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Remarks on L^2-wellposed Cauchy problem for some dispersive equations

By

Shigeo TARAMA

1. Introduction and results

We consider the Cauchy problem with data on line $t=0$ for the following operator A defind by

$$Au(t,x)=\frac{\partial}{\partial t}u(t,x)+\frac{\partial^3}{\partial x^3}u(t,x)+a(x)\frac{\partial}{\partial x}u(t,x)+b(x)u(t,x) \qquad (1.1)$$

with the complex-valued coeffcients $a(x)$ and $b(x)$ belonging to the space B^∞ consisting of all bounded smooth functions whose derivative of any order is also bounded on real line $\mathbf{R}$.

If the coefficients $a(x)$ and $b(x)$ are constant, we see by Fourier transformation that, when the imaginary part of the coefficent $a(x)$ is not zero, the Cauchy problem for A is not L^2-wellposed.

This implies that the Cauchy problem for A is not always L^2-wellposed. Indeed we see by the construction of asymptotic solutions that the following condition on the imaginary part of the coefficent $a(x)$, which is denoted by $a_I(x)$: there exists a constant K such that we have for any x and $y\in\mathbf{R}$

$$\left|\int_x^y a_I(s)\,ds\right|\leq K|x-y|^{\frac{1}{2}} \qquad (\mathrm{N})$$

is necessary for L^2-wellosedness.

Our main interest is the sufficiency of (N). We show in this paper that the condition (N) implies L^2-wellposedness.

Now we formulate the Cauchy problem. Let T be some given positive number. For given functions $g(x)$ and $f(t,x)$ find a solution $u(t,x)$ satisfying

$$\begin{cases} Au(t,x)=f(t,x) & \text{on } [0,T]\times\mathbf{R} \\ u(0,x)=g(x) & \text{on } \mathbf{R} \end{cases} \qquad (\mathrm{C})$$

Let X be a subspace of the space of temperate distributions on $\mathbf{R}$. We say that the above problem (C) is X-wellposed if for any $g(x)\in X$ and $f(t,x)$

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belonging to the space consisting of X-valued continuous functions on $[0. T]$, which is denoted by $C([0, T], X)$, there exists one and only one solution $u(t, x) \in C([0, T], X)$ satisfying following estimates: for any continuous semi-norme $\rho(\cdot)$ in X, there is a continuous semi-norm $\rho_1(\cdot)$ such that for any $t \in [0, T]$

$$\rho(u(t, x)) \leq C\Big(\rho_1(g(x)) + \int_0^t \rho_1(f(s, x))ds\Big), \tag{1.2}$$

where the constant C is independent of $g(x)$, $f(t, x)$, and t.

Proposition 1.1. *The condition* (N) *is necessary for L^2-wellposedness of the Cauchy problem* (C).

Proof. We follow the method of S. Mizohata [4]. We assume that the condition (N) is not satisfied and the Cauchy problem (C) is still L^2-wellposed. Then we draw some contradiction. If the condition (N) is not satisfied, for any integer $n>0$ there exist $y_{n,1}$ and $y_{n,2}$ satisfying $y_{n,2}-y_{n,1} \to +\infty$ as $n \to +\infty$ and

$$\left|\int_{y_{n,1}}^{y_{n,2}} a_I(s)ds\right| \geq n|y_{n,1}-y_{n,2}|^{\frac{1}{2}}.$$

On the other hand, if the Cauchy problem is L^2-wellposed, we have form (1.2), for a solution $u(t, x)$ of the problem (C)

$$\|u(t, x)\| \leq C\Big(\|g(x)\| + \int_0^t \|f(s, x)\|ds\Big), \tag{1.3}$$

where $\|v(x)\| = (\int_{-\infty}^{+\infty} |v(x)|^2 dx)^{\frac{1}{2}}$.

Let $u(t, x)$ be given by

$$u(t, x) = \exp\Big(it\xi^3 + ix\xi - \frac{i}{3\xi}\int_x^{x+3t\xi^2} a(y)dy\Big)|\xi|^{-\frac{1}{2}} v_0\Big(\frac{x-x_0+3t\xi^2}{|\xi|}\Big),$$

where $v_0(x)$ is a non-negative smooth funcition satisfying

$$v_0(x) = 0 \quad \text{for } |x| \geq 1$$

and

$$\int_{-\infty}^{+\infty} |v_0(x)|^2 dx = 1,$$

on the other hand ξ and x_0 are real constants to be specified later. Then we have $\|u(0, x)\| = 1$ and

$$Au(t, x) = \exp\Big(it\xi^3 + ix\xi - \frac{i}{3\xi}\int_x^{x+3t\xi^2} a(y)dy\Big) \times$$
$$\Big\{(a'(x+3t\xi^2) - a'(x) + b(x))|\xi|^{-\frac{1}{2}} v_0\Big(\frac{x-x_0+3t\xi^2}{|\xi|}\Big) + R(t, x, \xi)\Big\} \tag{1.4}$$

where the last term $R(t, x, \xi)$ satisfies for $|\xi|\geq 1$

$$|R(t, x, \xi)|\leq C_0\frac{1}{|\xi|}\sum_{k=0,1,2,3}|\xi|^{-\frac{1}{2}}\left|v_0^{(k)}\left(\frac{x-x_0+3t\xi^2}{|\xi|}\right)\right|,$$

with some positive constant C_0

According to the properties of $y_{n,1}$ and $y_{n,2}$, we can take ξ and x_0, which depend on n, as follows

$$x_0-\frac{3\xi^2}{n}=y_{n,1} \text{ and } x_0=y_{n,2}$$

and

$$\frac{1}{\xi}\int_{x_0-\frac{3\xi^2}{n}}^{x_0} a_I(y)\,dy\geq\sqrt{3n}.$$

For such ξ and x_0. let t_0 be an element in $[0, \frac{1}{n}]$ that maximizes the function $\frac{1}{\xi}\int_{x_0-3t\xi^2}^{x_0}a_I(y)\,dy$ on $[0, \frac{1}{n}]$. Then on $[0,t_0]$ $\|Au(t, x)\|$ is estimated by

$$C_1 \exp\left(\frac{1}{3\xi}\int_{x_0-3t_0\xi^2}^{x_0}a_I(y)\,dy\right)\left(1+C_2\frac{1}{|\xi|}\right),$$

for

$$\left|\exp\left(it\xi^3+ix\xi-\frac{i}{3\xi}\int_x^{x+3t\xi^2} a(y)\,dy\right)\right|=\exp\left(\frac{1}{3\xi}\int_x^{x+3t\xi^2} a_I(y)\,dy\right)$$

and on the support of $Au(t, x)$,

$$\frac{1}{3\xi}\int_x^{x+3t\xi^2} a_I(y)\,dy\leq\frac{1}{3\xi}\int_{x_0-3t\xi^2}^{x_0}a_I(y)\,dy+\frac{2}{3}\max_{x\in\mathbf{R}}|a_I(x)|.$$

Hence

$$\|u(0, x)\|+\int_0^{t_0}\|Au(s, x)\|ds\leq 1+C_1t_0 \exp\left(\frac{1}{3\xi}\int_{x_0-3t_0\xi^2}^{x_0}a_I(y)\,dy\right)\left(1+C_2\frac{1}{|\xi|}\right). \tag{1.5-a}$$

On the other hand, by similar arguments,

$$\|u(t_0, x)\|\geq\exp\left(\frac{1}{3\xi}\int_{x_0-3t_0\xi^2}^{x_0}a_I(y)\,dy-\frac{2}{3}\max_{x\in\mathbf{R}}|a_I(x)|\right). \tag{1.5-b}$$

Since

$$\frac{1}{3\xi}\int_{x_0-3t_0\xi^2}^{x_0}a_I(y)\,dy\geq\sqrt{\frac{n}{3}}$$

and

$$t_0\leq\frac{1}{n} \text{ and } |\xi|=\sqrt{\frac{n}{3}}\sqrt{y_{n,2}-y_{n,1}}\longrightarrow+\infty \text{ as } n\longrightarrow+\infty,$$

these estimates (1.5-a and -b) and the inequality (1.3) contradict each other for a large n. For $u(t, x)$ is a solution of the Cauchy problem (C) with $g(x)=u(0, x)$ and $f(t, x)=Au(t, x)$

Remark. Since the solution of

$$\frac{\partial}{\partial t}v(t, x)-3\xi^2\frac{\partial}{\partial x}v(t, x)=a'(x+3t\xi^2)-a'(x)+b(x)$$

$$v(0, x)=0$$

is given by

$$v(t, x)=ta'(x+3t\xi^2)-\frac{1}{3\xi^2}\left(-a(x)+a(x+3t\xi^2)-\int_x^{x+3t\xi^2}b(s)ds\right),$$

we can eliminate the term $a'(x+3t\xi^2)-a'(x)+b(x)$ in (1.4) but the term $-3it\xi a^{(3)}(x+3t\xi^2)$ appears if $u(t, x)$ is replaced by $e^{-v(t,x)}u(t, x)$. On the other hand we must consider the estimate of (1.4) in the relatively large time interval which contains $\frac{C}{|\xi|}$ with a large C in order that the term $\frac{1}{3\xi}\int_x^{x+3t\xi^2}a_I(y)dy$ becomes effecitve in the constructed solution. Thus it is difficult to improve the estimate.

Our main result is the following.

Theorem 1.2. *If the imaginary part of the coefficient $a(x)$ of A satisfies the condition* (N) *then the Cauchy problem* (C) *is L^2-wellposed.*

The proof is given in the next section.

Now we explain the notation used in the follwing section. The inner product $(\cdot,\cdot)$ of L^2 is defined by $(v(x), w(x))=\int_{-\infty}^{+\infty}v(x)\overline{w(x)}dx$.

We use the function space $H_{(k)}$ with $k\geq 0$ which is a space consisting of all of $u(x)\in L^2$ satisfying that

$$\|u(x)\|_{(k)}=\sqrt{\int_{-\infty}^{+\infty}(1+\xi^2)^k|\hat{v}(\xi)|^2d\xi}$$

is finite where $\hat{v}(\xi)$ is the Fourier transformation of $v(x)$ given by $\hat{v}(\xi)=\int_{-\infty}^{+\infty}e^{-i\xi x}v(x)dx$. Then $H_{(k)}$ with the norm $\|u(x)\|_{(k)}$ is a Banach space. Plancherel's theorem implies that $L^2=H_{(0)}$. By $\mathcal{S}$ we denote the space of all $f(x)\in B^\infty$ such that $x^kf(x)\in B^\infty$ for any $k\in\mathbf{N}$ where $\mathbf{N}$ is the set of all non-negative integers.

We use the symbol class S^m, which is the set of symbols with a parameter $l\geq 1$ $a_l(x, \xi)$ such that

$$\left|\frac{\partial^{j+k}}{\partial x^j\partial\xi^k}a_l(x, \xi)\right|\leq C_{j,k}(l+|\xi|)^{m-k}$$

for any $j,k \in \mathbf{N}$, and $l \geq 1$ and any x, $\xi \in \mathbf{R}$, where the constants $C_{j,k}$ are independent of l.

For a symbol $a_l(x, \xi)$, we denote by $a_l(x, D)$ p.d.o., this is to say, the peudodifferential operator, defined by

$$a_l(x, D)u(x) = \frac{1}{2\pi}\int \exp(ix\xi)a_l(x, \xi)\hat{u}(\xi)d\xi.$$

We say that the order of a p.d.o. $a_l(x, D)$ is m, if $a_l(x, \xi)$ is in S^m. For the calculus and properties of p.d.o. see H. Kumano-go [2]

The symbo $<\xi>_l$ denote $(\xi^2+l^2)^{\frac{1}{2}}$.

Concerning the constants, they mey be different in the different formulas even if the same letters are used.

2. The proof of Theorem 1.2

From now on we assume that the hypothesis of Theorem 1.2, that is to say, the condition (N) is satisfied.

We define the function $\Phi(x, \xi)$ by

$$\Phi(x, \xi) = \frac{-\xi}{3<\xi>_l^2}\int_{-\infty}^{x}\chi\left(\frac{y-x}{<\xi>_l^2}\right)a_l(y)dy \tag{2.1}$$

with a smooth function $\chi(x)$ satisfying

$$\chi(x) = \begin{cases} 1, & \text{for} \quad |x| \leq 1 \\ 0, & \text{for} \quad |x| \geq 2 \end{cases}.$$

Then we have the following.

Propsition 2.1.

$$\Phi(x, \xi) \in S^0, \tag{2.2}$$

$$\frac{\partial}{\partial x}\Phi(x, \xi) + \frac{\xi}{3<\xi>_l^2}a_l(x) \in S^{-2} \tag{2.3}$$

Proof. We have from the definition

$$\frac{\partial}{\partial x}\Phi(x, \xi) = -\frac{\xi}{3<\xi>_l^2}a_l(x) + \frac{\xi}{3<\xi>_l^4}\int_{-\infty}^{x}\chi'\left(\frac{y-x}{<\xi>_l^2}\right)a_l(y)dy,$$

more generally, since $\chi^{(k)}(0) = 0$ for $k > 0$.

$$\frac{\partial^k}{\partial x^k}\Phi(x, \xi) = -\frac{\xi}{3<\xi>_l^2}a_l^{(k-1)}(x) +$$
$$(-1)^{k+1}\frac{\xi}{3<\xi>_l^{2(l+k)}}\int_{-\infty}^{x}\chi^{(k)}\left(\frac{y-x}{<\xi>_l^2}\right)a_l(y)dy,$$

for $k\geq 1$. Hence the assertions (2.2) and (2.3) follow from the following lemma.

Lemma 2.2. *For any compacty supported smooth function $\psi(x)$ the function $F(x, \xi)$ defined by*

$$F(x, \xi)=\int_{-\infty}^{x}\psi\left(\frac{y-x}{<\xi>_l^2}\right)a_I(y)\,dy,$$

satisfies

$$|F(x, \xi)|\leq C<\xi>_l$$

where the constant C does not depend on $l\geq 1$.

Proof. Noting

$$\frac{\partial}{\partial y}\int_x^y a_I(w)\,dw=a_I(y),$$

we obtain by the integration by parts

$$F(x, \xi)=-\int_{-\infty}^{x}\psi'\left(\frac{y-x}{<\xi>_l^2}\right)\frac{1}{<\xi>_l^2}\left(\int_x^y a_I(w)\,dw\right)dy.$$

It follows from (N) that on the support of $\psi'(\frac{y-x}{<\xi>_l^2})$

$$\left|\int_x^y a_I(w)\,dw\right|\leq C<\xi>_l.$$

Thus

$$\begin{aligned}|F(x, \xi)|&\leq C<\xi>_l\int_{-\infty}^{+\infty}\left|\psi'\left(\frac{y-x}{<\xi>_l^2}\right)\right|\frac{1}{<\xi>_l^2}dy\\ &\leq C<\xi>_l.\end{aligned}$$

We denote by $e^{\Phi(x,D)}$ [resp. $e^{-\Phi(x,D)}$] the p.d.o. whose symbol is $e^{\Phi(x,\xi)}$ [resp. $e^{-\Phi(x,\xi)}$]. Then we see from (2.2) that

$$e^{\Phi(x,D)}e^{-\Phi(x,D)}=I+R_1(x, D),$$

and

$$e^{-\Phi(x,D)}e^{\Phi(x,D)}=I+R_2(x, D),$$

where $R_1(x, D)$ and $R_2(x, D)$ are p.d.o of order -1. By the definition of the symbol classes we see by choosing a large l that for $j=1$ and 2

$$\|R_j(x, D)v(x)\|\leq\frac{1}{2}\|v(x)\|,$$

and

$$\| R_j(x, D) v(x) \|_{(3)} \leq \frac{1}{2} \| v(x) \|_{(3)}$$

for any $v(x) \in \mathcal{S}$ (see H. Kumano-go [2, Ch. 2 § 4.]).
Therefore by choosing a large l we see that $e^{\Phi(x,D)}$ is an automorphism in L^2 and $H_{(3)}$. In the following we take and fix such l Then we put

$$E(x, D) = e^{\Phi(x,D)}$$

and its inverse will be denoted by $E^{-1}(x, D)$.

We define the operators G and G_1 whose domain is $H_{(3)}$ by

$$G = -\frac{\partial^3}{\partial x^3} - a(x) \frac{\partial}{\partial x} - b(x),$$

and

$$G_1 = -\frac{\partial^3}{\partial x^3} - a_R(x) \frac{\partial}{\partial x},$$

where $a_R(\mathrm{x})$ is a real part of $a(x)$. Then

Proposition 2.3.

$$GE(x, D) = E(x, D) G_1 + B(x, D)$$

where $B(x, D)$ is a L^2-bounded operator.

Proof.

In the following, we write $P_1 \equiv P_2$ if the difference of the operotors P_1 and P_2, $P_1 - P_2$ is a L^2-bounded operator.

The (2.3) implies that $\frac{\partial}{\partial x} \Phi(x, \xi) \in S^{-1}$. Thus

$$\left[\frac{\partial^3}{\partial x^3}, E(x, D) \right] \equiv 3P(x, D) E(x, D)$$

where the symbol of $P(x, D)$ is $-\xi^2 \frac{\partial}{\partial x} \Phi(x, \xi)$. The (2.3) implies aiso that

$$i a_I(x) \frac{\partial}{\partial x} \equiv -3P(x, D).$$

Therefore we see that

$$\left(\frac{\partial^3}{\partial x^3} + i a_I(x) \frac{\partial}{\partial x} \right) E(x, D) \equiv E(x, D) \frac{\partial^3}{\partial x^3}$$

On the other hand, since $i a_R(x) \xi \in S^1$, $[a_R(x) \frac{\partial}{\partial x}, E(x, D)] \equiv 0$. Hence, by noting $b(x) E(x, D) \equiv 0$, we obtain the assertion of Proposition 2.3.

Since for any $v(x) \in S$ and any real λ

$$\Re(\lambda v(x) - G_1 v(x), v(x)) \geq (\lambda - C)\|v(x)\|^2,$$

we obtain

$$\|\lambda v(x) - G_1 v(x)\| \geq (\lambda - C)\|v(x)\|,$$

and

$$\|\lambda v(x) - G_1^* v(x)\| \geq (\lambda - C)\|v(x)\|,$$

where G_1^* is the formal adjoint of G_1. On the other hand the ellipticity of G_1 implies that, if $v(x) \in L^2$ satisfes $G_1 v(x) \in L^2$, then $v(x) \in H_{(3)}$. Hence we see that G_1 is a generator of a C^0 semi-group on L^2 (see for example S. Mizohata [3. Ch. 6. See. 4.] or S. Tarama [5]). Thanks to Proposition 2.3, we have

$$G = E(x, D)(G_1 + E^{-1}(x, D)B(x, D))E^{-1}(x, D),$$

where $E^{-1}(x, D)B(x, D)$ is a L^2-bounded operator. Since the operator $G_2 = G_1 + E^{-1}(x, D)B(x, D)$ is also a generator of a C^0 semi-group $\exp(tG_2)$ on L^2 (see E. Davis [1, Ch. 3. See. 1.), the operator G is a generator of the C^0 semi-group $E(x, D)\ \exp(tG_2)\ E^{-1}(x, D)$ on L^2.

Therefore we see that the Cauchy problem (C) is $H_{(3)}$ wellposed. That is to say, when $g(x) \in H_{(3)}$ and $f(t, x) \in C([0. T], H_{(3)})$, there exists one and only solution $u(t, x) \in C([0. T], H_{(3)})$, satisfying $\frac{\partial}{\partial t}u(t, x) \in C([0. T], L^2)$, of the Cauchy problem (C) and this solution $u(t, x)$ satisfies

$$\|u(t, x)\| \leq C_1(\|g(x)\| + \int_0^t \|f(s, x)\| ds).$$

The estimate above implies the existence of a solution $u(t, x)$ of the (C) with $g(x) \in L^2$ and $f(t, x) \in C([0. T], L^2)$.

By using the arguments above we can show also that under the condition (N) the backward Cauchy problem for the adjoint operator:

$$\begin{cases} A^* u(t, x) = f(t, x) & \text{on } [0, T] \times \mathbf{R} \\ u(T, x) = g(x) & \text{on } \mathbf{R} \end{cases}$$

where

$$A^* = -\frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^3} - \overline{a(x)}\frac{\partial}{\partial x} - \overline{a'(x)} + \overline{b(x)},$$

is also $H_{(3)}$-wellposed. Hence we see that the uniqueness of solutions in $C([0. T], L^2)$ to the problem (C). Thus the proof of Theorem 1.2 is completed.

DEPARTMENT OF APPLIED MATHEMATICS AND PHYSICS
FACULTY OF ENGINEERING
KYOTO UNIVERSITY

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