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On dynamics of hyperbolic rational semigroups

By

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1. Introduction

For a Riemann surface S, let $\mathrm{End}(S)$ denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of functions. A *rational semigroup* is a subsemigroup of $\mathrm{End}(\bar{\mathbf{C}})$ without any constant elements. Similarly, an *entire semigroup* is a subsemigroup of $\mathrm{End}(\mathbf{C})$ without any constant elements. A rational semigroup G is called a *polynomial semigroup* if each $g \in G$ is a polynomial. When a rational or entire semigroup G is generated by $\{f_1, f_2, \dots f_n, \dots\}$, we denote this situation by

$$G = \langle f_1, f_2, \dots f_n, \dots \rangle,$$

The rational or entire semigrop generated by a single function g is denoted by $\langle g \rangle$. We denote the n-th iterate of f by f^n.

The study of rational semigroups is a generalization of the study of Kleinian groups, iteration of rational functions and systems of contraction maps related to self- similar sets in $\mathbf{C}$ in fractal geometry. D. Sullivan pointed out that there are many points of similarity between Kleinian groups and iteration of rational functions in [Sul]. In view of the study of rational semigroups, we can show some basic results similar between Kleinian groups and iteration of rational functions. For example, limit sets of Kleinian groups, Julia sets of rational functions and self-similar sets in $\mathbf{C}$ are *Julia sets* of rational semigroups. By Lemma 1.1.5.6, which is a result by A. Hinkkanen and G. J. Martin, the fixed points are dense in these sets. Several properties of dynamics of rational semigroups have been shown in [ZR], [GR], [HM1], [HM2], [S1] and [S2]. In 1992, the first study was investigated by W. Zhou and F. Ren ([ZR]). In 1996, the study of infinitely generated semigroup of meromorphic functions was investigated by Z. Gong and F. Ren ([GR]). In 1996, A. Hinkkanen and G. J. Martin studied about nearly abelian rational semigroups ([HM1]). They showed that Julia sets of finitely generated rational semigroups are uniformly perfect ([HM2]).

In this paper, we use the notations in [HM1], [HM2], [S1] and [S2]. We

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will show the following results. The Julia sets of finitely generated rational semigroups have the backward self-similarity (Lemma 1.1.4). If the hyperbolic rational semigroup is finitely generated and satisfies some conditions, the limit functions of the semigroup on the Fatou set are only constant functions that take their values on postcrical set (Theorem 2.2.8). When the generators of a finitely generated hyperbolic rational semigroup are perturbed, the hyperbolicity is kept and the Jilia set depends cotinuously on the generators of the semigroup (Theorem 2.4.1). Furthermore, if the finitely generated rational semigroup is hyperbolic and if the inverse images by the generators of the Julia set are mutually disjoint, then the Julia set moves by holomorphic motion (Theorem 2.4.1).

Because of the backward self-similarity, if the postcritical set is included in a Fatou component, then the Julia set has a property which is like usual self-similarity (Theorem 2.5.1), and moreover, if the inverse images by the generators of the Julia set are mutually disjoint, then the Julia set is a Cantor set (Theorem 2.5.2).

In [S3], it is shown that the hyperbolicity and the expandingness are equivalent if the semigroup is finitely generated and contains an element with the degree at least two. In that paper, the study of a construction of conformal measures and Hausdorff dimension of Julia sets of hyperbolic rational semigroups will be given. The study of generalized Brolin-Lyubich's invariant measures and estimates of Hausdorff dimension of Julia sets will be given in [S4].

The author will discuss about the existance and uniqueness of conformal measures and self-similar measures in more general cases in [S5].

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1.1. preliminaries

Definition 1.1.1. Let G be a rational semigroup.

$$F(G) \overset{\text{def}}{=} \{z \in \bar{\mathbf{C}} \mid G \text{ is normal in a neighborhood of } z\}$$

$$J(G) \overset{\text{def}}{=} \bar{\mathbf{C}} \setminus F(G)$$

$F(G)$ is called Fatou set for G and $J(G)$ is called Julia set for G. Similarly. Fatou set and Julia set for entire semigroup are defined.

Definition 1.1.2. Let G be a rational semigroup and z a point of $\bar{\mathbf{C}}$. The backward orbit $O^{-}(z)$ of z and the set of exceptional points $E(G)$ are defined by:

$$O^-(z) \stackrel{\text{def}}{=} \{w\in\bar{\mathbf{C}} \mid \text{there is some } g\in G \text{ such that } g(w)=z\},$$

$$E(G) \stackrel{\text{def}}{=} \{z\in\bar{\mathbf{C}} \mid \# O^-(z) \le 2\}.$$

Definition 1.1.3. A subsemigroup H of a semigroup G is said to be finite index if there is a finite collection of elements $\{g_1, g_2, \ldots, g_n\}$ of G such that $G=\cup_{i=1}^n g_iH$. Similarly we say that a subsemigroup H of G has cofinite index if there is a finite collection of elements $\{g_1, g_2, \ldots, g_n\}$ of G such that for every $g\in G$ there is a $j\in\{1, 2, \ldots, n\}$ such that $g_jg\in H$.

Lemma 1.1.4. *Let G be a rational semigroup.*

1. For any $f\in G$.

$$f(F(G))\subset F(G),\ f^{-1}(J(G))\subset J(G),$$
$$F(G)\subset F(\langle f\rangle),\ J(\langle f\rangle)\subset J(G)$$

2, *If* $G=\langle f_1, \ldots, f_n\rangle$, *then*

$$F(G)=\cap_{i=1}^n f_i^{-1}(F(G)),\ J(G)=\cup_{i=1}^n f_i^{-1}(G))$$

Proof. By definition, it is easy to show 1. We show 2. By 1,

$$F(G)\subset \cap_{i=1}^n f_i^{-1}(F(G)).$$

Now take any point $z_0\in \cap_{j=1}^n f_j^{-1}(F(G)))$ and set $w_j=f_j(z_0)\in F(G)$.

For any $\varepsilon>0$, there is some $\delta>0$ such that if $g\in G$, $1\le j\le n$, and $d(w, w_j) <\delta$, then

$$d(g(w), g(w_j))<\varepsilon.$$

For this δ, there is some $\eta>0$ such that if $d(z, z_0)<\eta$ then

$$d(f_j(z), f_j(z_0)<\delta,\ j=1, \ldots, n.$$

So if $g\in G$, $1\le j\le n$, and $d(z, z_0)<\eta$ then

$$d(gf_j(z), gf_j(z_0))<\varepsilon.$$

G is equal to $\cup_{j=1}^n\{G\cup\{id\}\}\circ f_j$, so G is equicontinuous at z_0, and

$$\cap_{j=1}^n f_j^{-1}(F(G))\subset F(G).$$

If a set K satisfies that $K=\cup_{i=1}^n f_j^{-1}(K)$, we say that K has backward self-similarity.

Next lemma was shown in [HM1], [ZR].

Lemma 1.1.5. *Let G be a rational semigroup.*

1. *If a subsemigroup H of G is of finite or cofinite index, then*

$$J(H)=J(G).$$

In particular, when G is rational semigroup generated by finite elements $\{f_1, f_2, \ldots f_n\}$ *and m is an integer, if we set*

$$H_m=\{g=f_{j_1}..f_{j_k}\in G | m \text{ devides } k\},$$
$$I_m=\{g\in G | g \text{ is a product of some elements of word length } m\}$$

then

$$J(G)=J(H_m)=J(I_m).$$

Here we say an element $f\in G$ *is of word length m if m is the minimum integer such that*

$$f=f_{j_1}..f_{j_m}.$$

2. *If* $J(G)$ *contains at least three points, then* $J(G)$ *is a perfect set.*

3. *If there is an element* $g\in G$ *such that* $\deg(g)\geq 2$ *or there is an element* $g\in G$ *such that* $\deg(g)=1$ *and the order of g is infinite, then*

$$E(G)=\{z\in\bar{\mathbf{C}} | \# O^-(z)<\infty\},\quad \# E(G)\leq 2.$$

4. *If a point z is not in* $E(G)$, *then for every* $x\in J(G)$, *x belongs to* $O^-(z)$. *In particular if a point z belongs to* $J(G)\setminus E(G)$, *then*

$$\overline{O^-(z)}=J(G).$$

5. *If there is an element* $g\in G$ *such that* $\deg(g)\geq 2$ *or there is an element* $g\in G$ *such that* $\deg(g)=1$ *and the order of g is infinite and* $J(G)$ *contains at least three points, then* $J(G)$ *is the smallest closed backward invariant set containing at least three points. Here we say that a set A is bacward invariant under G if for each* $g\in G$, $g^{-1}(A)\subset A$.

6. *If* $J(G)$ *contains at least three points, then*

$$J(G)=\overline{\{z\in\bar{\mathbf{C}} | z \text{ is a repelling fixed point of some } g\in G\}}$$

Proof. [HM1].

Remark. A similar result of *6* for entire semigroups can also be stated.

Proposition 1.1.6. *Let* $\{Q_\lambda\}$ *be a family of polynomials that are not of degree one and G a polynomial semigroup generated by* $\{Q_\lambda\}$.

If a transformation $\sigma(z)=\mu z+\gamma\in \mathrm{Aut}\mathbf{C}$, $\mu=\exp(\frac{2\pi i}{k})$, $k\in\mathbf{N}$ *satisfies for every* λ

$$\sigma(J(\langle Q_\lambda\rangle))=J(\langle Q_\lambda\rangle),$$

then

$$\sigma(J(G))=J(G).$$

Proof. For every polynomial Q that is not of degree one, $J(Q)$ is completely invariant under a transformation $z \mapsto (\exp(\frac{2\pi i}{k}))(z)$ if and only if $Q = az^d P(z^k)$, where P is a polynomial, a is a number, and d is an integer ([Bel]). So it is easy to see the stagement using Lemma 1.1.5.6.

Example 1.1.7. For a regular triangle $p_1 p_2 p_3$, we set $g_i(z) = 2(z - p_i) + p_i$, $i = 1,2,3$. And let G be a rational semigroup generated by $\{g_i\}$, not as a group. Then $J(G)$ is the Sierpiński Gasket.

2. Dynamics of hyperbolic rational semigroups

2.1. Limit Functions. First, we will give some comments about limit functions of semigroups. The study of limit functions plays a very important role in the study of complex dynamical systems. The forward invariant domains of iteration of rational functions are classified into five types by the limit functions ([Bel], [Mi]).

Let S be a hyperbolic Riemann surface, S_∞ the one point compactification of S, and H a subsemigroup of End(S).

Definition 2.1.1.

$\overline{\mathscr{L}}_H(S) \overset{\text{def}}{=} \{\varphi : S \to S_\infty |$there is a sequence (g_j) of mutually distinct elements of H such that $g_j \to \varphi$ locally uniformly on S as $j \to \infty\}$.

Remark. Every family A of elements of End(S) contains a sequence that converges to an element of End(S) or ∞. ([Mi]).

Lemma 2.1.2. *Let S be a hyperbolic Riemann surface and H a subsemigroup of* End(S). *If $g \in H$ is non-constant and φ belongs to $\overline{\mathscr{L}}_H(S)$, then $\varphi g \in \overline{\mathscr{L}}_H(S)$. Moreover if φ also belongs to* End(S), *then $g\varphi \in \overline{\mathscr{L}}_H(S)$.*

Proof. Let φ be an element of $\overline{\mathscr{L}}_H(S)$. There is a sequence (f_j) of mutually distinct elements of H such that $f_j \to \varphi$. Then the sequence $(f_j g)$ converges to φg and $\{f_j g\}$ are mutually distinct because g is non-constant. By definition φg belongs to $\overline{\mathscr{L}}_H(S)$.

Next assume φ also belongs to End(S). The sequence (gf_j) converges to $g\varphi$. We will show $\{gf_j\}$ contains infinitely many elements that are mutually distinct. For each number i, j, we set

$$C_{ij} = \{z \in S | f_i(z) = f_j(z)\}, \ C = \cup_{i \neq j} C_{ij}.$$

C is a countable set and we can take a point x of S which does not belong to C. Then $\{f_j(x)\}$ are mutually distinct and the sequence $(f_j(x))$ converges to $\varphi(x) \in S$. Now assume that there exists a subsequence (j_k) of (j) such that $j_k \to \infty$ as $k \to \infty$ and all elements of $\{gf_{j_k}\}$ are equal to an element $h \in$ End(S). Then for each k, $gf_{j_k}(x) = g\varphi(x)$ and this is a contradiction because g is nonconstant. So $\{gf_j\}$ contains infinitely many elements that are mutually distinct. By definition, it follows that $g\varphi$ belongs to $\bar{\mathscr{L}}_H(S)$.

Lemma 2.1.3. *Let S be a hyperbolic Riemann susrface and H a finitely generated subsemigroup of* End(S). *If there is a non-costant element $\varphi \in \bar{\mathscr{L}}_H(S)$, then at least one of these assersions is true:*

1. $\mathrm{Id}_S \in \bar{\mathscr{L}}_H(S)$ *and there is a generator $g_0 \in H$ such that g_0 is injective on S.*

2. There is a sequence (b_j) of elements of H such that for every j there is an element $h_j \in H$ such that $b_{j+1} = h_j b_j$ and (b_j) converges to ∞ locally uniformly on S.

Proof. We fix a generator system $\{g_1, \ldots, g_k\}$ of H. There is a sequence (f_j) of mutually distinct elements of H such that $f_j \to \varphi$ and word length of f_j strictly increases as $j \to \infty$. We represent each f_j by its reduced word. We take a subsequence $(f_{1,j})$ of (f_j) as follows. There is a generator g_{i_1} of H such that for each j

$$f_{1,j} = \cdots \circ g_{i_1}.$$

Inductively when we get a sequence $(f_{n,j})_j$, we take a subsequence $(f_{n+1,j})_j$ of it as follows. There is a generator $g_{i_{n+1}}$ of H such that for each j

$$f_{n+1,j} = \cdots \circ g_{i_{n+1}} \circ \cdots \circ g_{i_1}.$$

Now we get a sequence $(f_{n,n})_n$ and

$$f_{n,n} = \alpha_n \circ a_n, \text{ where } \alpha_n \in H,\ a_n = g_{i_n} \circ \cdots \circ g_{i_1}.$$

There are subsequences (α_{n_j}) of (α_n) and (a_{n_j}) of (a_n) and maps $\alpha, g : S \to S_\infty$ such that (α_{n_j}), (a_{n_j}) converge to α, g locally uniformly on S, respectively. Because $\{a_{n_j}\}$ are mutually distinct.

$$g \in \bar{\mathscr{L}}_H(S).$$

If g is not a constant, $g(S) \subset S$. If g is a constant, then $g = \infty$, for φ is not constant. In the former case, we can assume that for each j, there is an element $h_j \in H$ such that $\{h_j\}$ are mutually distinct, $a_{n_{j+1}} = h_j \circ a_{n_j}$, and h_j converges to a map h loally unformly on S as $j \to \infty$. Then $g = h \circ g$ and

$$h = Id_S.$$

We can also assume that there is a generator g_i such that for each j,

$$h_j = \cdots \circ g_i.$$

Then for $z, w \in S$, if we have $g_i(z) = g_i(w)$, then for each j, $h_j(z) = h_j(w)$ and so $z = w$. This implies that g_i is injective on S.

Next we define stable domains ([HM1]).

Definition 2.1.4. Let G be a rational semigroup and U a connected component of $F(G)$. We say that U is a stable domain if there is an element $g \in G \setminus \mathrm{Aut}\, \bar{\mathbf{C}}$ such that $g(U) \subset U$. And we set

$$G_U \overset{\text{def}}{=} \{g \in G \mid g(U) \subset U\}.$$

Similar definitions for entire semigroup can lso be given.

Defintion 2.1.5. Let U be a domain of $\bar{\mathbf{C}}$ and H a subsemigroup of $\mathrm{End}(U)$. Then we set

$\mathscr{L}_H(U) \overset{\text{def}}{=} \{\varphi : U \to \bar{U} \mid$ there is a sequence (g_j) of mutually distinct elements of H such that $g_j \to \varphi$ locally uniformly on U as $j \to \infty\}$.

Remark. If $g \in H$ is non-constant and φ belongs to $\mathscr{L}_H(U)$, then $\varphi \circ g \in \mathscr{L}_H(U)$. Moreover if φ also belongs to $\mathrm{End}(U)$, then $g \circ \varphi \in \mathscr{L}_H(U)$.

Now we consider a case such that there are only finitely many constant limit functions taking its value in a domain U. In this case $\mathscr{L}_H(U)$ has only finitely many elements.

Propositon 2.1.6. *Let G be a rational semigroup and U a subdomain of $F(G)$ and we set*

$$H = \{g \in G \mid g(U) \subset U\}, \ \mathscr{A} = \{\zeta \in U \mid \exists\, \varphi \in \mathscr{L}_H(U),\ \varphi \equiv \zeta\}.$$

If H is finitely generated and if $1 < \# \mathscr{A} < \infty$, then any $\varphi \in \mathscr{L}_H(U)$ is a constant map being its value $\in U$. And $M = H \cap \mathrm{Aut}(\bar{\mathbf{C}})$ has only finitely many elements.

Remarks. A similar result for entire semigroup also holds. And if we set

$$G = \langle z^2, e^{i\theta} z \rangle,\ \frac{\theta}{2\pi} \notin \mathbf{Q},\ U = \{|z| < 1\}.$$

then

$$\# \{\varphi \in \mathscr{L}_H(U) \mid \exists\, \zeta \in U,\ \varphi \equiv \zeta\} = 1,\ Id_U \in \mathscr{L}_H(U).$$

Next we consider a case such that there are infinitely many constant limit functions taking its value in a stable domain.

Proposition 2.1.7. *Let G be a rational (entire) semigroup, U a stable domain of G. We set*

$$H = G_U,$$
$$\mathscr{A} \overset{\text{def}}{=} \{\zeta \in U \mid \exists\, \varphi \equiv \mathscr{L}_H(U),\ \varphi \equiv \zeta\},\ \mathscr{B} \overset{\text{def}}{=} \{\zeta \in \bar{U} \mid \exists\, \varphi \in \mathscr{L}_H(U),\ \varphi \equiv \zeta\}.$$

If $\mathscr{A}$ has an accumulation point in U. then $\mathscr{B}$ is a perfect set, in particular an uncountable set.

Proof. First, it is easy to see that $\mathscr{B}$ is a closed subset of $\bar{U}$. Assume that $\mathscr{A}$ has an accumulation point in U and $\zeta \in \mathscr{B}$ is an isolated point. There is a sequence (g_j) of H converging to ζ locally uniformly on U. By our assumpsion $\mathscr{A}$ is not empty and take a point $x \in \mathscr{A}$. Then $g_j(x) \to \zeta$ as $j \to \infty$ and $g_j(x) \in \mathscr{A}$ by the remark after Definition 2.1.5. So ζ belongs to U, for it is an isolated point. Now $g_j(\zeta) \to \zeta$ as $j \to \infty$ and $g_j(\zeta) = \zeta$ for large enough j because ζ is isolated. Also for each compact set K. g_j maps K into a small disc about ζ for large enough j. It follows that for large enough j, the point ζ is an attracting fixed point of g_j. Take a large enough number j and set $g = g_j$. For each $y \in \mathscr{A}$ the sequence $(g^n(y))$ converges to ζ as $n \to \infty$. Because ζ is an isolated point, $g^n(y) = \zeta$ for each large enough n. So $\mathscr{A} \subset \cup_n g^{-n}\{\zeta\}$, and each point of $\mathscr{A}$ is isolated in U because $\{g^n\}$ is normal in U. This is a contradiction.

If $\mathscr{A}$ has infinitely many points and there is no accumulation point of $\mathscr{A}$ in U, then by the proof of Proposition 2.1.7, for any $\zeta \in \mathscr{A}$ there is an element g of H such that $\mathscr{A} \subset \cup_n g^{-n}\{\zeta\}$. It is a problem whether this situation can occur or not.

Conjecture 2.1.8. *If $\mathscr{A}$ has infinitely many points, then $\mathscr{A}$ has an accumulation point in U.*

If this conjecture is true, by Proposition 2.1.7, it implies the following conjecture.

Conjecture 2.1.9. *If $\mathscr{A}$ has infinitely many points, then $\mathscr{B}$ is a perfect set.*

Next we consider the nearly abelian semigroup in [HM1] and the limit functions as an example.

Defintion 2.1.10. Let G be a rational semigroup containing an element g with $\deg(g) \geq 2$. We say that G is nearly abelian if there is a compct family of Möbius (or linear fractional) transformations $\Phi = \{\varphi\}$ with the following properties.

- $\varphi(F(G)) = F(G)$ for all $\varphi \in \Phi$
- for all $f, g \in G$ there is a $\varphi \in \Phi$ such that $f\,g = \varphi g\,f$

Then by [HM1], if $g \in G$ is of degree at least two, then $J(G) = J(g)$. And it is also shown in [HM1] that in each stable domain U, the type of each element $g \in G_U$ such that $\deg(g)$ is at least two coincides. Here we define by the type of $g \in G_U$ the type of the connected component of $F(g)$ containing U.

Let X be a subset of $\mathbf{C}$ that is not a round circle. We set

$$G = \{g \mid g \text{ is a polynomial}, J(g) = X\}.$$

If G contains an element g such that $\deg(g)$ is at least two, then G is nearly abelian and we can take a family Φ of Definition 2.1.10 so that it contains only finitely many elements.

Proposition 2.1.11. *Let G be a nearly abelian rational semigroup, Φ the* *family in Definition* 2.1.10 *and* U *a stable domain. We set* $H=G_U$ *and* $\mathscr{B}=\{\zeta\in \bar{U}\mid \exists\ \varphi\in\mathscr{L}_H(U),\ \varphi\equiv\zeta\}$. *If* Φ *has only finitely many elements, then for any element* g *of* H,

$$\mathscr{B}\subset \bigcup_{n,m,\geq 1, n+m\leq \#\Phi+1} g^{-m}\{\text{fixed point of } g^n\},$$

in particular, $\mathscr{B}$ *has at most finitely many elements, Morover if* $\mathscr{B}$ *is not empty, either all points of* $\mathscr{B}$ *belong to* U *or all points of* $\mathscr{B}$ *belong to* ∂U.

Proof. Let α be an element of $\mathscr{L}_H(U)$. Then there is a sequence (g_j) of mutually distinct elements of H converging to α locally uniformly on U. Let g be any element of H. For every j there is an element $\varphi_j\in\Phi$ such that

$$gg_j=\varphi_j g_j g.$$

We can assume that (φ_j) converges to an element φ of Φ. Then

$$g\alpha=g\lim_{j\to\infty} g_j=\lim_{j\to\infty}\varphi_j g_j g=\varphi\alpha g.$$

If α is identically equal to a constant value $\zeta\in\bar{U}$, then

$$g(\zeta)=\varphi(\zeta).$$

There are some positive integers n, m with $n+m\leq \#\Phi+1$ such that $g^m(\zeta)$ is a fixed point of g^n. Now assume that $\mathscr{B}\cap U\neq\emptyset$ and $\mathscr{B}\cap\partial U\neq\emptyset$. Let x, y be points of $\mathscr{B}\cap U$, $\mathscr{B}\cap\partial U$ respectively. Then there is a sequence (h_j) of mustually distinct elements of H converging to y locally uniformly on U. The sequence $(h_j(x))$ converges to y as $j\to\infty$ and $h_j(x)$ belongs to $\mathscr{B}$ for each j, this implies that $\mathscr{B}$ has infinitely many elements.

Example 2.1.12. Let n be integer such that $n\geq 2$ and we set $f(z)=z^n+c$, $\sigma(z)=\exp(\frac{2\pi i}{n})z$, and $G=\langle f, \sigma f, \ldots, \sigma^{n-1}f\rangle$. Then G is nearly abelian. If $|c|$ is small enough, then 0 belongs to $F(G)$. Let U be the stable domain containing 0. Then

$$\mathscr{L}_H(U)=\{\sigma^j(z_0),\ j=0,\ldots,n-1\}.$$

where z_0 is an attracting fixed point of f in U and $\#\mathscr{L}_H(U)=n$. Also there is a number c such that each element of $\mathscr{L}_H(U)$ is a constant value of ∂U and $\#\mathscr{L}_H(U)=n$.

Example 2.1.13. Let m, n be integers greater than 1. We set $f(z)=z^m(z-c)$, $g(z)=z^n(z-c)+c$, $G=\langle f, g\rangle$. If $|c|$ is small enough, then 0 and c belong to the same connected componnt U of $F(G)$. Now $f(0), f(c)=0$ and $g(0)$, $g(c)=c$ and it implies that

$$\mathscr{L}_G(U)=\{\varphi_0, \varphi_c\},\ \text{where}\ \varphi_0\equiv 0,\ \varphi_c\equiv c.$$

Also G is not nearly abelian, for, the type of f in U is super attracting and different from that of g.

2.2. No wandering domains. Now we consider hyperbolic rational semigrpups.

Defintion 2.2.1. Let G be a rational semigroup. We set

$$P(G)=\overline{\bigcup_{g\in G}\{\text{critical values of } g\}}$$

and we say that G is hyperbolic if $P(G)\subset F(G)$.

Remark. In [S3], it will be shown that the hyperbolicity and the expandingness are equivalent if the semigroup is finitely generated and satisfies that it contains an elememt with the degree at least two and each Möbius transformation in it is not elliptic.

Definition 2.2.2. Let G a rational semigroup and U a component of $F(G)$. For every element of G, we denote by U_g the connected component of $F(G)$ containing $g(U)$. We say that U is a wandering domain if $\{U_g\}$ is infinite.

Theorem 2.2.3 *Let G a rational semigroup and U a wandering domain. Then there is a constant limit function φ of G on U taking its value ζ in $J(G)$.*

Proof. We have a sequence (g_j) in G such that it converges to a map φ locally uniformly on U and each U_{g_j} is mutually disjoint. Now we assume φ is nonconstant. Then $\varphi(U)$ is an open subset of $F(G)$ and this is a contradiction because (g_j) converges to φ and each U_{g_j} is mutually disjoint. So φ is constant. Now we assume the value ζ is in $F(G)$. But this is also a contradiction because for each large j component U_{g_j} is included in the component of $F(G)$ containing ζ.

Now we show a sufficient condition so that there is no wandering daomain.

Theorem 2.2.4. *Let G be a rational semigroup and U a wandering domain. Also let φ be a constant limit function of G on U taking its value ζ in $J(G)$. If there is an element of G such that the degree is at least two, then the value ζ is in $P(G)$.*

Corollary 2.2.5. *If G is a hyperbolic rational semigroup containing an element of degree at least two, then there is no wandering domain of $F(G)$.*

Proof of Theorem We assume that there is an element of G such that the degree is at least two. We will show that the value ζ is in $P(G)$. We can assume that $P(G)$ contains at least three points. Assuming that ζ is not in $P(G)$, there is a simply connected neighborhood V of ζ disjoint from $P(G)$. Then for every $g\in G$, we can take all branches of g^{-1} that are well defined on V. We denote by $\mathscr{A}$ the family of meromorphic functions on V such that each element of $\mathscr{A}$ is a branch of the inverse of an element of G. Then $\mathscr{A}$ is a normal family on V. Let (g_j) be a sequence with $g_j|_U\rightarrow\zeta$ compact uniformly and $g_j(U) \subset V$ for large j. Now we take a curve γ in U containing at least two points. For large j, we take a branch h_j of g_j^{-1} on V such that it maps $g_j(\gamma)$ to γ. Now $(g_j(\gamma))$ converges to ζ and so for any neighborhood W of ζ there is a number j such that $h_j(W)$ contains γ. But this is a contradiction because (h_j) is equicontinous.

Similarly we can show the following result.

Theorem 2.2.6. *In the same situation as Theorem 1.2.3, assume that every element of G is of degree one. For every point $x \in \bar{\mathbf{C}}$, we denote the closure of G orbit of x by $A(x)$. Then for all $x \in \bar{\mathbf{C}}$ but at most two points of G-fixed points, ζ belongs to $A(x)$.*

Corollary 2.2.7. *If every element of G is of degree one and there is a point $x \in \bar{\mathbf{C}}$ such that $A(x)$ contains at least two points and is included in $F(G)$, then there is no wandering domain of $F(G)$.*

Next we consider limit functions of a hyperbolic rational semigroup on the Fatou set.

Theorem 2.2.8. *Let G be a finitely generated hyperbolic rational semigroup which contains an element of degree at least two and assume that each Möbius transformation in G is neither the identity nor an elliptic element. Then for every compact subset K of $F(G)$, the G-orbit of K can accumulate only to $P(G)$ and every limit function of G on $F(G)$ is a localy constant function that takes its value in $P(G)$.*

Proof. We denote by A the union of all components each of which has a non-empty intersection with $P(G)$. Let U be a component of $F(G)$. By Corollary 2.2.5. there are only finitely many elements in $\{U_g\}_{g\in G}$. Let h be an element of G such that the degree is at least two. Let V be a component of $F(G)$ and suppose $h(V) \subset V$. Then the component of $F(\langle h\rangle)$ that contains V is an attracting basin of $\langle h\rangle$ and contains a critical point of h because G is hyperbolic. So V has a non-empty intersection with $P(G)$. We fix a system of generators of G. It follows that for large positive integer m, if $g \in G$ is a product of m generators of G, then $U_g \subset A$. And so we have only to consider the dynamics of G on A. We take the hyperbolic metric in each component of A. For large positive integer m, every element of G which is a product of m generators of G is a contraction map from A to A and the contraction rate is bounded by a constant strictly less than one in each fixed compact subset of A. Now the statement of the theorem follows immediately.

Proposition 2.2.9. *Let G be a hyperbolic rational semigroup, U a stable domain of G. We set*

$$H = G_U,\ \mathcal{A} \overset{\text{def}}{=} \{\zeta \in U \mid \exists\, \varphi \in \mathcal{L}_H(U),\ \varphi \equiv \zeta\}.$$

If $\mathscr{A}$ has infinitely many points, then $\mathscr{A}$ is a perfect set.

Proof. Because U is a stable domain, by definition, there is an element g of H with the degree at least two. If we denote by V the connected component of $F(g)$ containing U, there is a critical point $x \in V$ of g and for large enough n, the point $g^n(x)$ belongs to U. So $P(G) \cap U \neq \emptyset$. Assume that $\mathscr{A} \cap \partial U \neq \emptyset$. Then $P(G) \cap \partial U \neq \emptyset$ and this is a contradiction because G is hyperbolic. So $\mathscr{A} \cap \partial U = \emptyset$ and $\mathscr{A}$ has an accumulation point in U. By Proposition 2.1.7. the statement follows.

2.3. Continuity of Julia sets.

Definition 2.3.1. Let E be a metric space. We denote by $\mathrm{Comp}^*(E)$ the set of non-empty compact subsets of E. For every $A, B \in \mathrm{Comp}^*(E)$ we set

$$\partial(A,B) = \sup\{d(x,B) \mid x \in A\}$$

and

$$d_H(A,B) = \max\{\partial(A,B), \partial(B,A)\}.$$

It is well known that d_H is a distance on $\mathrm{Comp}^*(E)$. We call it the Hausdorff metric.

Next we consider if a Julia set depends continuously on the generators. For the case of iterations of rational functions, see [D], [MSS] and [Mc].

Definition 2.3.2. Let M be a complex manifold. Suppose the map

$$(z,a) \in \bar{\mathbf{C}} \times M \mapsto f_{j,a}(z) \in \bar{\mathbf{C}}$$

is holomorphic for each $j = 1, \ldots, n$. We set $G_a = \langle f_{1,a}, \ldots, f_{n,a} \rangle$. Then we say that $\{G_a\}_{a \in M}$ is a holomorphic family of rational semigroups.

Remark. If a map $F : \bar{\mathbf{C}} \times M \to \bar{\mathbf{C}}$ is holomorphic, then for each $a \in M$ the map $F(\ , a)$ is a rational map and $\deg(F(\ , a))$ is a constant function on M when M is connected. For, if two maps f, g from S^2 to S^2 are continuous and homotopic, then $\deg(f) = \deg(g)$. Holomorphic families of usual iteration of rational functions have been studied in [MSS]. It is well known that the set of J-stable parameters is open and dense in the parameter space ([MSS], [Mc]).

Definition 2.3.3. Let G be a rational semigroup. We say that a compact subset K of $F(G)$ is a confinement set of G if for every $z \in F(G)$, for all but finitely many elements g of G the point $g(z)$ is included in K.

Theorem 2.3.4. *Let $\{G_a\}_{a \in M}$ be a holomorphic family of rational semigroups where $G_a = \langle f_{1,a}, \ldots, f_{n,a} \rangle$. We assume that for a point $b \in M$ there is a confinement set K of G_b. Then the map*

$$a \mapsto J(G_a) \in \mathrm{Comp}^*(\bar{\mathbf{C}})$$

is cotinuous at the point $a=b$ with respect to the Hausdorff metric.

Proof. By Section 1. Lemma 1.1.5.6, for any $\varepsilon<0$ there is a finite set

$$X_b=\{x_{1,b}, \ldots, x_{l,b}\} \subset J(G_b)$$

of repelling fixed points of G_b such that

$$\partial(J(G_b), X_b)) \leq \varepsilon/2.$$

By the implicit function theorem, there is a neighgborhood W of b in M such that for every $a \in W$ and for every $j=1, \ldots, l$ there is a repelling fixed point $x_{j,a}$ of G_a such that

$$d(x_{j,b}, x_{j,a}) \leq \varepsilon/2.$$

For each $a \in W$ we set $X_a=\{x_{1,a}, \ldots, x_{l,a}\}$. Then

$$\partial(X_b, J(G_a)) \leq \partial(X_b, X_a) \in \varepsilon/2.$$

So

$$\partial(J(G_b), J(G_a)) \leq \partial(J(G_b), X_b) + \partial(X_b, J(G_a)) \leq \varepsilon.$$

Next, for every $a \in M$ we fix the generator system $\{f_{j,a}\}$ of G_a. We denote by A the union of all components of $F(G_b)$ that have a non empty intersection with K and we take the hyperbolic metric in each coponent of A. Let α be a positive number and K_2 the compact 2α neighborhood of K in A and K_1 be the compact α neighborhood of K in A. Then if we take the neighborhood W of b smaller, there is an integer m such that for every $a \in W$ and for every integer t satisfying $m \leq t \leq 2m$ every element $g \in G_a$ of a product of t generators of G_a satisfies

$$g(K_2) \subset K_1$$

So for every $a \in W$ and for every integer t satisfying $m \leq t$ every element $g \in G_a$ of a product of t generatosrs of G_a satisfies the above. Now we take the ε neighborhood O of $J(G_b)$ with respect to the chordal metric and we denote by L the set $\bar{\mathbf{C}} \backslash O$. And if we take W smaller again there is an integer u such that for every $a \in W$ every element $g \in G_a$ of a product of u generators of G_a satisfies that $g(L) \subset K_2$ and so L is included in $F(G_a)$. So

$$\partial(J(G_a), J(G_b)) \leq \varepsilon.$$

Hence $a \mapsto J(G_a)$ is continuous at the point b with respect to the Hausdorff metric.

2.4. Strucural stability of hyperbolic rational semigroups.

Theorem 2.4.1. *Let $\{G_a\}_{a \in M}$ be a holomorphic family of rational semigroups where $G_a=\langle f_{1,a}, \ldots, f_{n,a}\rangle$. Then*

1. Let b be a point of M. Assume that G_b is hyperbolic. And also assume that $\deg(f_{1,b})$ *is at least two and each Möbus transformation in G_b is neither the identity nor an elliptic element. Then there is an open neighborhood W of b such that for every $a \in W$ the rational semigroup G_a is hyperbolic and the map $a \mapsto J(G_a)$ is continuous with respect* to the Hausdorff metric.

2. Under the same assumption as 1, if the sets $(f_{j,b}^{-1}(J(G_b)),)_j$ are mutually disjoint, then there is an open neighborhood V of b and a continuous map i : $\bar{\mathbf{C}} \times V \to \bar{\mathbf{C}}$ *such that for every $z \in \bar{\mathbf{C}}$* the map $a \mapsto i(z, a)$ *is holomorphic*, and for every $z \mapsto i(z, a)$ *is a quasiconformal homeomorphism of $\bar{\mathbf{C}}$ mapping $J(G_b)$ onto $J(G_a)$.*

Proof of 1. For every $a \in M$ we fix the generator system $\{f_{j,a}\}$ of G_a. We denote by A the union of all components of $F(G_b)$ that have a non empty intersection with $K = P(G_b)$ and we take the hyperbolic metric in each component of A. Let α be a positive number and K_2 the compact 2α neighborhood of K in A and K_1 the compact α neighborhood of K in A. Then if we take a amall neighborhood W of b there is an interger m such that for every $a \in W$ and for every integer t satisfying $m \leq t \leq 2m$ every element $g \in G_a$ of a product of t generators of G_a satisfies

$$g(K_2) \subset K_1.$$

So for every $a \in W$ and for every integer t satisfying $m \leq t$ every element $g \in G_a$ of a product of t generators of G_a satisfies the above. Now let Q_a denote the union of all critical points of all generators of G_a. Let L be a relatively compact neighborhood of Q_b in $F(G_b)$. If we take W smaller, for every $a \in W$ the set Q_a is in L. And we can assume that there is a positive integer u such that for every $a \in W$ every element $g \in G_a$ of word length u satisfies $g(L) \subset K_2$. So for every $a \in W$ the set $P(G_a)$ is included in $F(G_a)$ and so G_a is hyperbolic. And from this fact combind with theorems 2.2.8, 2.3.4, it folows that the map $a \mapsto J(G_a)$ is continuous in W.

Proof of 2. We take a neighborhood W *of b* as above. We can assume that W ia a polydisc and for each $a \in W$ the sets $(f_{j,a}^{-1}(J(G_a)))_j$ are mutually disjoint. Let c be a point of W and x a repelling fixed point of $g_c = f_{j_1,c} \circ \cdots \circ f_{j_m,c}$ where the number m is the word length of g_c. Then there is an analytic function $x(a)$ in a small neithborhood U of c in W such that $x(a)$ is a repelling fixed point of g_a and $x(c) = x$. If a_0 is a point of $\partial U \cap W$, then $x(a_0)$ is a repelling fixed point of g_{a_0} because G_{a_0} is hyperbolic. So we can take an analytic continuation of $x(a)$ throughout W such that $x(a)$ is a repelling fixed point of g_a. Next if h_a is an element of G_a such that the word length is at most m and $x(a)$ is a fixet point of it then h_a is equal to g_a *because* G_a is hyperbolic and the sets $(f_{j,a}^{-1}(J(G_a)))_j$ are mutually disjoint. So by the λ lemma ([MSS], [BR], [ST]) and Lemma 1.1.5.6 the statement follows immediately.

2.5. Self-similarity of Julia sets. When G is genarated by a single rational function f, we know that if all the critical points and in the immediate attractive basin of a fixed point, then the Julia set is a Cantor set. Now we consider the following situation similar to that.

Theorem 2.5.1. *Let $G = \langle f_1, \ldots, f_n \rangle$ be a finitely generated rational semigroup. Assume that G contains an element with the degree at least two and each Möbius trnsformation in G is neither the identity nor an eliptic element. If $P(G)$ is included in a connected component U of $F(G)$, then there are simply connected domains $V_1, \ldots, V_k$ and mappings $h_1, \ldots, h_s$ from $W = \cup_j V_j$ to W such that for each j, i the map h_j is a contraction map from V_i to a domain $V_{j'}$ with respect to the hyperbolic metric with the rate of contraction bounded by a constant strictly less than one throughout V_i and*

$$J(G) \subset W, \ \bigcup_j h_j(J(G)) = J(G).$$

Proof. There is a relativery compact subfomain V of U including $P(G)$. For each positive integer m we denote by G_m the subsemigroup of G generated by all elements $g_1, \ldots, g_l$ of word length m. If we take a number m large enough, then for each $g \in G_m$, g maps the closure of V into V. So the closure of $g^{-1}(\bar{\mathbf{C}} \setminus V)$ is included in $\bar{\mathbf{C}} \setminus V$. Each connected component of $\bar{\mathbf{C}} \setminus \bar{V}$ is simply connected because V is connected. For each component of $\bar{\mathbf{C}} \setminus \bar{V}$ we take all branches of g^{-1} on it. Then each branch is a contraction map on each component of $\bar{\mathbf{C}} \setminus \bar{V}$ with respect to the hyperbolic metric with the rate of contraction bounded by a constant strictly less than one. Now from Lemma 1.1.4.2 and Lemma 1.1.5.1.

$$J(G) = J(G_m) = \bigcup_{j=1}^{l} g_j^{-1}(J(G_m)),$$

so the statement follows.

Remark. In the above proof, if we can take V as a simply connected domain, then the Julia set is a self-similar set in $\bar{\mathbf{C}} \setminus \bar{V}$ with respect to the hyperbolic metric.

By Theorem 2.5.1 and the proof, we can show the following result.

Theorem 2.5.2 *Let $G = \langle f_1, \ldots, f_n \rangle$ be a finitely generated rational semigroup. Assume that* $\deg(f_1)$ *is at least two. If $P(G)$ is included in a connected coponent U of $F(G)$ and the sets $\{f_j^{-1}(J(G))\}_{j=1,\ldots,n}$ are mutually disjoint, then the Julia set $J(G)$ is a Cantot set.*

Example 2.5.3. Let $G_c = \langle z^2 + c, z^2 + ci \rangle$. Then $J(G_c)$ is a Cantor set for sufficiently large positive number c.

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