial^F} \cdots.$$

The F-Adams spectral sequence $\{{}_F E_r^{s,u}(X),\ d_r^F\}$ is the spectral sequence induced from the exact couple consisting of the above long exact sequence. We have

$${}_F E_1^{s,u}(X) = F_u(\overline{F}^s \wedge X),\ d_1^F = (\tau^F \wedge 1)_* \circ (pr \wedge 1)_* \text{ and } {}_F E_2^{s,u}(X) = H^s({}_F E_1^{*,u}(X);\ d_1^F)$$

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(c.f., (2.1)). Under certain conditions, ${}_{F}E_{\infty}(X)$ is isomorphic to the associated graded group of the above filtration. If $F_*F=F_*(F)$ is flat over $F_*=F_*(S^0)$ then ${}_{F}E_2^{s,u}(X)=\mathrm{Ext}_{F_*F}^{s,u}(F_*, F_*(X))$.

Suppose F_*F is flat over F_* and consider an ideal I of F_* invarant under the coacton of F_*F. Then the filtration

$$F_*(X)\supset IF_*(X)\supset\cdots\supset I^tF_*(X)\supset I^{t+1}F_*(X)\supset\cdots$$

induces a spectral sequence

$$\{{}_{I}E_r^{s,t,u}(X),\ d_r^I\}\ \text{abutting to}\ \mathrm{Ext}_{F_*F}^{t,u}(F_*, F_*(X)).$$

We call it *the algebraic Novikov spectral sequence* for I. If $F=BP$, $X=S^0$ and $I=(p, v_1, v_2,\cdots)$ then this is so called by D. C. Ravenel (c.f. [8]).

Next consider another ring spectrum G. The cofibration $S^0\xrightarrow{\tau^G}G\xrightarrow{pr}\overline{G}$ induces a boundary homomorphism

$$\partial_G\colon \pi_{u+t}(\overline{G}^t\wedge F\wedge X)\to\pi_{u+t-1}(\overline{G}^{t-1}\wedge F\wedge X).$$

We define a filtration $\{V_{u+t}^t(X)\}_t$ on $\pi_u(F\wedge X)=F_u(X)$ by

(1.1) $$V_{u+t}^t(X)=\mathrm{Im}[(\partial_G)^t\colon \pi_{u+t}(\overline{G}^t\wedge F\wedge X)\to\pi_u(F\wedge X)].$$

This induces a spectral sequence

$$\{{}_{FG}E_r^{s,t,u}(X),\ d_r^{FG}\}\ \text{abutting to}\ {}_{F}E_2^{s,u-t}(X).$$

We call it *the May spectral sequence* (c.f. [3, 5]).

We assume that there exists a map $\lambda\colon F\to G$ of ring spectra. Let I be the kernel of $\lambda_*\colon F_*\to G_*$. In Lemma 3.1, we prove that if $\lambda_*\colon F_*(G)\to G_*(G)$ is monomorphic then $I^tF_*(X)\subset V_{*+t}^t(X)$. Then we have a homomorphism

(1.2) $$\rho\colon I^tF_*(X)/I^{t+1}F_*(X)\to V_{*+t}^t(X)/V_{*+t+1}^{t+1}(X).$$

Now Lemma 3.1 implies the following.

Theorem 1.1. *Assume that there exists a map $\lambda\colon F\to G$ of ring spectra. Let I be the kernel of $\lambda_*\colon F_*\to G_*$. If $\lambda_*\colon F_*(G)\to G_*(G)$ is monomorphic then there exists a homomorphism*

$$\phi\colon \{{}_{I}E_r^{s,t,u}(X),\ d_r^I\}\to\{{}_{FG}E_r^{s,t,u}(X),\ d_r^{FG}\}$$

of spectral sequence.

Moreover if ρ is isomorphic then ϕ is isomorphic.

For example, consider $F=BP$, $G=HZ/(p)$ and $X=S^0$. Then

$$I=(v_0=p, v_1, v_2, v_3,\cdots),\ \oplus_t I^t/I^{t+1}=Z/(p)[v_0, v_1,\cdots]\ \text{and}$$
$$\oplus_t V_{*+t}^t(S^0)/V_{*+t+1}^{t+1}(S^0)=\oplus_{tHZ/(p)}E_2^{t,*}(BP)=\oplus_t\mathrm{Ext}_{A_*}^{t,*}(Z/(p), P_*)=Z/(p)[a_0, a_1,\cdots],$$

where A_* is the dual of the Steenrod algebra and $P_* \subset A_*$ is the polynomial part of A_* when p is odd and the subalgebra generated by $\xi_1^2, \xi_2^2, \cdots$ when $p=2$. We see that ρ is isomorphic, and so the algebraic Novikov spectral sequence is isomorphic to the May spectral sequence.

Next, we study some other conditions under which ψ is an isomorphism.

Theorem 1.2. *We assume that $\lambda_*: F_*(G) \to G_*(G)$ is monomorphic. If*

$$\mathrm{Ker}[\lambda_*: F_*(\overline{G}^t \wedge X) \to G_*(\overline{G}^t \wedge X)] = IF_*(\overline{G}^t \wedge X),$$

then ψ is an isomorphism.

Theorem 1.3. *Let X be a CW spectrum, $\lambda: F \to G$ a map of ring spectra such that $\lambda_*: G_*(F) \to G_*(G)$ is monomorphic and let $I = \mathrm{Ker}[\lambda_*: F_* \to G_*]$. We assume that $F_*(X)$ is flat over F_* and G_*G is flat over G_* and G-Adams spectral sequence $\{_GE_2^{t,u}(F)\}$ converges to $\pi_*(F)$ and collapses.*

If the ring $\mathrm{Ext}_{G,G}^{*,*}(G_*, G_*(F))$ is generated by $\mathrm{Ext}_{G_*G}^{0,*}(G_*, G_*(F))$ and $\mathrm{Ext}_{G_*G}^{1,*}(G_*, G_*(F))$ and there exists an integer $t(u)$ for each integer u such that $\mathrm{Ext}_{G_*G}^{t,u+t}(G_*, G_*(F)) = 0$ for $t > t(u)$, then the homomorphism ψ is isomorphic.

In §2, we don't assume that F_*F is flat and define the above spectral sequences for any ring spectrum. The above theorems are proved in §3.

2. Definitions of spectral sequences

In this section, we define and discuss the May and algebraic Novikov spectral sequences according to [2, 3, 5, 8] for arbitrary ring spectra.

In the rest of this paper, we denote $X_n^F = \overline{F}_n \wedge X$. For the F-Adams spectral sequence $\{_FE_r^{s,u}(X); d_r^F\}$, we have

(2.1)
$$_FE_1^{s,u}(X) = \pi_u(F \wedge X_s^F),\ d_1^F = (\tau^F \wedge id)_* \circ (pr \wedge id)_*$$
$$_FE_2^{s,u}(X) = H^s(_FE^{s,u}(X); d_1^F) = \mathrm{Ker}\, d_1^F / \mathrm{Im}\, d_1^F.$$

For any ring spectrum F, consider a cochain complex $\{C_F^{s,u}(X), \delta^F\}$ defined by

(2.2)
$$C_F^{s,u}(X) = \pi_u(F^{s+1} \wedge X)\ (F^{s+1} = F \wedge \cdots \wedge F\ (s+1)\text{-times}) \text{ and}$$
$$\delta^F = \sum_{i=0}^{s+1} (-1)^i \delta_{i*}: C_F^{s,u}(X) \to C_F^{s+1,u}(X), \text{ where}$$

$$\delta_i = 1 \wedge \tau^F \wedge 1: F^{s+1} \wedge X = F^{s+1-i} \wedge S^0 \wedge F^i \wedge X \to F^{s+1-i} \wedge F \wedge F^i \wedge X = F^{s+2} \wedge X.$$

Proposition 2.1 ([3, Theorem 2.3]. *For any ring spectrum F, the following holds.*

i) ${}_FE_2^{s,u}(X)=H^s(C_F^{*,u}(X);\,\delta^F)$.
ii) *If* $F_*(F)$ *is flat over* $F_*=F_*(S^0)$, *then*

$$H^s(C_F^{*,u}(X);\,\delta^F)=\mathrm{Ext}_{F_*F}^{s,u}(F_*,\,F_*(X)).$$

Now we construct a concrete isomorphism of i). The map $pr\colon F\to\bar{F}$ induces a cochain homomorphism

$$(2.3)\qquad \phi^F=(pr\wedge\cdots\wedge pr\wedge 1)_*\colon C_F^{s,u}(X)=F_u(F^s\wedge X)\to F_u(\bar{F}^s\wedge X)={}_FE_1^{s,u}(X).$$

Proposition 2.2. ϕ^F *induces an isomorphism*

$$\phi_*^F\colon H^s(C_F^{*,u}(X);\,\delta^F)\xrightarrow{\cong}{}_FE_2^{s,u}(X)=H^s({}_FE_1^{s,u}(X);\,d_1^F).$$

Proof. The proof proceeds in the similar way as the one of [3, Lemma 4.10 iv)]. We consider the cochain complexes

$$M(r)_u^*=\{M(r)_u^s,\,\delta(r)_M^s\}\text{ and }K(r)_u^*=\{K(r)_u^s,\,\delta(r)_K^s\}\text{ for }r\geq 0$$

given as follows:

$$M(r)_u^s=\begin{cases}{}_FE_1^{s,u}(X)=\pi_u(F\wedge\bar{F}^s\wedge X) & \text{if } s\leq r\\ C_F^{s-r,u}{}_F(X_r^F)=\pi_u(F^{s-r}\wedge\bar{F}^r\wedge X) & \text{if } s>r,\end{cases}$$

$$\delta(r)_M^s=\begin{cases}d_1^F & \text{if } 0\leq s<r\\ \delta^F(X=X_r^F) & \text{if } s\geq r,\end{cases}$$

$$K(r)_u^s=\begin{cases}0 & \text{if } s\leq r\\ C_F^{s-r-1,u}(X_r^F)=\pi_u(F^{s-r-1}\wedge\bar{F}^r\wedge X) & \text{if } s>r,\end{cases}$$

$$\delta(r)_K^s=\begin{cases}0 & \text{if } s\leq r\\ (\tau^F\wedge 1)_* & \text{if } s=r+1\\ \delta^F\ (X=F\wedge X_r^F) & s\geq r+2.\end{cases}$$

Furthermore, we have cochain maps

$$i(r)=\{i(r)^*\}\colon K(r)_u^*\to M(r)_u^*\text{ and }j(r)=\{j(r)^*\}\colon M(r)_u^*\to M(r+1)_u^*$$

defined by

$$i(r)^s=\begin{cases}0 & \text{if } s\leq r\\ (\tau^F\wedge 1)_*\colon C_F^{s-r-1,u}(S^0\wedge X_r^F)\to C_F^{s-r-1,u}(F\wedge X_r^F) & \text{if } s>r\end{cases}\quad\text{and}$$

$$j(r)^s=\begin{cases}id & \text{if } s\leq r\\ (-1)^{s-r}(pr\wedge 1)_*\colon C_F^{s-r-1,u}(F\wedge X_r^F)\to C_F^{s-r-1,u}(\bar{F}\wedge X_r^F) & \text{if } s>r.\end{cases}$$

Then we have a short exact sequence

$$0\longrightarrow K(r)_u^*\xrightarrow{i(r)}M(r)_u^*\xrightarrow{j(r)}M(r+1)_u^*\longrightarrow 0.$$

By [3, Lemma 2.2], $H^s(K(r)_u^*)=0$ for any s; hence $j(r)_*$ is isomorphic on cohomology groups, and so is ϕ_*^E since $M(0)_u^s=C_F^{s,u}(X)$ and

$$\phi^E=(-1)^\varepsilon j(s)^s\circ\cdots\circ j(0)^s\colon M(0)_u^s\to M(s+1)_u^s={}_FE_1^{s,u}(X)\left(\varepsilon=\frac{s(s+1)}{2}\right).$$

Now a short exact sequence

$$0\to C_F^{s,u}(X_n^F)\to C_F^{s,u}(F\wedge X_n^F)\to C_F^{s,u}(X_{n+1}^F)\to 0$$

induces a long exact sequence

$$\cdots\xrightarrow{\delta^C}H^s(C_F^{*,u}(X_n^F))\to H^s(C_F^{*,u}(F\wedge X_n^F))\to H^s(C_F^{*,u}(X_{n+1}^F))\xrightarrow{\delta^C} H^{s+1}(C_F^{*,u}(X_n^F))\to\cdots.$$

Let $h^F\colon \pi_*(X)\to H^0(C_F^{*,u}(X))$ be the Hurewicz homomorphism. We denote

$$\overline{h^F}=(\delta^C)^s\circ h^F\colon \pi_*(X_s^F)\xrightarrow{h^F}H^0(C_F^{*,u}(X_s^F))\xrightarrow{\delta^C}\cdots\xrightarrow{\delta^C}H^s(C_F^{*,u}(X_0^F))=H^s(C_F^{*,u}(X)).$$

We notice the following

Lemma 2.3 [3, Corollary 1.7]. i) *$x^F\in H^s(C_F^{*,u}(X))$ converges to $x\in\pi_{u-s}(X)$ if and only if there exists an element $x_s\in\pi_u(X_s^F)$ such that $\overline{h^F}(x_s)=x^F$ and $(\partial^F)^s(x_s)=x$.*

ii) *We assume the F-Adams spectral sequence $\{{}_FE_r^{s,u}(X)\}$ converges and collapses at the E_2-term. Then $\overline{h^F}\colon \pi_*(X_s^F)\to H^s(C_F^{*,u}(X))$ is epimorphic for any s. If $(\partial^F)^s x_s=0$ for $x_s\in\pi_u(X_s^F)$ then $\partial^F x_s=0$.*

Lemma 2.4. *If $\overline{h^F}(x)=0$ for $x\in\pi_u(X_s^F)$ then there exists an element $y\in\pi_{u+1}(X_{s+1}^F)$ such that $(\partial^F)^2(y)=\partial^F(x)$.*

Proof. We notice that

$H^s(C_F^{*,u}(F\wedge X))=0$ for $s>0$ and $h^F\colon \pi_*(F\wedge X)\to H^0(C_F^{*,u}(F\wedge X))$ is isomorphic

by [3, Lemma 2.2]. Then we have an exact sequence

$$0\to H^0(C_F^{*,u}(X_n^F))\to H^0(C_F^{*,u}(F\wedge X_n^F))\xrightarrow{pr_*}H^0(C_F^{*,u}(X_{n+1}^F))\xrightarrow{\delta^C}H^1(C_F^{*,u}(X_n^F))\to 0$$

and isomorphisms

$$\delta^C\colon H^s(C_F^{*,u}(X_{n+1}^F))\cong H^{s+1}(C_F^{*,u}(X_n^F))\ \text{for}\ s\geq 1.$$

Then $\overline{h^F}(x)=(\delta^C)^s\circ h^F(x)=0$ implies $\delta^C\circ h^F(x)=0$, and so we have $\theta\in\pi_*(F\wedge X_{s-1}^F)$ with $pr_*\circ h^F(\theta)=h^F(x)$. By the commutative diagram

$$\begin{array}{ccccc} \pi_{u+1}(X_{s+1}^F) & \xrightarrow{\partial^F} & \pi_u(X_s^F) & \xrightarrow{(\tau^F\wedge id)_*} & \pi_u(F\wedge X_s^F) \\ & & \downarrow h^F & & \downarrow h^F \cong \\ 0 & \longrightarrow & H^0(C_F^{*,u}(X_s^F)) & \longrightarrow & H^0(C_F^{*,u}(F\wedge X_s^F)), \end{array}$$

$h^F(x-pr_*(\theta))=h^F(x)-pr_*\circ h^F(\theta)=0$ implies that $(\tau^F\wedge id)_*(x-pr_*(\theta))=0$, and so we have $y\in\pi_{u+1}(X_{s+1}^F)$ with $\partial^F(y)=x-pr_*(\theta)$. Then $(\partial^F)^2(y)=\partial^F(x)$.

Next we discuss the algebraic Novikov spectral sequence. For an ideal I of F_*, we assume that

$$\delta_{s+1}(I^tC_F^{s,*}(X))\subset I^tC_F^{s+1,*}(X).$$

Then the filtration

$$(2.4)\qquad C_F^{s,*}(X)\supset\cdots\supset I^t\cdot C_F^{s,*}(X)\supset I^{t+1}\cdot C_F^{s,*}(X)\supset\cdots$$

induces a spectral sequence

$$\{{}_IE_r^{s,t,u}(X),\ d_r^I\}\ \text{abutting to}\ {}_FE_2^{s,u}(X)=H^s(C_F^{*,u}(X);\ \delta^F).$$

We call it *the generalized algebraic Novikov spectral sequence.*

Let x be an element of $I^t\,C_F^{s,u}(X)$ and

$$p_I\colon I^tC_F^{s,u}(X)\to I^tC_F^{s,u}(X)/I^{t+1}C_F^{s,u}(X)$$

the projection. If $\delta^F(x)\in I^{t+1}C_F^{s+1,u}(X)$ then $p_I(x)$ is a cocycle, and so we have an element

$$[p_I(x)]\in H^s(I^tC_F^{*,u}(X)/I^{t+1}C_F^{*,u}(X)).$$

According to the filtration (2.4), the differential d_r^I is induced from the correspondence

$$\begin{aligned} {}_IE_1^{s,t,u}(X)=H^s(I^tC_F^{*,u}(X)/I^{t+1}C_F^{*,u}(X))\to H^{s+1}(I^{t+1}C_F^{*,u}(X)) \\ \leftarrow H^{s+1}(I^{t+r}C_F^{*,u}(X))\to H^{s+1}(I^{t+r}C_F^{*,u}(X)/I^{t+r+1}C_F^{*,u}(X))={}_IE_1^{s+1,t+r,u}(X). \end{aligned}$$

Moreover the coboundary

$$H^s(I^tC_F^{*,u}(X)/I^{t+1}C_F^{*,u}(X))\to H^{s+1}(I^{t+1}C_F^{*,u}(X))$$

is induced from the correspondence

$$I^tC_F^{s,u}(X)/I^{t+1}C_F^{s,u}(X)\leftarrow I^tC_F^{s,u}(X)\xrightarrow{\delta^F}I^tC_F^{s+1,u}(X)\supset I^{t+1}C_F^{s,u}(X).$$

By the above argument, we have the following.

Proposition 2.5. i) ${}_IE_1^{s,t,u}(X)=H^s(I^tC_F^{*,u}(X)/I^{t+1}C_F^{*,u}(X))$.
ii) *For* $x\in I^tC_F^{s,u}(X)$, *if* $\delta^F(x)\in I^{t+r}C_F^{s+1,u}(X)$ $(r\geq 1)$ *then*

$$[p_I(x)]\in {}_IE_1^{s,t,u}(X) \text{ and } d_{r'}^I[p_I(x)]=0 \text{ for } 1\leq r'\leq r-1,$$

and so $[p_I(x)]\in {}_IE_r^{s,t,u}(X)$. Moreover

$$[p_I(\delta^F(x))]\in {}_IE_r^{s+1,t+r,u}(X) \text{ and } d_r^I[p_I(x)]=[p_I(\delta^F(x))].$$

iii) *For* $x\in I^tC_F^{s,u}(X)$, *if* $\delta^F(x)=0$ *then* x *represents an element* $[x]\in {}_FE_2^{s,u}(X)=H^s(C_F^{*,u}(X))$ *and* $p_I(x)$ *represents* $[p_I(x)]\in {}_IE_1^{s,t,u}(X)$. *Moreover* $[p_I(x)]$ *converges to* $[x]$.

Next we discuss the May spectral sequence. Let F and G be ring spectra. The cofibration $S^0\xrightarrow{\tau^G}G\xrightarrow{pr}\overline{G}$ induces a boundary homomorphism

$$\partial^G\colon \pi_{u+t}(\overline{G}^t\wedge F\wedge\overline{F}^s\wedge X)\to\pi_{u+t-1}(\overline{G}^{t-1}\wedge F\wedge\overline{F}^s\wedge X).$$

We define

$$V_{u+t}^{s,t}(X)=\operatorname{Im}[(\partial^G)^t\colon \pi_{u+t}(\overline{G}^t\wedge F\wedge\overline{F}^s\wedge X)\to\pi_u(F\wedge\overline{F}^s\wedge X)]. \tag{2.5}$$

Now the filtration

(2.6)

$${}_FE_1^{s,u}(X)=\pi_u(F\wedge\overline{F}^s\wedge X)=V_u^{s,0}(X)\supset V_{u+1}^{s,1}(X)\supset\cdots\supset V_{u+t}^{s,t}(X)\supset V_{u+t+1}^{s,t+1}(X)\supset\cdots$$

induces a spectral sequence

$$\{{}_{FG}E_r^{s,t,u}(X);\, d_r^{FG}\} \text{ abutting to } {}_F\boldsymbol{E}_2^{s,u-t}(X).$$

We call it *the May spectral sequence* (c.f. [3, 5]).

We notice that the filtration (2.6) is the G-Adams filtration of $\pi_u(F\wedge X_s^F)={}_FE_1^{s,u}(X)$. If G-Adams spectral sequance

$${}_GE_2^{t,u}(F\wedge X_s^F)=H^t(C_G^*(F\wedge X_s^F);\,\delta^G)\Rightarrow\pi_{u-t}(F\wedge X_s^F)$$

converges and collapses at E_2, then the correspondence

$$V_u^{s,t}(X)/V_{u+1}^{s,t+1}(X)\leftarrow V_u^{s,t}(X)\leftarrow\pi_u(\overline{G}^t\wedge F\wedge X_s^F)\xrightarrow{\overline{h}^G}H^t(C_G^{*,u}(F\wedge X_s^F);\,\delta^G)$$

induces an isomorphism

$$V_u^{s,t}(X)/V_{u+1}^{s,t+1}(X)\cong H^t(C_G^{*,u}(F\wedge X_s^F);\,\delta^G)$$

by Lemma 2.3. Hence

$${}_{FG}E_1^{s,t,u}(X)=H^s(V_u^{*,t}(X)/V_{u+1}^{*,t+1}(X);\,d_{1*}^F)\cong H^s(H^x(C_G^{*,u}(F\wedge X_*^F);\,\delta^G);\,d_{1*}^F), \tag{2.7}$$

where

$$d_{1*}^F=(\tau^F\wedge 1)_*\circ(pr\wedge 1)_*\colon H^t(C_G^{*,u}(F\wedge X_s^F))\to H^t(C_G^{*,u}(X_{s+1}^F))\to H^t(C_G^{*,u}(F\wedge X_{s+1}^F)).$$

For the description of this E_1-term, we define a double cochain complex

$\{C^{s,t,u}_{FG}(X),\ \delta^F,\ \delta^G\}$ by

$$C^{s,t,u}_{FG}(X)=\pi_u(G^{t+1}\wedge F^{s+1}\wedge X) \text{ and}$$

$$\delta^F=\sum_{i=0}^{s+1}(-1)^i\delta^F_{i*}\colon C^{s,t,u}_{FG}(X)\to C^{s+1,t,u}_{FG}(X),\ \delta^G=\sum_{j=0}^{t+1}(-1)^j\delta^G_{j*}\colon C^{s,t,u}_{FG}(X)\to{}^{s,t+1,u}_{FG}(X),$$

where

$$\delta^F_i\colon G^{t+1}\wedge F^{s+1-i}\wedge S^0\wedge F^i\wedge X\xrightarrow{1\wedge\tau^F\wedge 1}G^{t+1}\wedge F^{s+1-i}\wedge F\wedge F^i\wedge X$$
$$\delta^G_j\colon G^{t+1-j}\wedge S^0\wedge G^j\wedge F^{s+1}\wedge X\xrightarrow{1\wedge\tau^G\wedge 1}G^{t+1-j}\wedge G\wedge G^j\wedge F^{s+1}\wedge X.$$

Now $(pr)^s\wedge 1\colon F^s\wedge X\to\overline{F}^s\wedge X=X^F_s$ induces a cochain map

$$\phi^F\colon\ \{C^{s,t,u}_{FG}(X),\ \delta^G\}=\{C^{t,u}_G(F\wedge F^s\wedge X),\ \delta^G\}\to\{C^{t,u}_G(F\wedge X^F_s),\ \delta^G\} \text{ and}$$
$$\phi^F_*\colon\ \{H^t(C^{s,*,u}_{FG}(X);\ \delta^G),\ \delta^F_*\}\to\{H^t(C^{*,u}_G(F\wedge X^F_s);\ \delta^G),\ d^F_{1*}\}.$$

By the same way as in Proposition 2.2, we can see that ϕ^F_* induces an isomorphism

(2.8)
$$\phi^F_{**}\colon H^s(H^t(C^{*,*,u}_{FG}(X);\ \delta^G);\ \delta^F_*)\to H^s(H^t(C^{*,u}_G(F\wedge X^F_s);\ \delta^G);\ d^F_{1*})={}_{FG}E^{s,t,u}_1(X).$$

Hence we have the following.

Proposition 2.6 ([5],[3, Theorem 5.8]. *We assume that*
(2.6.1) *G-Adams spectral sequence*

$${}_GE^{t,u}_2(F\wedge X^F_s)=H^t(C^{*,*,u}_G(F\wedge X^F_s);\ \delta^G)\Rightarrow\pi_{u-t}(F\wedge X^F_s)$$

converges and collapses at E_2 for any s.
Then ${}_{FG}E^{s,t,u}_1(X)=H^s(H^t(C^{*,*,u}_{FG}(X);\ \delta^G);\ \delta^F_*)$.
ii) *We assume that*
(2.6.2) *F_*F and $F_*(X)$ are flat over F_* and*
(2.6.3) *G-Adams spectral sequence*

$${}_GE^{t,u}_2(F)=H^t(C^{*,u}_G(F))\Rightarrow\pi_{u-t}(F)$$

converges and collapses at E_2.
Then ${}_{FG}E^{s,t,u}_1(X)=\mathrm{Ext}^{s,u}_{F_*F}(F_*,\ H^t(C^{*,*}_G(F\wedge X)))$.

The assumptions of the above proposition ii) hold for the following cases:

(2.9) $F=BP,\ G=HZ_p,\ X=S^0,$
(2.10) $F=BP,\ G=E(n),\ X=L_nS^0.$

For the second case, we refer [4, Theorem 3.18] and [7, Theorem 6.2].

3. Isomorphisms between spectral sequences

In this section, we prove Theorem 1.1-3.
In the first place, we argue on products:
For a ring spectrum G, the product μ^G: $G\wedge G\to G$ induces a product

$$C_G^{s,u}(X)\times C_G^{s',u'}(Y)\to C_G^{s+s',u+u'}(X\wedge Y)$$

by taking

$$xy\colon S^u\wedge S^{u'}\xrightarrow{x\wedge y}G^s\wedge G\wedge X\wedge G\wedge G^{s'}\wedge Y\xrightarrow{\approx}$$
$$G^s\wedge G\wedge G\wedge G^{s'}\wedge X\wedge Y\xrightarrow{1\wedge\mu^G\wedge 1}G^s\wedge G\wedge G^{s'}\wedge X\wedge Y$$

for $x\in C_G^{s,u}(X)$ and $y\in C_G^{s',u'}(Y)$ (c.f., (2.2)). Then we have a product

$$H^s(C_G^{*,u}(X))\times H^{s'}(C_G^{*,u}(Y))\to H^{s+s'}(C_G^{*,u+u'}(X\wedge Y)).$$

If G_*G is flat over G_*, then this is the original product on $\mathrm{Ext}_{G_*G}^{*,*}$-groups.

Consider a cofibering α: $X_1\to X_2\to X_3$ and a spectrum Y such that

$$0\to C_G^{s,u}(X_1)\to C_G^{s,u}(X_2)\to C_G^{s,u}(X_3)\to 0 \text{ and}$$
$$0\to C_G^{s,u}(X_1\wedge Y)\to C_G^{s,u}(X_2\wedge Y)\to C_G^{s,u}(X_3\wedge Y)\to 0$$

are exact. Then we have the coboundary homomorphisms

$$\delta^C\colon H^s(C_G^{*,u}(X_3))\to H^{s+1}(C_G^{*,u}(X_1)),$$
$$\delta^C\colon H^s(C_G^{*,u}(X_3\wedge Y))\to H^{s+1}(C_G^{*,u}(X_1\wedge Y)) \text{ and}$$
$$\delta^C\colon H^s(C_G^{*,u}(Y\wedge X_3))\to H^{s+1}(C_G^{*,u}(Y\wedge X_1)).$$

For $x\in H^s(C_G^{*,u}(X_3))$, $y\in H^{s'}(C_G^{*,u}(Y))$, we have

(3.1) $$\delta^C(xy)=(-1)^{s'}(\delta^C x)y \text{ and } \delta^C(yx)=y(\delta^C x).$$

For another ring spectrum F, the product μ^F: $F\wedge F\to F$ induces products

$$\pi_u(\overline{G}^s\wedge F)\otimes\pi_{u'}(\overline{G}^{s'}\wedge F)\to\pi_{u+u'}(\overline{G}^s\wedge\overline{G}^{s'}\wedge F\wedge F)\to\pi_{u+u'}(\overline{G}^{s+s'}\wedge F) \text{ and}$$
$$H^s(C_G^{*,u}(\overline{G}^t\wedge F))\otimes H^{s'}(C_G^{*,u}(\overline{G}^t\wedge F))\to H^{s+s'}(C_G^{*,u+u}(\overline{G}^t\wedge\overline{G}^{t'}\wedge F)).$$

For $x\in\pi_u(\overline{G}^s\wedge F)$, $y\in\pi_{u'}(\overline{G}^{s'}\wedge F)$,

(3.2)
$$\overline{h}^G(xy)=(\delta^C)^{s'}(\delta^G)^s(h(x)h(y))=((\delta^C)^s h^G(x))((\delta^C)^{s'}h^G(y))=\overline{h}^G(x)\overline{h}^G(y)$$

by (3.1).

Now we prove Theorem 1.1-3. Let λ: $F\to G$ be a map of ring spectra and $I=\mathrm{Ker}[F_*\to G_*]$. The following lemma implies Theorem 1.1.

Lemma 3.1. *If λ_*: $F_*(G)\to G_*(G)$ is monomorphic then the following holds*:

i) $\mathrm{Im}\{\partial^G: \pi_{*+1}(\bar{G}\wedge F)\to\pi_*(F)\}=I$.
ii) $\mathrm{Im}\{(\partial^G)^t: \pi_{u+*}(\bar{G}^t\wedge F\wedge X)\to\pi_*(F\wedge X)\}=V^t_{*+t}(X)\supset I^t\cdot F_*(X)$.

Proof. i) By the commutative diagram

$$\begin{array}{ccccc} \pi_{u+1}(\bar{G}\wedge F) & \xrightarrow{\partial^G} & \pi_u(S^0\wedge F) & \xrightarrow{(\tau^G\wedge 1)} & \pi_u(G\wedge F) \\ & & \downarrow{\scriptstyle (1\wedge\lambda)_*} & & \downarrow{\scriptstyle (1\wedge\lambda)_*} \\ & & \pi_u(S^0\wedge G) & \xrightarrow{(\tau^G\wedge 1)} & \pi_u(G\wedge G) \end{array}$$

in which the upper line is exact, we see i) since

$$(\tau^G\wedge 1)_*: \pi_u(S^0\wedge G)\to\pi_u(G\wedge G) \text{ and } (1\wedge\lambda)_*: \pi_u(G\wedge F)\to\pi_u(G\wedge G)$$

are monomorphic by the assumption.
ii) For

$$x=x_1x_2\cdots x_ty\in I^t\cdot F_*(X)\ (x_i\in I,\ y\in F_*(X)),$$

we have elements

$$x_i'\in\pi_{*+1}(\bar{G}\wedge F) \text{ with } \partial^G(x_i')=x_i \text{ and } x'=x_1'x_2'\cdots x_t'y\in\pi_{u+*}(\bar{G}^t\wedge F\wedge X)$$

by i). Then $(\partial^G)^t(x')=x$. This implies ii).

We notice that if

$$V^t_{*+t}(X)=I^tF_*(\mathrm{X}) \tag{3.3}$$

then the May and algebraic Novikov spectral sequences are isomorphic. Hence we prove Theorem 1.2 and 1.3 by showing (3.3).

Proof of Theorem 1.2. For any $x\in V^t_{u+t}(X)$, we have $x_t\in\pi_{u+t}(\bar{G}^t\wedge F\wedge X)$ with $(\partial^G)^t(x_t)=x$. Then

$$(\tau^G\wedge 1)_*\circ(1\wedge\lambda\wedge 1)_*(\partial^G x_t)=(1\wedge\lambda\wedge 1)_*\circ(\tau^G\wedge 1)_*(\partial^G x_t)=0.$$

Since

$$(\tau^G\wedge 1)_*: \pi_*(S^0\wedge\bar{G}^{t-1}\wedge G\wedge X)\to\pi_*(G\wedge\bar{G}^{t-1}\wedge G\wedge X)$$

is monomorphic, $(1\wedge\lambda\wedge 1)_*(\partial^G x_t)=0$. Hence

$$\partial^G x_t\in\mathrm{Ker}[\lambda_*: F_*(\bar{G}^{t-1}\wedge X)\to G_*(\bar{G}^{t-1}\wedge X)]=IF_*(\bar{G}^{t-1}\wedge X),$$

and so we have $x_{t-1}\in\pi_*(\bar{G}^{t-1}\wedge F\wedge X)$ and $\theta_t\in I$ with $\partial^G x_t=\theta_t x_{t-1}$. Inductively, we have elements

$$x_i\in\pi_*(\bar{G}^i\wedge F\wedge X)\ (i=t,\ t-1,\cdots,\ 0) \text{ and } \theta_i\in I\ (i=t,\ t\text{-}1,\cdots,\ 1)$$

with $\partial^G x_{i+1}=\theta_{i+1}x_i$. Hence

$$x=(\partial^G)^t(x_t)=(\partial^G)^{t-1}(\theta_t x_{t-1})=\theta_t(\partial^G)^{t-1}(x_{t-1})=\cdots=\theta_t\theta_{t-1}\cdots\theta_1 x_0.$$

This implies (3.3) and this theorem.

We prepare the following proposition for the proof of Theorem 1.3.

Proposition 3.2. *We assume that λ_*: $F_*(G)\to G_*(G)$ is monomorphic. If*

$$\rho: I^tF_*(X)/I^{t+1}F_*(X)\to V^t_{*+t}(X)/V^{t+1}_{*+t+1}(X)$$

is epimorphic and there exists an integer $t(u)$ for each integer u such that $V^t_{u+t}(X)=0$ for $t>t(u)$, then

$$\phi_*: \{{}_IE_r^{s,t,u}(X),\ d_r^I\}\to\{{}_{FG}E_r^{s,t,u}(X),\ d_r^{FG}\}$$

is an isomorphism.

Proof. We prove (3.3). Let

$$p_I: I^tF_*(X)\to I^tF_*(X)/I^{t+1}F_*(X),\ p_V: V^t_{*+t}(X)\to V^t_{*+t}(X)/V^{t+1}_{*+t+1}(X)$$

be projections. For any $x\in V^t_{u+t}(X)$, we have

$$x'\in I^tF_*(X) \text{ with } \rho\circ p_I(x')=p_V(x) \text{ and } \theta_1=x-x'\in V^{t+1}_{u+t+1}(X).$$

By induction, we have $\theta_i\in V^{t+i}_{u+t+i}(X)$ and $\theta_i'\in I^{t+i}F_*(X)$ $(i=1,2,\cdots)$ with

$$\rho\circ p_I(\theta_i')=p_V(\theta_i) \text{ and } \theta_{i+1}=\theta_i-\theta_i'\in V^{t+i+1}_{u+t+i+1}(X).$$

By the assumption, $\theta_i=0$ for $i>t(u)$, and so $x=x'+\sum_{i=1}^{t(u)}\theta_i'$. Hence $x\in I^tF_*(X)$.

Proof of Theorem 1.3. Since $F_*(X)$ is flat, the G-Adams spectral sequence $\{{}_GE_r^{t,u}(F\wedge X)\}$ is isomorphic to $\{{}_GE_r^{t,u}(F)\otimes_{F_*}F_*(X)\}$. Since G-Adams spectral sequence $\{{}_GE_r^{t,u}(F)\}$ converges and collapses at E_2 and G_*G is flat,

$$V^t_{*+t}(X)/V^{t+1}_{*+t+1}(X)={}_GE_2^{t,*}(F)\otimes_{F_*}F_*(X)=\mathrm{Ext}^{t,*}_{G_*G}(G_*,G_*(F))\otimes_{F_*}F_*(X).$$

Then, by Lemma 3.1 i),

$$I=V^1_{*+1}(S^0)\to V^1_{*+1}(S^0)/V^2_{*+2}(S^0)=\mathrm{Ext}^{1,*}_{G_*G}(G_*,G_*(F))$$

is epimorphic, and so is

$$I^t\cdot F_*(X)\to V^t_{*+t}(X)/V^{t+1}_{*+t+1}(X)=\mathrm{Ext}^{t,*}_{G_*G}(G_*,G_*(F))\otimes_{F_*}F_*(X)$$

by the assumption. By Proposition 3.2, this implies Theorem 1.3.

HIROSHIMA PREFECTURAL UNIVERSITY

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