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A remark on perturbation of hyperbounded semigroups for vector valued functions

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1. Introduction

Perturbation theory of hyperbounded semigroups has been usually developed in the framework of L^2 space. To obtain essential self-adjointness of perturbed infinitesimal generator, potential terms are often imposed on the condition that their exponentials have every (or large enough) moment.

On the other hand, Shigekawa [12] treated L^p semigroups for vector valued functions. He also discussed essential self-adjointness of a perturbed generator under the formulation applicable to L^p sense.

In this note, we discuss perturbation theory for (non-symmetric) semigroups for vector valued functions which are controlled by scalar valued hyperbounded semigroups, slightly modifying the setting in [12]. We give an explicit constant of moment sufficient for the stability of operator cores in L^p sense. This constant is expressed by p and the logarithmic Sobolev constant of the dominating semigroup.

2. Semigroups on L^p

We mainly refer to [12] to set up a framework. Let $(\Omega,\mathscr{B},m)$ be a probability space. Assume we are given a symmetric, strongly continuous, positivity-preserving semigroup $\{T_t\}$ on $L^2(\Omega,m)$. We denote its infinitesimal generator and resolvents by A and G_ν, respectively. Let K be a real or complex separable Hilbert space. We represent its inner product and norm by $(\cdot|\cdot)$ and $|\cdot|$, respectively. Suppose we are also given a strongly continuous semigroup $\{\vec{T}_t\}$ on $L^2(K) = L^2(\Omega,m;K)$. Its generator and resolvents are denoted by $\vec{A}$ and $\vec{G}_\nu$, respectively.

Theorem 2.1. *The following three conditions are mutually equivalent.*

(C.1) $|\vec{T}_t u| \leq e^{\lambda t} T_t|u|$ m-a.e. *for* $t>0$, $u\in L^2(K)$.

(C.2) $|\vec{G}_{\nu+\lambda}u| \leq G_\nu|u|$ m-a.e. *for sufficiently large* ν, $u\in L^2(K)$.

(C.3) $A|u| \geq \Re((\vec{A}-\lambda)u|\mathrm{sgn}\,u)$, $u\in \mathrm{Dom}(\vec{A})$,

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where

$$\mathrm{sgn}u=\begin{cases}u/|u|, & u\neq 0\\ 0, & u=0.\end{cases}$$

In (C.3), *A operates in the weak sense; precisely,* (C.3) *means*

$$\langle |u|, Ag\rangle \geq \langle \Re((\vec{A}-\lambda)u|\mathrm{sgn}u), g\rangle$$

for any $g\in \mathrm{Dom}(A)$ *with* $g\geq 0$. *Here* $\langle f,g\rangle=\int_\Omega f(x)g(x)m(dx)$.
Proof. See [13, 14]. See also [12].

Below, we omit 'm-a.e.' when we need not designate it. The coupling $\langle\cdot,\cdot\rangle$ is also used for K-valued functions. We assume one of (hence all of) the conditions in Theorem 2.1 with some $\lambda\geq 0$.

Let $\{\vec{T}_t^*\}$ be the dual semigroup of $\{\vec{T}_t\}$ and its resolvents, $\{\vec{G}_\nu^*\}$. Then (C.1) implies

$$|\vec{T}_t^* u|\leq e^{\lambda t}T_t|u| \quad \text{for any } t>0,\ u\in L^2(K). \tag{2.1}$$

Indeed, for any measurable set S of Ω,

$$\begin{aligned}\||\vec{T}_t^*u|1_S\|_2&=\sup_{\|f\|_2\leq 1}\langle|\vec{T}_t^*u|1_S,f\rangle=\sup_{\|f\|_2\leq 1}\langle u,\vec{T}_t(1_Sf\mathrm{sgn}(\vec{T}_t^*u))\rangle\\ &\leq\sup_{\|f\|_2\leq 1}\langle|u|,|\vec{T}_t(1_Sf\mathrm{sgn}(\vec{T}_t^*u))|\rangle\leq\sup_{\|f\|_2\leq 1}\langle|u|,e^{\lambda t}T_t|1_Sf|\rangle\\ &=\sup_{\|f\|_2\leq 1}\langle e^{\lambda t}T_t|u|,1_S|f|\rangle\leq\|e^{\lambda t}T_t|u|1_S\|_2,\end{aligned}$$

which implies (2.1). Hence the roles of $\{\vec{T}_t\}$ and $\{\vec{T}_t^*\}$ are equivalent; the claims about $\{\vec{T}_t\}$ as are shown below also hold when replacing $\{\vec{T}_t\}$ with $\{\vec{T}_t^*\}$.

From now on, we also suppose that $\{T_t\}$ is sub-Markovian; that is, for any $t>0$,

$$0\leq T_tf\leq 1 \quad \text{if } 0\leq f\leq 1.$$

By Riesz-Thorin's interpolation theorem and dual argument, $\{T_t\}$ can be considered as a semigroup on $L^p=L^p(\Omega,m)$ for $p\in[1,\infty]$. Moreover it is strongly continuous if $p\in[1,\infty)$. We denote its infinitesimal generator on L^p by A_p. Since $A_{p_1}\supset A_{p_2}$ if $1\leq p_1<p_2<\infty$, we often omit the subscript. Also we have

Proposition 2.2 ([12, Proposition 2.6]). *For any* $p\in[1,\infty)$, $\{\vec{T}_t\}$ (*resp.* $\{\vec{T}_t^*\}$) *can be seen as a strongly continuous semigroup on* $L^p(K)=L^p(\Omega,m;K)$.

So we can define the generator $\vec{A}_p$, $\vec{A}_p^*$ of $\{\vec{T}_t\}$, $\{\vec{T}_t^*\}$ on $L^p(K)$, $p\in[1,\infty)$, respectively. Also, (C.1) holds for any $u\in L^1(K)$.

We assume furthermore $\{T_t\}$ is hyperbounded in the following sense; for some $\alpha>0$, $\beta\geq 0$,

$$\|T_t\|_{p\to q}\leq \exp\left\{\beta\left(\frac{1}{p}-\frac{1}{q}\right)\right\}$$

for $t>0$, $1<p<q<\infty$ with $(q-1)/(p-1)\leq e^{4t/\alpha}$. This assumption holds if and only if the defective logarithmic Sobolev inequality holds:

$$\int_\Omega f^2\log f^2/\|f\|_2^2 dm\leq \alpha\mathscr{E}(f,f)+\beta\|f\|_2^2,\quad f\in \mathrm{Dom}(\mathscr{E}),$$

where $\mathscr{E}$ is the symmetric bilinear form associated with $\{T_t\}$. For the proof, see [2, Theorem 6.1.14] and [5, Lemma 5.5].

Now we state the main theorem in this note. In general, we say an operator A on a Banach space belongs to $G(M,\xi)$ if A is the infinitesimal generator of a strongly continuous semigroup $\{T_t\}$ satisfying $\|T_t\|\leq Me^{\xi t}$ for all $t>0$. Let $\mathscr{L}(K)$ be the space of bounded linear operators on K, and the norm in $\mathscr{L}(K)$ the operator norm $\|\cdot\|_{\mathrm{op}}$.

Theorem 2.3. *Let $1<p<\infty$ and let R be an $\mathscr{L}(K)$-valued measurable function on Ω. Suppose $\exp(\|R\|_{\mathrm{op}})\in L^r$ for some $r>\frac{\alpha p^2}{4(p-1)}$. If $p\neq 2$, we also assume $\{T_t\}$ is conservative: $T_t 1=1$ for every $t>0$. Then we have the following.*

(1) $\mathrm{Dom}(\vec{A}_p)\subset \mathrm{Dom}(R)$, *where we regard R as an operator on $L^p(K)$. Hence $\vec{A}_p-R$ can be defined on $\mathrm{Dom}(\vec{A}_p)$ as an operator sum.*

(2) $(\vec{A}_p-R,\mathrm{Dom}(\vec{A}_p))$ *is closable and the closure (which is denoted by the same notation) belongs to $G(1,\xi)$ for some ξ. Moreover the semigroup is consistent with respect to p, that is, for $p\wedge q\leq p_1\leq p_2\leq p\vee q$, we have $\{\vec{T}^R_{t,(p_1)}|L^{p_2}(K)\}=\{\vec{T}^R_{t,(p_2)}\}$. Here q is the conjugate exponent of p, and $\{\vec{T}^R_{t,(p_i)}\}$ is the semigroup generated by $\vec{A}_{p_i}-R$ on $L^{p_i}(K)$.*

(3) *If $\mathscr{C}$ is a core of $\vec{A}_p$, $\mathscr{C}$ is also a core of $\vec{A}_p-R$.*

We make a comment on (2). The function $f(x)=\frac{\alpha x^2}{4(x-1)}$ is monotone increasing for $x\geq 2$ and satisfies $f(p)=f(q)$ when $p^{-1}+q^{-1}=1$. Hence when R satisfies the assumption for some p, R also satisfies it for any numbers in $[p\wedge q, p\vee q]$.

To prove this theorem, we need a little more preparations. The following theorem is originally due to Bakry and Meyer [1].

Theorem 2.4. *Assume $\{T_t\}$ is conservative. For any $p\in(1,\infty)$, $r\in\mathbf{R}$, $s>0$ and $\nu>\lambda$, $(\nu-\vec{A})^{-s}$ is a bounded operator from $L^p\log^r L(K)$ to $L^p\log^{r+ps}L(K)$.*

As for the definition of the Orlicz space $L^p\log^r L(K)$ and the proof of Theorem 2.4, see [7, Appendix] and [1]. In the following, we only use the fact that $\vec{G}_\nu$ is bounded from $L^p(K)$ to $L^p\log^p L(K)$; in this case, the conservativeness of $\{T_t\}$ is not necessary if $p=2$, as we see from the proof of the theorem.

Proposition 2.5. *Suppose we are given a real valued measurable function V on Ω and an $\mathcal{L}(K)$-valued measurable function R on Ω. Assume both V and R are bounded and satisfy that*

$$V(x)|k|^2 \leq \Re(R(x)k|k) \tag{2.2}$$

for all $k \in K$, m-a.e. x. Let $\{T_t^V\}$, $\{\vec{T}_t^R\}$ be the associated semigroups of $A-V$ and of $\vec{A}-R$, respectively. Then $\{T_t^V\}$ is a positivity preserving semigroup and it holds that

$$|\vec{T}_t^R u| \leq e^{\lambda t} T_t^V |u|. \tag{2.3}$$

Proof. The proof is seen in [12, Proposition 4.1], but here we give an alternative one. The positivity preserving property of $\{T_t^V\}$ follows from the Beurling-Deny criterion (see e.g. [11, Theorem XIII.50]). To prove (2.3), first we note that (2.2) is equivalent to

$$V(x)|k| \leq \Re(R(x)k|\mathrm{sgn}k).$$

Hence for $u \in \mathrm{Dom}(\vec{A}_2)$ and $g \in \mathrm{Dom}(A_2)$ with $g \geq 0$, we have

$$\begin{aligned}\langle |u|, (A-V)g\rangle &= \langle |u|, Ag\rangle - \langle V|u|, g\rangle \geq \langle \Re((\vec{A}-\lambda)u|\mathrm{sgn}u), g\rangle - \langle \Re(Ru|\mathrm{sgn}u), g\rangle \\ &= \langle \Re((\vec{A}-R-\lambda)u|\mathrm{sgn}u), g\rangle.\end{aligned}$$

By Theorem 2.1, we obtain (2.3).

We also quote the following proposition in [12].

Proposition 2.6 ([12, Proposition 4.2]). *Let V be a bounded real measurable function. For $p \in (1,\infty)$, the semigroup $\{T_t^V\}$ corresponding to $A_p - V$ satisfies*

$$\|T_t^V\|_{p\to p} \leq e^{\zeta t},$$

where $\zeta = \log(\|e^{-V}\|_r) + 4\beta/\alpha$ with $r = \alpha p^2/4(p-1)$.

As a consequence of Proposition 2.5 and Proposition 2.6, by taking $V(x) = -\|R(x)\|_{\mathrm{op}}$, we have a following

Corollary 2.7. *Let R be an $\mathcal{L}(K)$-valued bounded measurable function. For $p \in (1,\infty)$, the semigroup $\{\vec{T}_t^R\}$ corresponding to $\vec{A}_p - R$ satisfies*

$$\|\vec{T}_t^R\|_{p\to p} \leq e^{\xi t},$$

where $\xi = \log(\|e^{\ R\|_{\mathrm{op}}}\|_r) + 4\beta/\alpha + \lambda$ with $r = \alpha p^2/4(p-1)$. Moreover, $\kappa > \xi$ belongs to the resolvent set of $\vec{A}_p - R$ and it holds that

$$\|(\kappa - \vec{A}_p + R)^{-1}\|_{p\to p} \leq (\kappa - \xi)^{-1}.$$

3. Proof of the main theorem

Proof of Theorem 2.3. To prove (1), we need only $\exp(a\|R\|_{op})\in L^1$ for some $a>0$. Take $u\in \mathrm{Dom}(\vec{A}_p)$. By Theorem 2.4, $u\in L^p\log^p L(K)$. Using Young's inequality, $st\leq e^s-t\log t+t$, $s\in\mathbf{R}$, $t>0$, we have for $\varepsilon>0$,

$$|Ru|\leq\varepsilon^{-1}\varepsilon\|R\|_{op}|u|\leq\varepsilon^{-1}(|u|\log|u|-|u|+e^{\varepsilon\|R\|_{op}})\leq\varepsilon^{-1}(|u|\log^+|u|+e^{\varepsilon\|R\|_{op}}),$$

where $\log^+t=(\log t)\vee 0$. Hence

$$|Ru|^p\leq\varepsilon^{-p}\cdot 2^p(|u|^p(\log^+|u|)^p+e^{p\varepsilon\|R\|_{op}}).$$

Taking $\varepsilon=a/p$, we see $Ru\in L^p(K)$ and moreover, R is a bounded operator from $L^p\log^p L(K)$ to $L^p(K)$.

Next we prove (2) and (3). We follow the argument of Wu ([15, Theorem 2.5]). Take $R_n=R\cdot 1_{\{\|R\|_{op}\leq n\}}$. We denote the associated semigroup of $\vec{A}-R_n$ by $\{\vec{T}_t^{R_n}\}$. Since $\|R_n(x)\|_{op}\leq\|R(x)\|_{op}$, each $\vec{A}-R_n$ belongs to $G(1,\xi)$, $\xi=\log(\|e^{\|R\|_{op}}\|_r)+4\beta/\alpha+\lambda$ by Corollary 2.7. By a version of Trotter-Kato's theorem (see e.g. [9, Chapter 3, Theorem 4.5]), it is enough to verify the following:

(a) For any $u\in\mathscr{C}$, $(\vec{A}-R_n)u$ converges to $(\vec{A}-R)u$ in $L^p(K)$,

(b) For some $\kappa>\xi$, $(\kappa-(\vec{A}-R))(\mathscr{C})$ is dense in $L^p(K)$.

Take $u\in\mathscr{C}$. We have

$$\|(\vec{A}-R_n)u-(\vec{A}-R)u\|_p=\|Ru\cdot 1_{\{\|R\|_{op}>n\}}\|_p.$$

Since $Ru\in L^p(K)$, this converges to 0 as $n\to\infty$. Hence (a) holds. Let us prove (b). Let $v\in L^q(K)$ satisfy

$$\langle v,(\xi+1-\vec{A}+R)u\rangle=0\quad\text{for every } u\in\mathscr{C}.$$

Define R^* by $R^*(x)=$ the dual of $R(x)$. Since R^* sends $(L^p(K))^*=L^q(K)$ to $(L^p\log^p L(K))^*$ by (1), we have $R^*v\in(L^p\log^p L(K))^*$. By Theorem 2.4, $\vec{G}^*_{\xi+1}(R^*v)\in(L^p(K))^*=L^q(K)$. Thus for $u\in\mathscr{C}$, we have

$$\langle v+\vec{G}^*_{\xi+1}(R^*v),(\xi+1-\vec{A})u\rangle=\langle v,(\xi+1-\vec{A})u\rangle+\langle v,Ru\rangle=0.$$

Since $(\xi+1-\vec{A})(\mathscr{C})$ is dense in $L^p(K)$, we conclude $v+\vec{G}^*_{\xi+1}(R^*v)=0$; that is, $(\xi+1-\vec{A}^*+R^*)v=0$. Now take $s<q$ such that $\frac{\alpha s^2}{4(s-1)}\leq r$. Since $v\in L^q(K)\subset L^s(K)$,

$$\begin{aligned}\|v\|_s&=\|(\xi+1-\vec{A}^*+R_n^*)^{-1}(R_n^*-R^*)v\|_s\leq\|(\xi+1-\vec{A}^*+R_n^*)^{-1}\|_{s\to s}\|(R_n^*-R^*)v\|_s\\&\leq\|R^*v\cdot 1_{\{\|R^*\|_{op}>n\}}\|_s\qquad\text{(By Corollary 2.7)}\\&\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Therefore $v=0$, which implies (b). We have proved the first half of (2), and

(3). To show the latter half of (2), it is enough to notice that $\vec{T}_t^{R_n}$ converges to $\vec{T}_{t,(p)}^R$ in strong sense in $L^p(K)$.

Remark 3.1. We proved that $\vec{A}-R_n$ converges to $\vec{A}-R$ in strong resolvent sense. In the same way of [12, Proposition 4.6], we see the convergence is in fact the norm resolvent sense.

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References

[1] D. Bakry and P. A. Meyer, Sur les inegalites de Sobolev logarithmiques, I, II, Séminaire de Prob. XVI, Lecture Notes in Math., vol.920, 138-150.
[2] J.-D. Deuschel and D. W. Stroock, Large deviations, Academic Press, San Diego, 1989.
[3] G. F. Feissner, Hypercontractive semigroups and Sobolev's inequality, Trans. Amer. Math. Soc., **210** (1975), 51-62.
[4] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math., **97** (1975), 1061-1083.
[5] L. Gross, Logarithmic Sobolev inequalities on loop groups, J. Funct. Anal., **102** (1991), 268-313.
[6] J. Gustavsson and J. Peetre, Interpolation of Orlicz spaces, Studia Math., **60** (1977), 33-59.
[7] M. Hino, Spectral properties of Laplacians on an abstract Wiener space with a weighted Wiener measure, to appear in J. Funct. Anal.
[8] P. A. Meyer, Interpolation entre espaces d'Orlicz, Séminaire de Prob. XVI, Lecture Note in Math. vol.920, 153-158.
[9] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences 44, Springer-Verlag, 1983.
[10] M. Reed and B. Simon, Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness, Academic Press, New York, 1975.
[11] M. Reed and B. Simon, Methods of Modern Mathematical Physics IV: Analysis of Operators, Academic Press, New York, 1978.
[12] I. Shigekawa, L^p-contraction semigroup for vector valued functions, to appear in J. Funct. Anal.
[13] B. Simon, An abstract Kato's inequality for generators of positivity preserving semigroups, Indiana Univ. Math. J., **26** (1977), 1069-1073.
[14] B. Simon, Kato's inequality and the comparison of semigroups, J. Funct. Anal., **32** (1979), 97-101.
[15] L. M. Wu, Feynman-Kac Semigroups, Ground State Diffusions, and Large Deviations, J. Funct. Anal., **123** (1994), 202-231.