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Tight closure in graded rings

By

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In its principal setting, tight closure is an operation perfomed on ideals in a commutative, Noetherian ring of prime characteristic. This operation was introduced by Hochster and Huneke in [HH1], and has had applications to several disparate but classical problems in commutative algebra such as the Syzygy problem, the local cohomological conjectures, and the Briançon-Skoda theorems. Tight closure appears to be giving information about the singularities of a local ring. For example, with mild hypotheses, the property that all ideals of a ring are tightly closed implies that ring is normal and Cohen-Macaulay [HH1] and even pseudo-rational [S1], which amounts to rational singularities in characteristic zero. Tight closure also sheds light on log terminal and log canonical singularities [W] [H]. However, a serious difficulty in this theory remains: how does one compute the tight of closure of a given ideal in a given ring?

This paper attacks the problem of computing the tight closure of homogeneous ideals in a graded ring. Because of the subtle information tight closure provides about both the ring and the ideal, an actual algorithm for computing tight closure seems much too to hope for. However, it is of interest to at least narrow the search. In this paper, the problem is confronted from both ends. A general lower bound on the degrees of elements in I^* is proven (Theorems 2.2, 2.4): (with mild assumptions on R) any element in I^*-I must have degree strictly larger than the smallest degree of any of the minimal generators for I. For an m-primary ideal I, two upper bounds are given (Propositions 3.1, 3.3), such that elements exceeding this degree are always in I^*: any element of degree larger than N is always in I^*, where N is the smaller of the sum of the degrees of a minimal set of generators for I or of the dimersion of R times the largest degree of any minimal generator for I. These bounds have been useful to the author in computing tight closures.

Section 2 deals with the lower bounds. Though these results are quite useful in practice, one of the main points of this section is to introduce a new method for studying tight closure. This method is *differential operators.*

In our setting, the union of the endomorphism rings of R as an R^{p^e} module,

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as e ranges through all non-negative integers, is a ring of differential operators on R. These operators operate on the equations that define tight closure (see Definition 1.1), and can be used to manipulate these equations to great effect. The author believes that this "differential operator" point of view on tight closure will have further applications, and hopes that deeper connections will eventually be revealed.

There is another method for proving some of the results in Section 2, which involves the use of *test elements* for tight closure. The theory of test elements is one of the most important and deepest aspects of tight closure. To illustrate this approach, the proof of Theorem 2.4 is written using test elements, though it can also be deduced using differential operators. The differential operator point of view is self-contained; it does not require test elements, nor indeed, any knowledge of tight closure beyond the definition.

In section 3, Briançon-Skoda type theorems are used to prove that all forms of degree greater than a certain constant (explicitly described in terms of the degrees of the generators of the ideal I) are in the tight closure of I, for m-primary I. For computational purposes, this is quite useful, since it gives an upper bound on the degrees of homogeneous elements that need to be considered for inclusion in the tight closure of a particular ideal. The results in this section actually give methods for checking that all elements of high enough degree are in the "plus closure" of I. When R is a domain, this is simply the ideal $IR^+ \cap R$, where R^+ is the integral closure of R in an algebraic closure of its fraction field. Whether or not the plus closure is the same as the tight closure remains an open question, although this is the case for ideals generated by parameters [S1] (see also [Ab], where the class of ideals where this is known to hold is enlarged). The work in this paper was partially motivated by this question, and the results of Section 2 and 3 both offer further evidence for the equality $I^* = IR^+ \cap R$.

Aside from their use in computing tight closures, the results of Sections 2 and 3 have several interesting consequences, which are recorded in the final section of the paper. For example, we deduce a sufficient condition for a standard K-algebra to have the property that all parameter ideals are tightly closed in terms of its a-invariant alone; see Theorem 4.1. This ring property is called *F-rationality* because of its close connection with rational singularities; indeed, by the main result of [S3], we deduce the same test for pseudorational rings, and therefore for rational singularities when K has characteristic zero; see 4.4. An immediate consequence is that F-rationality and pseudorationality are equivalent for two-dimensional standard algebras in *all characteristics*. The characteristic zero case of this theorem was proved first by Fedder [F]. The question of whether or not F-rationality is equivalent to pseudorationality in general has persisted since the inception of the theory of tight closure; see, for example, the nice summary of the progress on this problem as of 1989 in [FW]. For the (more recent) proof that F-rational implies pseudorational for excellent local rings, consult [S3].

The suggestions of several people have helped to make this a better paper. I am grateful to Amnon Yekutieli for teaching me about differential operators. Helpful discussions with Mel Hochster improved Section 2. Donna Glassrenner provided a nice example to show the hypothesis in Theorem 2.4 are best possible. Comments of Will Traves exposed unclear arguments that have since been improved. The author is especially grateful to Craig Huneke, whose idea it was to use test elements to give an alternate proof of Theorem 2.4; furthermore, much of the research for this paper was conducted while the author was visiting Purdue University and enjoying his kind hospitality. And finally, the referee made several good suggestions that improved the paper.

1. Preliminaries

This section can be used as a reference for the rest of the paper.

Throughout this paper R always denotes an $\mathbf{N}$-graded K-algebra, $R=\bigoplus_{i\in\mathbf{N}}R_i$, where $R_0=K$ is a field. We always assume that R is Noetherian, and if we further wish to indicate that R is generated by its homogeneous elements of degree one, we will say that R is *standard*. The unique homogeneous maximal ideal of R, $\bigoplus_{i\geq 1}R_i$, is denoted by m.

Tight Closure. We summarize some definitions and elementary facts about tight closure. We treat here only the case of tight closure for ideals, although it is more generally defined for modules as well [HH1].

Let A be any commutative, Noetherian ring of characteristic $p>0$.

1.1. Definition. The tight closure I^* of an ideal $I\subset A$ is the ideal defined by

$$z\in I^*$$

if there exists some c not in any minimal prime of A and some non-negative integer N such that

$$cz^{[pe]}\in I^{[pe]} \quad \text{for all integers } e\geq\mathbf{N}$$

where $I^{[pe]}$ denotes the ideal of A generated by the p^{eth} powers of the elements (equivalently, the generators) of I.

Tight closure is a true closure operation in the sense that $(I^*)^*=I^*$. We refer the reader to [HH1, Proposition 4.1] for a few of its elementary properties.

Most questions about and applications of tight closure reduce easily to the domain case. In fact, its very definition does:

(1.1.1) $\quad z\in I^*$ if and only if $z \bmod \mathcal{P}\in (I\frac{A}{\mathcal{P}})^*$ in $\frac{A}{\mathcal{P}}$

for all minimal primes $\mathscr{P}$ of A [HH1, Prop 6.25]. We will therefore restrict our attention primarily to *domains* throughout this paper. In the domain case, the only restriction on the element c in Definition 1.1 is that c is non-zero, and furthermore, in this case, one may always take N to be 0 [HH1, Prop 4.1c].

Our main concern is with homogeneous ideals in graded rings. Fortunately, it turns out that the tight closure I^* of a homogeneous ideal I in the graded ring R is itself a homogeneous ideal. In addition, for a graded ring R, the element c in Definition 1.1 can always be chosen to be homogeneous, whether or not I and z are assumed to be homogeneous. Both these assertions are proved in [HH2, Theorem 4.2].

In many settings, it turns out that actually a much (*a priori*) weaker definition of tight closure is available in many situations. In particular, this is the case for graded rings.

1.2. Proposition. *Let I be an arbitrary ideal in the graded ring R and let z be an arbitrary element of R. Suppose that for infinitely many $e \in \mathbf{N}$, there exists some c_e not in any minimal prime and of fixed degree (i.e. c_e depends on e but its degree is a constant independent of e) such that*

$$c_e z^{p^e} \in I^{[p^e]}.$$

Then $z \in I^*$.

Proof. This is essentially Theorem 6.9 of [HH1]. In our case, the concept of degree (of leading terms) replaces that of the "order of the norm".

2. Elements excluded from the tight closure (an application of differential operators)

Throughout this section we will assume that the graded algebra $R = \bigoplus_{i \in \mathbf{N}} R_i$ is finitely generated over $R_0 = K$, of characteristic $p > 0$. The letters q, q', Q, *et cetera*, will denote various positive integer powers of p.

Remark. There is a notion of tight closure for finitely generated algebras over a field of characteristic zero, as well; see [HH4]. The definition is somewhat involved, but is a standard application of the idea of reduction to characteristic p. We do not state this definition here, but the reader familiar with it will recognize that all theorems stated in this section are valid also in characteristic zero. This follows from the definition of tight closure in characteristic zero combined, of course, with the fact that these theorems hold incharacteristic p.

Let I be an ideal generated by forms $\mu_1, \mu_2, \cdots, \mu_k$ all of degree at least δ.

We seek restrictions on the degree of a non-zero homogeneous element z which can be in the tignt closure I^* of I. Quite trivially, for a reduced graded ring R, we see that the degree of z is at least δ. Otherwise, for any homogeneous test element c, the equations

$$cz^q \in (\mu_1^q, \mu_2^q, \cdots, \mu_k^q) R$$

would quickly yield a contradiction as q gets very large. Indeed, if deg $z < \delta$, then for $q \gg 0$, the element cz^q is of degree

$$\deg c + q(\deg z) \ll q\delta.$$

Because q^δ is the degree of the generators for $I^{[q]}$, the element cz^q can not be contained in $I^{[q]}$ unless $cz^q = 0$. Since R is reduced, the fact that c is not in any minimal prime implies it is not a zero-divisor, whence $z=0$. In fact,

2.1. Proposition. *Any homogeneous element of degree less than the degree of the generators of a homogeneous ideal I can not be in I^* unless it is nilpotent.*

If R is not reduced, any element $z \in I^*$ having degree smaller than the degrees of the generators of I must be nilpotent, since z is in the tight closure of the image of I modulo every minimal prime, so that the preceding arugument shows that z is zero modulo every minimal prime.

We now prove a much harder fact.

2.2. Theorem. *Let R be a normal graded domain finitely generated over a perfect field $K = R_0$. Let I be an ideal generated by forms all of degree at least δ. If z has degree δ and is in I^*, then z must be in I itself.*

This theorem has an amusing corollary (which we challenge the reader to prove by elementary methods):

2.3. Corollary. *Let K be a perfect field with algebraic closure $\overline{K}$ and suppose that R is a normal $\mathbf{N}$-graded domain over $R_0 = K$. Let Z be any form of positive degree in the graded domain $\overline{K} \otimes_K R$. Then the ring $R[Z]$ is not normal unless Z is in R.*

Proof. Suppose $Z = \lambda_1 m_1 + \lambda_2 m_2 + \cdots \lambda_t m_t$ where $\lambda_i \in \overline{K}$ and $m_i \in R$ are forms all of the same degree. Denoting $R[Z]$ by S, we see that $Z \in (m_1, m_2, \cdots, m_t) S^+ \cap S \subset (m_1, \cdots, m_t) S^*$. By Theorem 2.2, $Z \in (m_1, m_2, \cdots, m_t) S$, whence Z is a K combination of the m_i. This forces Z to be in R.

Before embarking on the proof of Theorem 2.2, we first give a technical improvement. Recall that a K-algebra R is said to be *geometrically reduced* if $R \otimes_K K^\infty$ is reduced, where K^∞ is the perfect closure of K. Since the induced map on spectra is an isomorphism, it follows that if R is a geometrically reduced domain, then in fact $R \otimes_K K^\infty$ is also a domain.

2.4. Theorem. *Let R be a geometrically reduced graded domain over the field $K = R_0$. Assume that K is algebraically closed in the fraction field of R. Let I be an ideal generated by forms all of degree at least δ. If z has degree δ and is in I^*, then z must be in I itself.*

2.4.1. Remark. The two hypotheses above, that R is geometrically reduced and that K is algebraically closed in fraction field Frac (R) together are equivalent to the assumption that $K \subset$ Frac (R) is a regular extension, where Frac (R) denotes the fraction field of R [ZS II, p226, p230]. Recall that an extension of fields $K \subset L$ is said to be a *regular extension* if

(1) L is separable over K; and

(2) K is algebraically closed in L.

Recall also that the extension $K \subset \mathrm{Frac}(R)$ is a regular extension if and only if $R \otimes_K L$ is a domain for *every* field extension L of K [ZS II p230]. Of course, if R is a finitely generated normal domain over a perfect field, then $K \subset \mathrm{Frac}(R)$ is trivially a regular extension.

Proof of 2.4. The map $R \to R \otimes_K K^\infty = S$ is faithfully flat and S is again a domain. The inclusion $z \in I^*$ in R persists into S. If the theorem holds in S, then $z \in IS \cap R = I$ by faithfully flatness. Note that if $R_0 = K$ is algebraically closed in the fraction field of R, then $S_0 = K^\infty$ is algebraically closed in the fraction field of S. Indeed, suppose that some $\frac{a}{b} \in U^{-1}S$ where $U \subset S$ is the multiplicative system of all non-zero-divisors on S satisfies an equation of integral dependence over K^∞. This equation has the form

$$(\frac{a}{b})^N + r_1 (\frac{a}{b})^{N-1} + \cdots + r_N = 0, \tag{2.4.2.}$$

where each $r_i \in K^\infty$. Because each element of S has a q^{th} power in R, equation 2.4.2 gives an equation

$$(\frac{a^q}{b^q})^N + (r_1)^q (\frac{a^q}{b^q})^{N-1} + \cdots + r_N^q = 0, \tag{2.4.3.}$$

of integral dependence of $\frac{a^q}{b^q}$ over K. Because K is algebraically closed in the fraction field of R, we conclude that $\frac{a^q}{b^q}$ is in $K = R_0$, where $\frac{a}{b} \in K^\infty$. We may therefore assume that without loss of generality that $R_0 = K$ is perfect.

We will reduce to R normal, whence Theorem 2.2 will apply. Let S denote the normalization of R. Note that $S_0 = R_0 = K$. For if $a/b \in S_0$, then a/b would satisfy a homogeneous equation of integral dependence over R. The coefficients of this equation would all necessarily lie in $R_0 = K$, whence a/b would lie in $R_0 = K$ because of the assumption that K is a algebraically closed in the fraction field of R.

Now suppose we have an inclusion $z \in (\mu_1, \cdots, \mu_n)^*$ in R where the μ_i and z all have the same degree. This equation persists after expanding to S, where all the hypotheses of 2.2 hold. We conclude that $z \in (\mu_1, \cdots, \mu_n)S$. But then

$$z = \alpha_1 \mu_1 + \alpha_2 \mu_2 + \cdots + \alpha_n \mu_n,$$

where the α_i, are homogeneous elements of degree zero in S. But then the α_i are actually in R, and we conclude that $z \in (\mu_1, \cdots, \mu_n)R$.

The proof of Theorem 2.2 will ocuupy most of this section. The technique will be to consider the ring

$$D(R) = \bigcup_{e \in \mathbf{N}} \mathrm{Hom}_{R^{p^e}}(R,R).$$

Note that the elements of R that are q^{th} powers from a subring R^q of R, over which we may consider R as a module. The ring R^q is also a graded algebra over the perfect field $R_0^q = K$, and the R^q modules $\mathrm{Hom}_{R^q}(R,R)$ are graded R^q modules. Since $R^q \subset R^Q$ for all $q \geq Q$, we have that

$$\mathrm{Hom}_{R^Q}(R,R) \subset \mathrm{Hom}_{R^q}(R,R)$$

in a natural degree preserving way, so that the ring $D(R)$ inherits natural $\mathbf{Z}$ grading. These inclusions also give us a natural (increasing) filtration of R by graded subrings which we call the Frobenius filtration on $D(R)$.

The ring R is a left $D(R)$ module in an obvious way: each element $\theta \in \mathrm{Hom}_{R^q}(R,R)$ acts on R. The reason for the notation $D(R)$ is that this is actually the ring of K linear differential operators on R (see [Y]).

Although it is possible to conduct the ensuing analysis without explicit reference to differential operators, this interesting connection is worth pointing out; it may eventually yield future insight into tight closure. Some connections between tight closure and the structure of a ring as a left module over its ring of differential operators are discussed in [S4].

We now established several lemmas.

2.5. Lemma. *Let A be a normal Noetherian domain (of characteristic $p>0$) with fraction field L, and let $u \in L-\{0\}$. If $u \in \bigcap_{e \in \mathbf{N}} L^{p^e}$, then u is a unit of A.*

Proof. Since u and $\frac{1}{u}$ satisfy the same hypothesis, it is enough to show that $u \in A$.

Because A is normal, we know

$$A = \bigcap_{\substack{\mathcal{P} \in \mathrm{Spec} A \\ \mathrm{ht}\mathcal{P}=1}} A_{\mathcal{P}}.$$

If $u \notin A$, we can therefore choose some height one $\mathcal{P} \in \mathrm{Spec} A$ such that $u \notin A_{\mathcal{P}}$. Denote by ν the valuation associated to the discrete valuation ring $A_{\mathcal{P}}$. By hypothesis, we know that for each $e \in \mathbf{N}$, there exists some $v_e \in L$ such that $u = v_e^q$, where $q = p^e$. It follows that $\nu(u) = q\nu(v_e)$ for all q, whence $\nu(u)$ is divisible by all $q = p^e$. This is impossible unlessh $\nu(u) = 0$, whence $u \in A_{\mathcal{P}}$.

2.6. Lemma. *Let $R = \bigoplus_{i \in \mathbf{N}} R_i$ be a normal graded $K = R_0$ algebra, where K is a perfect field of characteristic p. Suppose that α and β are two forms of R of the same degree, linearly independent over K. Then there exists a (homogeneous) differential operator*

$$\theta \in D(R)$$

for which $\theta(\alpha)=0$ *but* $\theta(\beta)\neq 0$.

Of course, the element $\theta \in D(R)$ arising in Lemma 2.6 will be in $\mathrm{Hom}_{R^q}(R,R)$ for all $q \gg 0$.

Proof. To ease the notation, fix any $e \in \mathbf{N}$, and let S be the subring R^q of R, where $q=p^e$. Since R is a domain, R is a torsion-free S module. Moreover, R is a finitely generated S module, as one easily checks from the fact that R is finitely generated over a perfect field. Therefore, in order to prove the existence of θ, it suffices to find such a θ after tensoring with the fraction field F of S. That is, (abusing notation slightly idetifying R with its image in $F\otimes_S R$), we seeky

$$\phi \in \mathrm{Hom}_F(F\otimes_S R, F\otimes_S R)$$

such that $\phi(\alpha)=0$ but $\phi(\beta)\neq 0$.

As $F\otimes_S R$ is a finite dimensional F vector space, there is an F module splitting

$$\phi\colon F\otimes_S R \to F$$
$$\phi\colon \beta \mapsto 1.$$

The map ϕ projects the space spanned by β onto F, so that we can choose ϕ such that $\phi(\alpha)=0$ unless $\alpha \in F\beta$.

Therefore, the only way for Lemma 2.6 to fail is if for every $q=p^e$, the fraction $\frac{\beta}{\alpha}$ is in the fraction field of R^q. But according to Lemma 2.5, if $\frac{\beta}{\alpha}$ is in the fraction field of R^q for all q, then $\frac{\beta}{\alpha}$ is a unit in R, whence $\frac{\beta}{\alpha}\in K$ and α and β are not linearly independent over K after all.

We can now prove the main theorem of this section.

Proof of Theorem 2.2. We first use Lemma 2.6 to conclude something even stronger about differential operators on R. Let α be any homogeneous element of R of degree d. We claim that the only degree d elements of R that are annihilated by all the differential operators annihilating α are the K-multiples of α. In other words, the K vectorspace

$$\mathrm{Ann}_R(\mathrm{Ann}_{D(R)}\alpha)\cap R_d \subset R$$

is exactly $K\alpha$. (Here $\mathrm{Ann}_{D(R)}\alpha$ denotes that the left ideal in $D(R)$ of elements θ such that $\theta(\alpha)=0$; likewise for any left ideal $\mathcal{L}\subset D(R)$, $\mathrm{Ann}_R\mathcal{L}$ denotes the set of elements of R annihilated by every element of $\mathcal{L}$.)

To establish this claim, note that the Frobenius filtration on $D(R)$ induces a decreasing chain of K subvector spaces of R:

$$\mathrm{Ann}_R(\mathrm{Ann}_{D_1}\alpha)\supset \mathrm{Ann}_R(\mathrm{Ann}_{D_2}\alpha)\supset\cdots\supset \mathrm{Ann}_R(\mathrm{Ann}_{D(R)}\alpha),$$

where D_e denotes the subring $\mathrm{Hom}_{R^{p^e}}(R,R)$ of $D(R)$. Since the d forms of R

form a finite dimensional K vectorspace, this decreasing chain must eventually stablize in each particular degree. In degree d, this stable vectorspace, which clearly contains $K\alpha$, is $[\mathrm{Ann}_R(\mathrm{Ann}_{D(R)}\alpha)]_d$. Lemma 2.6 then implies that it is exactly $K\alpha$, for if β is any homogeneous element of degree d not in $K\alpha$, then Lemma 2.6 implies that some $\theta \in D(R)$ kills α but not β, so that

(2.6.1.) $$[\mathrm{Ann}_R(\mathrm{Ann}_{D(R)}\alpha)]_d = K\alpha.$$

We now use this fact to prove Theorem 2.2. Assume that the theorem is false and choose an ideal I generated by *the minimal possible number* n of elements, $\mu_1, \mu_2, \cdots, \mu_n$ (all degree δ), such that the ideal I contradicts the theorem. Note that n is at least 2, since all principal ideas are tightly closed in a normal ring.

Observe that in order to prove theorem 2.2, there is no loss of generality in assuming that K is infinite. For instance, K can be replaced by the field extension $L = K(t)^{\infty}$, the perfect closure of $K(t)$. All hypothesis are preserved upon passing to the faithfully flat extension $R \otimes_K L$. And because $R \subset R \otimes_K L$ is faithfully flat, for any ideal $I \subset R$, we have that $I(R \otimes_K L) \cap R = I$.

Let w be a form of degree δ which is in I^* but not I. There exists a homogeneous element c such that for all q,

(2.6.2.) $$cw^q = a_{1q}\mu_1^q + a_{2q}\mu_2^q + \cdots + a_{nq}\mu_n^q,$$

for some homogeneous $a_{iq} \in R$, necessarily of degree equal to the degree of c, which we assume to be d.

First suppose that for some fixed q, each $a_{iq} = k_{iq}c$ for some $k_{iq} \in K$. In this case, we could divid out c from equation (2.6.2) to conclude that

(2.6.3.) $$w^q = k_{1q}\mu_1^q + k_{2q}\mu_2^q + \cdots + k_{nq}\mu_n^q,$$

whence $w = k_{1q}^{\frac{1}{q}}\mu_1 + k_{2q}^{\frac{1}{q}}\mu_2 + \cdots + k_{nq}^{\frac{1}{q}}\mu_n \in I$, as K is perfect. Thus, we may assume that for each q, some a_{iq} in not in the K-span of c.

Choose a homogeneous $\theta \in D(R)$ vanishing on c but not vanishing on any d-form not in the one-dimensional vectorspace Kc. This is possible, assuming K is infinite, by 2.6.1 above. In particular, for all $q \gg 0$, at least one of the elements $\theta(a_{1q}), \theta(a_{2q}), \cdots, \theta(a_{nq})$ in non-zero. Thus for some fixed index i, we must have that $\theta(a_{iq})$ is non-zero, for infinitely many q. Changing notation if necessary, say $i=1$.

For all $q \geq Q$, we apply θ to equations (2.6.2.) to get

(2.6.4.) $$0 = \theta(a_{1q})\mu_1^q + \theta(a_{2q})\mu_2^q + \cdots + \theta(a_{nq})\mu_n^q,$$

and then rearrange to conclude that

$$\theta(a_{1q})\mu_1^q \in (\mu_2^q, \cdots, \mu_n^q); \quad \theta(a_{1q}) \neq 0$$

for infinity many $q = p^e$. Since $\theta(a_{1q})$ has fixed degree $(= d + \mathrm{degree}\theta)$, we conclude, via Proposition 1.2, that $\mu_1 \in (\mu_2, \cdots \mu_n)^*$. Our minimality assumption thus forces μ_1 into $(\mu_2, \cdots, \mu_n) R$, whence I was not minimally generated by the n elements $\mu_1, \mu_2, \cdots, \mu_n$, as assumed.

It is worth noting that the assumption that R is geometrically reduced is necessary in Theorem 2.4. Otherwise, we could obviously have a relation as in equation (2.6.3.) holding, so that w would be in $(\mu_1, \cdots, \mu_n) R^{\frac{1}{q}} \cap R \subset (\mu_1, \cdots, \mu_n)^*$ but not in $(\mu_1, \cdots, \mu_n)$ itself. In this case, however, $w - k_{1q}^{\frac{1}{q}}\mu_1 + k_{2q}^{\frac{1}{q}}\mu_2 + \cdots + k_{nq}^{\frac{1}{q}}\mu_n$ is nilpotent in $R \otimes_K K^{\infty}$.

Although Theorem 2.4 can be used to help compute tight closure in non-domains (*c.f.* 1.1.1), the theorem itself is quite false when R is not a domain. If $R = \frac{k[x,y]}{(xy)}$, the element $x \in (x-y)^*$ but $x \in (x-y) R$, as is easily verified by killing the minimal primes.

We prove a similar result that is useful for computing tight closures in practice. Theorem 2.7 can be deduced easily from Theorem 2.2, but we supply a different proof using the existence of test elements.

2.7. Theorem. *Let R be a domain finitely generated over a finite field K, and assume that K is algebraically closed in the fraction field of R. Let I be a homogeneous ideal of R. Fix an integer δ and write I as $I = I_1 + I_2$ where I_1 is the ideal generated by all the elements of I whose degree is strictly less than δ, and I_2 is the ideal $I \cap \bigoplus_{i \geq \delta} R_i$ of elements of I of degree δ or more. Then*

$$[I^*]_\delta = [I_1^*]_\delta + [I_2]_\delta.$$

That is, any element of degree δ in I^* is actually in the (a priori smaller) ideal $I_1^* + I_2$.

Proof. We first reduce to the case where R is a normal domain. Let S denote the normalization of R; recall from the proof of Theorem 2.4 that $S_0 = R_0) = K$. Suppose that $z \in R$ has degree δ and that $z \in R$ has degree δ and that $z \in (I_1 + I_2)^*$ in R. This also holds in S, so assuming the result for normal domains, we have $z \in (I_1 S)^* + I_2 S$. Write z as $z' + w$ where $z' \in (I_1 S)^*$ and $w \in I_2 S$ are homogeneous. Since all the generators of I_2 have degree at least as large as the degree of z, w is a $K = S_0 = R_0$ combination of the generators of I_2. This means that $w \in R$ so $z' \in R$. It is easy to check that in general, $(IS)^* \cap R = I^*$ for any ideal of a Noetherian domain R and any integral extension domain S of R. Thus $z' \in I_1^*$ in R. This argument shows that we can assume without loss of generality that R is normal.

Now because R is a normal garded domain, the defining ideal for its non-regular locus is a homogeneous ideal of height at least two. Because R is normal, we can find a regular sequence c_1, c_2 contained in this height two ideal. Replacing the c_i by powers, if necessary, the elements c_1 and c_2 may be assumed to be test elements for R, by [HH3, 6.2], since R_{c_i} is a regular ring. This means that if $z \in I^*$, then $c_i z^q \in I^{[q]}$ for all q.

Suppose that $z \in I^*$ where I is generated by forms $\mu_1, \mu_2, \cdots, \mu_k$. Using the same sort of arguments described at the beginning of this section, we see that if μ_i has degree strictly greater than δ, then the homogeneous tight closure equations

$$cz^q = a_{1q}\mu_1^q + a_{2q}\mu_2^q + \cdots + a_{kq}\mu_k^q$$

reveal that the coefficients a_{iq} of μ_i^q must all be zero for large q. Thus we are immediately reduced to the case where I is generated by homogeneous elements μ_i all of degree less than or equal to δ.

Assume that z is not in the tight closure of any ideal generated by a proper subset of the μ_i's, and that some μ_i, say μ_1, has degree δ. For each q, we have a homogeneous equations

$$c_1 z^q = a_{1q}\mu_1^q + a_{2q}\mu_2^q + \cdots + a_{kq}\mu_k^q,$$
$$c_2 z_q = b_{1q}\mu_1^q + b_{2q}\mu_2^q + \cdots + b_{kq}\mu_k^q,$$

where a_{1q}, b_{1q} has the same degree as c_i. For each q, we multiplying the first of these equations by b_{1q} and the second by a_{1q} to achieve an equation

$$(c_1 b_{1q} - c_2 a_{1q}) z^q \in (\mu_2^q, \cdots, \mu_k^q).$$

There are several cases to consider. First, if $c_1 b_{1q} - c_2 a_{1q}$ is not zero, then because the degree of this element is constant in q, we see that $z \in (\mu_2, \cdots, \mu_k)^*$ as in the proof of 2.2. In this case the proof is now complete by induction.

On the other hand, suppose that $c_1 b_{1q} - c_2 a_{1q}$ is zero. Consider indices $1 \leq i \leq r$ where μ_i are the generators for I_i, all of degree the same as the degree of z. If any coefficient $a_{iq} b_{1q} - b_{iq} a_{1q}$ is non-zero for infinitely many q, then we have

$$(a_{iq} b_{1q} - b_{iq} a_{1q}) \mu_i^q \in (\mu_1^q, \cdots, \hat{\mu}_i, \cdots, \mu_k^q),$$

so that $\mu_i \in (\mu_1, \cdots, \hat{\mu}_i, \cdots, \mu_k)^*$. This says that $z \in (\mu_1, \cdots, \hat{\mu}_i, \cdots, \mu_k)^*$, and again we are done by induction on k.

Finally, we must consider the case where for all i, $1 \leq i \leq r$,

(2.7.1.) $$a_{iq} b_{1q} = b_{iq} a_{1q}$$

and also $c_1 b_{1q} = c_2 a_{1q}$. Because c_1, c_2 form a regular sequence and have the same degree, we can find $\lambda_q \in K$ such that $b_{1q} = \lambda_q c_2$ and $a_{1q} = \lambda_q c_1$. Plugging these into the equations 2.7.1 and again using the fact that c_1, c_2 form a regular sequence, find that we can write each a_{iq} as $\lambda_{iq} c_1$ for some $\lambda_{iq} \in K$. We get equations of the form

$$c_1 (z^q - \lambda_{1q}\mu_1^q - \cdots - \lambda_{rq}\mu_r^q) = a_{r+1q}\mu_{r+1}^q + \cdots + \alpha_{kq}\mu_k^q.$$

Thus $c_1 (z - \lambda_{1q}^{1/q}\mu_1 - \cdots - \lambda_{rq}^{1/q}\mu_r)^q$ is in $I_2^{[q]}$ for infinitely many q. Now, assuming that K is finite, there must be some r-tuples of elements in K, $\lambda_1, \cdots, \lambda_r$ that appears infinitely often among the r-tuples $-\lambda_{1q}^{1/q}, \cdots, -\lambda_{rq}^{1/q}$, so we get an element $z+\lambda_1\mu_1+\cdots+\lambda_r\mu_r\in I_2^*$. We conclude that z is in $I_1+I_2^*$, and the proof is complete.

Remark. Presumably, the assumption that K is finite above is unnecessary, but I do not know an argument for an arbitrary perfect field K.

3. Elements forced into the tight closure

In this section we record some useful observations about the tight closure of homogeneous, m primary ideals. Let I be an m primary ideal of a graded algebra (R,m). Assume that I is generated by forms all of degree δ. Of course, since I^* is also m primary, every element of sufficiently high degree in R will be in I^*. For computational purposes, it is useful to know explicitely what "sufficiently high" is. In this section we derive bounds on this degree.

The method is to use the Briançon-Skoda theorem. The Briançon-Skoda theorem asserts that the integral closure of the n^{th} power of an n-generated ideal I is contained in I^*; see [HH1]. It is quite elementary to prove directly from the definitions.

This method produces even stronger results. Namely, we produce elements in the *plus closure* of I, not just the tight closure. Given any domain R, the ring R^+ denotes the integral closure of R in an algebraic closure of its fraction field. If R happens to be graded, we may consider a homogeneous version: the subring R^{+gr} of R^+ consisting of all those elements which can be considered homogeneous of integral order in the sense that they satisfy an integral polynomial of the form $U^N+r_1U^{N-1}+r_2U^{N-2}+\cdots+r_N$, where r_i is homogeneous of degree id, for some non-negative integer d. It is easy to see that $IR^+\cap R\subset I^*$ (resp. $I^{+gr}\cap R\subset I^*$ in the graded case) for all ideals, but it has been a long standard open question whether or not the converse is true. This is known to be the case for ideals generated by part of a system of parameters (in an excellent local or graded domain) [S1], [S2]. All of the elements we produce in the tight closures of ideals in this section are actually, as we show, in the plus closure, further evidence (though far from a proof) that $I^*=IR^{+gr}\cap R$ for all ideals.

3.1. Proposition. *Let $I\subset R$ be an m-primary ideal generated by forms of degree less than or equal to δ in the graded algebra R of dimension d. Then any element z of degree greater than or equal to $d\ \delta$ is in the ideal I^*. If R is a domain, and $R_0=K$ is infinite, then $z\in IR^{+gr}\cap R$. If R is not a domain, but is equidimensional, then this holds modulo every minimal prime of R.*

Proof. Suppose that I is generated by the forms $\mu_1,\mu_2,\cdots\mu_k$, each of degree at most δ. Let S be the graded subring $K[\mu_1,\mu_2,\cdots,\mu_k]\subset R$. Since I is m-primary, the extension $S\rightarrow R$ is integral. Any $z\in R$ therefore satisfies a homogeneous equation of integral dependence:

(3.1.1.) $$z^N+a_1z^{N-1}+a_2z^{N-2}+\cdots+a_{N-1}z+a_N$$

over S. Since the equation is homogeneous, the degree of a_i is equal to $i\times\deg z$, so that $\deg a_i\geq id\delta$. Since a_i is a polynomial in the μ_i, this implies that

$$a_i\in(\mu_1,\cdots,\mu_k)^{id}.$$

It follows that Equation 3.1.1 forces

$$z\in\overline{(\mu_1,\cdots,\mu_k)^d},$$

the integral closure of the d^{th} power of the ideal I. By the Briançon-Skoda theorem (Theorem 5.1 of [HH1]), we conclude that $z\in I^*$.

Asume that R is a domain. The assumption that $K=R_0$ is infinite guarantees that I has a reduction generated by a homogeneous system of parameters, $x_1,\cdots,x_d$, for R. (The assumption that K is infinite is unnecessary if this is otherwise known to be the case.) Thus

$$z\in\overline{(\mu_1,\cdots,\mu_k)^d}=\overline{(x_1,\cdots,x_d)^d}$$

which puts $z\in(x_1,\cdots,x_d)^*$. By [S2], $z\in(x_1,\cdots,x_d)R^{+gr}\cap R$.

If R is equidimensional, the $x_1,\cdots,x_d$ will be a system of parameters module every minimal prime, so that the result holds modulo each minimal prime as claimed.

The next Lemma gives a better result when R is generated by a system of parameters. This result will be used to prove the same bound even when the ideal is not generated by a system of parameters.

3.2. Lemma. *Let $x_i,\cdots,x_d$ be a homogeneous system of parameters for the graded ring R and assume that R is equidimensional. If $\deg z\geq\sum_{i=1}^d\deg x_i$, then $z\in(x_1,\cdots,x_d)^*$. Consequently, $z\in(x_1,\cdots,x_d)S\cap R$, where S is some graded integral extension domain of $\frac{R}{P}$ where P is any minimal prime of R.*

Proof. We prove the final statement first: if $z\in(x_1,\cdots,x_d)^*$, why is $z\in(x_1,\cdots x_d)S\cap R$? The reason is that z will be in the tight closure of the ideal $(x_1,\cdots,x_d)\frac{R}{P}$, for every minimal prime of R, and the image ideal is still generated by a system of parameters. We then use [S3], to conclude that $z\in(x_1,\cdots,x_d)R^{+gr}\cap R$, so that $z\in(x_1,\cdots,x_d)S\cap R$, where S is some graded integral extension domain of $\frac{R}{P}$. Therefore, the proof of the proposition is complete, once we have shown that $z\in(x_1,\cdots,x_d)^*$.

If the x_i all have the same degree, this is just a special case of the previous theorem. We reduce to this case.

Let t_i be the degree of x_i and let $t=\prod_{i=1}^d t_i$. Consider the product

$$w=x_1^{\frac{t}{t_1}-1}x_2^{\frac{t}{t_2}-1}\cdots x_d^{\frac{t}{t_d}-1}z$$

and the ideal

$$I=(x_1^{\frac{t}{t_1}-1},x_2^{\frac{t}{t_2}-1},\cdots,x_d^{\frac{t}{t_d}-1})R.$$

The ideal I is generated by elements all of degree t and the element w has degree

$$\deg z+\sum_{i=1}^{d} t_i(t/t_i-1)=\deg z+\sum_{i=1}^{d}(t-t_i).$$

Since $\deg z\geq\sum_{i=1}^{d}(t_i)$, we see that $\deg w\geq td$, and therefore $w\in I^*$. We then use the "colon-capturing" properties of tight closure to conclude that $z\in(x_1,\cdots,x_d)^*$. (See [HH1, Thorem 1.15a] for the basic theorem on colon-capturing.)

3.2.1. Remark. An alternative (but less elementary) proof of Lemma 3.2 has become part of the "folklore" of tight closure and uses the machinery local cohomology and the notion of tight closure for modules. The tight closure of the zero module in the local cohomology module $H_m^d(R)$ of an equidimensional excellent local ring (R,m) of dimension d is well-understood: according to [S1, 2.5], it is precisely the largest proper submodule of $H_m^d(R)$ which is stable under the natural action of Frobenius on $H_m^d(R)$. In the graded case, the Frobenius action on $H_m^d(R)$ multiplies degrees by p, so that the non-negatively graded part of the local $H_m^d(R)$ is stable under the Frobenius action, and hence contained in the tight closure of zero. That is, $[H_m^d(R)]_{\geq 0}\subset 0^*_{H_m^d(R)}$. By using the Koszul complex on a system of parameters $x_1,\cdots,x_d$ to represent elements of $H_m^d(R)$, elements of $0^*_{H_m^d(R)}$ may be represented by inclusions of elements $z\in(x_1,\cdots x_d)^*$. The fact that $[H_m^d(R)]_{\geq 0}\subset 0^*_{H_m^d(R)}$ translates into the statement of Lemma 3.2. This is carefully explained in Section 2 of [S1].

There are two advantages to using the Briançon-Skoda theorem to prove Lemma 3.2. The obvious one is that it avoids the introduction of the technical tools of local cohomology, tight closure of modules in an overmodule, and the re-interpretation of properties of local cohomology in terms of properties of parameter ideals. But more importantly, in addition to its simplicity, this argument immediately generalizes to arbitrary m-primary ideals, as the next result shows.

3.3. Proposition. *Let R be a graded ring and suppose that I is any m-primary ideal of R generated by the homogeneous elements $\mu_1,\mu_2,\cdots,\mu_k$. Let z be any form of degree greater than or equal to $\sum_{i=1}^{k}\deg\mu_i$. Then $z\in I^*$. In fact, if R is a domain, then $z\in IR^{+gr}\cap R$, where R^{+gr} denotes the graded integral closure of R in an algebraic closure of its fraction field.*

Proof. In light of the remarks following Definition 1.1, we may assume that R is a domain.

Let S be the graded subalgebra of R generated over K by $\mu_1,\mu_2,\cdots,\mu_k$. Since the ideal I is m-primary, R is integral over S, so that z satisfies a homogeneous equation of integral dependence of the form

$$z^N+a_1z^{N-1}+a_2z^{N-2}+\cdots+a_{N-1}z+a_N=0$$

where each a_j is a polynomial in the μ_i's of degree $j\times\deg z$.

Let $U_1,U_2,\cdots,U_k$, Z be indeterminates and define the quotient ring

$$T=\frac{K[U_1,U_2,\cdots,U_k,Z]}{Z^N+A_1Z^{N-1}+\cdots+A_N}$$

where A_j is the same polynomial in the U_i's that a_j is in the μ_i's. Note that T is a graded Cohen-Macaulay K-algebra, with the degree of U_i defined to be the same as the degree of μ_i, and the degree of Z defined to be equal to the degree of z. The elements $U_1,U_2,\cdots,U_k$ clearly form a *s.o.p.* for T, whence it follows from the previous corollary that

$$Z\in(U_1,U_2,\cdots,U_k)^* \quad \text{in} \quad T.$$

Because T is equidimensional, the images of the elements $U_1,U_2,\cdots,U_k$ are a system of parameters in the quotient $\overline{T}$ of T by any of its minimal primes. Identifying elements of T with their images in $\overline{T}$, we thus have that $Z\in(U_1,U_2,\cdots,U_k)^*$ in $\overline{T}$. If we denote by $\overline{T}^{gr+}$ the graded integral closure of $\overline{T}$ in an algebraic closure of its fraction field, we recall that

$$(U_1,U_2,\cdots,U_k)\overline{T}^*=(U_1,U_2,\cdots,U_k)\overline{T}^{+gr}\cap\overline{T}$$

(This is the main theorem of [S3]; the non-graded version is Theorem 5.1 of [S2]).

We have an obvious map of T to R sending Z to z and U_i to μ_i. Since R is a domain, the map passes to a well defined map of $\overline{T}$ to R. The map $\overline{T}\to R$ extends to a map $\overline{T}^{+gr}\to R^{+gr}$. Therefore the inclusion $Z\in(U_1,U_2,\cdots,U_k)\overline{T}^{+gr}\cap\overline{T}$ maps to an inclusion

$$z\in(\mu_1,\mu_2,\cdots,\mu_k)R^{+gr}\cap R.$$

Since $JR^{+gr}\cap R\subset J^*$ for all ideals J of an arbitrary domain R, we conclude that

$$z\in(\mu_1,\mu_2,\cdots,\mu_k)^* \quad \text{in} \quad R.$$

This completes the proof.

4. Applications

The result of preceding two sections have been useful to the author in computing tight closure of homogeneous ideals, or for simply understanding better the structure of a graded ring. The results in this paper are not an algorithm for computing tight closure! However, because they give partial information about the tight closure of ideals in graded rings and bound the degree of elements that must be checked, in practice they are helpful. Because tight closure can be useful for determining whether a particular ring has rational singularities or is Cohen-Macaulay, these results may also be used as a tool for these purposes.

Example: Two Dimensional Rings. Let R be any standard normal ring of dimension 2 over a perfect field. Suppose that x,y is a linear system of parameters (*i.e.* of degree 1). Then

$$(x,y)^* = (x,y) + \bigoplus_{i\geq 2} R_i,$$

the ideal generated by all forms of degree 2 and the original elements x and y. Indeed, from Proposition 3.3, one sees that all homogeneous elements of degree two or more are in $(x,y)^*$, whereas no element of degree one not already in (x,y) can be in $(x,y)^*$, by Theorem 2.2.

From this example, we gain insight into the structure of two dimensional graded *F-rational* rings. In particular, we see that the ideal (x,y) is tightly closed if and only if $R_2 \subset (x,y)R$. In this case, the multiplicity of the ideal (x,y) is $\delta - 1$, and the Hilbert Function of R is $H(n) := \text{length}_K(R_n) = n(\delta-1)+1$, where $\delta = \text{length}_K(R_1)$ is the embedding dimension of R. In this case the Hilbert function agrees with its Hilbert polynomial right from $n=0$. This example may be interpreted as a tight closure analogue of M. Artin's results for rational singularities; see Theorem 4 of [Ar]. The reason for this similarity is illuminated by Corollaries 4.3 and 4.4 below.

Even in higher dimensions, one obtains similar (but not complete) information for standard graded rings that are F-rational. For example, in dimension 3, we would see that $R_3 \subset (x,y,z)R_2$, where x,y,z is a system of parameters all of degree one. We see again that the Hilbert function agrees with its polynomial right from the start, so it is completely determined by its values at $n=1$ and 2.

We now apply the results of the previous two sections to the study of F-rational and pseudorational rings. Recall that a ring is F-rational if all parameter ideals are tightly closed. For a local (or graded) domain, this is equivalent to the property that some ideal generated by a (homogeneous) system of parameters is tightly closed. Pseudorationality is a characteristic-free analog of rational singularities. For the formal definition, see [LT].

Recall that the a-invariant [GW] of a graded ring (R,m) is the integer

$$a(R) = \max_{n\in\mathbf{Z}}\{[H_m^{\dim R}(R)]_n \neq 0\}.$$

4.1. Theorem. *Let R be graded Cohen-Macaulay domain over a field K such that the fraction field of R is a regular extension of K, and assume that R has* *a system of parameters consisting of one-forms (at least after possibly extending the ground field K). If the a-invariant $a(R) \leq 1-d$ where is the dimension of R, then R is F-rational.*

4.1.1. Remark. The assumption that R has a system of parameters of degree one after extending the ground field is always satisfied for any standard graded domain. The assumption that the fraction field of R is a regular extension of K is equivalent to the assumption that R remains a domain after any extension of the ground field K, and is therefore trivially satisfied when K is algebraically closed. See 2.4.1.

Proof. Let $x_1,\cdots,x_d$ be a system of parameters (*s.o.p.*) of one-forms. (If necessary, we make the faithfully flat base change $R \subset L \otimes_K R$ extending K to a field extension L; all hypothesis, as well as the presumed failure of the conclusion, are preserved). We need only check that the ideal generated by $x_1, \cdots, x_d$ is tightly closed. Suppose that z is a homogeneous element in $(x_1,\cdots,x_d)^*$ but not in $(x_1,\cdots,x_d)R$. Recall that $H^d_m(R)$ is isomorphic to the direct limit $\varinjlim_t \frac{R}{(x_1^t,\cdots,x_d^t)}$ where the maps are given by multiplication by the products of the x_i's. When R is Cohen-Macaulay, these maps are injective, and the element $\eta = [z+(x_1,\cdots,x_d)] \in H^d_m(R)$ is non-zero. Thus η necessarily has degree less than or equal to $1-d$, which forces the degree of z to be less than or equal to 1. Theorem 2.2 then implies that $z \notin (x_1,\cdots,x_d)^*$ unless it is already in $(x_1,\cdots,x_d)R$.

Theorem 4.1 has an amusing consequence.

4.2. Corollary. *A graded Cohen-Macaulay domain R over an algebraically closed field which admits a system of parameters of degree one is normal if $a(R) \leq 1-d$.*

Proof. F-rational rings are normal.

4.3. Corollary. *Let R be a graded Cohen-Macaulay domain over a field K such that the fraction field of R is a regular extension of K, and assume that R has a system of parameters of degree one (at least after possibly extending the fround field). Then*

(1) *If $a(R) \leq 1-d$, where d is the dimension of R, then R is pseudorational.*
(2) *If $a(R) \leq 1-d$ and the ground field is of characteristic zero, then R has rational singularities.*

Proof. This is an immediate consequence of Theorem 4.1 above and the main theorems of [S1], that locally excellent F-rational rings are pseudorational (in char p) and that F-rational type algebras have rational singularities (in characteristic zero).

4.3.1. Remark. If R is actually *generated* by its elements of degree one, then the assumption that $a(R) \leq 1-d$ is well understood. Indeed, in this case, $a(R)$ is at least $-d$, and it is exactly $-d$ if and only if R is a polynomial ring. If $a(R)=1-d$, then the degree of the projective scheme Proj$R$ is precisely $n-d+1$, where n is the dimension of $[R]_1$ (which is one more than dimension of the ambient projective space). These so-called *varieties of minimal degree* are completely classified: they consist of quadric hypersurfaces, cones over Veronese surfaces, and rational normal scrolls [Ha, p48]. The cones over all of these varieties are easily checked to have rational singularities. Indeed, their coordinate rings are all either quadric hypersurfaces (which are Gorenstein and F-rational by 4.1) or toric varieties (which are direct summands of regular rings), so in fact, these rings have the property that *all* ideals are tightly closed in all characteristics.

This corollary immediately yields the following.

4.4. Corollary. *A two-dimensional graded algebra of prime characteristic which admits a system of parameters of degree one is F-rational if and only if it is pseudorational. A two dimensional graded algebra of characteristic zero which admits a system of parameters of degree one is F-rational type if and only if it has rational singularities.*

Proof. It suffices to verify the characteristic p statement. The characteristic zero analog is explained in [S1].

F-rational implies pseudorational in general [S1], so suppose that R is a two dimensional pseudorational ring. Pseudorationality is preserved upon tensoring with the infinite field extension $L=K(X)$, where X is an indeterminate. Therefore, it suffices to show that $R\otimes_K L$ is F-rational, since $R\rightarrow R\otimes_K L$ is faithfully flat. We henceforth assume that R is graded over R_0, an infinite field.

Let x_1, x_2 be a homogeneous *s.o.p.* for R of degree one. Since R is pseudorational, it is normal and Cohen-Macaulay, with $a(R)<0$ [FW]. Therefore, $a(R)\leq 1-2$, and R is F-rational by Corollary 4.1.

The equivalence of pseudorationality and F-rationality has been studied both by Fedder and by Watanabe. In particular, Fedder proves that two dimensional graded rings with rational singularities are F-rational (in characteristic zero) using different methods.

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